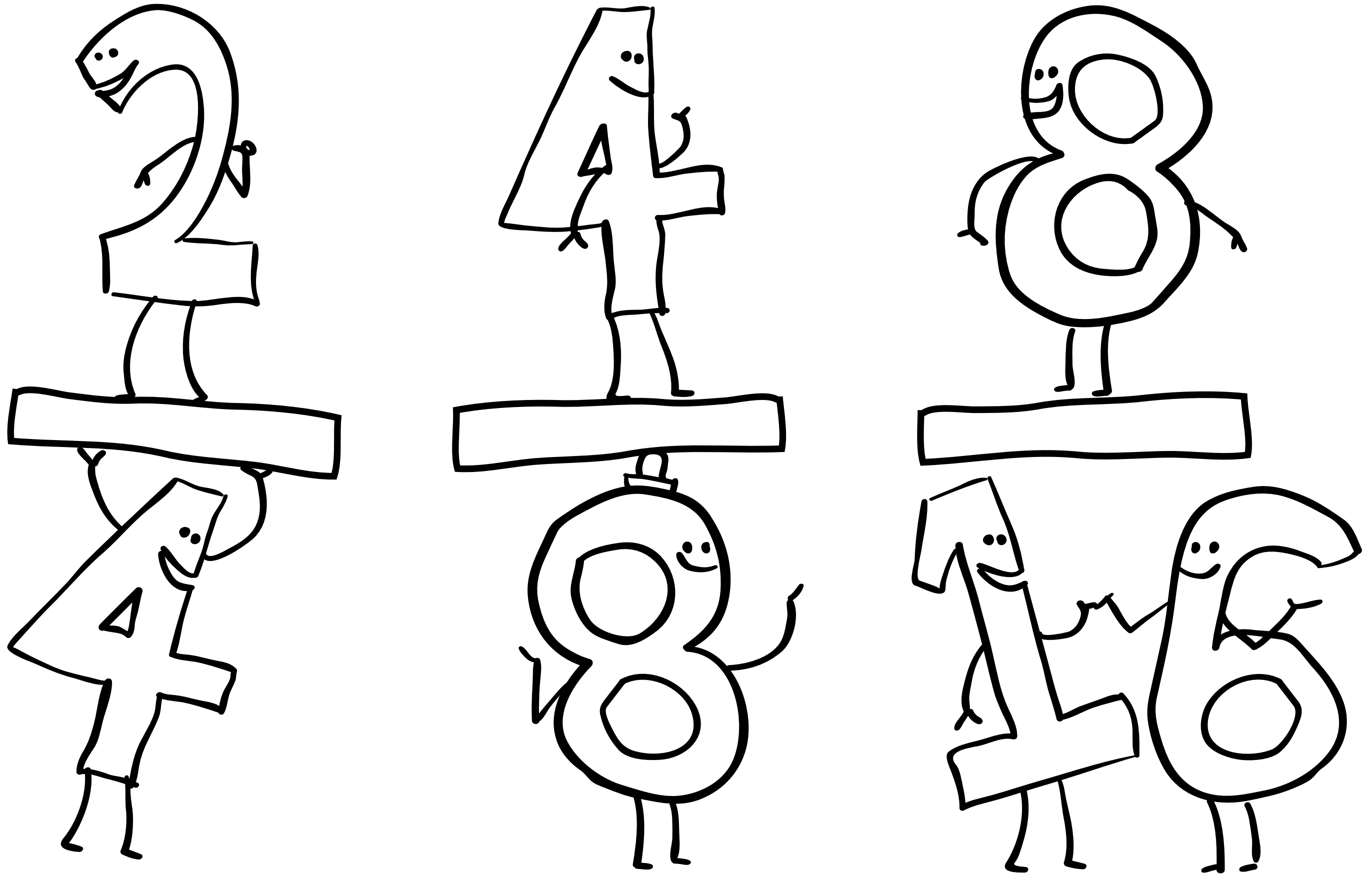
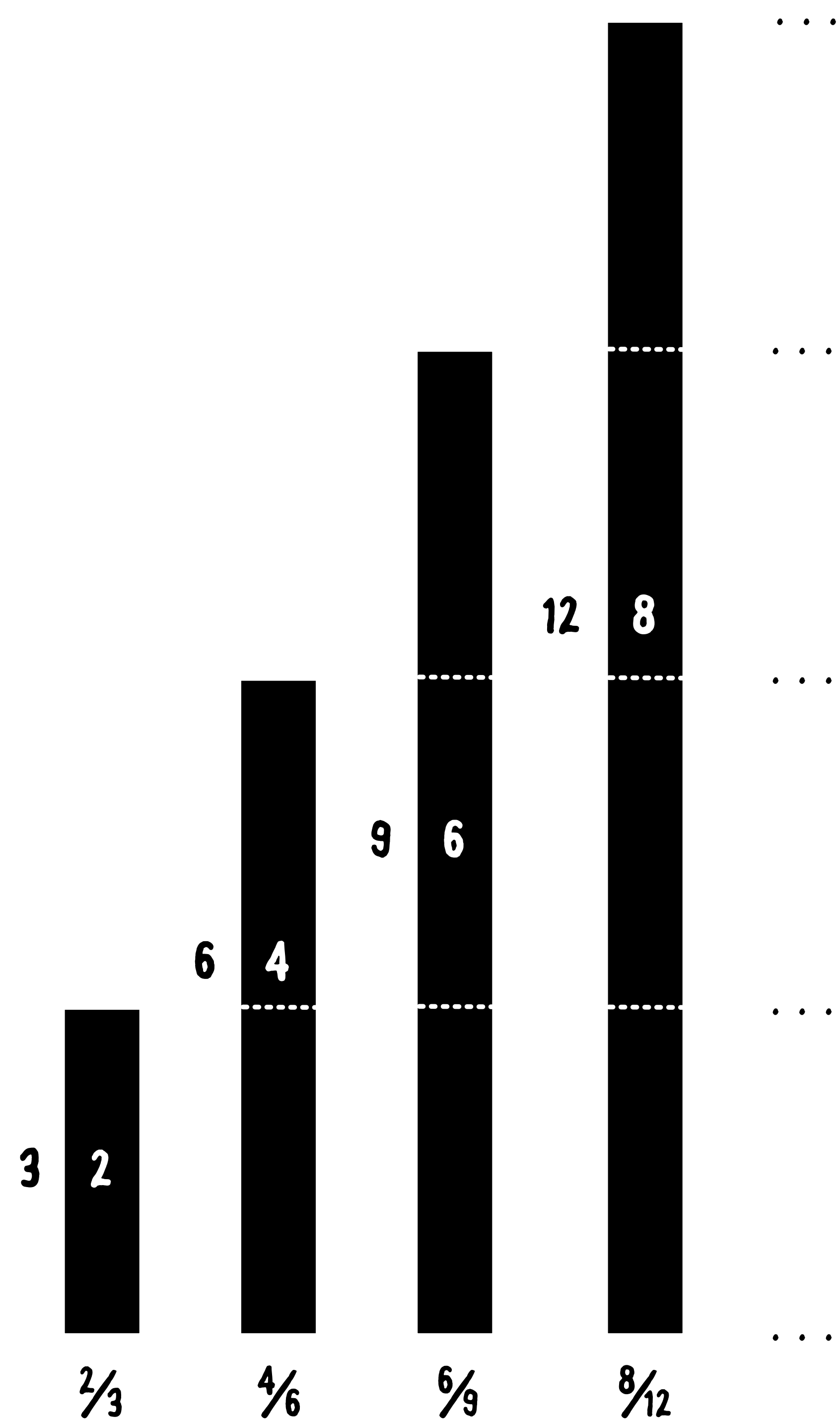


EQUIVALENT FRACTIONS

The tenth scene in a series of articles
on elementary mathematics.

written by Eugene Maier
designed and illustrated by Tyson Smith

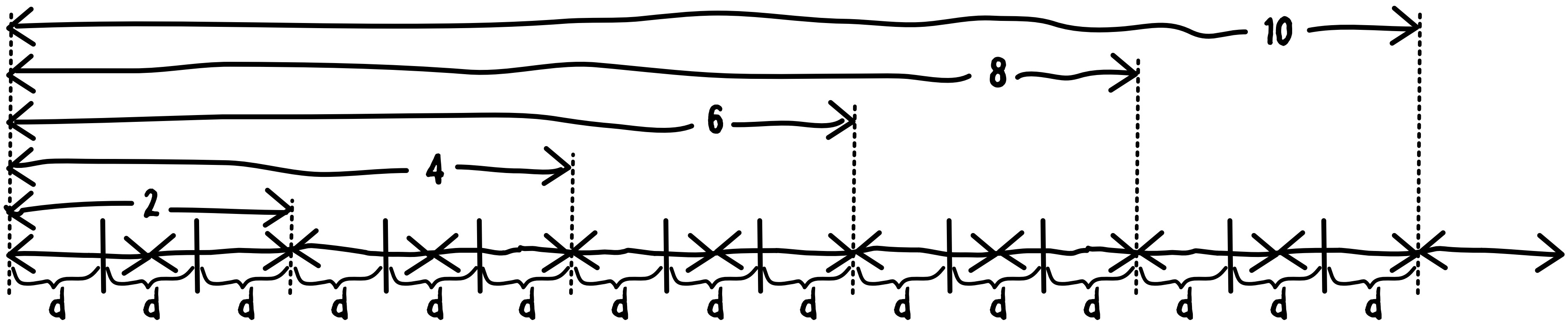
Consider a sequence of rectangles in which the first rectangle has area 2 and height 3, the second rectangle consists of 2 copies of the first rectangle stacked atop one another, the third rectangle consists of 3 copies of the first rectangle stacked atop one another, and so on. Given the manner in which the rectangles have been constructed, they all have the same base. But the base of a rectangle is its area divided by its height. Thus the bases of the rectangles are, successively, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, $\frac{10}{15}$, ... Since the bases are all equal, we have the string of equalities: $\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \ldots$ Notice that any fraction of the form $\frac{2 \times \boldsymbol{k}}{3 \times \boldsymbol{k}}$, where k is a positive integer will be in this string of equalities since $\frac{2 \times \boldsymbol{k}}{3 \times \boldsymbol{k}}$ is the base of a rectangle which consists of k copies of the first rectangle stacked atop one another.



Fractions that have equal values are said to be ***equivalent***. The above set of fractions, that is, the set of all fractions of the form $\frac{2 \times \boldsymbol{k}}{3 \times \boldsymbol{k}}$, is a family of equivalent fractions.

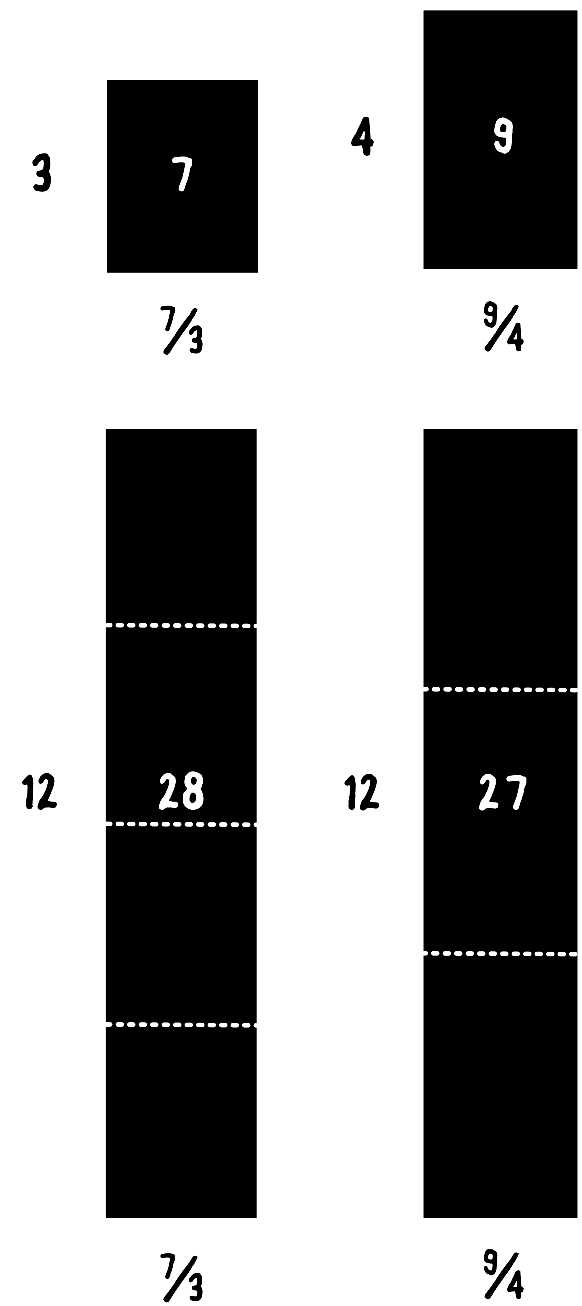
The smallest fraction of those listed above, namely $\frac{2}{3}$, is in ***lowest terms***, that is, there is no positive integer which divides both the numerator 2 and the denominator 3. It is the only fraction among those listed that is in lowest terms.

Gene says: We began this scene by viewing fractions as the bases of rectangles. As we shall see in the next scene, viewing fractions in this way is useful when discussing fraction operations. But other ways of viewing fractions can also be helpful. For example, thinking of a fraction as the length obtained when an interval is divided into equal parts, one sees in the following diagram that the distance ***d*** is the length obtained when an interval of length 2 is divided into 3 equal parts. Thus, $\boldsymbol{d} = \frac{2}{3}$. But ***d*** is also the length obtained when an interval of length 4 is divided into 6 equal parts, so $\boldsymbol{d} = \frac{4}{6}$. Similarly, it is the length obtained when an interval of length 6 is divided into 9 equal parts, an interval of length 8 is divided into 12 equal parts and so on. This gives us another way of viewing the equivalence of the fractions, $\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots$.

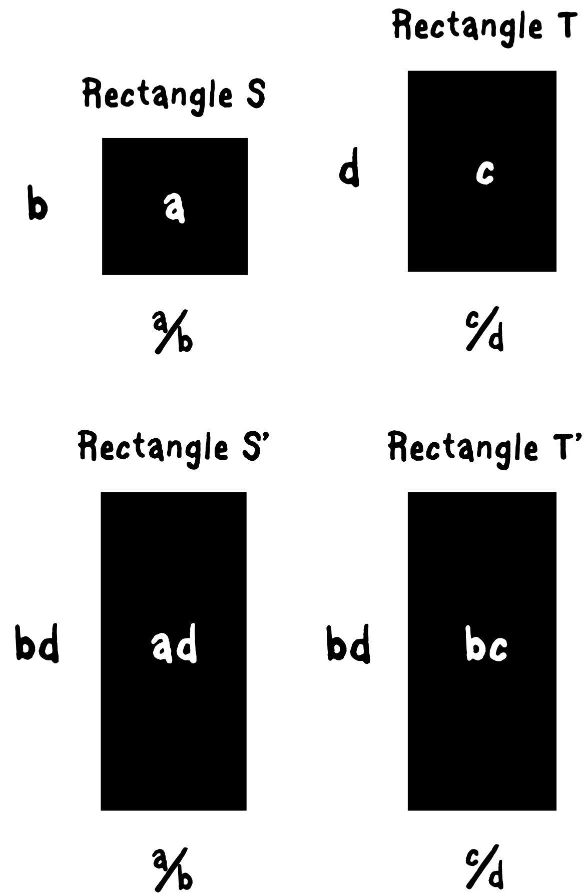


Suppose we want to determine if two fractions, say $\frac{7}{3}$ and $\frac{9}{4}$, are equivalent. We can view $\frac{7}{3}$ as the base of a rectangle of area 7 and height 3 and $\frac{9}{4}$ as the base of a rectangle of area 9 and height 4.

Stacking 4 of the first rectangle atop one another, we get a rectangle of area 28 whose height is 12 and base is $\frac{7}{3}$. Stacking 3 of the latter rectangle atop another, we get a rectangle of area 27 whose height is 12 and base is $\frac{9}{4}$. The two rectangles have the same height, but the area of the second is smaller than the area of the first, Hence its base must be the smaller of the two. Hence, $\frac{7}{3} > \frac{9}{4}$.



The previous example suggests a general method for comparing two positive fractions $\frac{a}{b}$ and $\frac{c}{d}$. Consider the two rectangles shown in the sketch. Rectangle ***S*** has area ***a*** and height ***b*** so its base is $\frac{a}{b}$, and Rectangle ***T*** has area ***c*** and height ***d*** so its base is $\frac{c}{d}$. (The sketches are intended to aid our thinking. No attempt has been made to draw the rectangles to scale.)



Now stack ***d*** copies of Rectangle ***S*** atop one another to obtain Rectangle ***S***' which has the same base as Rectangle ***S*** but has height and area which are ***d*** times that of Rectangle ***S***. Then stack ***b*** copies of Rectangle ***T*** atop one another to obtain Rectangle ***T***' which has the same base as Rectangle ***T*** but has height and area which are ***b*** times that of Rectangle ***T***.

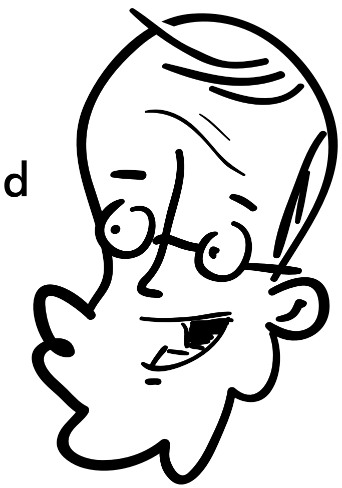
Rectangles ***S***' and ***T***' have equal heights. Thus their bases will be equal if, and only if, their areas are equal, that is, $\frac{a}{b} = \frac{c}{d}$ if, and only, if $ad = bc$. Also, the base of ***S***' will be larger than the base of ***T***' if, and only if, the area of ***S***' is larger than the area of ***T***', that is, $\frac{a}{b} > \frac{c}{d}$, if and only if $ad > bc$. Similarly, the base of ***S***' will be smaller than the base of ***T***' if, and only if, the area of ***S***' is smaller than the area of ***T***', that is, $\frac{a}{b} < \frac{c}{d}$, if and only if $ad < bc$.

To summarize: (1) $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad = bc$
(2) $\frac{a}{b} > \frac{c}{d}$ if, and only if, $ad > bc$
(3) $\frac{a}{b} < \frac{c}{d}$ if, and only if, $ad < bc$

The above relationships can be used to determine the relative size of two fractions. Consider, for example, $\frac{5}{7}$ and $\frac{8}{11}$. Since $5 \times 11 = 55 < 56 = 7 \times 8$, $\frac{5}{7} < \frac{8}{11}$.

One consequence of statement (1) above is that dividing the numerator and denominator of a fraction by the same number doesn't change the value of the fraction: $\frac{ak}{bk} = \frac{a}{b}$ since $ak \times b = bk \times a$. For example, $\frac{30}{36} = \frac{3 \times 10}{3 \times 12} = \frac{10}{12} = \frac{2 \times 5}{2 \times 6} = \frac{5}{6}$, the latter fraction being in lowest terms since 5 and 6 have no common factors. The reduction of $\frac{30}{36}$ to lowest terms could have been accomplished in a single step by dividing the numerator and denominator by 6, the largest integer which divides 30 and 36, referred to variously as the ***greatest common divisor*** (***gcd***) of 30 and 36 or the ***highest common factor*** (***hcf***) of 30 and 36. Dividing the numerator and denominator of a fraction by their ***gcd*** always reduces that fraction to an equivalent fraction which is in lowest terms.

Gene says: Students are sometimes instructed that fractions ought always be reduced to lowest terms. As with turning improper fractions into proper fractions, I see no particular urgency to do this. Reducing a fraction to lowest terms may simplify subsequent calculations involving that fraction. On the other hand, in many applications, fractions are approximated by decimals and finding a decimal approximation of a fraction is just as readily done on a calculator for fractions that aren't reduced as for those that are.

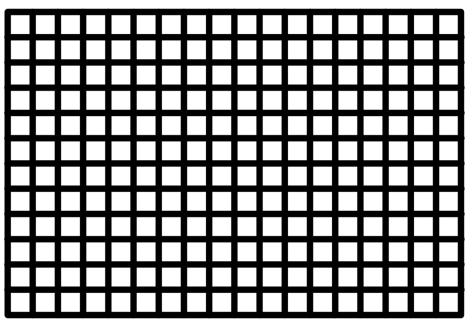


Also reducing fractions is not as simple as it's made out to be in most fifth grade textbooks. Students are generally instructed to find common divisors of the numerator and denominator of a fraction by inspection, but doing so may not be straightforward. What, for instance, are the common divisors of 874 and 1406? There is a process called the ***Euclidean algorithm***, for finding the greatest common divisor of two numbers, but it generally is not part of the school curriculum. Click here for a discussion of the ***Euclidean algorithm***.

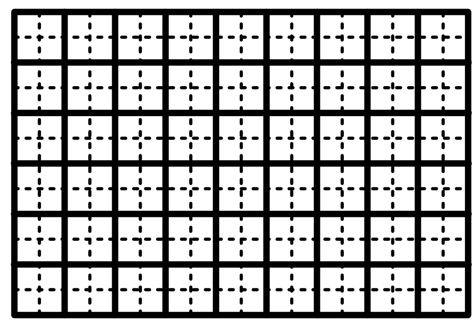
The Euclidean Algorithm:

Finding the greatest common factor of two numbers is analogous to the geometric problem of finding the largest square of integral dimension that will tile a rectangle. Consider, for example, a 12 x 18 rectangle. This rectangle can be tiled by 1 x 1, 2 x 2, 3 x 3, and 6 x 6 squares. Since the dimension, of a tiling square must divide both dimensions of the rectangle, 1, 2, 3, 4 and 6 are the common divisors of 12 and 18. The largest of these is 6.

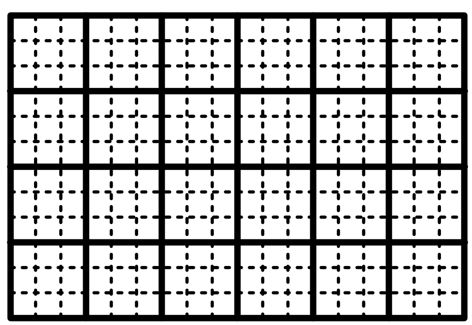
A 12 x 18 rectangle tiled by:



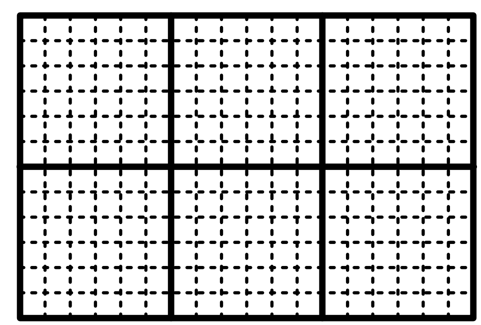
1 x 1 squares



2 x 2 squares



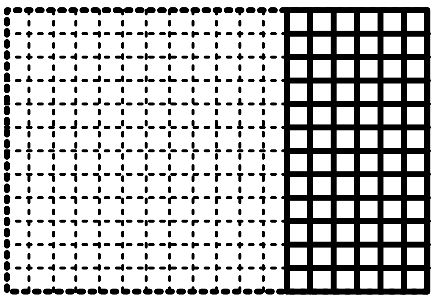
3 x 3 squares



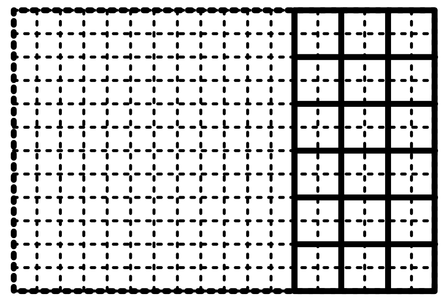
6 x 6 squares

Finding the squares which tile a rectangle is simplified by the following observation: If a square is cut off the end of a rectangle, the remaining rectangle is tiled by exactly the same set of squares as the original rectangle. For example, cutting a square off the end of a 12 x 18 rectangle produces a 6 x 12 rectangles. This rectangle is also tiled by a 1 x 1, 2 x 2, 3 x 3, and 6 x 6 squares.

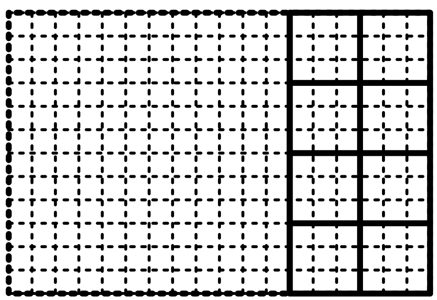
A 12 x 18 rectangle, with a square removed, tiled by:



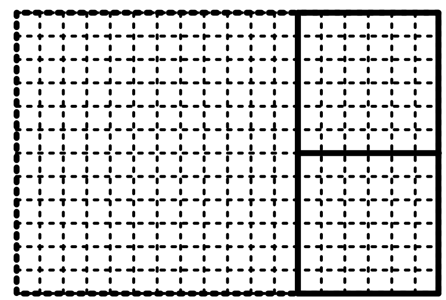
1 x 1 squares



2 x 2 squares

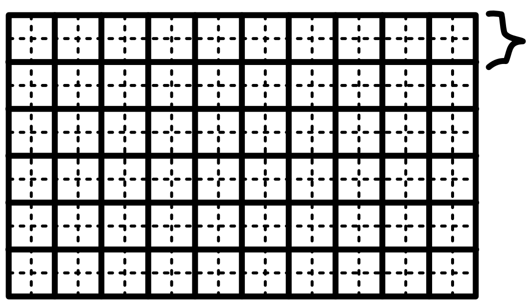


3 x 3 squares

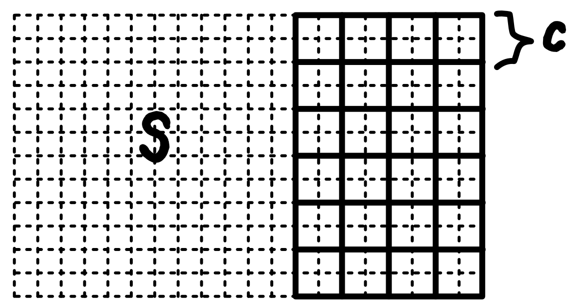


6 x 6 squares

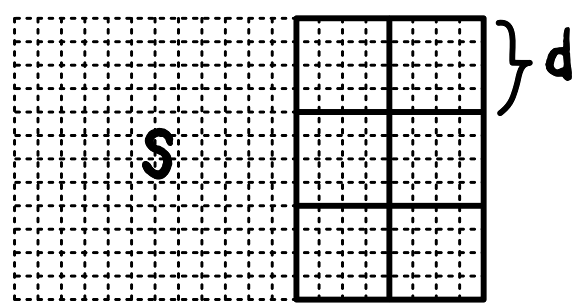
In general, if rectangle ***B*** is obtained from rectangle ***A*** by cutting a square ***S*** off the end of ***A***, then every square which tiles ***A*** also tiles ***B*** and conversely, every square which tiles ***B*** also tiles ***A***. For suppose a ***c*** x ***c*** square tiles rectangle ***A*** , then ***c*** evenly divides the side of square ***S*** and hence tiles ***S***. Thus when ***S*** is cut off ***A***, the cut goes along the edges of the tiling squares so the remaining portion of ***A***, which is rectangle ***B***, remains tiled. Conversely suppose a ***d*** x ***d*** square tiles rectangle ***B***. Then ***d*** evenly divides the side of ***B*** which is the side of square ***S***. Hence the ***d*** x ***d*** square tiles square ***S*** and since it tiles both ***B*** and ***S***, it tiles ***A***.



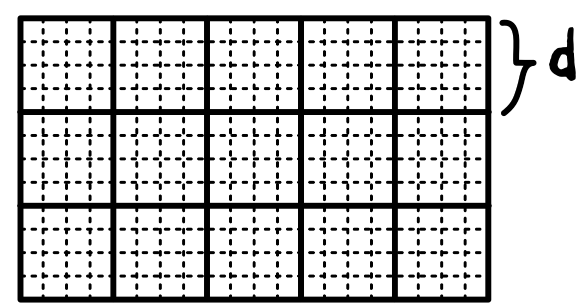
If a square tiles rectangle A,



it also tiles rectangle B.



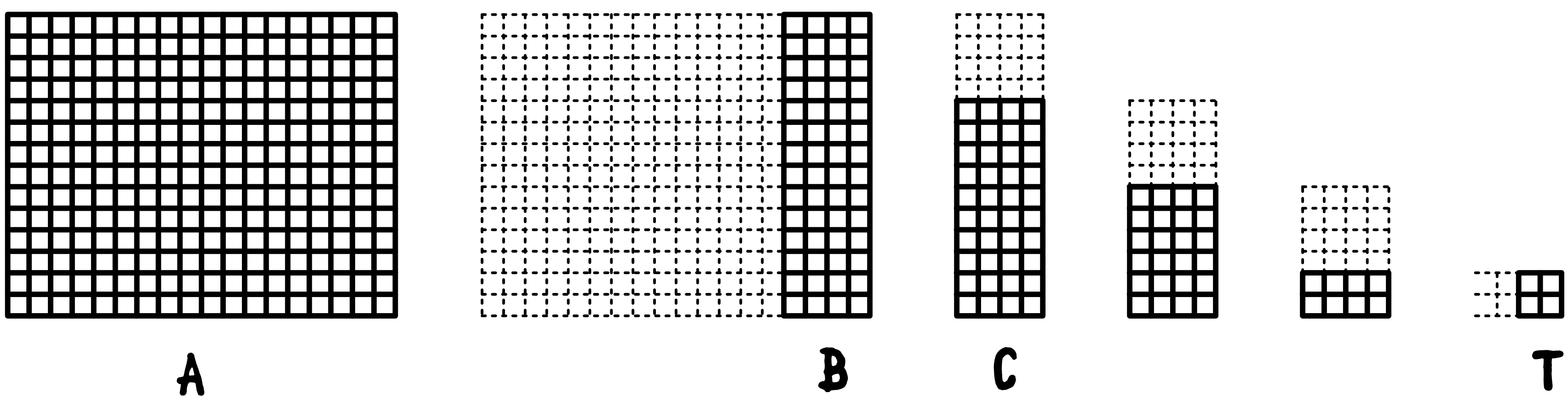
If a square tiles rectangle B,



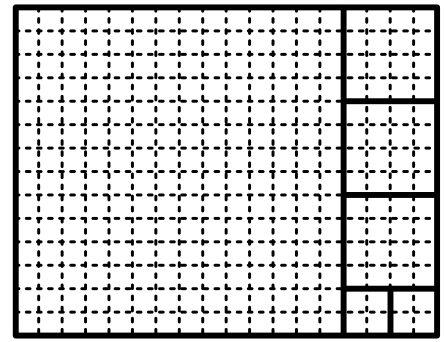
it also tiles rectangle A.

Now if one starts with a rectangle ***A***, cuts off a square from the end of ***A*** to get a rectangle ***B***, then cuts off a square from the end of ***B*** to get a rectangle ***C***, and continues this process, one will ultimately obtain a rectangle which is a square, call it ***T***. (It may be that the process continues until all that remains is a 1 x 1 square.) Now by the above results, all the rectangles obtained, including ***T***, are tiled by the same set of squares. Thus the largest square which tiles ***A*** is the same as the largest square which tiles ***T***, which is ***T*** itself. Thus the ***gcd*** of the dimensions of ***A*** is the dimension of the square ***T***.

As an example, starting with a 14 by 18 rectangle and carrying out the above process, we arrive at a 2 x 2 square as shown below. Hence the ***gcd*** of 14 and 18 is 2.

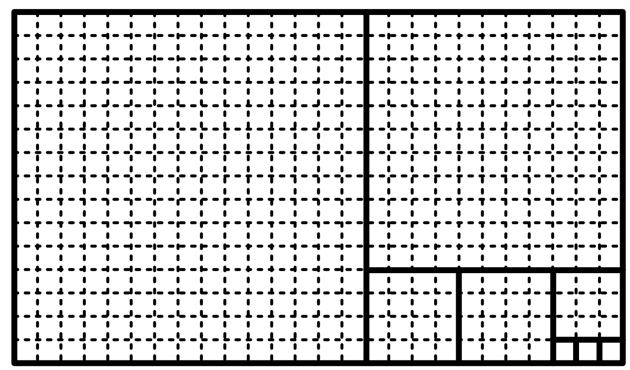


Instead of showing all the rectangles individually, as on the previous page, we can simply show where the cuts occur.



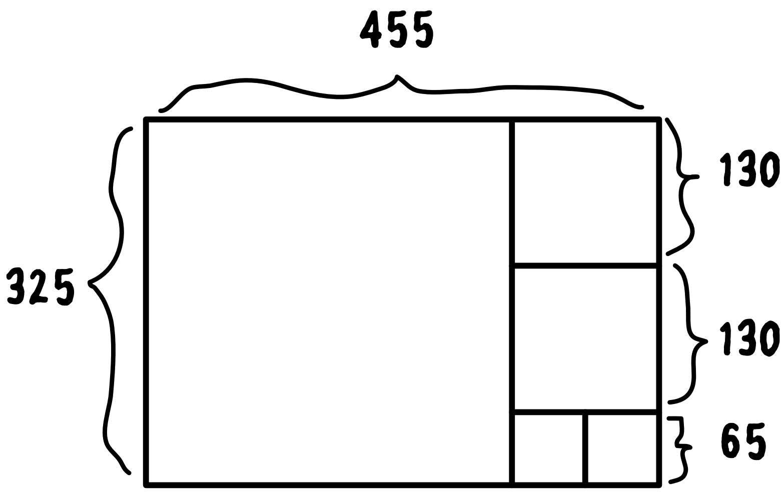
the gcd of 14 and 18 is 2

Here is another example, showing the ***gcd*** of 15 and 26 is 1. Two numbers, such as 15 and 26, whose ***gcd*** is 1 are said to be ***relatively prime***.

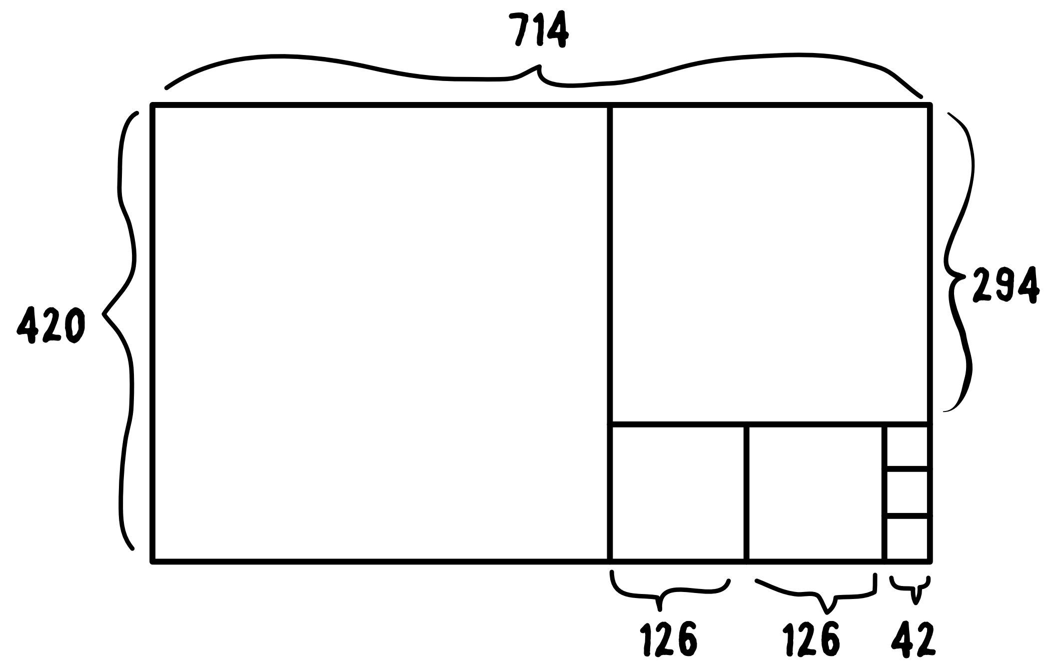


the gcd of 15 and 26 is 1

For larger numbers, we can omit the grid squares and write dimensions alongside the various rectangles, as shown in the following sketches.



the gcd of 325 and 455 is 65



the gcd of 420 and 714 is 42

Rather than drawing the sequence of rectangles, one can simply record their dimensions. Notice that if the larger dimension of a rectangle is ***t*** and the smaller dimension is ***s***, then the dimensions of the next rectangle in the sequence are ***s*** and ***t*** – ***s***. Which of these is the larger depends on the values of ***t*** and ***s***.

Below is a table showing the dimensions of the rectangles in a sequence of rectangles determining the ***gcd*** of 1406 and 874:

Dimensions		
1406	874	1406 – 874 = 532
874	532	874 – 532 = 342
532	342	532 – 342 = 190
342	190	342 – 190 = 152
190	152	190 – 152 = 38
152	38	152 – 38 = 114
114	38	114 – 38 = 76
76	38	76 – 38 = 38
38	38	

The process ends in a 38 x 38 square. Hence the ***gcd*** of 1406 and 874 is 38.

END of SCENE 10: EQUIVALENT FRACTIONS

For comments and questions
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