visual mathematics

Linda Cooper Foreman and Albert B. Bennett, Jr.

COURSE I

Visual Mathematics, Course I
by Linda Cooper Foreman and Albert B. Bennett Jr.

Visual Mathematics, Course I is followed by:
Visual Mathematics, Course II
Math Alive! Visual Mathematics, Course III

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THIS BOOK IS DEDICATED TO our dear friend, mentor, and colleague, Michael J. Arcidiacono. Your ability to arouse in students of all ages a sense of wonder and fascination with mathematics, your gentle spirit, and your unfaltering belief in children and their teachers will forever influence our lives and work.

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Oregon Trail Elementary School – Frazer Boergadine

Eugene; Roosevelt Middle School – Audrey Manning

Gresham; Clear Creek Middle School – Nicole Miller

Hood River; Hood River Middle School – Trudy Mitchell

Oakridge; Oakridge Junior–Senior High School – Fred Board

(Continues on next page)

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Wallingford
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Strath Haven High School – Regina Keller

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Shelburne; Shelburne Middle School – Ann Rainey
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Introduction

Visual Mathematics is an outgrowth of *Math and the Mind's Eye*, a curriculum development project that began in 1984 with support from the National Science Foundation. We began work in 1987 on the first version of *Visual Mathematics*, which was a collection of suggestions for implementing the *Math and the Mind's Eye* philosophy and teaching activities. After extensive field testing in our own and other teachers' classrooms, our vision for *Visual Mathematics* has expanded to a series of four one-year courses. With support from the National Science Foundation and The Math Learning Center, these courses will be part of a new seamless K-8 curriculum that will include *Visual Mathematics* for grades 5-8*, *Opening Eyes to Mathematics* for grades 3 and 4, and a K-2 program based on *Box It or Bag It Mathematics*. The entire K-8 curriculum will be grounded in the visual thinking philosophy and spirit of *Math and the Mind's Eye*. The first *Visual Mathematics* course and *Opening Eyes to Mathematics* are available now (1995). Courses II-IV of *Visual Mathematics* will be available in subsequent years, and the primary grades program will begin development in the coming year.

Throughout the 45 lessons that comprise *Visual Mathematics, Course I*, there are many suggestions for implementing the curriculum. In addition, the teachers' resource book, *Starting Points for Implementing Visual Mathematics*, provides an overview of the philosophy and goals of *Visual Mathematics* courses, together with extensive suggestions for: organizing materials; planning, pacing, and assessing lesson activities; working with parents and the community outside your classroom; finding support as you seek changes in your teaching practice; and creating a classroom climate that invites risk-taking and nourishes the mathematician within each student. The ideas in *Starting Points* are based on our own classroom experiences and comments we have received from many other teachers field testing *Visual Mathematics*.

Teaching *Visual Mathematics* ourselves has affirmed our beliefs in the potential within each student, enriched our views of mathematics and the art of teaching mathematics, and reinforced our commitment to support teachers in their efforts to change the way mathematics is learned and taught. It is our hope that teaching *Visual Mathematics* will be equally fulfilling for you.

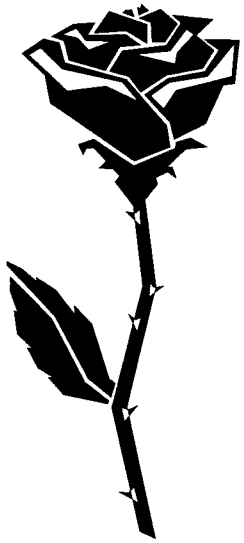
* *Visual Mathematics, Course I*, is designed to follow and extend ideas from *Opening Eyes to Mathematics*, Volume 3, but it is also appropriate as students' first experience with the visual thinking philosophy. Although most students in *Visual Mathematics, Course I*, will be in grades 5 or 6, many of the activities provide meaningful first experiences in visual thinking for older students as well.

Introduction to Visual Mathematics

Lesson 1

THE BIG IDEA

Sensory experiences help students develop mental images that aid in understanding and recalling relationships and information. Hence, math explorations that involve the senses and provide students with models and images for concepts enhance the learning of math. Using hands-on activities to educe and nurture each student's inner mathematician is central to all *Visual Mathematics* lessons.



CONNECTOR

OVERVIEW

Students recognize and discuss ways in which they already use mental imagery to help recall facts, ideas, or relationships in aspects of life outside of math.

MATERIALS FOR TEACHER ACTIVITY

- ✔ *Mathematics and Visual Thinking* and *Mathematical Swindling*, teacher background readings, see *Starting Points for Implementing Visual Mathematics.*
- ✔ *Math and the Mind's Eye* and *A Change of Course* (two optional videos), teacher background information, available from The Math Learning Center.

FOCUS

OVERVIEW

The philosophy and spirit of *Visual Mathematics* are introduced by discussing the goals for the course and the process of learning math.

MATERIALS FOR TEACHER ACTIVITY

- ✔ *Visual Mathematics* Philosophy (see *Blackline Masters*), 1 copy per group.
- ✔ Philosophy web statements (see *Blackline Masters*) and string (see Comment 1), 1 set per class.
- ✔ *Visual Mathematics* Goals (optional, see *Blackline Masters*), 1 transparency or 1 poster.
- ✔ Student journals (optional), see *Starting Points.*
- ✔ Scissors, 1 pair per student.
- ✔ An assortment of colored markers, glue or tape, colored paper, old magazines, for each group.
- ✔ Paper "clouds" (see Comment 1), 4-6 per group.

FOLLOW-UP

OVERVIEW

Students write a mathography, describing their experiences in and feelings about math.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 1.1, 1 copy per student.

LESSON IDEAS

TIMING

It is helpful to use the level of student engagement as a guide for deciding whether to pursue ideas and/or questions that emerge during *Visual Mathematics* activities.

Each *Visual Mathematics* lesson is intended to take from 2-5 hours of class time. This first lesson can be completed in about 2 hours depending on the extent of discussions.

It is important to introduce students to aspects of the learning process and to goals for the course so that, from the outset of class, students can use these as a basis for monitoring the development of their knowledge and disposition about math. However, discussion needs to be brief since experiences with the *Visual Mathematics* philosophy are what give it real meaning.

FOLLOW-UP

To promote higher quality work on Mathographies read excerpts from your own Mathography or from ones saved from previous classes (be sure to keep these anonymous). Some points to illustrate might include: depth of feelings (both positive and negative); specific examples of "important mathematical moments;" thoughtful statements about a student's goals for math class; etc.

ASSESSMENT

Completed Mathographies (*see Follow-up Student Activity 1.1*) provide important baseline information about students' mathematical dispositions. It is important to keep these Mathographies, either as the opening entry in students' journals or as a Growth-folio piece (see *Starting Points*), for making comparisons later regarding the development of their feelings and attitudes about math.

PARENT INVOLVEMENT

The background readings and the *Math and the Mind's Eye* video listed in the Connector can be useful to share with parents and/or administrators whose experiences in mathematics have been very different from activities in this course.

The *Visual Mathematics* Philosophy and *Visual Mathematics* Goals (see *Blackline Masters*) could be used as hand-outs for an open house or other parent meeting.

TEACHER SUPPORT

The video, *A Change of Course*, is available from The Math Learning Center and contains interviews with several *Visual Mathematics* teachers regarding ways they deal with implementation issues such as planning and organizing for instruction and working with parents. These and other issues are also addressed in the teacher's resource book, *Starting Points for Implementing Visual Mathematics*.

SELECTED ANSWERS

Responses will vary.

Connector Teacher Activity

OVERVIEW & PURPOSE

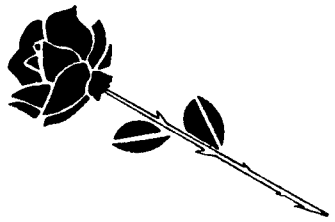
Students recognize and discuss ways in which they already use mental imagery to help them recall facts, ideas, or relationships in aspects of life outside of math. This discussion relates to the major purpose of this course—to engage students in sensory experiences that provide them with meaningful mental images for important mathematical ideas.

MATERIALS

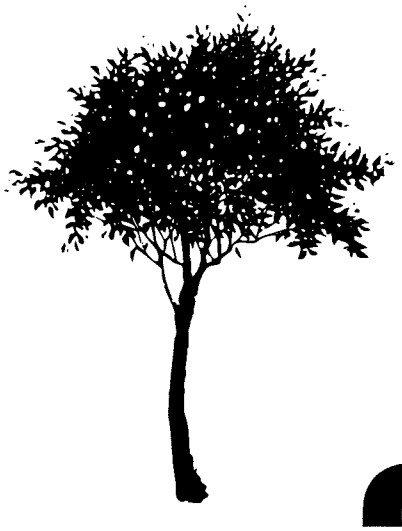
- ✔ *Mathematics and Visual Thinking* and *Mathematical Swindling*, teacher background readings, see *Starting Points for Implementing Visual Mathematics.*
- ✔ *Math and the Mind's Eye* and *A Change of Course* (two optional videos), teacher background information, available from The Math Learning Center.

ACTIONS

1 Point out to students that one of the authors of their math program lives in the city of Portland, Oregon, also known as the City of Roses. Ask them to share what they see in their mind's eye when you say the word "rose."



2 Repeat Action 1 for "tree" and then "thirty-two."



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COMMENTS

1 Invite students to close their eyes if they wish. Allow a few moments for students to think privately and then ask volunteers to describe their mental images. Students generally have vivid images to report—a single long-stemmed rose, a favorite rose garden or bush, a stained-glass window depicting a rose, a favorite aunt named Rose, etc. Some report imagining the fragrance of a rose or the feel of touching it.

2 Students' images for "tree" are usually vivid. For students whose mathematical experiences have been primarily symbolic, images for "thirty-two" are often quite different. They are mostly composed of symbols—the numeral 32, or the numerals for four and eight with a multiplication sign between them, or perhaps a favorite ball player's jersey with the numeral 32 on it. If the quality of people's images of "rose" were of the same nature, their only image would be of the word itself. Thus, conceiving a design for an emblem for the Portland Rose Festival would be a daunting task!

To develop trust and promote sharing, it is important to acknowledge all images and to avoid suggesting one is better or more meaningful than the other. You may get clues, however, about the nature of some students' prior experiences in math. When asked to think of the number 32, students who have had a more hands-on, visual experience in math may report images such as 3 strips of 10 and 2 units or a 4 by 8 array of tiles. Such images carry meaning and can be useful in thinking about mathematical situations. Providing experiences that will enable students to develop a range of such useful images is the basis of this course.

Connector Teacher Activity (cont.)

ACTIONS

3 Ask the students if they know of any examples of sports or other activities in which they or other people visualize or use their mind's eye to aid their performance, understanding, or recall.

4 Ask the students to mentally recall the method they use to tie a shoe and/or which way to turn a jar lid to loosen it. Discuss their methods for recalling this information.

5 Ask volunteers to share examples of how kinesthetic memories help them to remember.

6 Tell the students that Albert Einstein, the famous scientist, said he relied primarily on visual and kinesthetic thinking in his work. In fact, he told that his most important thought was accomplished with visual and kinesthetic images rather than with words and numbers. Point out that the intent of this course is to provide sensory experiences that will enable the students to develop a collection of meaningful mental and kinesthetic images that will help them (like Einstein!) understand, retain, recall, and invent mathematical ideas and methods.

COMMENTS

3 Students may remember watching athletes (swimmers, skiers, runners, weight lifters, gymnasts, etc.) on television or at sporting events as they practice visualization in preparation for competition. These athletes believe that in order to win, they must first visualize themselves successfully completing each stage of competition. For example, before he actually hits the ball, professional golfer Jack Nicklaus always imagines hitting it and, in his mind's eye, "watches" the ball through its flight until it lands.

Prior experiences with objects provide a basis for recalling many facts and relationships. Students may enjoy predicting the following by picturing the objects in their mind's eye: On the American flag, is the top stripe red or white? Which direction does Lincoln's head face on a penny? How many sides are on a wooden pencil? On a traffic light, is the top light red or green? In which hand is the Statue of Liberty's torch? etc.

4 Certain information is easier for some people to recall by combining a mental image with "kinesthetic memories." For example, people often remember which way to turn a jar lid to loosen it by re-creating the action in their mind's eye while twisting their hand in midair.

5 Some possibilities are: opening a combination lock, loosening a screw, shooting a basket, or shifting gears on a bicycle.

6 Students may enjoy investigating and reporting on other famous mathematicians, scientists, artists, or writers who "thought in images." A few examples are provided in the teacher background readings, *Mathematics and Visual Thinking* and *Mathematical Swindling*, which are included in *Starting Points*.

The video *Math and the Mind's Eye*, which is available from The Math Learning Center, provides additional teacher background information regarding the role of visual thinking in mathematics.

Focus Teacher Activity

OVERVIEW & PURPOSE

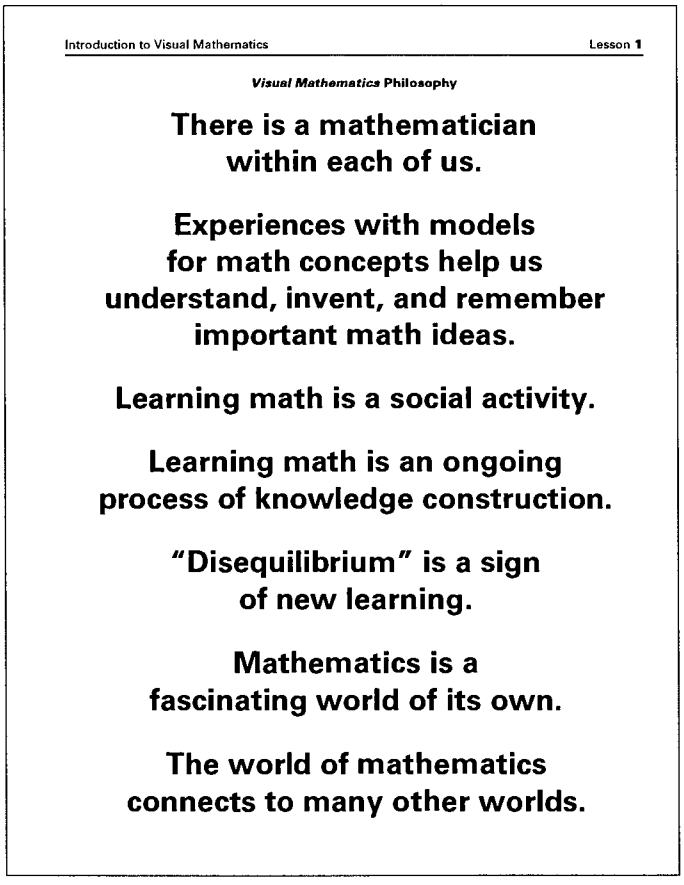
The philosophy and spirit of Visual Mathematics are introduced by discussing the process of learning math and the goals for the course. Emphasis is placed on acknowledging students' current views of themselves as mathematicians and their hopes and concerns regarding the class.

MATERIALS

- ✔ *Visual Mathematics* Philosophy (see *Blackline Masters*), 1 copy per group.
- ✔ Philosophy web statements (see *Blackline Masters*) and string (see Comment 1), 1 set per class.
- ✔ *Visual Mathematics* Goals (optional, see *Blackline Masters*), 1 transparency or 1 poster.
- ✔ Student journals (optional), see *Starting Points.*
- ✔ Scissors, 1 pair per student.
- ✔ An assortment of colored markers, glue or tape, colored paper, old magazines, for each group.
- ✔ Paper "clouds" (see Comment 1), 4-6 per group.

ACTIONS

1 Arrange the students in groups and distribute a copy of the *Visual Mathematics* Philosophy (see *Blackline Masters*) to each group. Ask each group to spend about 5 minutes talking over what these statements mean to them. Then have the class create a philosophy web as described on the next page.



Introduction to Visual Mathematics Lesson 1

***Visual Mathematics* Philosophy**

There is a mathematician within each of us.

Experiences with models for math concepts help us understand, invent, and remember important math ideas.

Learning math is a social activity.

Learning math is an ongoing process of knowledge construction.

"Disequilibrium" is a sign of new learning.

Mathematics is a fascinating world of its own.

The world of mathematics connects to many other worlds.

COMMENTS

1 These statements are fundamental to the philosophy of *Visual Mathematics*. Although the language or ideas in these may be unfamiliar to students now, as you proceed through *Visual Mathematics* lessons you can help bring meaning to the statements by relating them to moments when you see students access their inner mathematician, construct new knowledge, struggle with an idea, make a connection among ideas, etc.

This activity is only the beginning of a year-long exploration of what it means to be a mathematician and it provides a reference point for future discussions. Note that many of the journal thought starters and other assessment activities in *Starting Points for Implementing Visual Mathematics* relate to these philosophy statements.

If you have more than one math class, each class' web could be mounted on a sheet of butcher paper and then taken down during other classes. Later you could consolidate their statements into one web.

(Continued next page.)

Focus Teacher Activity (cont.)

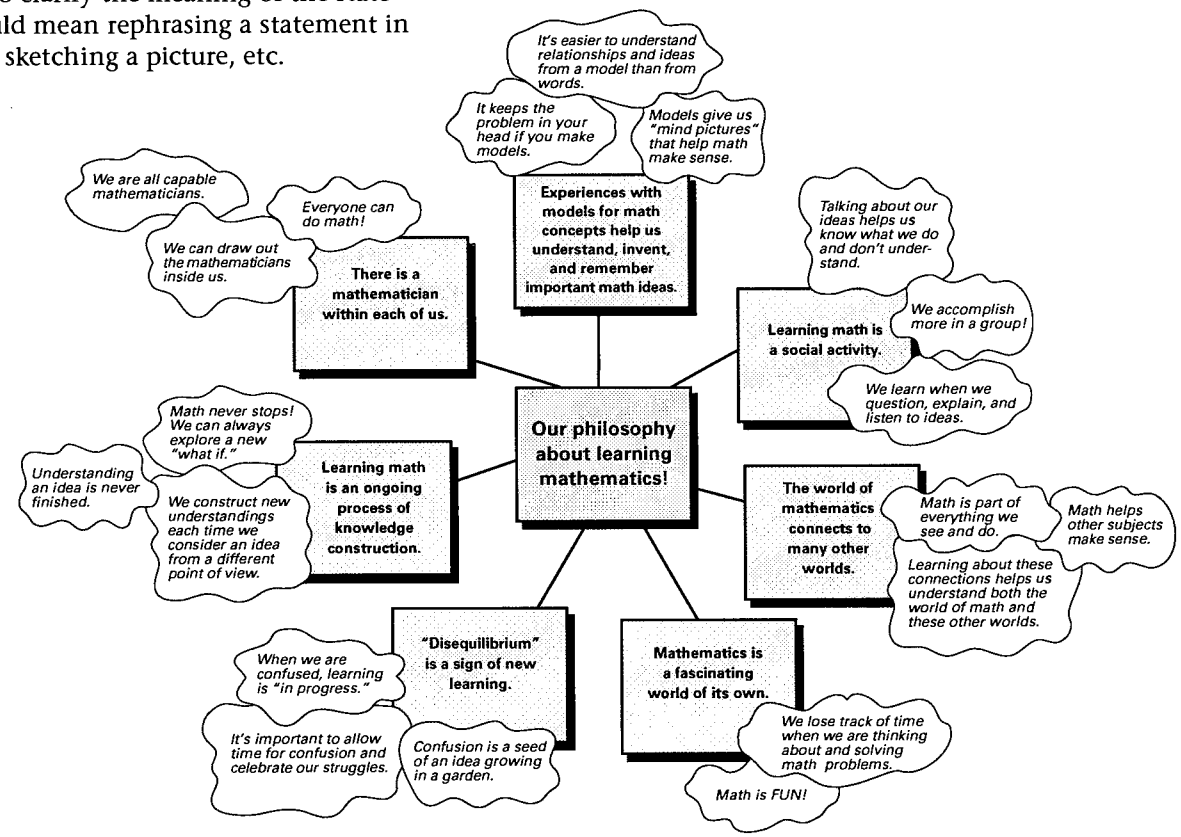
ACTIONS

1 (continued.)
To create a class philosophy web:

a) Prior to class, use string and butcher paper or construction paper to make the central (shaded) portion of the display shown below (a copy of each statement is in *Blackline Masters*). Also cut out 4-6 blank "clouds" for each group. Using one color of paper for the statements and another color for the clouds makes an attractive display.

b) Give each group a marking pen and 4-6 blank clouds. Assign 1 or 2 of the statements from the *Visual Mathematics* Philosophy to each group. Ask the groups to write on their clouds phrases, sentences, or diagrams they think help to clarify the meaning of the statement(s). This could mean rephrasing a statement in their own words, sketching a picture, etc.

COMMENTS

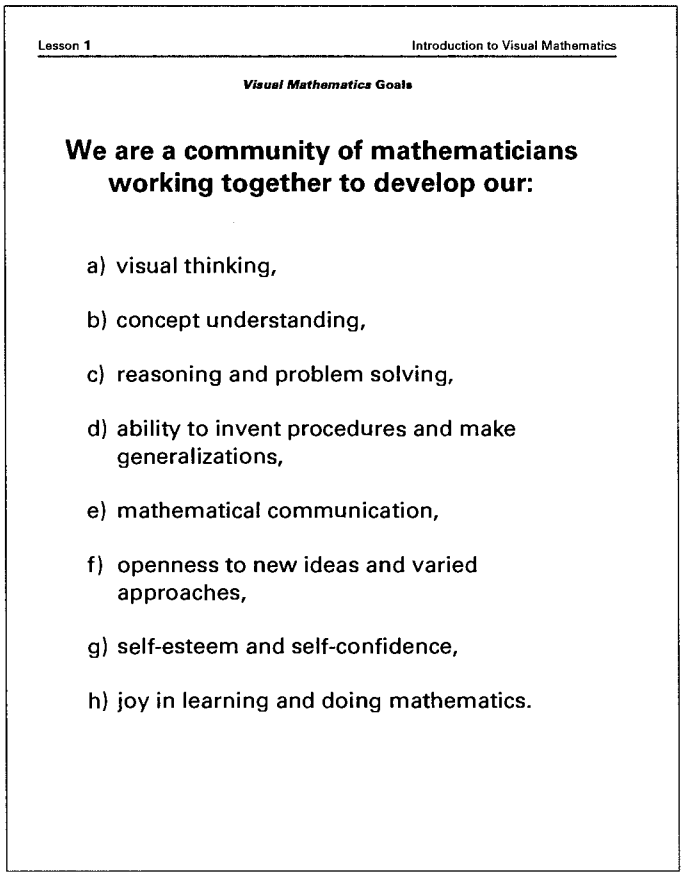


c) Have the groups, one at a time, attach their clouds to the statements on the wall and briefly explain their ideas. Some examples are shown above.

d) Invite students to add clouds containing other ideas that emerge during discussion. Have blank clouds available to add throughout the year, as you and/or students find ways to clarify or expand the meanings of the statements.

Focus Teacher Activity (cont.)

ACTIONS



Lesson 1 Introduction to Visual Mathematics

***Visual Mathematics* Goals**

We are a community of mathematicians working together to develop our:

a) visual thinking,

b) concept understanding,

c) reasoning and problem solving,

d) ability to invent procedures and make generalizations,

e) mathematical communication,

f) openness to new ideas and varied approaches,

g) self-esteem and self-confidence,

h) joy in learning and doing mathematics.

2 Ask students to think privately about the previous discussion and then to spend 2 or 3 minutes writing completions for the thought starter shown in bold print below. Encourage students to keep writing for the entire time, to not worry about spelling or grammar, and to get down as many thoughts as they can. Tell them you will announce when it is time to stop writing.

What excites me most about our plans for this math class is...

because...

3 Repeat Action 2 for the following statement:

What worries me most about math this year is...

because...

COMMENTS

This is also an appropriate time to point out your goals for the course. However, to keep student interest and involvement high, limit "teacher talk" time to 2 or 3 minutes. The *Visual Mathematics* Goals considered by the authors during the development of this course are in *Blackline Masters* (see reduced copy at left), in case you would like to share them with parents.

Bringing these goals up now lets students know that you view mathematics as much more than computation, and lends credibility to the use of writing and other "nontraditional" forms of assessment used in *Visual Mathematics*. However, discussion needs to be brief since it is actual experiences with the lessons that will bring meaning to these goals. A poster listing your goals is convenient for on-going reference. For example, if during another lesson a student gives a particularly clear explanation of their thinking, draw attention to the quality of their *mathematical communication*. Or, if a student effectively uses a model or diagram to solve a problem, use this as an example of *visual thinking* at work.

You may wish to have students periodically give you feedback (e.g., in their journals) regarding the progress they feel the class is making in achieving the goals.

2 Asking students to write for a set period of time without concern for issues like grammar or spelling focuses attention on their thoughts and encourages them to consider more than one idea.

If you plan to have students keep journals, this is a good time to introduce your plans for using them. The written responses in this action make an appropriate journal entry. See *Starting Points* for suggestions about organizing and using journals.

3 Because students' written responses provide important baseline information about attitudes, it is useful to save these (e.g., in journals) for comparison to later comments.

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
4 Ask for volunteers to share ideas that excite them and ideas that worry them regarding math this year.	**4** Acknowledge their joys and concerns without judgment. Simply having worries and fears recognized—or perhaps recognizing that someone else has the same feelings—can help to allay those concerns and to set a supportive tone to the class.
5 (Optional) Provide the students with colored pencils, marking pens, colored paper, scissors, glue, magazines (for cutting out pictures), etc., and ask each of them to create a portrait of their inner mathematician. Encourage them to find ways to show on their portrait what their inner mathematician thinks mathematics "is," and how their inner mathematician feels regarding math.	**5** These portraits can contribute to the baseline information provided in Actions 3 and 4 and can be useful for making comparisons throughout the year in the development of students' mathematical dispositions. The portraits and responses could be included in students' journals or portfolios. To examine ways in which the students' views develop, later in the year invite them to create new or adjusted self-portraits. A bulletin board display of portraits can be fun for students to view. You could include portraits of famous mathematicians.

Follow-up Student Activity 1.1

NAME _____ **DATE** _____

Write a one to two page Mathography that describes your past feelings and experiences in math and that explains your hopes for this math class. Include:

- how you feel about math;

- situations both in and out of school that were "important moments" for you because they affected how you feel about math; and

- what you hope to gain from this class and what you hope to contribute.

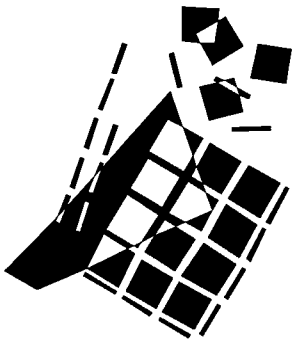
My Mathography

Basic Operations

Lesson 2

THE BIG IDEA

Visual models of the four basic operations (addition, subtraction, multiplication, and division) reveal the meanings of the operations and relationships between operations. In *Visual Mathematics* courses, these models are the basis for operating with whole numbers, fractions, decimals, integers, and complex numbers, as well as with algebraic expressions.



CONNECTOR

OVERVIEW

Students write their perceptions of the meaning and purposes of math.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional), see *Starting Points.*

FOCUS

OVERVIEW

Students use tile and linear pieces to express their perceptions of the four basic arithmetic operations. Models which are based on the meanings of the operations are introduced.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces (with lengths that match the edge of a tile), 30 per student (see Focus Comment 3).
- ✔ Tile and linear pieces for use at the overhead.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.

FOLLOW-UP

OVERVIEW

Students draw diagrams to illustrate the meanings of the basic operations; they write word problems modeled by their diagrams; and they write a letter to a friend, describing their understandings of the operations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 2.1, 1 copy per student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student.

LESSON IDEAS

QUOTE

If conceptual understandings are linked to procedures, children will not perceive of mathematics as an arbitrary set of rules; will not need to learn or memorize as many procedures; and will have the foundation to apply, recreate, and invent new ones when needed.

NCTM Standards

TIMING

Most *Visual Mathematics* lessons are intended to take from 2-5 hours of class time, depending on students' backgrounds and student or teacher generated extensions that you choose to pursue. About 3 or 4 hours are appropriate for this lesson.

Models for the four basic operations will be explored throughout this course. Hence, it isn't expected that students "master" or memorize models, or that they explore every situation in Action 13 of the Focus Activity.

FOLLOW-UP

Students could respond to Problem 7 in their journals or this Follow-up could be a "Teacher Choice" piece for their Growth-folios (see *Starting Points*). Some things to watch for in their letters and the other Follow-up problems: Did students create word problems that match the models they sketched? Did students use certain models more than others? Are there explanations, models and/or problems from some students' papers that other students might benefit from seeing?

QUOTE

Students can write a letter to tell a friend about something they have learned in mathematics class. This type of activity allows the students to consider mathematics for a new purpose. If letters are exchanged, then students learn from the thought processes of their peers.

NCTM Standards

HOMEWORK

It is important to assign problems from a Follow-up before the Focus activity is completed. However, select only problems that are related to ideas explored in class so far. (Let students know they are free to investigate other Follow-up problems and formulate questions that may focus their attention during the next day's activities.)

Having Follow-up assignments span more than 1 day gives students opportunities to ask questions and to refine and revise their work before turning it in. See *Starting Points* for additional discussion of Follow-ups.

LOOKING AHEAD

It will be helpful for you to read the Lesson 6 Connector activity now. Between now and Lesson 6, make regular reference to specific ways you see students behaving like mathematicians. This will make the Lesson 6 Connector richer and more meaningful to students.

ASSESSMENT

An important goal of *Visual Mathematics* is to help students develop a broadened view of math and a sense that math has value in their lives. Therefore, collecting early information about students' views of math will facilitate the assessment of their growth later in the year or in subsequent *Visual Mathematics* courses.

You could introduce students to portfolios now by having them label their Work- and Growth-folios and by briefly discussing your plans for using them as a showcase of the students' work and growth. Remind students to date all of their work (including all journal entries) and to keep all completed work in their Work-folios. See *Starting Points for Implementing Visual Mathematics* for detailed suggestions regarding organizing and managing Work- and Growth-folios.

SELECTED ANSWERS

Responses will vary.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students write their perceptions of the meaning and purposes of math. This provides insights about the beliefs and attitudes of the class as a whole and about individual students' views.

MATERIALS

✔ Student journals (optional), see *Starting Points*.

ACTIONS

1 Write the following thought starters on the overhead and ask students to spend 2 or 3 minutes writing a response to each. Remind them that you are interested in their thoughts and ideas more than grammar, spelling, etc. Encourage them to write for the entire time.

The way I would define math is...

I think we study math because...

COMMENTS

1 If students are keeping journals, ask them to record their responses in them. This will provide important baseline information regarding students' views of the nature, value, and purposes of mathematics. By posing similar journal questions later in the year, it will be possible to make comparisons as you and/or the students look for evidence of growth.

Rather than discussing their views now, you may wish to move on to the Focus activity, noting it will be interesting to see how their views of math develop during the year. Depending on their prior math experiences, many students may view mathematics primarily as computation. In *Visual Mathematics* students develop a much broader perspective of math and an appreciation for understanding the meanings of concepts and inventing procedures.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use tile and linear pieces to express their perceptions of the four basic arithmetic operations. Models which are based on the meanings of the operations are introduced. These models are explored again in many subsequent Visual Mathematics activities and will provide students with a conceptual basis for performing operations with whole numbers, fractions, decimals, integers, and complex numbers, as well as with algebraic expressions.

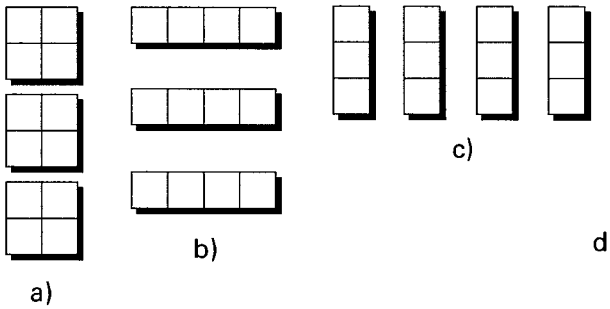
MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces (with lengths that match the edge of a tile), 30 per student (see Focus Comment 3).
- ✔ Tile and linear pieces for use at the overhead.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.

ACTIONS

1 Arrange the students in pairs or small groups and distribute tile to each student. Write "3 × 4" on the overhead. Ask each student to use tile to show what this expression means to them. Emphasize there is no *right* way to do this and you expect a variety of representations.

2 Call attention to the various representations. Ask for volunteers to form their models at the overhead and to tell how their models relate to their view of the meaning of the symbols "3 × 4."



COMMENTS

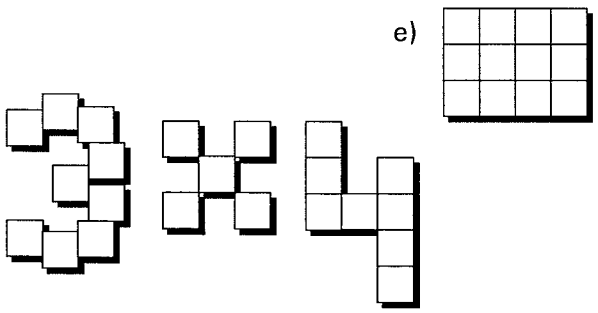
1 Any size tile will work; however, 1-inch size are easy to handle and work well in subsequent activities involving tile. It is also convenient if the tile are colored.

Students who have experienced the *Opening Eyes to Mathematics* curriculum may move quickly through this lesson. However, this activity allows you to make note of their present conceptions of each operation and gives students an opportunity to expand or strengthen their understanding. To encourage many ideas to emerge, you may wish to ask students to take some personal "think time" and build their models before noting what others build.

Writing "3 × 4" rather than saying "3 times 4," "the product of 3 and 4," "3 multiplied by 4," or some other phrase, allows the students to focus on the symbols and to interpret the meaning of the symbols for themselves.

2 Transparent colored tile are useful at the overhead, particularly if students use colors to help illustrate their thinking.

Students' representations can give you insights about the ways in which they view multiplication. Some possible representations are shown below.



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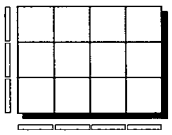
Focus Teacher Activity (cont.)

ACTIONS

3 Draw attention to the *rectangular region* e) shown in Comment 2. If no one makes this model, make it yourself and add it to the collection of representations of 3×4. Place linear pieces along two sides of the rectangle to indicate its dimensions, as in the diagram below. Discuss the distinction between the *dimensions* and the *area* of a rectangle.



4 Distribute linear pieces to each student. Tell them that for now the length of 1 linear piece is 1 linear unit and the area of 1 tile is 1 area (square) unit. Ask the students to use tile and linear units to form a rectangular region that shows an *area* representation of 4×6. Ask for volunteers to show their models at the overhead and to show how the *repeated addition* representation of 4×6 is also contained in their rectangles.

COMMENTS

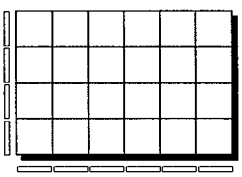
2 (continued.)
Some students may think of 3×4 as 3 sets of 4 as in diagrams a) and b) above. Others will think of it as 4 sets of 3 as in c). These are *repeated addition* representations of the meaning of multiplication. Those students who form symbols, as in d), may think of arithmetic primarily as symbols and symbol manipulation without other meaning. The rectangular region e) is a particularly useful way of viewing the *product* of 2 numbers. It is discussed further in the next Action and recurs frequently in many settings throughout *Visual Mathematics*.

3 You could cut toothpicks to match the length of the tile's edge. Manufactured linear pieces are available from The Math Learning Center.

Note that models b) and c) in Action 2 are easily converted to rectangular regions by pushing the rows or columns together. Thus, a 3×4 rectangle may be viewed as either 3 sets of 4 or 4 sets of 3. This illustrates the commutative property for multiplication which is discussed in more detail in Lesson 3.

The area of a rectangle is the number of area units used to form it. The dimensions of a rectangle are the lengths of its sides. Area is most commonly measured in square units (other possibilities will be explored in later lessons). Dimensions are measured in terms of linear units. For the rectangle shown at the left, if each tile represents 1 square unit and each linear piece represents 1 linear unit, the area of the rectangle is 12 square units and its dimensions are 3 linear units by 4 linear units. Thus, for the mathematical statement $3 \times 4 = 12$, the 3 and 4 represent lengths and the 12 represents area.

4 This rectangle has area 24 square units and dimensions 4 linear units by 6 linear units, as shown here:



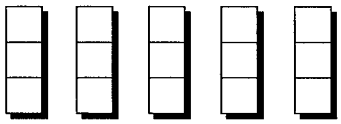
Pulling apart the rows reveals 4 sets of 6 and pulling apart the columns reveals 6 sets of 4. Thus, the repeated additions $6 + 6 + 6 + 6$ and $4 + 4 + 4 + 4 + 4 + 4$ are also illustrated in the rectangle.

Focus Teacher Activity (cont.)

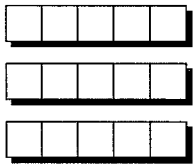
ACTIONS

5 Write "15 ÷ 3" on the overhead or board. Ask the students to use tile to represent what this expression means to them. Call attention to the various representations and discuss them with your students.

6 For each method of division modeled below, ask the students to work with their groupmates to create a word problem whose solution could be found using that method. Discuss.

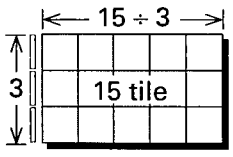


15 ÷ 3: grouping method



15 ÷ 3: sharing method

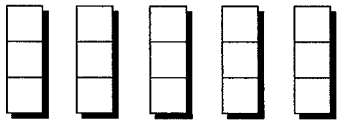
7 If it hasn't come up, ask students to discuss ways they think a rectangle and its dimensions could be used to show the meaning of 15 ÷ 3. Discuss the students' ideas about ways that multiplication and division are related in the area method.



COMMENTS

5 Accept all representations without judgment since your purpose is to learn how students think about the meanings of the symbols. Both symbolic and non-symbolic representations may occur. Two common non-symbolic models are:

5 groups of 3



or 3 groups of 5

6 Note that it is easy to confuse the names of the methods. Keep emphasis on understanding the different meanings of division rather than vocabulary. One way to facilitate discussion is to invite volunteers to read one of their group's problems and then have the rest of the class identify the division method that is associated with that problem.

Representing 15 ÷ 3 by 5 groups of 3 is an example of the *grouping* method of division. In this case, 15 ÷ 3 is thought of as the number of groups of 3 into which 15 can be divided, as in the question: "If each student is to receive 3 pencils, how many students will 15 pencils supply?"

Representing 15 ÷ 3 by 3 sets of 5 is an example of the *sharing* method of division. Here, 15 ÷ 3 is thought of as the number of objects in each of 3 sets when 15 objects are shared equally among them, as in the question: "If 15 pencils are divided equally among 3 students, how many will each student get?"

The grouping method of division is also called the *subtractive* or *measurement* method since groups are subtracted away or measured off. The sharing method is also called the *dealing* or *partitive* method since objects are dealt or partitioned into sets.

7 Notice that both tile arrangements in Action 6 become the same rectangle (15 tile in 3 rows and 5 columns) when rows or columns are pushed together. Linear units can be placed to indicate the dimensions of the rectangle. For the symbols 15 ÷ 3, the 15 represents an area of 15 tile (15 square units) and the 3 represents one dimension (3 linear units). The quotient 15 ÷ 3 is the other dimension (in linear units).

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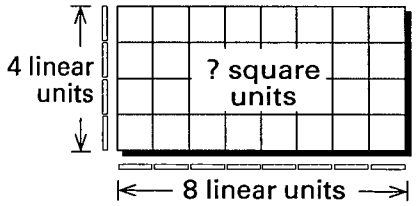
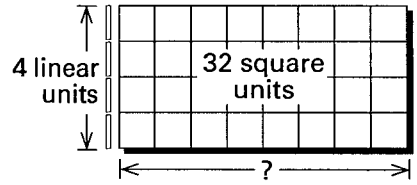
Focus Teacher Activity (cont.)

ACTIONS



4×8 is the area



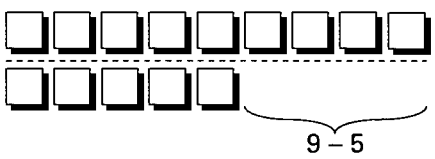
$32 \div 4$ is the missing dimension

8 Write "9 + 5" on the overhead and ask students to use their tile to represent the meaning of this expression. Discuss.

9 Write "9 – 5" on the overhead and ask students to use tile to show their ideas about the meaning of this expression. Discuss.

10 Sketch the following diagrams on the overhead. Ask your students to work with their groupmates to create word problems whose solutions could be found using each method.

Take-away method



Difference method

COMMENTS

7 (continued.)
Hence, the process of multiplication may be thought of as determining the area of a rectangle when the dimensions are given, as shown at the left.

The process of division may be thought of as determining one dimension of a rectangle (the *quotient*) given the other dimension and the area, as shown at the left. This is called the *area method* for division.

8 Most students will count out 9 tile and 5 tile and combine the groups to show addition as the process of *putting together* sets.

9 Many students will form a group of 9 tile and take 5 tile from this group. Other students may determine how many tile must be added to a group of 5 tile to match the group of 9 tile. Still others may compare the two groups of tile and state that 9 – 5 is the difference between the two groups.

10 You may wish to have volunteers read their word problems aloud and then have classmates identify the subtraction method associated with each problem.

Carrying out 9 – 5 by taking 5 tile from 9 tile is an example of the *take-away* method of subtraction. This method is appropriate in responding to the question: "If a student has 9 pencils and gives 5 of them away, how many pencils does the student have left?"

Determining 9 – 5 by finding how many tile must be added to 5 tile to match a group of 9 tile is an example of the *difference* method of subtraction. This method is appropriate for the question: "If one student has 9 pencils and another has 5, how many more pencils does the first student have than the second?"

Focus Teacher Activity (cont.)

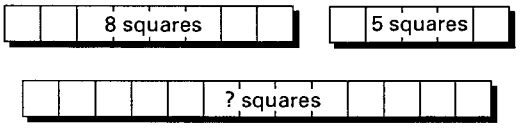
ACTIONS

11 If it hasn't come up, ask students to use their tile to show ways they think the operations of addition and subtraction are related. Discuss.

12 (Optional) Ask the students to work in groups to explore and determine ways linear pieces can be used to model the meanings of addition, subtraction, multiplication, and division. Have volunteers show and explain their groups' ideas at the overhead.

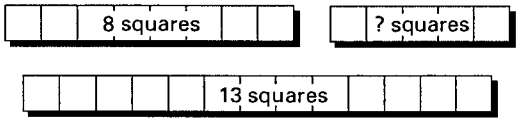
COMMENTS

11 In the process of addition, one knows the number of tile in *2* sets of tile and seeks the total number of tile as in the model of 8 + 5 shown here:



? = 8 + 5

In the process of subtraction, one knows the number of tile in one of the sets of tile and the total number of tile and seeks the number of tile in the other group, as illustrated here for 13 – 8.



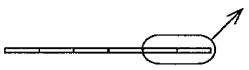
? = 13 – 8

12 You may wish to have the students make a poster or an Eyewitness Book (see Extended Projects in *Starting Points*) summarizing their observations and conclusions. These could be completed at home with an extended timeline for completion.

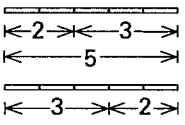
As with area units, all four basic operations can be modeled with linear units. Following are a few examples of arithmetic that can be carried out with lengths:

the difference between 6 linear units and 2 linear units

Subtraction. The difference between a length of 6 linear units and a length of 2 linear units is modeled at the left. This illustrates the *difference method for subtraction*. A numerical statement that represents this model is 6 – 2, or 6 – 2 = 4, or 6 linear units – 2 linear units = 4 linear units.



If 2 linear units are removed from a length of 6 linear units, a length of 4 linear units is left. This illustrates the *take-away method* for the subtraction 6 – 2 = 4, as shown at the left.



Addition. 2 linear units and 3 linear units placed end-to-end form a length of 5 linear units, as shown at the left. Placing the lengths together in the reverse order yields the same total length (this illustrates the *commutative property for addition* which is explored in Lesson 3). Both of these models reflect the *put-together* method of addition and show 2 + 3 = 3 + 2 = 5.

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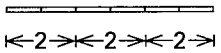
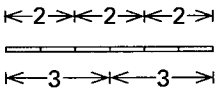
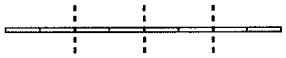
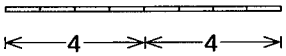
Focus Teacher Activity (cont.)

ACTIONS









13 Distribute 2 sheets of 1-cm grid paper to each group. Place a transparency of Focus Master A on the overhead, reveal only Situation a), and ask the groups to:

a) use their tile and/or linear pieces to build a model of the situation (or use the grid paper to sketch tile and linear pieces);

b) make one or more observations about mathematical relationships they notice in their model and write those observations in sentences;

c) if not included in their sentences above, write mathematical equations (number statements) to describe the mathematical relationships listed in b).

Have volunteers share their models, observations, and equations at the overhead.

COMMENTS

12 (continued.)
Multiplication. If 3 groups of 2 linear units are placed end-to-end, the result is a length of 6 linear units. This can be recorded as $3 \times 2 = 6$, or $2 + 2 + 2 = 6$, and illustrates the *repeated addition method* of multiplication.

Notice that 3 copies of 2 linear units have total length equal to 2 copies of 3 linear units. This illustrates the *commutative property for multiplication* (see Lesson 3).

In general, the process of multiplication with lengths can be thought of as determining the total length formed by repeated copies of a given length.

Division. If a length of 8 linear units is divided into 4 equal parts, the length of each part is $8 \div 4$. This illustrates the *sharing method* of division and is shown at the left.

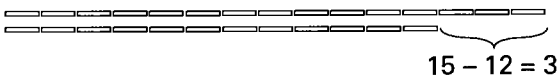
If a length of 8 linear units is divided into groups of 4 linear units, the number of groups is $8 \div 4$. This illustrates the *grouping method* of division and is modeled at the left.

13 To conceal the remaining situations on the overhead, cover them with a sheet of paper. It may be helpful to give instructions for part b) after students have formed their models, and for part c) after completing b). Suggest that number statements contain only numbers and arithmetic symbols such as +, –, ×, ÷, ( ), =.

Since students are asked only to model and observe the relationships in this situation and there are no questions to answer, there are many possible observations. Following are a few examples.

Students may place groups of 12 and 15 tile or linear units end-to-end and observe that the total number is $12 + 15 = 27$.

Or, they may arrange one group of linear pieces above the other and compare to show the difference between the numbers is $15 - 12 = 3$:



Focus Teacher Activity (cont.)

ACTIONS

Basic Operations Lesson 2

Focus Master A

Situations to Model

a) Erin has 12 compact discs and Malia has 15 compact discs.

b) Rosa sold 7 packages of gift wrap to each of her 5 neighbors.

c) Nick sold a total of 48 packages of gift wrap to 6 of his neighbors. All of Nick's neighbors bought equal amounts of gift wrap.

d) Ramona earned $18 mowing Mr. Jones' lawn. Mr. Jones paid Ramona $4 per hour.

e) Laraina jumped 10 feet and 6 inches in the long jump, which is 18 inches further than Willis jumped.

f) The 6th grade class planted a flower garden at the entrance to the school. The area of the garden is 24 square feet.

g) Julian built a rectangular pen for his dog. The perimeter (distance around the outside) of the pen is 48 feet and one dimension of the pen is 6 feet.

h) Miranda had 15 pairs of socks until her sister borrowed 12 pairs when she went to camp.

i) Jamie and Alex baked 50 cookies for the bake sale and packaged them 6 cookies per bag.

j) The 50 choir students rode in vans to the concert. 6 students can ride in each van.

k) Travis cut 50 feet of rope into 6 equal parts.

l) The art teacher stores her supplies in boxes that are 6 inches high. The boxes are stacked in a cabinet which has 50 inches between each pair of shelves.

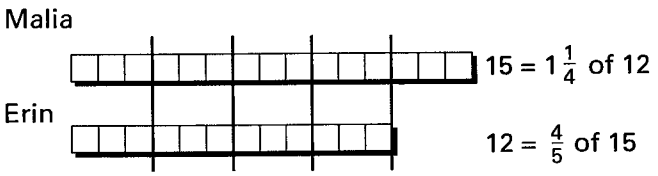
m) Anna has grown 7 inches since 5th grade. Andrew has grown twice as much.

n) Carola made a square picture frame from a piece of wood that is 40 inches long and 3 inches wide.

14 Repeat Action 13 for one or more of Situations b)-n), depending on the needs and interest of your students.

COMMENTS

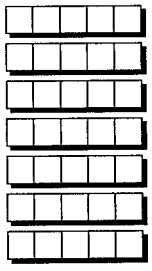
Still others may show that there are $1\frac{1}{4}$ as many CD's in Malia's collection as in Erin's, or that Erin has $\frac{4}{5}$ as many CD's as Malia (see diagram below). If these fractional relationships don't come up, you may wish to postpone discussion of them until Lessons 8 and 9 when fractions are introduced.



14 Some of these could be assigned as homework, as "Eyewitness Book" projects (see Extended Projects in *Starting Points*), as journal entries, or on other days as "warm-ups" at the beginning of class.

Models and observations will vary. Accept all reasonable models and conclusions. Note students will have many more opportunities to model situations in this course. Hence, the nature of their responses here will provide baseline information that will be useful for measuring growth later. Here are examples of models and mathematical observations that could be made about some of the situations:

b) Rosa sold a total of $7 \times 5 = 35$, or $5 + 5 + 5 + 5 + 5 + 5 + 5 = 35$ packages of giftwrap.



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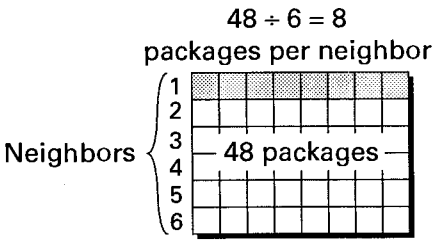
Focus Teacher Activity (cont.)

ACTIONS

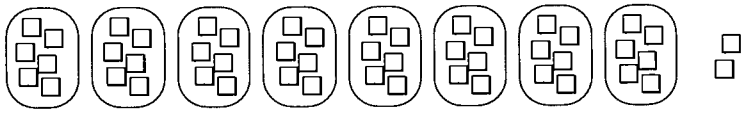


COMMENTS

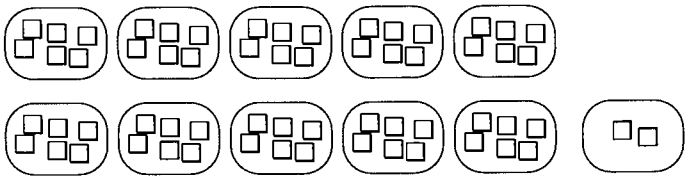
14 (continued.)
c) Arranging the 48 tile (representing packages) into 6 equal groups shows that 48 ÷ 6 = 8 packages were purchased by each neighbor. Also illustrated in the model are the facts that 6 groups of 8 packages total 6 × 8 = 48 packages and that when 48 packages are divided into groups of 8 packages, there are 48 ÷ 8 = 6 neighbors.

Notice that one observation that could be made about relationships in each of the Situations i)-l) involves division of 50 by 6, but the remainder needs to be treated differently in each case. You might ask students to explore these situations and describe ways they are the same and ways they differ. Another possibility is to assign each group one situation from i)-l) to model. Collect sketches of each group's model (caution groups not to give clues like pictures of cookies or vans). Photocopy the sketches and distribute copies to all the groups who try to predict which situation is represented by each sketch.

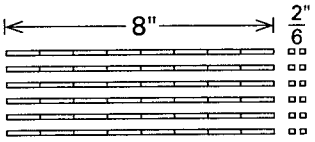
i) There are 50 ÷ 6 = 8 bags of 6 cookies with 2 cookies left over for Jamie and Alex to share.



j) They took 50 ÷ 6 = 8 vans, but since that left 2 students, they took 1 more van, for a total of 9 vans.



k) Each piece of rope has length 50 ÷ 6 = 8⅓ inches:



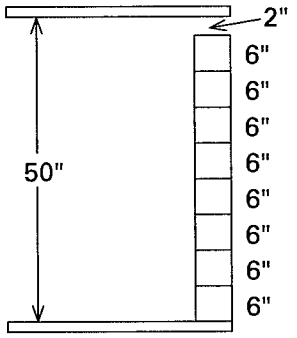
Or, some students may suggest first cutting off 2 inches of rope and then dividing so that each piece has length 48 ÷ 6 = 8 inches.

Focus Teacher Activity (cont.)

ACTIONS

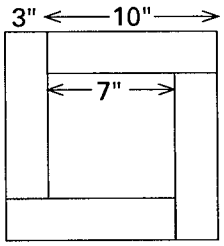
COMMENTS

l) Each stack holds $50 \div 6 = 8$ boxes, with 2" of space remaining.

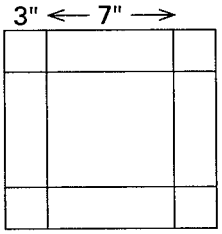


m) Together they have grown a total of $3 \times 7 = 21$ inches. Or, the difference in their amounts of growth is $14 - 7 = 7$ inches, as shown below.

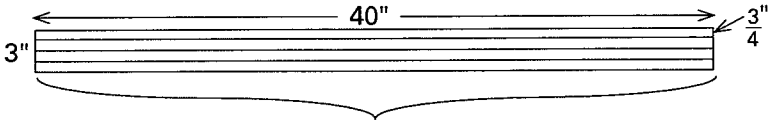
n) Groups may determine more than one way of forming the "frame." Three possibilities are shown below.



Cut four
3" × 10" pieces.



Cut four 3" × 7"
pieces and four
3" × 3" pieces.



Arrange these pieces to form a
40¾" × 40¾" square frame.

TEACHER NOTES:

Follow-up Student Activity 2.1

NAME _____ DATE _____

Problem	Draw a picture of tile and/or linear units to show the meaning of the problem.	Write a word problem whose solution is modeled by your picture.
1 $7 + 9 = 16$		
2 $15 - 6 = 9$		
3 $8 \times 5 = 40$		
4 $24 \div 6 = 4$		

(Continued on back.)

Follow-up Student Activity (cont.)

5 On grid paper, draw a diagram of tile or linear pieces to model the mathematical relationships in each of these situations.

a) Lewis saved $23 last week, which is $8 more than Joanne saved.

b) Adela sold 3 times as many cookies as Josh, who sold 13 boxes.

c) LaTina planted a rectangular garden with area 32 square feet. One side of the garden has length 8 feet.

6 Next to each situation you modeled in Problem 5 write a math question about the situation that could be answered by looking at your model. Then give the answer to your question.

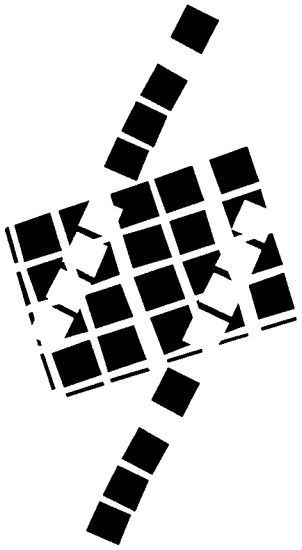
7 On a separate sheet write a letter to a friend who isn't in your math class and tell him or her about the models your class explored for the four basic operations (add, subtract, multiply, and divide). Use clear diagrams and careful explanations to help them understand the meanings of each operation.

Visualizing Number Relationships

Lesson 3

THE BIG IDEA

Models for the meanings of the basic operations provide a basis for visualizing the commutative, associative, and distributive properties. The use of parentheses in mathematical expressions provides a way to communicate the order of one's thoughts and actions.



CONNECTOR

OVERVIEW

Students use tile to explore the commutative property.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Connector Master A (optional), 1 transparency.
- ✔ Transparent tile for use at the overhead.

FOCUS

OVERVIEW

Students use tile to explore the meaning of several number properties and are introduced to the use of parentheses as a way of recording their thought processes and actions with tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Tile and linear pieces for use at the overhead.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.

FOLLOW-UP

OVERVIEW

Students use diagrams of tile to illustrate number properties and arithmetic expressions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 3.1, 1 copy per student.

LESSON IDEAS

LOOKING AHEAD

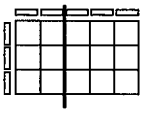
Although it is not the major focus of this lesson, you can promote the development of students' mathematical vocabulary by pointing out formal terms that relate to examples and informal language used by students. For convenient reference, you might have students create a poster that names each number property explored in this lesson and shows three or more examples that illustrate how and why each property works. To do this each group could be responsible (in or out of class) for one example of each property, or each group could prepare three examples for one of the properties.

FOLLOW-UP

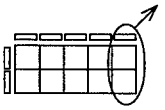
If you use the Follow-up as homework you could pick (or let students pick) parts of each problem to assign the first day. Then assign the remaining parts on the following day(s). This gives students a chance to explore, ask questions, and then refine their ideas. See *Starting Points* for ideas regarding assigning and evaluating Follow-ups.

SELECTED ANSWERS

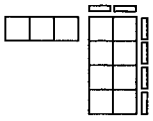
1. d) One possibility:



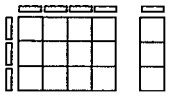
e) One possibility:



2. b) One possibility:



d) One possibility:



TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

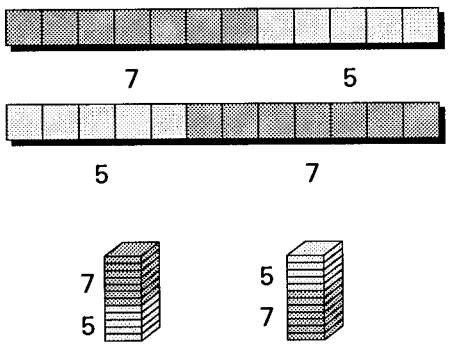
Students use tile to explore the commutative property. Models for the commutative property and other properties in this lesson are intended to help students communicate their thoughts and actions and will be used to develop mental computation and estimation strategies in later lessons.

MATERIALS

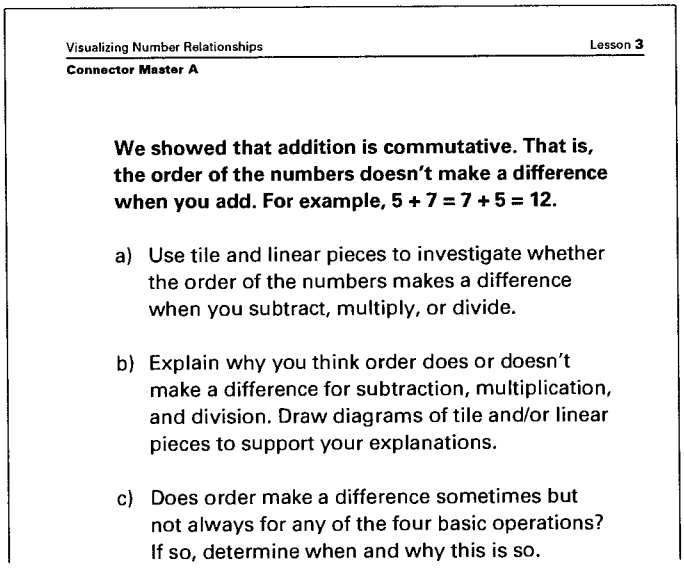
- ✔ Tile, 30 per student.
- ✔ Connector Master A (optional), 1 transparency.
- ✔ Transparent tile for use at the overhead.

ACTIONS

1 Place the students in small groups and give tile to each student. Write 7 + 5 and 5 + 7 on the overhead and ask the groups of students to use their tile to represent each expression in a way that shows how the expressions are different and how they are the same. Ask volunteers to show their ideas to the class.



2 (Optional) Give each group a sheet of 1-cm grid paper. Place a transparency of Connector Master A on the overhead for groups to explore. Discuss their conclusions.



Visualizing Number Relationships Lesson 3

Connector Master A

We showed that addition is commutative. That is, the order of the numbers doesn't make a difference when you add. For example, 5 + 7 = 7 + 5 = 12.

a) Use tile and linear pieces to investigate whether the order of the numbers makes a difference when you subtract, multiply, or divide.

b) Explain why you think order does or doesn't make a difference for subtraction, multiplication, and division. Draw diagrams of tile and/or linear pieces to support your explanations.

c) Does order make a difference sometimes but not always for any of the four basic operations? If so, determine when and why this is so.

COMMENTS

1 There are many possibilities. If colored tile are not available, the backs and fronts of most ceramic tile can provide two different shades. At the left are two arrangements which illustrate that 7 + 5 = 5 + 7.

These arrangements or similar ones can be used to call the students' attention to the fact that the total number of tile in each group is the same. This shows that regardless of the order in which the tile (numbers) are combined (added), the total number of tile is the same. This is an example of the *commutative property* for addition.

Throughout this lesson, emphasis is placed on understanding relationships such as the commutative property. Students may be interested in the names of properties but memorizing names isn't necessary.

2 Copies of Connector Master A and 1-cm grid paper are in *Blackline Masters*.

You may find it helpful to read the Lesson 6 Connector prior to this action so that during this activity you can point out "important mathematical moments" (i.e., moments when students talk or act like mathematicians) that you witness. This will make the Lesson 6 Connector a richer activity.

Allow plenty of time for groups to grapple with the questions on Master A. However, to keep interest high and avoid investing too much time, you may want to reveal only one part of Master A at a time and ask for oral rather than written explanations. Note that several questions may emerge which are best left unresolved for now. For example, some students may experience "disequilibrium" over the meaning of 3 ÷ 12 as compared to 12 ÷ 3 or over 5 – 8 as compared to 8 – 5. After listening to and acknowledging students' ideas and questions (this information can be useful for future

(Continued next page.)

Connector Teacher Activity (cont.)

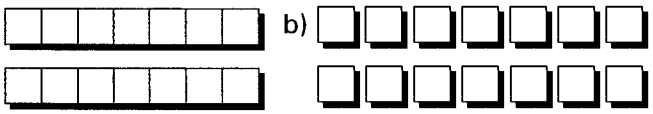
ACTIONS

COMMENTS

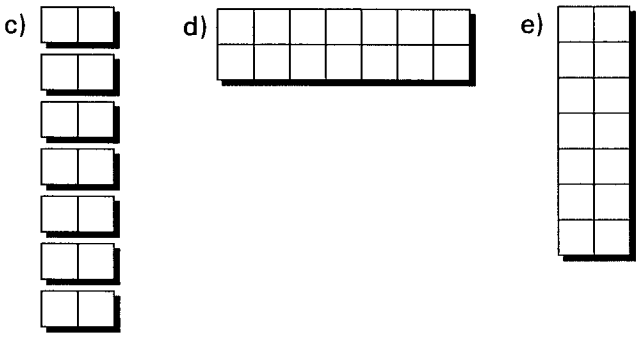
2 (continued.)
planning), you may wish to suggest they record their unanswered questions in their journals. Encourage them to watch for and record AHA's regarding those questions in future lessons.

The commutative property holds for addition and multiplication, but it does not hold for subtraction and division. The tile arrangements that follow illustrate the *commutative property for multiplication*. They can each be viewed as 7×2 or 2×7 depending on how the model is interpreted (e.g., as 7 groups of 2, 2 groups of 7, 2 by 7, 7 by 2, etc.). In each model the total number of tile is 14.

a) b)



c) d) e)



Focus Teacher Activity

OVERVIEW & PURPOSE

Students use tile to explore the meaning of several number properties and are introduced to the use of parentheses as a way of recording their thought processes and actions with tile. Throughout Visual Mathematics students have opportunities to develop comfort with using numbers and arithmetic symbols as a way of communicating their thoughts about number relationships.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Tile and linear pieces for use at the overhead.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.

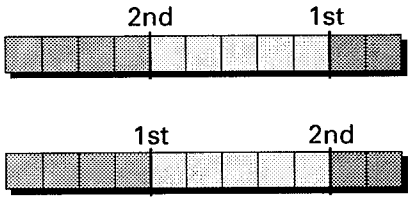
ACTIONS

1 Arrange the students in groups and give tile to each student. Write 4 + 5 + 2 on the overhead and ask each student to privately determine the sum. Then ask for volunteers to, without talking, use tile at the overhead to represent their methods of mentally arriving at the total. Have other class members speculate about the thought processes suggested by each volunteer's actions with the tile.

2 If it hasn't come up, tell the students that one method of recording the order in which their tile in Action 1 were combined is to use parentheses in a mathematical expression. When parentheses appear in an expression, the operations inside the parentheses are carried out first. Write the expression 4 + 5 + 2 on the overhead and ask for volunteers to place parentheses in the expression to show the order in which they combined the tile.

COMMENTS

1 The intent here is to have students use the tile to represent the steps in their mental processes. There are many possibilities: a pile of tile, a tower of tile, a row of tile, etc., each formed by joining together sets of 4, 5, and 2 tile. The following diagram illustrates two possibilities for adding 4 + 5 + 2. The "1st" and "2nd" above the subdividing lines indicate the order in which the tile were combined (i.e., the numbers were added). For example, in the first figure, the 5 tile and 2 tile were joined and then the 4 tile were added. In the second figure the 4 tile were combined with the 5 tile and then 2 tile were added.



2 A major purpose of this action and subsequent actions is to encourage students to view numbers and arithmetic symbols (e.g., +, –, ×, ÷, ( ), =) as a way of recording their actions and thoughts. Some students will record (4 + 5) + 2 and others will record 4 + (5 + 2).

Lining up the models illustrated in Comment 1 shows their lengths are the same and therefore 4 + (5 + 2) = (4 + 5) + 2. The fact that 5 could be associated with 4 or 2 without changing the sum [i.e., that (4 + 5) + 2 = 4 + (5 + 2)] is an illustration of the *associative property for addition.*

Some students may combine the 4 tile and 2 tile first. If so, the commutative property allows them to switch the order of the 5 and 2 before associating the 4 and 2. One way of recording these actions is 4 + (5 + 2) = 4 + (2 + 5) = (4 + 2) + 5 = 11.

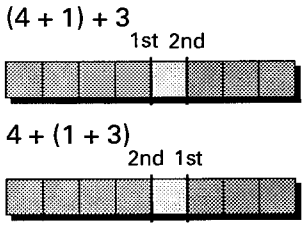
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

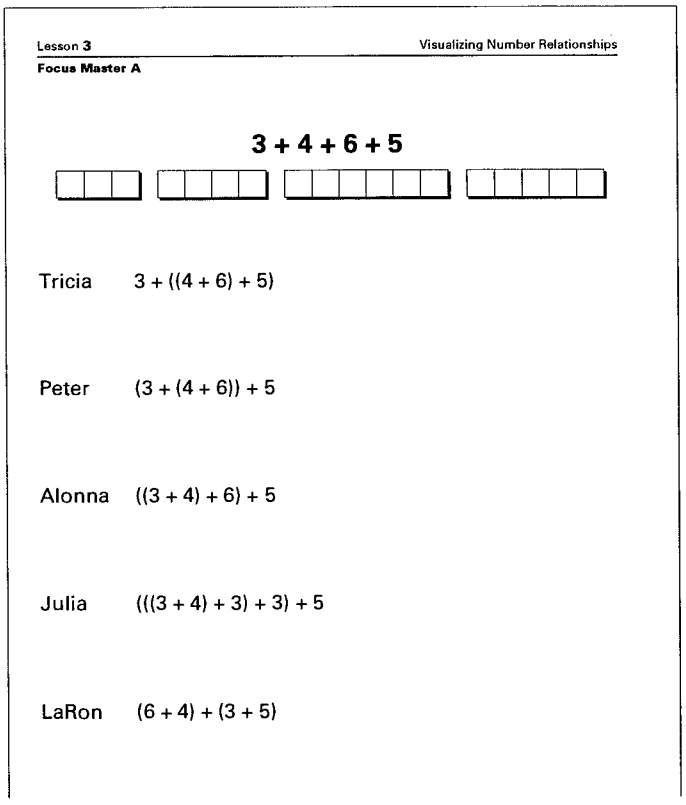
COMMENTS

2 (continued.)
You may wish to ask the groups to build or sketch models to show the associative property is true for other sums. One possibility is shown below:



The associative property also holds for multiplication but not for division or subtraction. These relationships are explored in *Visual Mathematics, Course II.*

3 Place a transparency of Focus Master A on the overhead. Tell the students that each statement reflects the way in which another student combined collections of 3, 4, 6, and 5 tile to compute 3 + 4 + 6 + 5. Ask the groups to write a sentence that describes what they think were each student's actions with tile. Have volunteers show their group's ideas and reasoning at the overhead.



Lesson 3 Visualizing Number Relationships
Focus Master A

3 + 4 + 6 + 5

Tricia 3 + ((4 + 6) + 5)

Peter (3 + (4 + 6)) + 5

Alonna ((3 + 4) + 6) + 5

Julia (((3 + 4) + 3) + 3) + 5

LaRon (6 + 4) + (3 + 5)

3 Each statement shows a different way of associating the numbers to get 3 + 4 + 6 + 5 = 18. The *nested* parentheses in Tricia's recording suggest that she determined 4 + 6 = 10, then computed 10 + 5 = 15, and finally 3 + 15 = 18.

Peter combined 4 + 6 = 10, then found 3 + 10 = 13, and finally computed 13 + 5 = 18.

Alonna determined 3 + 4 = 7, then 7 + 6 = 13, and finally 13 + 5 = 18. Some students may point out that Alonna's method could be written using no parentheses and it would still suggest the same order of additions. This is because of the rules for *order of operations* which are explored in Lesson 4. Unless a student brings them up, you may wish to postpone discussion of these rules for now, and revisit Master B after completing Lesson 4.

Julia computed 3 + 4 = 7, changed the 6 to 3 + 3, then computed 7 + 3 = 10, 10 + 3 = 13, and finally 13 + 5 = 18.

LaRon changed the order of the numbers (i.e., he "commuted" numbers) before adding. Unless students are familiar with the order of operations, it isn't clear whether he first added 6 + 4 or 3 + 5. You may want to leave this ambiguous for now, noting that it will be clarified during discussion of order of operations in Lesson 4.

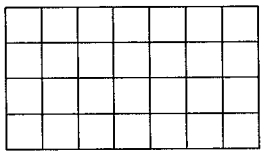
Notice that Tricia, Peter, LaRon, and Julia all used combinations that totaled 10. The strategy of making tens (or twenties, hundreds, thousands, etc.) is useful for mental computation.

Focus Teacher Activity (cont.)

ACTIONS

4 Give each student 1 sheet of 1-cm grid paper. Use tile to form the following rectangle at the overhead and ask the groups to do the same.

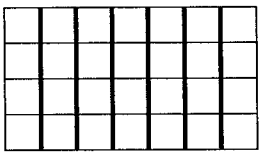
Mention that one way of finding the area of the rectangle is to count the tile one by one. Ask each group to determine several other ways of counting the total number of tile in the rectangle and to make sketches that illustrate their different counting strategies. Ask them to write a number statement next to each sketch to describe their counting methods. Discuss.



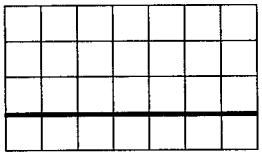
COMMENTS

You may wish to pose a few problems for mental addition and then ask the students to write mathematical expressions using parentheses to show the order of their thoughts. Here are some sample problems: 5 + 7 + 8 + 2; 9 + 2 + 19 + 7; 13 + 11 + 4 + 12. See Lessons 26 and 31 for further explorations of mental computation.

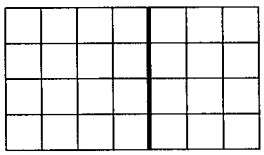
4 It may be helpful to demonstrate one example at the overhead. Counting methods can be illustrated on grid paper by drawing lines on rectangles as shown below. In each of these examples, the 4 by 7 rectangle has been subdivided into smaller regions. The sum of the areas of the small regions is equal to 4×7.



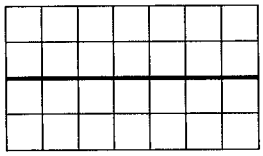
$4 + 4 + 4 + 4 + 4 + 4 + 4 = 28$



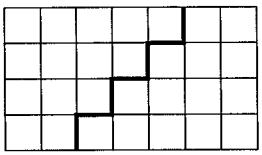
$(3 \times 7) + (1 \times 7)$



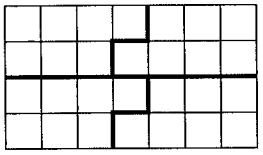
$(4 \times 4) + (4 \times 3)$



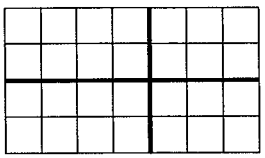
$(2 \times 7) + (2 \times 7)$



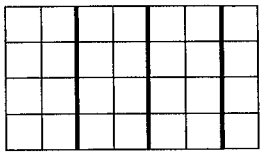
$2(2 + 3 + 4 + 5) = 2(14) = 28$



$7 + 7 + 7 + 7 = 28$



$(2 \times 4) + (2 \times 4) + (2 \times 3) + (2 \times 3)$
or
$2(2 \times 4) + 2(2 \times 3)$



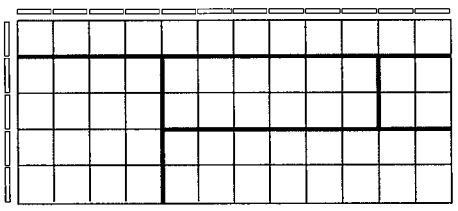
$(4 \times 2) + (4 \times 2) + (4 \times 2) + (4 \times 1)$
or
$3(4 \times 2) + (4 \times 1)$

Note that it is common practice to leave out the multiplication sign when a number is multiplied by an expression in parentheses. For example, the expression $3(4 \times 2)$ is equivalent to $3 \times (4 \times 2)$.

Focus Teacher Activity (cont.)

ACTIONS

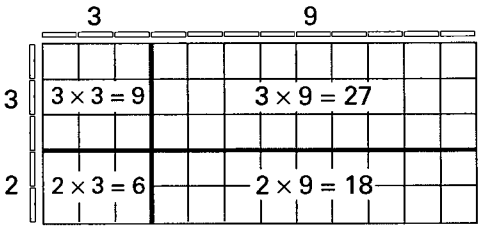
5 Give linear pieces to each student. Use tile and linear pieces to form a 5 × 12 rectangle on the overhead and have the groups do the same. Ask the groups to each identify several ways of "seeing" the total of 60 tile by separating the rectangle into smaller rectangles and adding their areas. Have them draw diagrams and write number statements that show their methods. Discuss students feelings about which methods of subdividing they prefer. Repeat for other products.



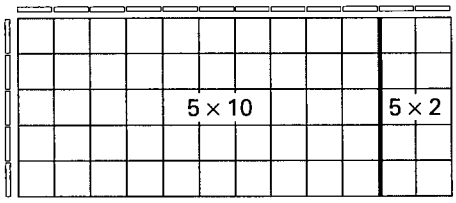
COMMENTS

5 Encouraging students to visualize "hidden" rectangles when calculating products can reinforce basic facts and ways to use known facts to determine other products. (The activities called "Hidden Arrays" in *Opening Eyes to Mathematics* give additional ideas for developing students abilities to do this.)

Following are two ways to subdivide a 5 × 12 rectangle into smaller rectangles in order to compute the product 5 × 12. Using linear pieces may help students identify dimensions (factors, see Lesson 7) of each smaller rectangle.



$5 \times 12 = (3 + 2) \times (3 + 9) =$
$(3 \times 3) + (3 \times 9) + (2 \times 3) + (2 \times 9)$



$5 \times 12 = 5 \times (10 + 2) = (5 \times 10) + (5 \times 2)$

Note that the above two methods are examples of the *distributive property of multiplication*, which is used extensively in *Visual Mathematics* courses to compute products involving whole numbers, decimals, fractions, and algebraic expressions. The intent here is to develop understanding of this property and to build students' comfort with using numbers and symbols to represent their thought processes, rather than to develop vocabulary.

Some students may subdivide the rectangle more randomly as shown at the left.

Some other possible products to pose for students to represent include: 9 × 15, 14 × 11, 6 × 17, 13 × 17, and 8 × 21. (Note that the multiplication methods explored in this activity are examined in more detail and extended to larger numbers in Lesson 37.)

Focus Teacher Activity (cont.)

ACTIONS

6 Ask the groups to determine methods of finding the product 5×9 using:

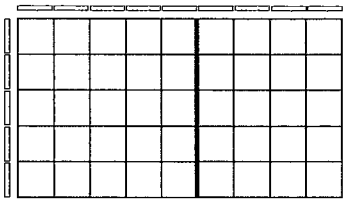
a) the sum of the areas of 2 rectangles, and

b) the difference of the areas of 2 rectangles.

7 Repeat Action 6 for several other products. Discuss students ideas about products they think are easier to compute using differences and ones that are easier using sums.

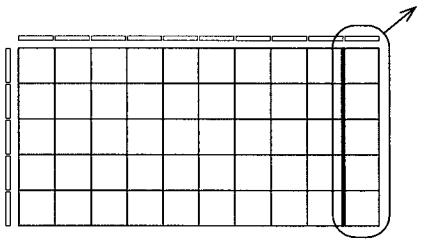
COMMENTS

6 Have additional grid paper available as needed. The distributive property for multiplication over addition could be used to determine 5×9 in many ways. Here is one example:



$5 \times 9 = 5 \times (5 + 4) = (5 \times 5) + (5 \times 4)$

The following diagram shows that $(5 \times 9) = 5 \times (10 - 1) = (5 \times 10) - (5 \times 1)$ and illustrates the *distributive property for multiplication over subtraction.*



Note that some students may be interested in exploring whether the distributive property holds for division [e.g., Does $24 \div (6 + 2) = (24 \div 6) + (24 \div 2)$? Why or Why not? Are there special cases when it does hold? etc.].

7 Here are some possibilities: 7×9; 19×12; 8×13; 17×14; 17×9; 23×7.

TEACHER NOTES:

Follow-up Student Activity 3.1

NAME _____ DATE _____

1 For each of the following equations, draw diagrams of tile that show the meaning of the expression on each side of the equals sign.

a) $4 \times 5 = 5 \times 4$

b) $(2 + 6) + 3 = 2 + (6 + 3)$

c) $4 + 3 = 3 + 4$

d) $3 \times (2 + 3) = (3 \times 2) + (3 \times 3)$

e) $2 \times (5 - 1) = (2 \times 5) - (2 \times 1)$

(Continued on back.)

Follow-up Student Activity (cont.)

2 Jamie wrote each of the following computations to describe his actions with tile. For each computation, draw a diagram to show what you think Jamie's actions were in the order he did them. Next to each diagram, write an explanation of Jamie's actions.

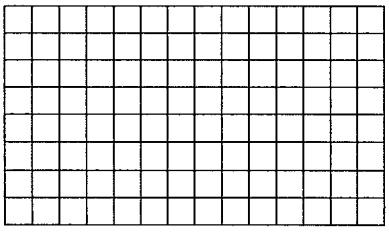
a) $(7 + 9) - (3 + 8)$

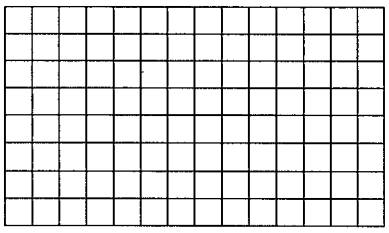
b) $3 + (4 \times 2)$

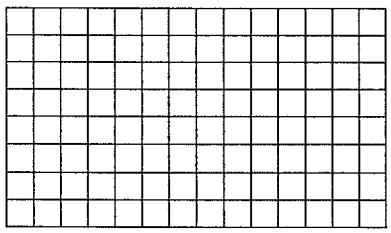
c) $(3 \times (5 - 2)) + 1$

d) $3 \times (4 + 1)$

3 Separate each of these 8×14 rectangles into smaller rectangles to show *3 different* ways to "see" that $8 \times 14 = 112$. Find the area of each 8×14 rectangle by adding the areas of the small rectangles.

a) 

b) 

c) 

Complete these number statements to show how you "saw" and computed the area of each rectangle above.

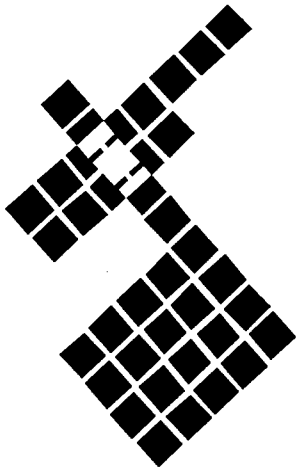
a) $8 \times 14 =$ _____ b) $8 \times 14 =$ _____ c) $8 \times 14 =$ _____

Communicating Mathematics

Lesson 4

THE BIG IDEA

Mathematical communication relies on agreement about the order of operations. Without such agreement, many mathematical expressions have more than one logical interpretation. Having students observe and compare computations carried out according to the standard order of operations provides a problem-solving setting for determining that order.



CONNECTOR

OVERVIEW

Students use parentheses as a way of recording the order in which computations are carried out.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Tile and linear pieces for use at the overhead.

FOCUS

OVERVIEW

Situations are posed that motivate the need for agreement concerning the order of operations when parentheses are not used.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Butcher paper, 1 sheet per group of students.
- ✔ Marking pen, 1 for each group of students.
- ✔ Focus Student Activity 4.1, 1 copy per group and 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.

FOLLOW-UP

OVERVIEW

Students use the order of operations to evaluate several mathematical expressions and write expressions to represent diagrams of tile.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 4.2, 1 copy per student.

LESSON IDEAS

PHILOSOPHY

Visual Mathematics courses provide many opportunities over time to explore each mathematical concept or idea that is introduced. This practice is based on the philosophy that learning is an ongoing process of knowledge construction. New insights can emerge each time students revisit an idea or see it in a new context or from a different point of view. Because of this philosophy, it is not intended that students "master" each lesson. It is hoped that students develop the viewpoint that learning about an idea is never finished and hence, they continue to seek new insights.

Journals and other classroom writing activities (see *Starting Points*) are useful for helping you and the students monitor ways their mathematical understanding develops over time.

FOLLOW-UP

For Problem 2, some students will use parentheses to emphasize their methods even when unnecessary because of order of operations. This is okay as long as their recordings reflect their thought processes. You could ask students who do this to also write each computation using as few parentheses as possible.

MATH VOCABULARY

One way to encourage the development of a mathematical vocabulary is to have students keep a glossary in their journals. This glossary could be an ongoing project. As unfamiliar terms come up during class discussions students could add them to their glossary and write an explanation and/or draw a diagram that shows their understanding of each term. A classroom (poster size) list of glossary terms could also be kept. As math terms come up, the teacher or students could add them to the list. Rather than asking students to memorize lists or take vocabulary tests, the intent in *Visual Mathematics* is to develop mathematical language by modeling relationships associated with terminology, by using formal terms in conjunction with informal explanations, and by making reference to different contexts in which the same terms are used.

SELECTED ANSWERS

1. Both are correct since $(4 + 2) + 5 = 4 + (2 + 5) = 11$. Carlos and Maria have only associated the numbers differently to add.

2. There are many possibilities for these. Here are a few ways of viewing the number of tile in each model.

   a)-c) $4 \times 3 = 12$; $6 + 6 = 12$; $(3 + 3) + (3 + 3) = 12$; $6 \times 2 = 12$; $9 + 3 = 12$; etc.

   d)-f) $(3 \times 2) + (3 \times 5) = 21$; $3 \times (2 + 5) = 21$; $3 \times 7 = 21$; $(2 \times 7) + (1 \times 7) = 21$; etc. Note that $(3 \times 2) + (3 \times 5)$ for example, may be written $3 \times 2 + 3 \times 5$ since order of operations requires multiplying first.

   g)-i) $2 \times 7 - 4 = 10$; $7 + 3 = 10$; $2 \times 3 + 4 = 10$. Some students may prefer to use parentheses and write $(2 \times 7) - 4$ and $(2 \times 3) + 4$.

3. a) 10 d) 25
   b) 12 e) 12
   c) 13 f) 34

Connector Teacher Activity

OVERVIEW & PURPOSE

Students use parentheses as a way of recording the order in which computations are carried out.

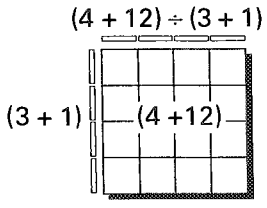
MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Tile and linear pieces for use at the overhead.

ACTIONS

1 Place the students in pairs or small groups and distribute 30 tile and 30 linear pieces to each student. Write the statement $7 \times 4 + 2 - 2 = 40$ on the overhead and ask the students to show how one or more sets of parentheses could be placed in the statement to make it true. Then ask them to show with tile possible actions their statement could illustrate.

2 Repeat Action 1 for the following statement: $4 + 12 \div 3 + 1 = 4$.



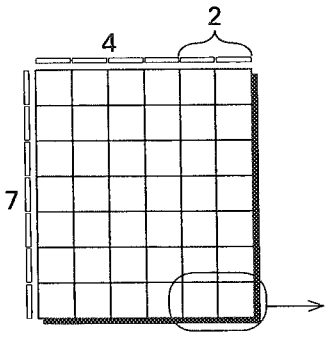
COMMENTS

1 Both expressions $7 \times (4 + 2) - 2$ and $(7 \times (4 + 2)) - 2$ equal 40 and could be representations of the following actions:

a) add lengths of 4 linear units and 2 linear units;

b) form a $7 \times (4 + 2)$ rectangle of 42 tile;

c) remove 2 tile, leaving 40 tile.



Ways to interpret expressions according to rules for the order of operations are explored in the Focus activity.

2 The placement of parentheses is $(4 + 12) \div (3 + 1) = 4$.

Students often suggest the following actions (carrying out operations in parentheses first):

a) add 4 tile and 12 tile to get 16 tile;

b) separate 16 tile into $3 + 1 = 4$ equal groups. Each group has $(12 + 4) \div (3 + 1) = 4$ tile.

Some students may suggest this set of actions:

a) add 4 tile and 12 tile to get 16 tile;

b) add lengths of 3 linear units and 1 linear unit to get a length of 4 linear units;

c) form an array of $4 + 12 = 16$ tile with one dimension $3 + 1 = 4$. The other dimension is $(4 + 12) \div (3 + 1) = 16 \div 4 = 4$ linear units, as shown at left.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Situations are posed that motivate the need for agreement concerning the order of operations when parentheses are not used. Emphasis is placed on viewing the use of numbers and math symbols as a means of communicating thoughts and actions.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces, 30 per student.
- ✔ Butcher paper, 1 sheet per group of students.
- ✔ Marking pen, 1 for each group of students.
- ✔ Focus Student Activity 4.1, 1 copy per group and 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.

ACTIONS

1 Place the students in small groups and distribute tile and linear pieces to each student. Write $3 + 4 \times 2$ on the overhead and ask the students to use the tile to model different ways this expression could be interpreted. Ask volunteers to show and explain their models at the overhead. Acknowledge all logical interpretations.

2 Tell the students that there are many situations, as in Action 1, for which more than one interpretation is logical. Because of this, many years ago mathematicians decided there was a need for a set of rules, called *order of operations*, so there would be no confusion about which operations to do first when parentheses were not included in a problem. One of those rules is to multiply before adding and subtracting. Therefore, a correct representation of $3 + 4 \times 2$ contains $3 + (4 \times 2) = 11$ tile.

3 To practice the correct order of operations, ask the groups to use the tile to model and compute the following expressions:

a) $5 \times 2 + 3$

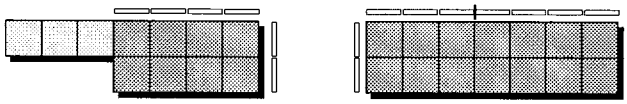
b) $6 + 4 \times 5$

c) $5 \times 6 - 2$

d) $17 - 4 \times 3$

COMMENTS

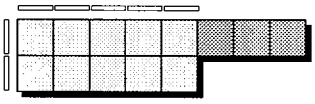
1 Two possible responses are shown below. Figure a) represents $3 + (4 \times 2) = 11$, found by multiplying 4×2 and then adding the product to 3. Figure b) represents $(3 + 4) \times 2$, the equally logical result found by adding 3 and 4 and then multiplying the sum by 2. The different numbers of tile in these figures show the need for an agreement regarding the order of carrying out the operations.



a) b)

2 It is important that students not feel discouraged if they don't come up with the correct interpretation, since both are logical and the correct one is established only by a rule.

3 a) A correct interpretation will have 13 tile and represent multiplying 5×2 and then adding 3. One possible arrangement is shown below:



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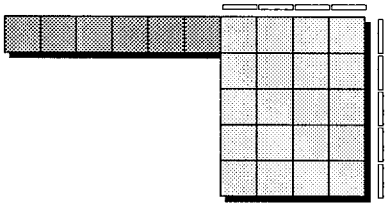
Focus Teacher Activity (cont.)

ACTIONS



4 Give each group of students a copy of Student Activity 4.1 to complete. Discuss their conclusions.

Communicating Mathematics Lesson 4

Focus Student Activity 4.1

NAME _____ DATE _____

1 Each of the following problems is worked correctly according to the rules for the *order of operations*. Determine the order in which you think each set of computations were carried out to get the answer shown. Then write a sentence or two to describe the order of operations for that problem.

a) $3 + 4 \times 2 = 11$

b) $(3 + 4) \times 2 = 14$

c) $8 - 4 + 2 + 15 = 21$

d) $3 \times 4 + 10 + 2 = 17$

e) $7 + 32 + 16 + 2 \times 0 = 9$

f) $5 \times 4 + 7 - 3 \times 2 = 21$

g) $6 + 7 + 8 + 1 - 3 - 2 = 17$

h) $12 - 7 + 6 - 4 = 7$

i) $12 + 4 + 2 \times 5 = 13$

j) $12 + (4 + 2) \times 5 = 10$

k) $3 \times 4 \times 5 \times 2 = 120$

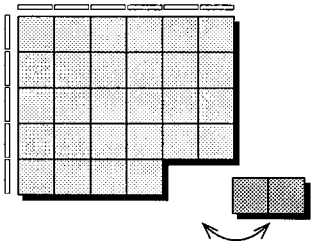
l) $12 + 4 - 3 + 2 \times 5 = 10$

m) $(5 \times 4 + (7 - 3)) \times 2 = 48$

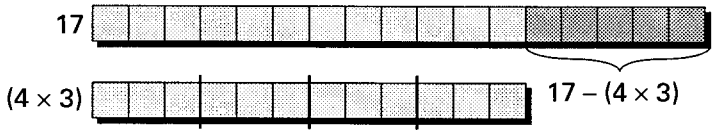
COMMENTS

3 (continued.)
b) A correct interpretation will have 26 tile and represent multiplying 4×5 and then adding that product to 6. One possible model is shown at the left (note the 6 tile could be attached anywhere to the 4×5 rectangle).

c) A correct model will have 28 tile and represent multiplying 5×6 and then subtracting 2. One possible model is shown here:



d) This computation can be illustrated by showing the difference between 17 and the product 4×3 as shown below:



4 As groups work, offer encouragement by pointing out ways that groups and/or individuals are revealing the mathematicians within them (this will help prepare students for the Lesson 6 Connector Activity).

This action provides an opportunity for the students to discover other rules for order of operations (see Comment 5). Some students may wish to insert parentheses to show how numbers were grouped for computations.

Focus Teacher Activity (cont.)

ACTIONS

5 Give each group of students a sheet of butcher paper and a marking pen. Ask them to write a summary of what they think are the rules for the order of operations. When their summaries are completed, ask the groups to post them on the wall and to explain their observations. Develop a class version of the rules for order of operations to post permanently in the classroom.

6 Write the following expressions on the overhead and ask the students to compute the value of each. Have volunteers share their methods with the class.

a) $4 \times (5 + 2) \div 2 \times (3 - 1)$

b) $3(16 - 5) + 7 - 5 \times 2$

c) $(9 + 1)2 \div ((6 - 2) \div 2) - 10$

COMMENTS

5 You may wish to point out that when students think they see a pattern in how the operations are carried out, their ideas about the pattern are *conjectures* or *hypotheses.* When they try out an idea to see if it works on other problems, students are *testing the conjecture.* If they decide the idea works in all situations, they make *generalizations* about how it works. These are important aspects of "doing mathematics."

If a group asks for more clues regarding the rules, you might show them other problems that are worked correctly according to the order of operations. (This lets students know you believe they can complete the task and encourages them to stay engaged.)

The order of operations requires that: computations within parentheses should be carried out first; then products and quotients should be computed in the order they occur from left to right; and finally sums and differences should be computed in the order they occur from left to right. Exponents, which are evaluated before multiplication or division, are introduced in Lesson 25. It is important that students understand these rules are arbitrary and were established to avoid confusion.

6 According to the order of operations these expressions have the following values:

a) $4 \times (5 + 2) \div 2 \times (3 - 1) = 4 \times 7 \div 2 \times 2$
$= 28 \div 2 \times 2$
$= 14 \times 2$
$= 28$

b) $3\,(16 - 5) + 7 - 5 \times 2 = 3 \times 11 + 7 - 10$
$= 33 + 7 - 10$
$= 40 - 10$
$= 30$

c) $(9 + 1)2 \div ((6 - 2) \div 2) - 10 = 10 \times 2 \div (4 \div 2) - 10$
$= 20 \div 2 - 10$
$= 10 - 10$
$= 0$

This is an appropriate time to investigate students' calculators to see whether order of operations are "built in" and to discuss how to use parentheses on the calculator. A simple test such as entering $2 + 3 \times 5$ will show if the calculator performs according to the order of operations.

TEACHER NOTES:

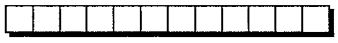
Follow-up Student Activity 4.2

NAME _____ DATE _____

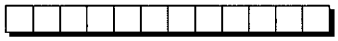
1 Carlos said that he determined the number of tile in the picture at the right by finding (4 + 2) + 5. Maria said she thought about it this way: 4 + (2 + 5). Who was right? Why?

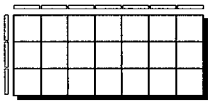
2 Next to each of the following diagrams:

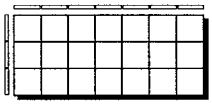
- Write an equation that represents a different method of "seeing" and counting the number of tile in the diagram.
- Subdivide each diagram to illustrate your methods.

a) 

b)

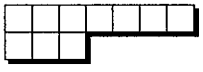
c) 

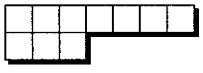
d) 

e) 

f)

g)

h) 

i) 

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Follow-up Student Activity (cont.)

3 Use the rules for order of operations that we discovered in class to solve these computations. Next to each computation, write the answer. Then write an explanation of each step you used to get the answer.

a) $10 \times 7 - 12 \times 5 =$

b) $4 + 2 \times 5 - 12 \div 6 =$

c) $8 \div 4 \times (2 + 3) - 4 + 7 =$

d) $12 \times (8 - 6) - 8 \times 0 + 1 =$

e) $13 - 6 \div 2 \div 3 =$

f) $24 + 8 \div 4 - 6 + 14 =$

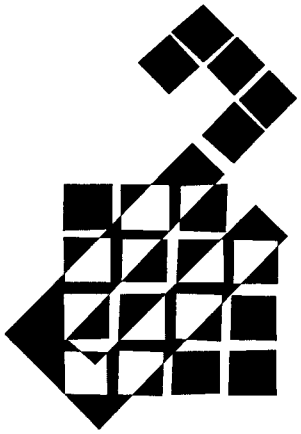
4 Invent a new set of rules for order of operations. On another sheet of paper, explain what your rules are, and then write the answers to d), e), and f) above using your rules.

Visual Reasoning

Lesson 5

THE BIG IDEA

Patterns are found throughout mathematics and can be used to help students understand concepts and solve problems. Studying patterns helps students to develop intuitions about algebra, in particular, about the concept of a variable.



CONNECTOR

OVERVIEW

Students explore ways of viewing and recording the number of tile in an arrangement of tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per group of students.
- ✔ Tile for use at the overhead.
- ✔ Connector Master A, 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

Students use visual reasoning to predict the size and shape of the 4th, 5th, 10th, and other figures in sequences of arrangements of tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Tile for use at the overhead.
- ✔ Focus Master A (optional), 1 transparency or 1 copy per group.

FOLLOW-UP

OVERVIEW

Students make observations, predictions, and generalizations about visual patterns.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 5.1, 1 copy per student.
- ✔ 2-cm grid paper (optional, see *Blackline Masters*), 2 sheets per student.

LESSON IDEAS

JOURNALS
Have students describe in their journals experiences with Problem 6 on the Follow-up.

PORTFOLIOS
This Follow-up could go in students' Growth-folios (see *Starting Points*) as baseline information about their abilities to generalize patterns and explain their thinking. Later, students can compare this to related activities and look for evidence of growth.

TIMING
It isn't necessary to explore every problem in this Lesson or Follow-up. Because visual patterning comes up in several lessons, you may want to move on after 3-4 hours of exploration.

QUOTE
Identifying patterns is a powerful problem-solving strategy. It is also the essence of inductive reasoning. As students explore problem situations appropriate to their grade level, they can often consider or generate a set of specific instances, organize them, and look for a pattern. These, in turn, can lead to conjectures about the problem. Students should be encouraged to validate these conjectures by constructing supporting arguments, which can be at many levels of sophistication.

NCTM Standards

FOLLOW-UP
In general, Follow-ups are intended to help students clarify their understanding and questions. Following are some suggestions for using Follow-ups:

- Assign 1 or 2 Follow-up problems as homework (or let students pick 2) after each day of the lesson.
- Instead of going over all problems in class, have students ask the class for clues about certain problems (without giving away answers). Sometimes give other clues during class, like: "Mary's observation may be helpful on Problem 3." "This activity could give you insights about Problem 5," etc.
- To encourage more detailed responses, have students write responses on other paper.
- Don't collect Follow-ups until the Lesson is completed. On other days have students turn in a daily report (see *Starting Points*) telling what they did and what they have questions about.
- Encourage students to refine and revise their work as they gain insights.
- Have students use a Follow-up Assessment Guide (see *Starting Points*) before turning in a completed Follow-up.

SELECTED ANSWERS

Each solution given is one of several possibilities:

1. a) The first figure has 1 tile; the 2nd has one tile in the center with 1 tile attached to each of its sides; the 3rd has a center tile and lines of 2 tile attached to each of its sides. So the 50th figure has a center tile with lines of 49 tile attached to each of its sides.

   b) $49 + 49 + 49 + 49 + 1$

   c) $(4 \times 50) - 3$ (4 "arms" of 50 tile; subtract 3 since the center tile is counted 4 times)

2. Method 1: A row of 36 tile across the bottom, with a column of 2×35 tile sitting on top of the far left tile;
   $36 + (2 \times 35) = 36 + 70 = 106$ tile.

   Method 2: An "L" formed by 1 corner tile plus 3 groups of 35 (1 extending to the right and 2 extending up);
   $1 + (3 \times 35) = 106$ tile.

3. a) The 70th figure has a 70×70 square with rows of 72 tile attached to the top and bottom sides of the square; this figure contains $4900 + 144 = 5044$ tile.

   b) The 70th figure is a 70×72 rectangle with 1 tile added on each corner; $70 \times 72 + 4 = 5044$ tile.

   c) The 20th figure.

Connector Teacher Activity

OVERVIEW & PURPOSE

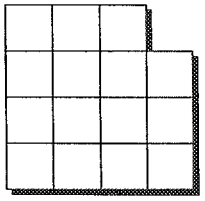
Students explore ways of viewing and recording the number of tile in an arrangement of tile. Discussion centers on recognizing a variety of counting methods and on writing number statements that reflect those methods.

MATERIALS

- ✔ Tile, 30 per group of students.
- ✔ Tile for use at the overhead.
- ✔ Connector Master A, 1 copy per group and 1 transparency.

ACTIONS

1 Place the students in pairs or small groups and distribute tile to each group. Form the following arrangement of tile at the overhead and ask the groups to form it at their tables.

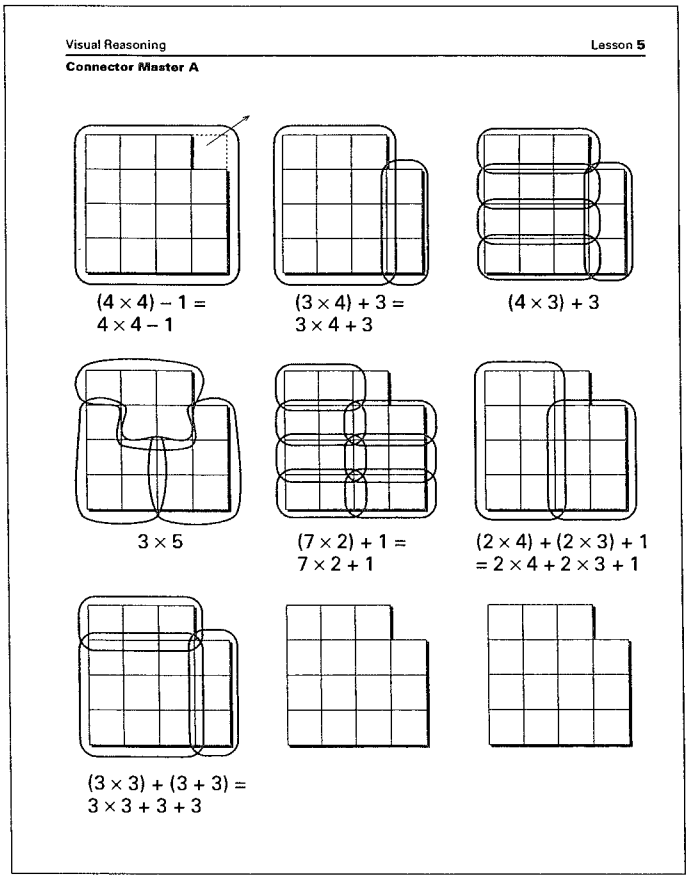


2 Point out that one way to determine the number of tile in the above arrangement is to count them 1 by 1. Ask the groups to find several other ways to "see" the total number of tiles. Give each group a copy of Connector Master A and ask them to record each of their methods and write a number statement to describe each method. Have volunteers show their methods on a transparency of Master A.

COMMENTS

1 To avoid influencing students' thinking by the order in which you place the tile on the overhead, turn the overhead projector off while you form the figure.

2 A few examples of ways students may view the arrangement are shown below on a copy of Master A. Students may emphasize the order of their counting methods by including more parentheses than is necessary (because of order of operations). This is okay.



Connector Teacher Activity (cont.)

ACTIONS

3 Ask the students to imagine expanding this arrangement of tile so that it is 10 tile high and 10 tile across and still has 1 tile missing in the corner. Have them determine the number of tile in this new arrangement. Repeat as appropriate for arrangements 20 tile high and 20 across, 100 by 100, etc.

COMMENTS

3 If students have difficulty with this, you might encourage them to try a variety of the methods which came up during Action 2. For example, using the first method in Comment 2, a similar arrangement with 10 tile high and 10 tile across would have $(10 \times 10) - 1 = 99$ tile.

Focus Teacher Activity

OVERVIEW & PURPOSE

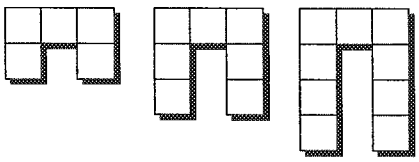
Students use visual reasoning to predict the size and shape of the 4th, 5th, 10th, and other figures in sequences of arrangements of tile. They examine these visual patterns from many points of view and have the opportunity to extend their own and their classmates' ways of thinking about the patterns. This is the first of many Visual Mathematics patterning activities that lay important groundwork for the study of algebra.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Tile for use at the overhead.
- ✔ Focus Master A (optional), 1 transparency or 1 copy per group.

ACTIONS

1 Arrange the students in groups and give tile to each student. Form the 3 tile figures shown below on the overhead.

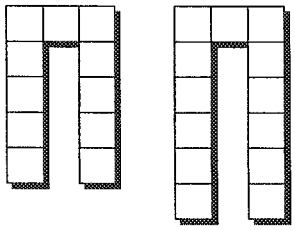


Tell the students the above figures are the first 3 figures in a sequence that is based on a pattern that you have in mind. Ask them to use their tile to form what they think are most likely to be the 4th and 5th figures in your sequence, based on what they observe about the first 3 figures. Have volunteers build their 4th and 5th figures at the overhead and describe how they decided their shape and size.

2 Ask the students to each imagine in their mind's eye what they believe the 10th figure in the sequence would look like, without building the intervening figures. Have volunteers build or sketch their 10th figure at the overhead and describe how they decided its shape and the number of tile it contains.

COMMENTS

1 Encourage students to think and work privately before sharing, thus allowing several ways of thinking about the pattern to emerge. The 4th and 5th figures most commonly formed by students are shown here:

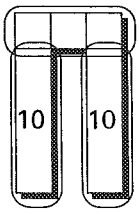


Asking students to guess what pattern you have in mind, allows you to acknowledge all ideas but focus on a specific pattern. You may want to extend an interesting alternate pattern suggested by a student.

2 Some students may find imagining larger figures difficult at first. However, this usually becomes easier after listening to others explain their reasoning. Thus, it is important to elicit a variety of approaches.

Here are some possible descriptions of the 10th figure, based on the pattern suggested in Comment 1:

"The 1st figure has 3 tile on top and 1 tile on each side; the 2nd has 3 on top and 2 on each side; the 3rd has 3 on top and 3 on each side. So the 10th has 3 tile on top and 10 on each side, or 10 + 10 + 3 = 23 tile."



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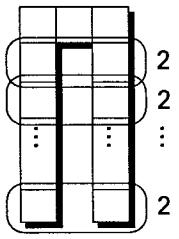
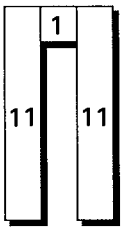
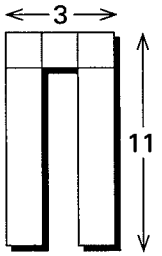
Focus Teacher Activity (cont.)

ACTIONS







3 Ask the students to imagine the 20th figure in the sequence, including its shape and the number of tile it contains. Ask for volunteers to describe their mental images of the 20th figure and their methods of determining the number of tile in it.

COMMENTS

2 (continued.)

"The 1st figure has 5 tile; for each subsequent figure, you add 2 tile (1 on each side); so the 10th figure has 5 plus 9 groups of 2 tile, or $5 + 9 \times 2 = 5 + 18 = 23$ tile."

"The 1st figure has 2 on each side and 1 in the middle of the top row; the 2nd has 3 on each side and 1 in the middle; and the 3rd has 4 on each side and 1 in the middle. So the 10th has 11 on each side and 1 in the middle of the top row, or 11 + 11 + 1 tile."

"The 1st is a 3 by 2 rectangle with 1 tile missing tile; the 2nd is a 3 by 3 rectangle with 2 missing tile; and the 3rd is a 3 by 4 rectangle with 3 missing. So the 10th is a 3 by 11 rectangle with 10 missing tile. Hence, it has $3 \times 11 - 10$ tile."

If at first students have difficulty describing their thinking, you could share how you "see" the 10th figure and how you think about the number of tile it contains. It is important to keep in mind throughout this lesson that this is the first of many lessons involving visual patterning, and therefore, it is not intended that students show "mastery" of the topic. (See *Opening Eyes to Mathematics*, for some introductory patterning activities.)

3 It is interesting to poll students to see how many adopted or adapted methods they heard others share in Action 2 to help them think about the 20th figure. This provides an opportunity to discuss the value of sharing ideas.

Focus Teacher Activity (cont.)

ACTIONS

4 Select one of your student's methods from Action 2 or 3. Ask the class to imagine the 50th (and/or 100th, 75th, etc.) figure using that student's method and to determine the number of tile the figure contains. Discuss. Repeat as appropriate.

5 (Optional) Pose one or more of the following questions for groups to investigate about the preceding sequence. Ask for volunteers to share their group's methods.

a) How many tile are in the figure that has 80 "empty spaces" (i.e., tile missing in the middle row)?

b) Which figure has 69 tile? 50 tile?

c) Two consecutive figures (one is right next to the other) have a total of 156 tile. Which figures are they?

COMMENTS

4 Choose the number of the figure according to the students' comfort level with the preceding actions.

This action engages students in generalizing methods and provides encouragement for trying more than one approach. Often when a student volunteers an especially interesting or useful method, the class attaches the student's name to that method. This can be very gratifying to the student who volunteered the method.

5 You may find it helpful to encourage students to use methods other than trial and error. While trial and error works, it doesn't promote the "algebraic thinking" that generally occurs when students base their thinking on visual relationships.

a) The 80th figure has 80 empty spaces and has $(2 \times 80) + 3 = 163$ tiles.

b) Notice that if the top row of 3 tile are removed from a figure, 2 columns are left and the number of tile in each column is the same as the number of the figure. Hence, the figure that contains 69 tile is the $^{(69-3)}/_{2} = 33$rd figure.

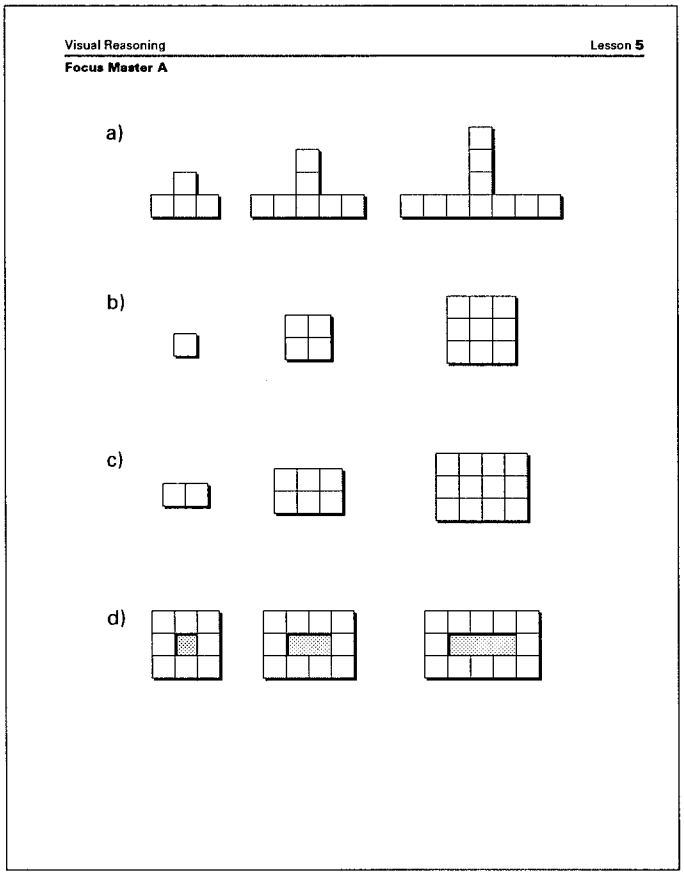
No figure has 50 tile. Notice that removing 3 tile leaves 47 tile, which can't be divided into 2 equal columns.

c) Here is one possible method: If you remove 2 tile from the larger figure, you are left with 2 copies of the smaller figure, each with $154 \div 2 = 77$ tile. Using methods such as those in a) and b) above, the 37th figure has 77 tile. Therefore, the 37th and 38th figures have a total of 156 tile.

Focus Teacher Activity (cont.)

ACTIONS

6 Have students investigate one or more of the tile patterns on Focus Master A. Discuss.



COMMENTS

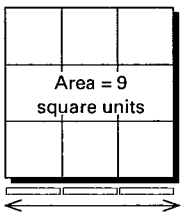
6 As an alternative to building the figures on the overhead, you could use a transparency of Focus Master A or give each group 1 copy.

One way to carry out this Action is to have small groups record on poster paper several different ways of viewing selected figures (the 4th, 10th, 20th, 100th, etc.) in each pattern you assign. Have them also describe ways of determining the total number of tile each figure contains. As you circulate while groups work, lay groundwork for a rich discussion during the Lesson 6 Connector by pointing out evidence you notice that students' inner mathematicians are at work.

Let students know that you are most interested in the quality of the mathematical content of their posters (as compared to their artistic qualities). You could provide a scoring guide, or have students help create one (see *Starting Points*), to use as a basis for evaluating their posters. One way to give feedback to groups is to write two "I appreciate..." and two "I wish..." comments about each poster. Groups can also provide this information to each other.

If appropriate when discussing pattern b), point out that the numbers 1, 4, 9, 16, ... which represent the numbers of tile in the figures of this pattern, are historically called *square* numbers. A square number of tile is any whole number of tile that can be arranged to form a perfect square with no overlaps or gaps. (To reinforce this idea you could have students select a handful of tile and, without counting the tile, give a "visual proof" that they do or do not have a square number of tile.)

Note that the length of each side of a square is called the *square root* of its area. For example, if the area of a square is 9 square units, then the length of its side is $\sqrt{9}$ (read "square root of 9") linear units, which also happens to be 3 linear units. Hence, the lengths of the edges of the squares in sequence b) are: $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt{25} = 5$, ...

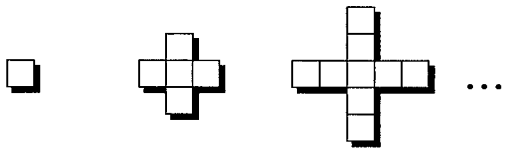


Length = $\sqrt{9} = 3$
linear units

Follow-up Student Activity 5.1

NAME _____ **DATE** _____

1 The first 3 figures in a pattern are shown below. Cut out squares and form what you think is the 4th figure. Sketch your 4th figure below.

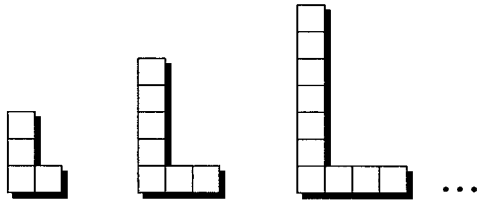


a) Assuming your pattern continues, explain how you think these 3 figures give you clues to what the 50th figure looks like.

b) Tell how (other than building the figure and counting tile) to find the total number of tile in the 50th figure.

c) Describe another method (other than building and counting) of finding the number of tile in the 50th figure of the pattern above.

2 The first 3 figures in another pattern are shown below. Form what you think is the 4th figure. Draw your 4th figure below.



Assuming your pattern continues, explain *two* different methods (other than building the figure and counting) of telling what the 35th figure looks like and how many tile it contains.

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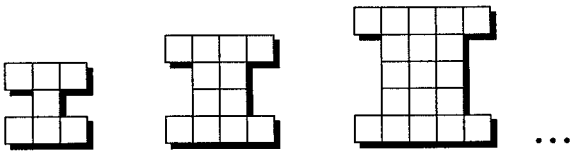
Follow-up Student Activity (cont.)

3 The first 3 figures in another pattern are shown below:



a) Tell how many tile you think are in the 70th figure and explain how you decided this number.

b) Tell another method (other than building and counting) of finding the number of tile in the 70th figure.

c) Suppose a certain figure in the above pattern has exactly 444 square tile in it. Which figure is it? Explain how you decided this.

4 Create the first 4 figures in an interesting pattern of tile figures. Sketch your 4 figures below.

5 Describe your pattern in Problem 4 and tell what the 20th figure in your pattern looks like.

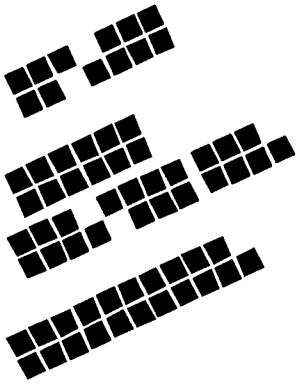
6 Cut out squares from the attached grid paper and build the first 3 figures in the pattern in Problem 1. Get an adult to share with you how they "see" the 20th figure in the pattern. Repeat this process for Problems 2 and 3. If needed, help the adult by sharing ways you "see" each 20th figure. On another sheet, describe what happened.

Odd & Even Numbers

Lesson 6

THE BIG IDEA

Explorations of certain patterns for odd and even numbers can lead to insights about the concepts of oddness and evenness and can enable students to make generalizations about operations with odd and even numbers.



CONNECTOR

OVERVIEW

Students identify and discuss ways they can show evidence they are learning and growing as mathematicians.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Butcher paper cut in narrow (3-4 inches) strips, 10 per group of students.
- ✔ Marking pens, 1 or more per group.
- ✔ Journals (optional).
- ✔ Connector Master A, 1 copy per group.

FOCUS

OVERVIEW

The concept of oddness and evenness of numbers becomes visually clear when models for odd and even numbers are developed from patterns of tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Odd and even demonstration cards (see Comment 14).
- ✔ Grid paper (1- or 2-cm size, see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ Scissors, 1 or more pairs per group.
- ✔ Tile for use at the overhead.

FOLLOW-UP

OVERVIEW

Students use models for odd and even numbers to make generalizations about the oddness or evenness of sums and differences.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 6.1, 1 copy per student.

LESSON IDEAS

JOURNALS

Prior to discussion of questions a)-g) from Focus Action 13, students could gather their thoughts by writing ideas in their journals. Then, after discussion, they could add new thoughts or insights to their journal entry. It is important to regularly encourage private "think time" before class discussion of an idea or problem and to allow time for private reflection after discussions. Having students write their "before and after" ideas in journals heightens their awareness of the development of their thinking.

PARENT INVOLVEMENT

The blackline masters *Visual Mathematics* Philosophy (see Lesson 1), and Evidence I'm Growing I, and II (see the blackline section of *Starting Points*) are useful to share with parents during an Open House or conference. Later parents could be invited to help a student look for particular evidence in a completed assignment.

ASSESSMENT

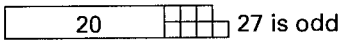
After the Connector, you could have students help you create an observation checklist (see *Starting Points*) that you (or groups or an observer) could complete during the Focus activity.

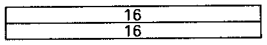
TEACHER NOTES:

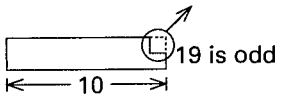
SELECTED ANSWERS

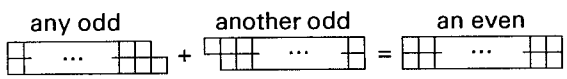
1. No. The student hasn't specified that one dimension of the rectangle must be 2. There could be an odd number of tile (e.g., a 3×5 rectangle).

2. Here are some possibilities:

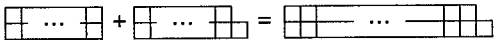
   a) 

   b) 

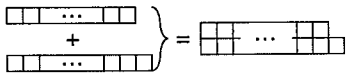
   c) 

3. a) 

4. Since any 2 consecutive counting numbers differ by 1, one is always odd and the other even. Their sum is odd as shown here:

   Or, the sum of 2 consecutive counting numbers can be shown as follows:

5. The 90th odd number contains a row of 89 tile on top of a row of 90 tile.

9. Even numbers formed by 2 rows of an odd number of tile (i.e., only the "odd-numbered even numbers" such as 2, 6, 10, 14, 18, 22, etc.).

Connector Teacher Activity

OVERVIEW & PURPOSE

Students identify and discuss ways they can show evidence they are learning and growing as mathematicians. As the emphasis of lessons shift to more group investigations, more complex mathematical ideas, and to students monitoring their own development, it is important for students to recognize actions, questions, difficulties, and insights that are signposts of mathematical growth.

MATERIALS

- ✔ Butcher paper cut in narrow (3-4 inches) strips, 10 per group of students.
- ✔ Marking pens, 1 or more per group.
- ✔ Journals (optional).
- ✔ Connector Master A, 1 copy per group.

ACTIONS

1 Read the following to the students and ask them to picture the situation in their minds as they listen. Pause briefly between sentences, giving the students time to create meaningful images.

> **Picture yourself hard at work during an activity in our math class. You are understanding what is going on and you want to learn more. You can feel the mathematician within you at work! What do you notice about yourself? What are you doing, saying, and feeling that show you think mathematically and understand the math?**
>
> **Visitors are in the classroom, watching you closely. They can't talk with you, but they can tell that you are a mathematician at work. And they can tell that you want to learn and grow in math. What are they seeing and hearing that makes them feel this way?**

2 Have each student spend 2-3 minutes writing about their images during Action 1. Then have the students in each group do a "round robin" sharing of their most vivid images. Ask them to be sure that each person gets equal time to share.

3 Distribute a copy of Connector Master A, 10 narrow strips of butcher paper, and some marking pens to each group of students. Ask each group to reach consensus on what they believe are the 8-10 most meaningful "I statements" that respond to the questions on Master A, and to write these statements on their strips. Tape a copy of Master A to the center of a large sheet of poster paper and have the groups attach their statements around it. Discuss.

COMMENTS

1 Create a relaxed setting that will encourage students to form detailed mental images. You might ask students to fold their hands and close their eyes so they will not be distracted while listening. Sometimes it helps to privately count to 10 to ensure a long enough pause between sentences. The questions posed are meant to stimulate images, and are not to be answered now, other than in each student's own mind.

2 Students could write about their images in their journals. To assure each student an opportunity to talk about their images, you might suggest that each person in a group gets uninterrupted sharing time (say ½-1 minute) with no questions or comments by others until everyone has shared.

3 Cut strips in widths of 3-4 inches. If the class is very large, you may wish to reduce the number of statements each group writes. Connector Master A contains a copy of the statement written in Action 1.

Discussing and writing "I statements" (e.g., "I volunteer my ideas during discussions" or "I honor my disequilibrium") can give students ownership of the ideas and

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Evidence I'm Growing I

Here are some ways that I show I think mathematically and understand the math we do:

a) I support my conclusions with valid mathematical reasoning.

b) I give more than one solution and/or more than one approach to problems.

c) My explanations show careful reasoning and deep thinking.

d) I use models and methods from class to solve problems.

e) I make connections among different ideas in math and between math and other subjects.

f) My explanations show that I am constructing deeper and broader understandings of math concepts.

g) I make and test conjectures, and I make generalizations about ideas and methods.

Evidence I'm Growing II

Here are some ways that I show I want to learn and grow in math:

a) I listen actively to others and I respect their views.

b) I volunteer my ideas in group discussions.

c) I ask thoughtful questions of my classmates, my teacher, and myself.

d) I willingly struggle with ideas or problems that seem challenging.

e) I share my difficulties with problems and ideas.

f) I show joy with my AHA!'s in math and honor my disequilibrium.

g) I extend problems and explore my own "what-if" questions.

h) I pay attention to my thought processes and to ways my understanding is developing.

COMMENTS

3 (continued.)
build confidence. Statements that groups record could be based on students' images from Action 1 or on other ideas that come up during group discussion.

Two important purposes of this math class are to enable students to get in touch with the mathematician within them and to nurture and help that mathematician grow (see Lesson 1). Throughout *Visual Mathematics* students are asked to monitor their mathematical growth. The lists generated during this activity describe evidence of such growth, and can be used as a basis for many assessment activities and discussions throughout the course.

You may wish to add statements to the poster. If so, it is less intimidating to students if your ideas are offered as ways you reveal that you are learning and growing as a mathematician. The sheets "Evidence I'm Growing I and II" shown at the left are in the blackline section of *Starting Points* and they describe behaviors that are emphasized throughout *Visual Mathematics*. (These sheets have been printed in this form in case you wish to share them with parents or administrators to illustrate the kinds of activity you hope to nurture in students.) It isn't necessary that all of these behaviors come up now. Rather, you and students can add to the poster later as new ideas emerge during math activities.

If you have more than one class of mathematics students, you could have each class create a poster. During this activity in one class, remove any other classes' posters. Later classes could compare their posters. You (or student volunteers) could consolidate and/or summarize the classes' statements on a single poster for permanent display in the classroom, and make regular reference to it by drawing attention to examples of "mathematical behavior" you observe in students (e.g., when a student poses an "I wonder" statement or "what if" question, makes a connection between math ideas, generalizes a relationship, etc.). Encourage students to also acknowledge such behaviors in themselves and each other.

Focus Teacher Activity

OVERVIEW & PURPOSE

The concept of oddness and evenness of numbers becomes visually clear when models for odd and even numbers are developed from certain patterns of tile. These models are used to make generalizations about the oddness and evenness of sums, differences, and products.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Odd and even demonstration cards (see Comment 13).
- ✔ Grid paper (1- or 2-cm size, see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ Scissors, 1 or more pairs per group.
- ✔ Tile for use at the overhead.

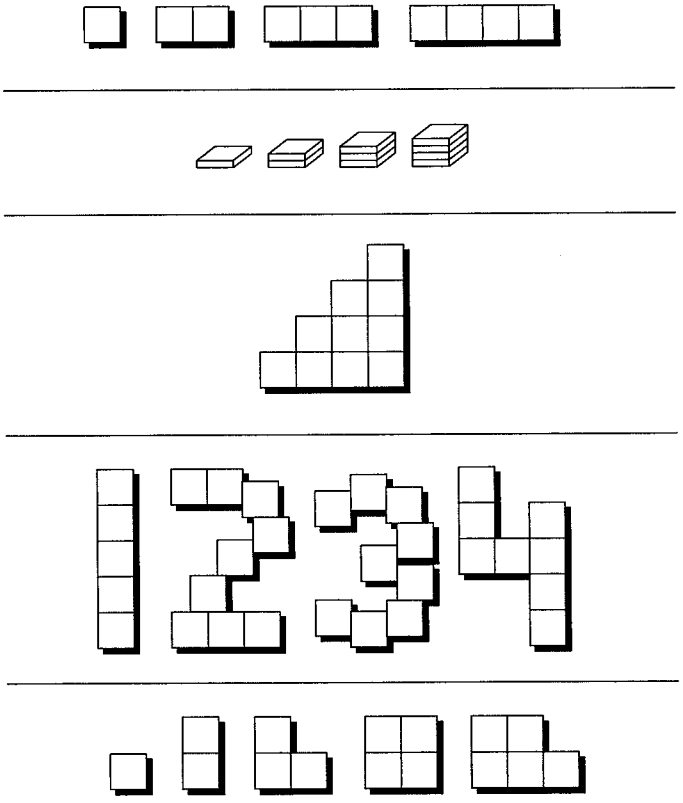
ACTIONS

1 Arrange the students in groups and distribute tile to each student. Write the first 7 counting numbers on the chalkboard or overhead. Tell the students to use their tile to represent each of these counting numbers. Encourage students to work privately so that a variety of ideas will emerge. Have the students observe the different ways that others have represented the numbers.

2 Select one student's representation of the numbers from 1 to 7 and, based on that representation, ask the other students to predict what the number 30 would look like. Discuss their reasoning. Repeat for other students' tile representations of the numbers from 1 to 7.

COMMENTS

1 There is no one "correct" way to represent the numbers. There are several ways for students to view others' ideas: have groups take "field trips" to other tables; have students share at the overhead; or you could summarize different ways on the overhead. Be prepared for variety of representations. Here are some examples:

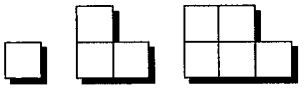


2 Encourage students to think of each others' representations as visual patterns for the counting numbers. (Note that the last example above is the basis for the odd and even number pattern explored in the rest of this lesson.)

Focus Teacher Activity (cont.)

ACTIONS

3 Place the following 3 figures on the overhead:



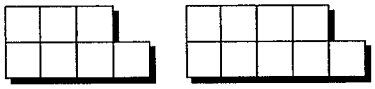
Tell the students that these are the first 3 figures in a sequence of figures that are formed according to a pattern that you have in mind. Ask them to each form these figures and to form what they feel are most likely to be figures 4 and 5 in this sequence. Have volunteers share their figures and reasoning at the overhead.

4 Ask the students to assume that figures 4 and 5 in the sequence are those illustrated in Comment 3 and, to imagine, based on that pattern, what the 20th figure looks like and the number of tile it contains. Discuss their images and reasoning.

5 Keep the first four or five figures of the previous pattern on the overhead. If it hasn't come up, point out that the numbers of tile in the figures of this pattern (1, 3, 5, 7, ...) are called *odd numbers*. Based on this visual pattern for odd numbers, ask the students to describe the 100th odd number. Repeat for the 23rd and 80th.

COMMENTS

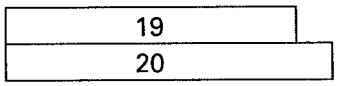
3 If the fifth example illustrated in Comment 1 was discussed, students may relate this pattern to that discussion. Acknowledge all reasonable ideas. Most students will form the following for figures 4 and 5:



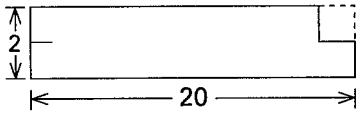
Throughout this activity, acknowledge ways students show evidence their inner mathematicians are at work, based on ideas from the Connector.

4 Some examples of student descriptions follow:

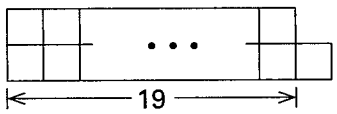
"The first figure has one tile on the bottom row and none on the top row; the second has two on the bottom row and one on the top; the third has three on the bottom and two on the top. So, the 20th has 20 tile on the bottom row and 19 on the top row, for a total of 20 + 19 = 39 tile."



"Each figure is a '2 by' rectangle with the top right corner tile missing. The length of the rectangle is always the same as the number of the figure. So, I see a 2 by 20 rectangle with the top right corner missing, or $(2 \times 20) - 1 = 39$ tile."



"To form each figure you add 2 tile to the previous figure (either to the left or right side), so the 20th figure is the first one plus 19 groups of 2 or $1 + (19 \times 2) = 39$."



5 One way to describe the 100th odd number is as "a 2 by 100 rectangle with one tile missing from the upper right corner," or $(2 \times 100) - 1 = 199$. Another description is a row of 100 tile with "a row of 99 tile on top of it," or 100 + 99 = 199. The 23rd and 80th odd numbers can be described similarly and have 45 and 159 tile, respectively.

Focus Teacher Activity (cont.)

ACTIONS

6 Ask the students to imagine the odd number that contains 55 tile and to determine which odd number it is. Have volunteers share their visual reasoning.

7 Repeat Action 6 for other odd numbers, such as 71, 77, 85.

8 Keep the first 5 figures in the sequence of odd numbers on the overhead, and ask the students to propose reasonable models for the missing numbers: 2, 4, 6, and 8, which are called *even numbers*. Have volunteers share their ideas.

COMMENTS

The models for odd and even numbers explored in this Lesson are introduced in *Opening Eyes to Mathematics*.

6 Three examples of students' reasoning include: "I added 1 tile to get 56, imagined them in a 2 by 28 rectangle, and then I removed the upper right corner. The dimension along the bottom is 28, so 55 must be the 28th odd number."

"I took away 1 tile to get 54 tile. I knew those would form $54 \div 2 = 27$ columns of 2 tile. I then replaced the 1 tile to get a total of 28 columns (27 columns of 2 tile plus 1 column of 1 tile). Therefore, 55 is the 28th odd number."

"I knew from Action 5 that 45 was the 23rd odd number. Each odd number is 2 more than the previous one. Since 45 is 10 less than 55 and there are 5 groups of 2 in 10, then 55 must be 5 figures after the 23rd. That is, it is the $23 + 5 = 28$th figure."

If students have difficulty with this, you could begin by posing smaller odd numbers; or share your ways of thinking about 55 and then pose other odd numbers for them to consider.

7 71, 77, and 85 are the 36th, 39th, and 43rd odd numbers, respectively.

8 Most students will suggest "filling in the missing corners," to form the following visual pattern for the even numbers:

It is important to acknowledge all reasonable suggestions. The preceding pattern for odd numbers and this pattern for even numbers are emphasized because they provide a visual way of distinguishing between odd and even numbers. These models reveal many important mathematical properties and relationships. They also extend conveniently to other mathematical ideas such as modeling operations with odd and even numbers.

Focus Teacher Activity (cont.)

ACTIONS

9 Place the first 4 even numbers, as illustrated in Comment 8, on the overhead. Ask the students to imagine and describe the 30th even number, based on this pattern. Repeat for the 25th, the 150th, and the 200th.

10 Ask the students to imagine the even number (based on the pattern in Comment 8) containing 42 tile and to predict which even number it is. Discuss their visual reasoning.

11 Repeat Action 10 for other even numbers such as 124, 206, 1000 (or ask students to suggest even numbers).

12 Distribute 1-2 sheets of grid paper and scissors to each group. Ask each group to cut out grid paper models to show visually, based on the preceding models for odd and even numbers and the meanings of the operations, why the following are odd or even.

a) 11 + 3

b) 14 + 9

c) 18 – 11

d) 3 × 7

e) 4 × 7

Have volunteers show their methods and results at the overhead.

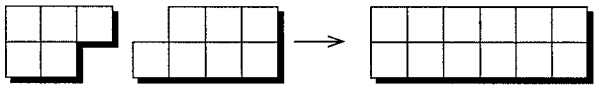
COMMENTS

9 Some descriptions of the 30th even number include: a 2 × 30 rectangle; 2 rows of 30 tile; and 30 stacks of 2 tile.

10 One possibility is: 42 tile can be arranged in 2 rows of 21, so it is the 21st even number.

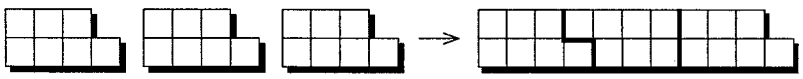
11 124, 206, and 1000 are the 62nd, 103rd, and 500th even numbers, respectively.

12 If students have difficulty, you could demonstrate adding 5 and 7 by pushing their corresponding pattern members together to get 6 groups of 2, or 12. Hence, combining the odd and even tile representations of 5 and 7 shows that 5 + 7 = 12 is an even number:

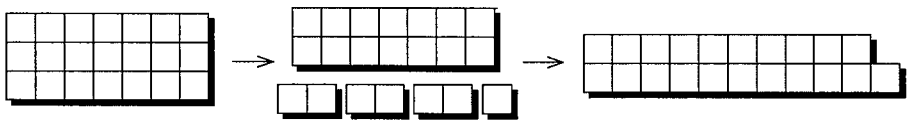


1- or 2-cm grid paper is appropriate (see *Blackline Masters*). Depending on the needs of your class, you may wish to select other examples for students to cut out and model.

d) Use the repeated addition method for multiplication to show 3 × 7 as 3 groups of 7 added together:



Or, a second way of viewing d) is to use the area model of multiplication to determine 3 × 7 = 21 tile; and then rearrange these to show an odd number of tile.



Focus Teacher Activity (cont.)

ACTIONS

13 Ask students to picture odd and even numbers in their minds as they reflect on one or more of the following questions. Have each group discuss their ideas and reach consensus on answers to the questions you pose. Have groups explain or discuss the reasons for their answers.

a) If you add two odd numbers will the sum be even or odd?

b) If you add two even numbers will the sum be even or odd?

c) If you add an even number and an odd number will the sum be even or odd?

d) What happens when you add three odd numbers? two odd numbers and an even number? two evens and an odd? five odds and one even? etc.

e) What happens when you multiply an odd number times an even number? an even times an even? an odd times and odd?

f) Is the result odd or even when you subtract an odd number from an odd number? an even from an even? an odd from an even? an even from an odd?

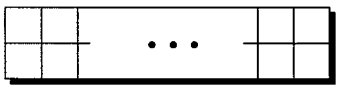
g) Why does the "rule" for determining whether a number is odd or even by looking at the ending digit work? (i.e., Why is it recessary to look only at the units digit to determine whether a number is even or odd?)

14 Ask students to use models for odd and even numbers to explain why the following are odd or even: 17, 42, 79, and 106.

COMMENTS

13 Some students may find the models they formed in Action 12 useful for picturing these sums and products.

It is helpful to have some generic odd and even patterns made out of cardstock to illustrate or reinforce the students' conclusions. These large demonstration cards have no particular number assigned and look as follows:



even card

odd card

You may want several copies of each demonstration card if you plan to discuss parts d) and e). To reinforce the meanings of subtraction, when discussing f) include both the take away and difference methods of subtraction that were introduced in Lesson 2.

One possible response to g) follows: When the value of the units digit is subtracted from any number, the result is a multiple of ten, which is even. Hence, adding the value of the units digit back will produce an even number if the units digit is even (e.g., see b) above) and an odd number if the units digit is odd (e.g., see c) above.

14 Here are two examples of students' visual reasoning that 79 is odd: "80 is even because it is 2 rows of 40, remove the upper right corner tile to get 79, an odd number;" or "10 is even (it's 2 rows of 5 tile), 79 is formed by 7 tens (which is even) plus 9 (which is odd) and an even plus and odd is always odd."

TEACHER NOTES:

Follow-up Student Activity 6.1

NAME _____ DATE _____

1 A student said that "any collection of tile that can be arranged to form a rectangle with no gaps or overlaps must contain an even number of tile." Is the student correct? Explain and draw diagrams to show why you think this way.

2 Draw diagrams of these numbers so that it is possible to "see" without counting whether each number is odd or even.

a) 27 b) 32 c) 19

3 For each of the following, draw a picture to show why you think the answer is *always* odd, *always* even, or *sometimes* odd and *sometimes* even.

a) the sum of 2 odd numbers

b) the sum of an even number and an odd number

c) the difference between an odd an even number

d) the difference between 2 odd numbers

4 Is the sum of two consecutive counting numbers *always* odd, *always* even, or *sometimes* odd and *sometimes* even? (Note: consecutive means one comes right after the other.)

Draw a diagram to show why you feel your answer is correct.

(Continued on back.)

Follow-up Student Activity (cont.)

5 According to the model explored in class, the first 4 odd numbers look like this:

Using this pattern, imagine the odd number that contains 179 tile. Write a description of what it looks like and explain how to tell which odd number it is (the 1st, 2nd, 3rd, etc.).

6 Sketch a diagram of the first 4 even numbers, based on the model explored in class.

7 Imagine the 50th even number in your pattern for Problem 6. Write a description of it and tell how many tile it contains.

8 Imagine the even number with 208 tile. Describe what it looks like and explain how to tell which even number it is (the 1st, 10th, 20th, etc.)

9 This diagram shows that 14 can be divided into 2 equal odd numbers.

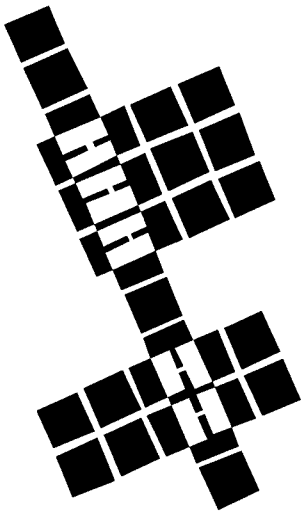
What are all the other numbers that can be divided into 2 equal odd numbers? Explain why this is so.

Factors & Primes

Lesson 7

THE BIG IDEA

Rectangular representations of counting numbers reveal the structures of prime and composite numbers. Throughout *Visual Mathematics* rectangles and their dimensions are used to represent products and their factors. These models provide a basis for understanding products of whole numbers, fractions, decimals, integers, and ultimately algebraic expressions.



CONNECTOR

OVERVIEW

Students examine and build several tile figures in order to clarify terminology used in the Focus activity.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student (at least 90 per group).
- ✔ Linear pieces, 30 per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.

FOCUS

OVERVIEW

Students use rectangular regions and their dimensions as models for counting numbers and their factors. By sorting and comparing these rectangular representations, students identify the unique characteristics of prime and composite numbers.

MATERIALS FOR TEACHER ACTIVITIES

Teacher Activity 1

- ✔ Tile, 30 per student.
- ✔ Linear pieces, about 30 per student.
- ✔ Focus Student Activity 7.1, 1 copy per student and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 3-4 sheets per group of students and 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.
- ✔ Butcher paper (optional), ten 6-inch strips per group.
- ✔ Marking pens (optional), 1 per group.

Teacher Activity 2 (optional)

- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Colored pencils, pens, or crayons, 1 each of 4 or more colors per student.
- ✔ Colored pens for the overhead (same colors as students have).

FOLLOW-UP

OVERVIEW

Students use rectangular regions to determine the factors of several numbers, to tell whether those numbers are prime or composite, and to solve a puzzle.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 7.2, 1 copy per student.

LESSON IDEAS

FOLLOW-UP
Keep in mind that, when used as homework assignments, Follow-ups are intended to span more than 1 day. To provide encouragement, you may wish to let students know that Problem 4 is intended to take an extended period to solve. One strategy they could use is to write down all the conclusions they can about the problem in 20 minutes, leave the problem for a while, and then return to the problem later (or on another day after talking over ideas with their groupmates).

JOURNALS
You could have students write their letters for Follow-up Problem 7 in their journals.

SELECTED ANSWERS

2. a) 1, 40, 2, 20, 4, 10, 5, 8
   b) 1, 32, 2, 16, 4, 8
   c) 1, 48, 2, 24, 3, 16, 4, 12, 6, 8
   d) 1, 23

3. 23 is prime because it has exactly 2 factors

4. Dimensions: 9×10
   Area: 90 square units
   Perimeter: 38 linear units

5. Clues a), b), d), g), and h) are sufficient to solve the puzzle.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students examine and build several tile figures in order to clarify the following: the meaning of "rectangular region;" relationships between a square and a rectangle; and the interpretation of "different" rectangles. This prepares the students for the Focus activity in which they use rectangular regions and their dimensions as models for counting numbers and their factors.

MATERIALS

- ✔ Tile, 30 per student (at least 90 per group).
- ✔ Linear pieces, 30 per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.

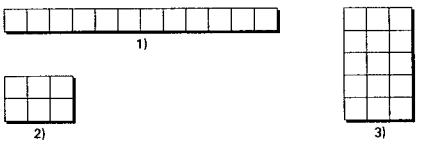
ACTIONS

1 Arrange the students in groups and distribute tile and linear pieces to each student. Place a transparency of Connector Master A on the overhead (or give each group a copy) and ask the groups to carry out the instructions. Discuss their results.

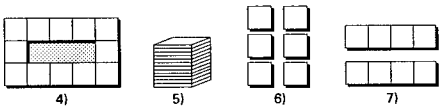
Factors and Primes Lesson 7

Connector Master A

These are rectangular regions:



These are not rectangular regions:



Use tile and linear pieces to form the following rectangular regions and to show their dimensions. Keep all of your rectangles on display for class discussion.

a) A rectangular region with dimensions 4 linear units by 6 linear units.

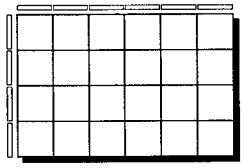
b) A rectangular region with 25 tile and one dimension 5 linear units.

c) All the possible *different* rectangular regions that each have exactly 12 tile.

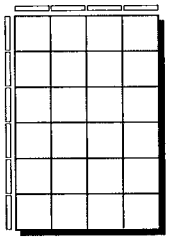
COMMENTS

1 Note that rectangular regions can be formed by sliding together rows, columns, or arrays of tile so they are arranged in the shape of a rectangle with no gaps or overlaps.

a) The region described looks like this:



or this:



b) Some students may suggest it is not possible to form this region based on a mistaken belief that squares are not rectangles. It is important to clarify that a square is a special kind of rectangle (but all rectangles are not squares).

c) In order to assure that students have enough tile to complete this, it is necessary to agree that for now "different" rectangular regions means no rectangle can be turned to exactly match the size and shape of another (i.e., they are not congruent). Thus, by disregarding orientation, there are three non-congruent 12-tile rectangles: 1 by 12, 2 by 6, and 3 by 4.

TEACHER NOTES:

Focus Teacher Activity 1

OVERVIEW & PURPOSE

Students use rectangular regions and their dimensions as models for counting numbers and their factors. By sorting and comparing these rectangular representations, students identify the unique characteristics of prime and composite numbers. This provides students with a meaningful mental image on which to base their thinking about numbers and their factors.

MATERIALS

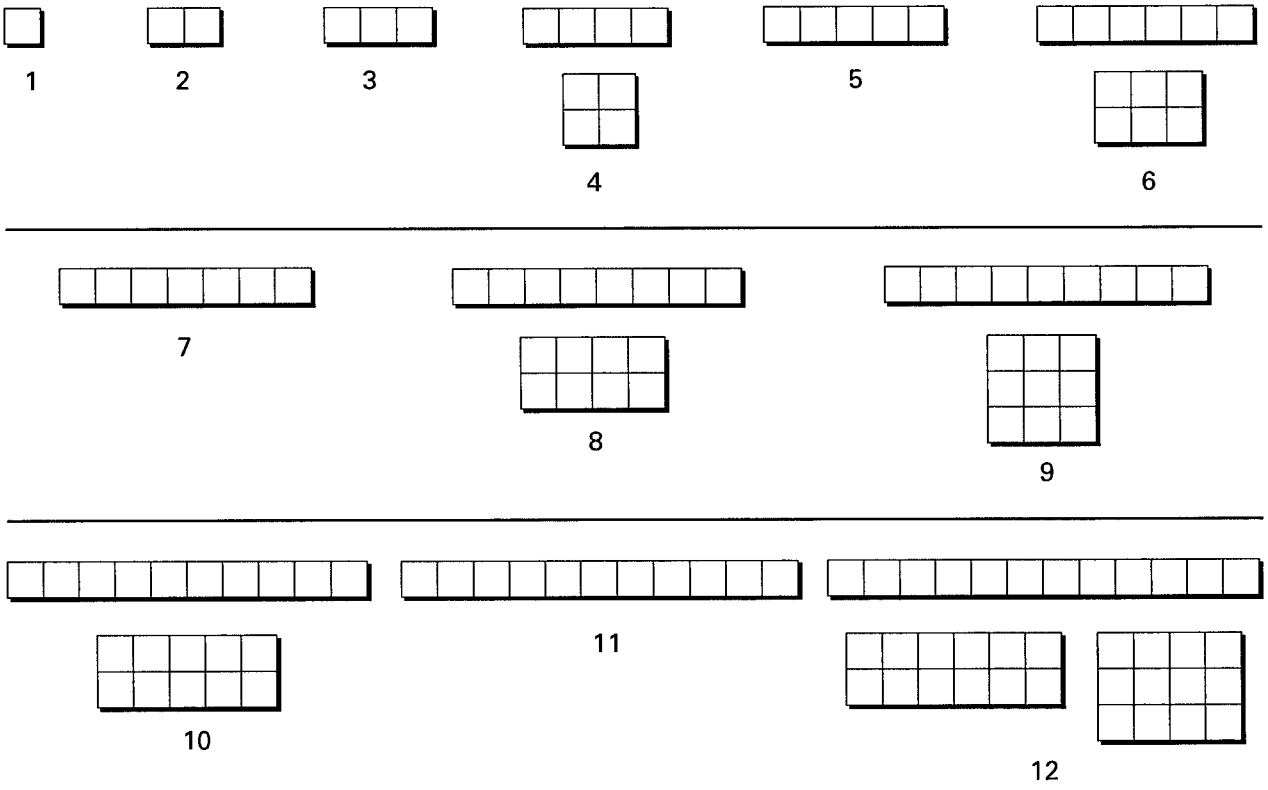
- ✔ Tile, 30 per student.
- ✔ Linear pieces, about 30 per student.
- ✔ Focus Student Activity 7.1, 1 copy per student and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 3-4 sheets per group of students and 1 transparency.
- ✔ Tile and linear pieces for use at the overhead.
- ✔ Butcher paper (optional), ten 6-inch strips per group.
- ✔ Marking pens (optional), 1 per group.

ACTIONS

1 Arrange the students in groups of 5 and distribute tile to each group. Ask the groups to form all of the different rectangular regions that can be made with 12 or fewer tile. Have them arrange the rectangles in order of the numbers they represent and keep the entire collection on display. Clarify directions as necessary.

COMMENTS

1 It will take a large space to arrange the tile. If tables are not available, you may wish to have students work on the floor. There will be 19 different rectangles using a total of 139 tile, as illustrated below.

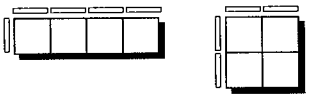


(Continued next page.)

Focus Teacher Activity 1 (cont.)

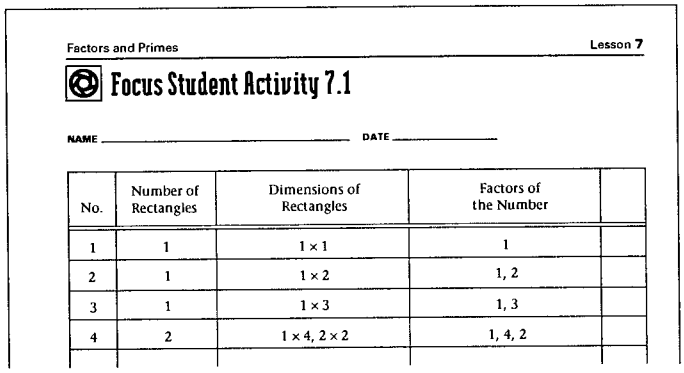
ACTIONS

2 Distribute linear pieces to each student. At the overhead, form the 2 different rectangles containing 4 tile and place linear pieces to indicate the dimensions of each rectangle, as shown below.



Define a *factor (divisor)* of a number as one of the dimensions of a rectangle representing that number. Point out that this suggests the factors of 4 are 1, 4, and 2. Have the students use their linear pieces to determine the factors of 8. Repeat for other numbers.

3 Give each student a copy of Focus Student Activity 7.1. Demonstrate the first few entries on a transparency of the activity. Ask students to complete the chart by referring to their rectangles and/or by imagining rectangles. Remind students not to destroy their rectangles.



Factors and Primes Lesson 7

Focus Student Activity 7.1

NAME _____ DATE _____

No.	Number of Rectangles	Dimensions of Rectangles	Factors of the Number	
1	1	1 × 1	1	
2	1	1 × 2	1, 2	
3	1	1 × 3	1, 3	
4	2	1 × 4, 2 × 2	1, 4, 2	

COMMENTS

1 (continued.)
You may need to remind students that during the Connector activity it was established that: squares are rectangles; rectangular regions are 2-dimensional and have no gaps or overlaps; and the orientation of a rectangle is not a distinguishing characteristic.

For ease of reference during the rest of this activity, it is helpful to have the students label the rectangles (e.g., with small numbered scraps of paper) to indicate the numbers they represent, as shown in the preceding diagram.

2 Caution students not to destroy any of their rectangles from Action 1 since they will be using them throughout the remainder of the activity.

This assumes that 1 square tile is a unit of area and the length of the side of a tile is 1 linear unit. (This model for factors is also introduced in *Opening Eyes to Mathematics*.) The factors of 8 are 1, 8, 2, and 4.

3 The first few entries for Focus Student Activity 7.1 are listed on the copy shown at the left.

Some students may wish to build or sketch rectangles for the numbers 13-20. Encourage this. The point of this activity is to provide a meaningful visual model that will facilitate the retention and recollection of factors rather than to promote the rote memorization of facts. Hands-on experiences forming and mentally picturing rectangles help to reinforce this model. Hence, even if students remember multiplication "facts" for numbers, encourage them to visualize rectangles representing the facts.

Focus Teacher Activity 1 (cont.)

ACTIONS

4 Ask the groups to make and record observations about the numbers 1-20 and their rectangles. What can they observe about the numbers by looking at their rectangles and at their charts? Discuss the groups' results.

5 Distribute 2-3 sheets of 1-cm grid paper to each group. Ask them to make diagrams of the rectangles for the numbers 1-20 and to arrange their diagrams so the rectangles form 3 groups: (1) those with exactly 1 factor, (2) those with exactly 2 factors, and (3) those with more than 2 factors. Discuss their results and ask them to find other numbers whose rectangles belong in groups (2) and (3).

6 Ask the students whether they think it is possible to find any other number, like 1, which has exactly 1 representation with identical dimensions, and to explain their reasoning.

7 Tell the students that numbers with exactly 2 factors are called *prime* numbers and that numbers with more than 2 factors are called *composite* numbers. The number 1 is therefore neither a prime number nor a composite number.

Have the students complete Student Activity 7.1 by placing a P or C in the last column to indicate if the number is prime or composite.

8 Give each group a blank sheet of paper. Ask the groups to make "rough sketches" of all possible rectangles for each of the numbers 24, 31, 72, 144 and to label their dimensions. Ask them to identify:

a) which, if any, of the numbers are prime, and

b) the factors of each number.

COMMENTS

4 It is impossible to predict what students might observe. One way to give credibility to student observations is to summarize and list them on the overhead or chalkboard as groups share. Or, you could give each group several 6-inch wide strips of butcher paper for recording their observations. Then have groups combine and post their strips to form a single chart on the classroom wall. Have blank strips available for other observations that come up during discussion. Observations like, "Some numbers have only one rectangle," and "Certain numbers have squares," are of particular interest.

5 The intent of this activity is to classify counting numbers according to their number of factors.

Some groups may prefer to cut out and arrange the rectangles, or to rearrange and add to the rectangles they formed in Action 1. However, recording the rectangles on grid paper will enable discussion and observations to continue on another day without rebuilding all of the rectangles.

6 There will be no other numbers grouped with 1. To be grouped with 1 and have identical dimensions the number would have to be square. But any other square number has at least 2 rectangular representations.

7 The factor approach is a precise way to define prime and composite numbers. Looking at rectangles and factors also makes it clear why the number 1 is neither prime nor composite and is the only such counting number.

8 Because students often want to include too much detail in their sketches, you may wish to discuss differences in expectations for a "rough sketch" and more accurate drawings. For example, a rough sketch won't show all the tile or linear pieces but can be labeled to indicate the area and dimensions (see next page).

(Continued next page.)

Focus Teacher Activity 1 (cont.)

ACTIONS

9 Pose questions like one or more of the following as challenges or ending activities for students to explore while looking at their tile rectangles, drawings, and Student Activity 7.1.

a) Some numbers are called *square numbers*. Looking at your tile rectangles, which numbers do you think they are? Will 72 be a square number? 48? 64?

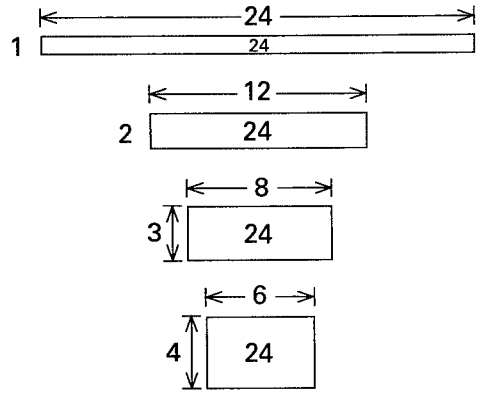
b) What is the first number that can be represented by exactly 4 different rectangles? 5 rectangles?

c) Which numbers have an odd number of factors? Why?

d) Look at the rectangular arrays of some numbers with exactly 3 factors. Can you find additional numbers with exactly 3 factors? What conclusions can you draw?

COMMENTS

8 (continued.)



24 is composite since it has more than 2 factors, as shown above.

31 is prime because it has exactly 2 factors, 1 and 31 (i.e., the only rectangle that can be formed from 31 tile has dimensions 1 by 31).

72 is composite and has factors 1, 72, 2, 36, 3, 24, 4, 18, 6, 12, 8, and 9.

The rectangles for 144 have dimensions of 1×144, 2×72, 3×48, 4×36, 6×24, 8×18, 9×16, and 12×12.

9 Many of these may have already come up during earlier discussions.

a) Those numbers which have a square among their rectangular regions are called *square numbers*: 1, 4, 9, 16, 25, etc. Square numbers were first discussed in Lesson 5.

b) 24 is the first number that has 4 rectangular regions. The first number with 5 rectangles is 36. There are other numbers with 4 or 5 rectangles.

c) Square numbers will always have an odd number of factors. Each rectangle, except the square, made from a square number of tile contributes 2 factors. The square contributes only 1 factor.

d) Only square numbers have an odd number of factors. Numbers like 4, 9, 25, and 49 have exactly 3 factors but numbers like 1, 16, 36, and 64 do not. Only numbers which are the squares of primes have exactly 3 factors.

Focus Teacher Activity 1 (cont.)

ACTIONS

e) On your chart (Student Activity 7.1) you recorded numbers with 1, 2, 3, 4, 5, and 6 factors. Do you think you could find a number with 8 factors? With *any* number of factors?

f) If two numbers have 7 as a factor, will their sum also have 7 as a factor? What about their difference? Why or why not?

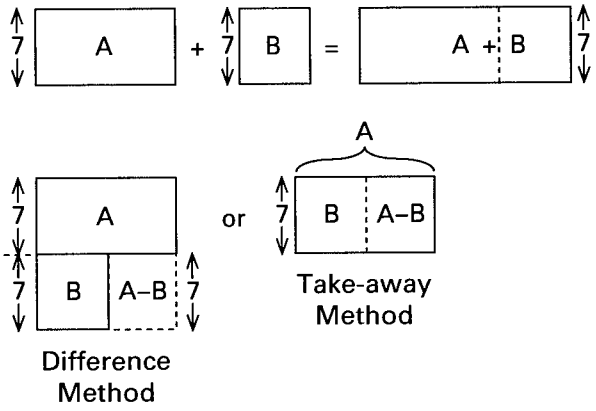
g) Suppose rectangles are sketched on grid paper, but dimensions aren't restricted to whole numbers. Now how many rectangles having area 36 can be drawn?

h) What are some other interesting questions you would like to explore related to your observations about relationships between numbers and their rectangles and factors?

COMMENTS

e) Given any positive counting number, it is possible to find numbers with that many factors. The following are all numbers with 8 factors: 24, 30, 40, 42, 54, and 56.

f) Since 7 is a factor of each number, a "7 by " rectangle (see rectangles A and B below) can be formed to represent each number. The following diagrams show why the sum and difference of the numbers must also form rectangles with dimension of 7.



g) Have extra grid paper available for experimentation. There are an infinite number of possibilities. For example, $\frac{1}{2} \times 72$, $\frac{1}{4} \times 144$, and $1\frac{1}{2} \times 24$ rectangles can be formed.

h) You may wish to select from students' ideas to explore now or later.

TEACHER NOTES:

Focus Teacher Activity 2 (optional)

OVERVIEW & PURPOSE

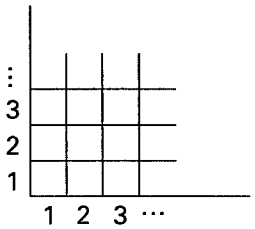
Students use logical reasoning and their knowledge of the relationship between the rectangular representations for numbers and the factors of those numbers to determine the size and position of various colored rectangles that cover a grid. This provides both a motivational and meaningful setting for developing competence with multiplication and division facts.

MATERIALS

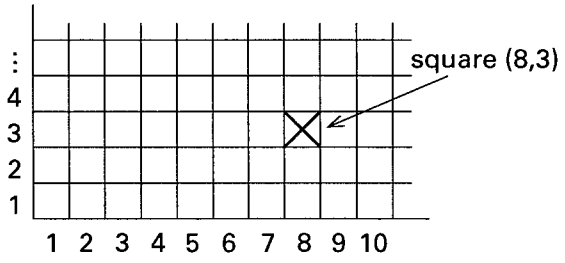
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Colored pencils, pens, or crayons, 1 each of 4 or more colors per student.
- ✔ Colored pens for the overhead (same colors as students have).

ACTIONS

1 Distribute a sheet of 1-cm grid paper and four colored pencils, pens, or crayons to each student. Place a transparency of the grid paper on the overhead, and outline and number a 10 x 10 square, as shown below. Ask the students to do the same on their grid paper.



2 Place an X on square (8,3), as shown below, and explain the method for naming squares according to coordinates. To be sure students understand this method, pose a few coordinates for volunteers to mark on the grid at the overhead.



COMMENTS

1 For Actions 1-6, students need only 4 colors. Red, blue, black, and green are used as examples in the following actions, but any 4 colors can be substituted. You may wish to add other colors in Action 7.

Notice that *spaces* are numbered, rather than grid lines.

2 When naming the coordinates of a square, the first number identifies the column containing the square and the second number identifies the row containing the square. This corresponds to conventions used in graphing coordinate points, in which case grid lines rather than spaces are numbered.

Focus Teacher Activity 2 (cont.)

ACTIONS

3 Tell the students that:

You have divided the 10 × 10 square into exactly 4 different rectangular regions, leaving no overlaps or gaps.

Each rectangular region is a different color.

One small square on the grid represents 1 area unit and the length of the edge of 1 square is 1 linear unit.

The object of the activity is for them to determine your coloring pattern while getting as few clues as possible from you.

4 Tell the students the area and color of each rectangular region in your coloring pattern, but give no clues about their locations or dimensions.

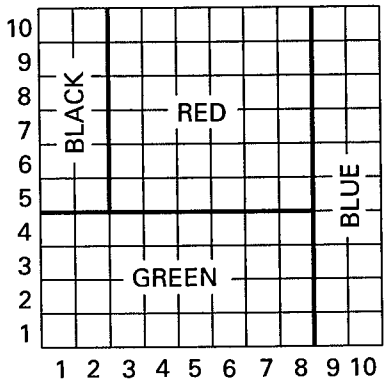
5 Have a volunteer give coordinates to identify 3 squares whose colors they would like to know. As the student names a square, tell its color. Mark the overhead grid as you respond and have the students mark their grids accordingly.

6 Now tell students that, to receive clues, they must make a prediction about the color of a square and give their reasons for making that prediction.

Ask for volunteers to make predictions. Respond with a "yes" or "no" to their predictions, coloring the overhead grid for correct ones. Have the students color in their grids accordingly. Repeat until someone can correctly identify your coloring pattern.

COMMENTS

3 Do not let the students see your colored grid. Here is one sample:



4 For example, using the example given in Comment 3, state that "the red rectangle has area 36 square units, the blue rectangle has area 20, the green rectangle has area 32, and the black rectangle has area 12." Give no other information; however, this is an appropriate time for students to speculate about possible dimensions of the rectangles. This provides meaningful reinforcement for multiplication and division facts.

5 With experience students will develop strategies for choosing these coordinates.

6 For example, in the sample coloring provided in Comment 3, if it had been established that (9,4) was blue, a student might incorrectly predict "I think (6,4) is blue. Since (9,4) is blue I think the blue rectangle could have dimensions 5 across by 4 high and have corners at (10,1) and (6,4)." In this case, while their reasoning is logical, it doesn't match your pattern. As more information is revealed students can eliminate possibilities. You may or may not wish to provide the correct color for any incorrect predictions.

When a student feels they know your entire coloring pattern, ask them to offer their rationale for locating each rectangle.

Focus Teacher Activity 2 (cont.)

ACTIONS

7 Place the students in pairs and give each pair several sheets of grid paper. Have one person in each pair design and conceal a coloring pattern for covering a grid with rectangles. Have their partner try to determine the pattern as done in Actions 4-6. Repeat by having partners switch roles.

COMMENTS

7 Some students may wish to tally the clues as they receive them. (Each response to a prediction is considered a clue.) Encouraging students to minimize the number of clues promotes their developing strategies and using reasoning based on the area and dimensions of rectangles.

Students may wish to make the activity more challenging by enlarging the grid and/or increasing the number of rectangles and colors. The grid could also be a rectangle that is not a square.

Later this activity could be repeated using fractions, decimals, and percentages by letting a 10 by 10 square represent 1 square unit.

TEACHER NOTES:

Follow-up Student Activity 7.2

NAME _____ DATE _____

1 Sketch all possible rectangles that can be formed with tile for each of the following numbers. Label the dimensions of each rectangle (it isn't necessary to show every tile in your sketches).

a) 40

c) 48

b) 32

d) 23

2 List all the factors for each number in Problem 1.

a) 40

c) 48

b) 32

d) 23

3 List any numbers in Problem 1 that are prime. Make another list of any that are composite.

Prime:

Composite:

How did you decide whether a number is prime or composite?

(Continued on back.)

Follow-up Student Activity (cont.)

4 Use clues a)-h) below to solve this puzzle. Keep notes of your methods and conclusions as you work (see Problem 6).

a) I am a rectangle.
b) My dimensions are consecutive counting numbers.
c) My perimeter (the total number of linear units around me) has 4 factors.
d) The sum of my length and width is a prime number.
e) One of my dimensions is an odd number and one is even.
f) My area is not a square number.
g) My perimeter is 3 less than a prime number and 1 more than another prime number.
h) My area is less than the 10th square number and greater than the 7th prime number.

Draw a picture of me and label my dimensions, area, and perimeter.

5 Are there any clues that aren't needed to solve the puzzle in Problem 4? List the fewest clues needed in order to be sure you have the correct rectangle.

6 On a separate sheet, describe the methods and reasoning that you used to solve problem 4.

7 On another sheet, write a letter to a friend explaining the meaning of the following: factor, prime number, composite number, and square number. Include pictures that will help your friend "see" and understand your explanations.

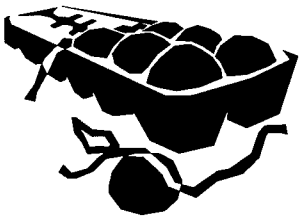
Fraction Concepts With Egg Cartons

Lesson 8

THE BIG IDEA

Egg cartons provide a setting for "seeing" the meaning of the part-to-whole and division concepts of fractions and for developing intuitions about fraction equivalence.

This is the first of several fraction models explored in *Visual Mathematics*.



CONNECTOR

OVERVIEW

Prior to exploring fraction concepts and procedures, students write descriptions of their feelings and understandings about fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).

FOCUS

OVERVIEW

Egg cartons are used to introduce fractions and fraction equivalence.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 3" × 5" index cards, 32 (see Action 8).
- ✔ Egg Carton Wall Chart (see Comment 9).
- ✔ Egg cartons, 12-egg size, 1 per student.
- ✔ "Eggs" (colored tile, 1½" diameter pom-poms, plastic eggs, etc.), 12 per student.
- ✔ Focus Master A, 1 transparency.
- ✔ Yarn or string, six 10" lengths per student.
- ✔ Egg Carton Recording Paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.

FOLLOW-UP

OVERVIEW

Students diagram and record equivalent fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 8.1, 1 copy per student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student.

LESSON IDEAS

JOURNALS

Have students write their responses for the Connector activity in their journals, or have students keep them in their Work-folios, or collect and retain them. These writings will be important for making comparisons later as you and the students look for evidence of their learning and growth.

After the lesson is completed, have the students complete the following in their journals: *One or more idea(s) about fractions that I understand better, or understand for the first time, is/are... What helped me to understand was...*

FOLLOW-UP

Problems 4 g), h), and i) could be used to motivate discussion of mixed numbers and improper fractions which are brought up again in Lesson 9. It may be helpful to remind students to tell what part of a dozen each represents.

TIMING

It is important to keep in mind that fraction concepts and operations are explored in several *Visual Mathematics, Course I* lessons (and in other *Visual Mathematics* courses). To keep interest high and allow understanding to develop over time, limit the time spent in class on this lesson to about 4 hours.

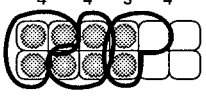
Questions from Action 13 that you don't get to in class could be posed over time for individual out-of-class exploration and/or for small group assessment activities.

SELECTED ANSWERS

3. $\frac{4}{6}$, $\frac{8}{12}$, $\frac{2}{3}$

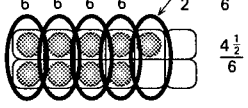
4. a) $\frac{5}{6}$, $\frac{10}{12}$
   b) $\frac{3}{12}$, $\frac{1}{4}$
   c) $\frac{8}{12}$, $\frac{2}{3}$, $\frac{4}{6}$
   d) $\frac{9}{12}$, $\frac{3}{4}$
   e) $\frac{4}{12}$, $\frac{1}{3}$, $\frac{2}{6}$
   f) $\frac{2}{12}$, $\frac{1}{6}$
   g) $\frac{22}{12}$, $1\frac{5}{6}$, $\frac{11}{6}$, $1\frac{10}{12}$
   h) $1\frac{1}{12}$, $\frac{13}{12}$
   i) $2\frac{1}{2}$, $2\frac{3}{6}$, $\frac{15}{6}$, $\frac{30}{12}$, $\frac{5}{2}$, etc.

5. $\frac{1}{4}$ $\frac{1}{4}$ $\frac{2}{3}$ of $\frac{1}{4}$



3 eggs represent $\frac{1}{4}$ and 2 eggs represent $\frac{2}{3}$ of $\frac{1}{4}$. So, 8 eggs are two and two-thirds fourths or $2\frac{2}{3}/4$.

6. $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{6}$ $\frac{1}{2}$ of $\frac{1}{6}$



$4\frac{1}{2}$ / 6

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students write descriptions of their feelings and understandings about fractions. This provides baseline information that is useful for instructional planning and for measuring student growth. It also enables students to connect their past knowledge and experiences to new experiences with fractions.

MATERIALS

✔ Student journals (optional).

ACTIONS

1 Write the following statements on the overhead or blackboard and ask the students to spend the next 5-7 minutes writing as much as they can about fractions. Ask them to use these statements as "starters" for their comments and to include specific examples wherever possible.

About fractions,

I feel ...

I know ...

I wish I understood ...

COMMENTS

1 Ask students to keep writing for the entire time period that you allot for this activity. If students are keeping journals, this is a meaningful journal entry. It provides baseline information regarding students' feelings and knowledge about fractions. This information is useful in planning the amount of time to spend on upcoming lessons and in measuring student growth over time.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Egg cartons are used to introduce fractions and fraction equivalence. Students represent fractions using the part-to-whole and division concepts and discuss ways their representations illustrate the meaning of fractions.

MATERIALS

- ✔ 3" × 5" index cards, 32 (see Action 8).
- ✔ Egg Carton Wall Chart (see Comment 8).
- ✔ Egg cartons, 12-egg size, 1 per student.
- ✔ "Eggs" (colored tile, 1½" diameter pom-poms, plastic eggs, etc.), 12 per student.
- ✔ Focus Master A, 1 transparency.
- ✔ Yarn or string, six 10" lengths per student.
- ✔ Egg Carton Recording Paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.

ACTIONS

1 Distribute an egg carton, a dozen "eggs," and 6 lengths of yarn or string to each student. Ask the students to use these materials to show how they think of the meaning of "one-half." Ask for volunteers to show and explain their models. Acknowledge all models without judgment.

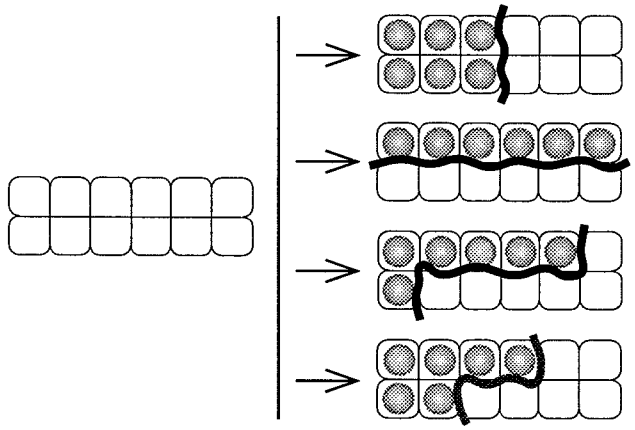
COMMENTS

1 Some helpful preliminary activities with fraction concepts are in *Opening Eyes to Mathematics.*

Items such as colored tile, colored pom-poms (about 1½" in diameter), or plastic eggs can be used to represent eggs. Use yarn or string cut in 10" lengths to show subdivisions of the carton.

The models students share will reveal information about their perceptions of the meanings of fractions. It is important to have students both show their models and explain how the models relate to the way they view the meaning of ½.

One method is to use yarn to subdivide the cartons into 2 equal parts and then to fill 1 part. This is referred to as the *part-to-whole* concept of a fraction. Four different ways to do this are shown here:



(Continued next page.)

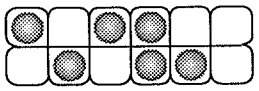
Focus Teacher Activity (cont.)

ACTIONS

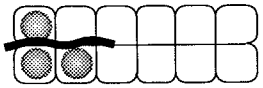
COMMENTS

1 (continued.)
Some students may pick a "whole" that is different from 1 carton and fill ½ of that whole. For example, they might fill 3 parts out of 6.

Other students may place 6 eggs in a carton and describe their method as filling 1 space for every 2 spaces in the carton. This illustrates the *ratio concept* of a fraction, which is explored in *Visual Mathematics, Course II.* Other students may focus on finding ½ of the eggs (rather than the carton) and randomly place 6 eggs in the carton, as shown below:

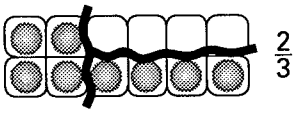


Occasionally, a student's model will reveal a misconception. For example, someone may form 2 unequal parts and fill 1 part. Or, some students may think of ½ only in symbols, showing "1 over 2" as in the diagram below:

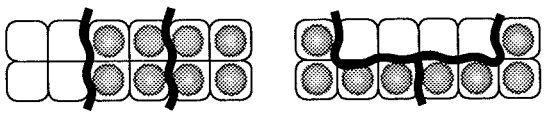


Since the purpose is to find out how students view the meaning of ½, acknowledge their ideas without suggesting they are right or wrong. The following actions provide opportunities for students to adjust their conceptions.

2 Subdivide and fill an egg carton to show ⅔ (one possibility is shown below). Discuss students' ideas about how the fraction symbol ⅔ relates to your model. Ask the students to find other ways to subdivide and fill the carton to show ⅔ of a dozen.



2 To show the part-to-whole concept of ⅔, subdivide the carton (the "whole") into 3 equal parts, and fill 2 of those parts with eggs. Here are two other possibilities.



The essential idea for this concept of the fraction a/b is to subdivide the whole carton into b equal parts (equal in the sense that each part holds the same number of eggs) and then fill a of these parts with eggs.

You may wish to introduce (or students may bring up) the terms *numerator* and *denominator.* For a fraction a/b, a is the numerator (the number of parts under consideration) and b is the denominator (the total number of parts into which the whole is subdivided).

Focus Teacher Activity (cont.)

ACTIONS

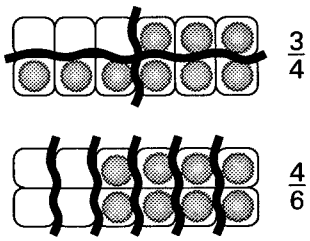
3 Ask the students to devise ways to subdivide the carton to show the meaning of 3/4 and 4/6 of a carton. Repeat as needed for other fractions. Have volunteers show their models to the class.

4 Distribute 2 sheets of Egg Carton Recording Paper to each group of students. Ask the groups to find all possible different egg carton fractions of a dozen, recording each fraction and its diagram. When the students finish, discuss their results. Remind students to draw the subdivision (yarn) lines, so one can be certain which fraction they have sketched by only observing their diagrams.

COMMENTS

The *division* concept of 2/3 (in this model 2/3 is viewed as the amount in 1 part when 2 cartons are subdivided into 3 equal parts) is explored in Action 9.

3 Note that there are many ways to do each of these, but here are some typical responses that are based on the part-to-whole concept:



Discuss enough examples so students can relate fractions to egg cartons. Refer to Action 9 if the division concept is suggested.

4 Students may find it helpful to discuss and explore possibilities in their groups using egg cartons and yarn before sketching on Egg Carton Recording Paper (see *Blackline Masters*).

For this activity 2 fractions are considered "different" if the numerator and/or denominator of one fraction are different from the numerator and/or denominator of the other fraction. For example, while 9/12 and 3/4 of a dozen represent the same number of eggs, as symbols they are different, and, hence, are called different egg carton fractions. The fact that some fractions are *equivalent* to others will be addressed in Action 5.

Here are 32 egg carton fractions that students usually discover:

0/12, 1/12, 2/12,...11/12, 12/12
0/6, 1/6, 2/6,...5/6, 6/6
0/4, 1/4, 2/4,...3/4, 4/4
0/3, 1/3, 2/3, 3/3
0/2, 1/2, 2/2

Sometimes students want to add 0/1 and 1/1 to the list of egg carton fractions. 0/1 and 1/1 are obtained by not subdividing the carton and filling it all or none of it. You may wish to add these two fractions to the list if your students discover them.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

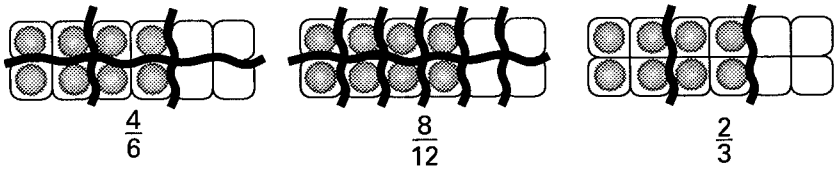
COMMENTS

4 (continued.)
The zeroes, like $0/4$, come from dividing an egg carton into 4 equal parts but not filling any of them.

Because the question is open-ended there may be some ingenious answers. Some students have obtained up to 90 egg carton fractions. For example, they get elevenths by removing one egg from the carton and then dividing the carton into eleven parts. Of course, they are no longer getting fractions of a dozen. These interesting approaches should be acknowledged.

5 Ask the students to place 8 eggs in their cartons, and to determine what fraction of a dozen eggs is in the carton. Have volunteers share their methods of determining the answer.

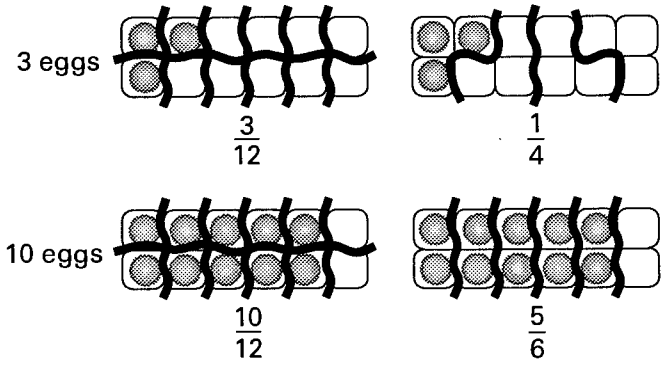
5 A transparency of Egg Carton Recording Paper is useful for having students show their ideas. The carton with 8 eggs can be described as $8/12$ of a dozen, $2/3$ of a dozen, or $4/6$ of a dozen depending upon how the subdivision is visualized. (Note that some students may suggest fractions like $1\frac{1}{3}/2$ dozen; see also Comment 12 g) and Lesson 9.)



Since $8/12$, $2/3$, and $4/6$ of a dozen all represent the same number of eggs, one can write $8/12 = 2/3 = 4/6$. Hence, $8/12$, $2/3$, and $4/6$ are called *equivalent* fractions. Note that $2/3$ is "reduced" or in *lowest terms* because the subdivisions of the egg carton for this fraction has the fewest number of parts. However, it is important to keep emphasis on how students "see" the fractions rather than on reducing fractions.

6 Repeat Action 5 for 3 eggs, 10 eggs, and 12 eggs. Discuss the equivalent fractions represented by these numbers of eggs.

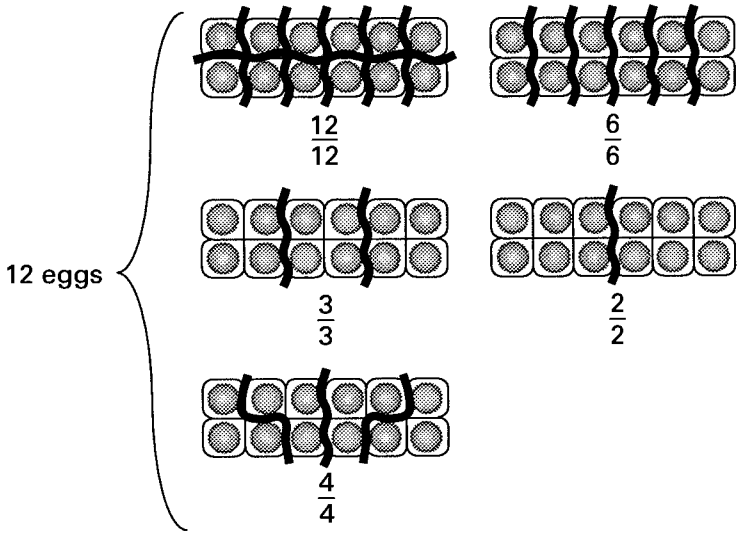
6 You may wish to add more examples. Here are some possibilities for each number of eggs listed in this action:



Focus Teacher Activity (cont.)

ACTIONS

COMMENTS



7 (Optional) Print each of the 32 egg carton fractions on separate index cards. Ask for a pair of volunteers. Have the pair select a card, without revealing the fraction named on the card, and use their egg cartons, eggs, and yarn to model the fraction. After showing their model to the class, have the pair select a volunteer to name the fraction represented and explain how they could identify it. Repeat with other pairs of students.

7 For example, if a pair of students selected the card "3/4" and modeled it, another student might say, "I think your card is 3/4 because you used yarn to separate the carton into 4 equal parts and you filled 3 of the parts."

It is helpful to prepare the index cards prior to class. Having students come to the front of the room in pairs makes this action less threatening. Another way to reduce student anxiety is to give pairs the option of replacing the card they selected with a different one. As an alternative to using egg cartons for this action, students could sketch the fractions on a transparency of Egg Carton Recording Paper.

8 Fill 13 egg cartons with a different number (0-12) of eggs in each. Mount the cartons on the wall in a column as shown in the diagram on the next page. Using fraction cards like those in Action 7, complete the top row of an Egg Carton Wall Chart by attaching the 0/12, 0/6, 0/4, 0/3, and 0/2 cards next to the empty egg carton, as shown in the diagram. Shuffle the remaining fraction cards and distribute them equally among the groups. Have each group discuss the correct placement of their cards and then select 1 or 2 group members to place their cards on the chart. Discuss the completed chart.

8 It is helpful to have the egg cartons attached to the wall and the cards prepared prior to class. To make patterns more immediately apparent, you may wish to suggest that students line up twelfths cards under the 0/12, sixths under the 0/6, etc. Tape can be used to attach cards. Or, a reusable chart can be made by mounting library card pockets on a sheet of large poster paper.

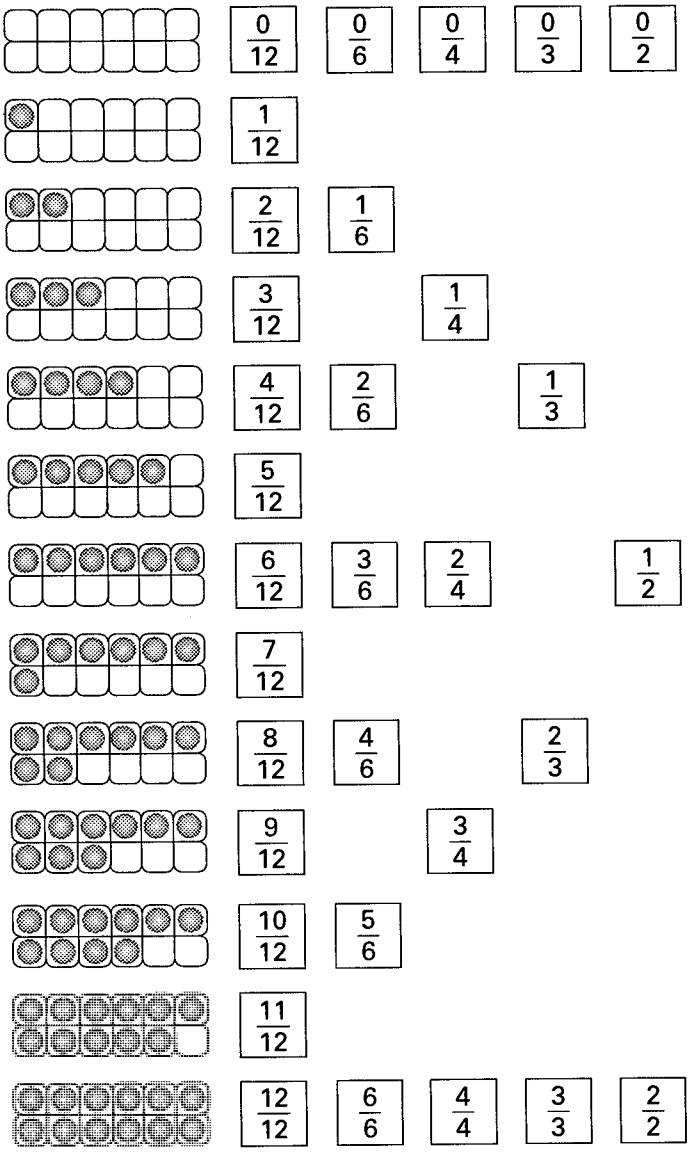
Leaving this chart on the wall is helpful for many students. It can be used for warm-up activities at the beginning of class. For example:

a) Before students arrive in class, scramble a few fraction cards to see if students can spot and correct the errors.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



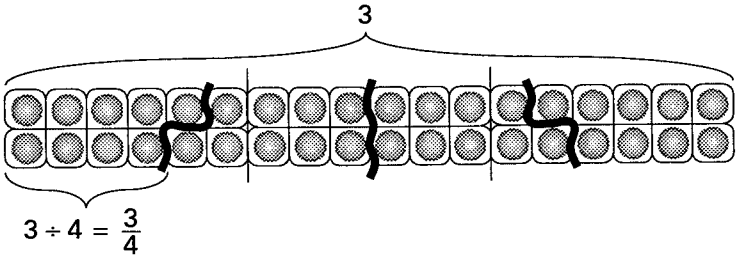
9 Ask the groups to fill 3 whole cartons. Have them divide this amount into 4 equal parts and determine the number of cartons and/or parts of a carton in *one* of those 4 parts. Discuss their methods.

COMMENTS

8 (continued.)
b) Take a few fraction cards off the wall and select students to correctly replace them.

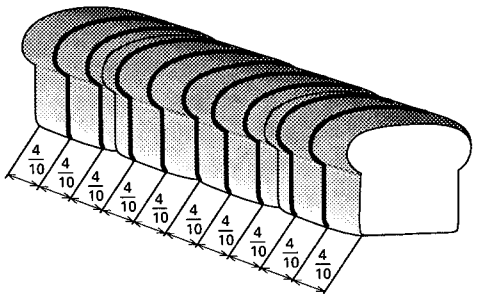
In both of the above warm-ups the students can be asked to "prove" the correct placement by sketching an egg carton diagram corresponding to their fraction.

9 The *division concept* of a fraction is another interpretation of the meaning of a fraction. According to this model $\frac{a}{b}$ is $a \div b$. In the following diagram 3 cartons are joined together and the total is divided into 4 equal parts that each contain $3 \div 4 = \frac{3}{4}$ of a carton. Some students may also describe this process as finding $\frac{1}{4}$ of 3.



Focus Teacher Activity (cont.)

ACTIONS

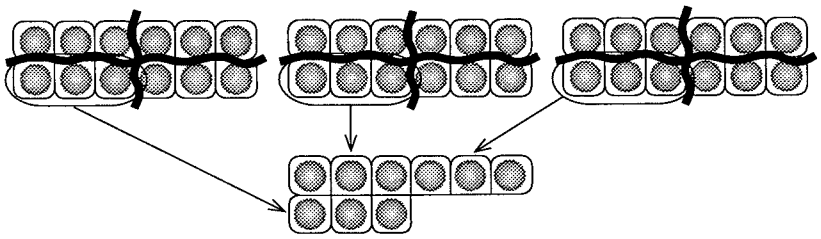


10 Ask students to use the division concept of a fraction discussed in Comment 9 to model the meaning of $\frac{2}{6}$. Repeat for $\frac{7}{12}$ and $\frac{5}{10}$.

11 Write the fractions $\frac{2}{3}$ and $\frac{3}{2}$ on the overhead and ask the groups to draw diagrams to show the meaning of each based on a) the part-to-whole concept, and b) the division concept of a fraction. Discuss.

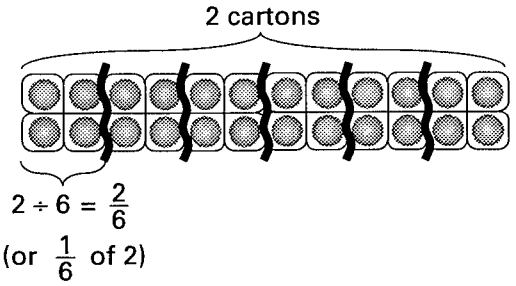
COMMENTS

Another method is illustrated below. Each whole carton is divided into 4 equal parts which each contain $\frac{1}{4}$ of a carton. These parts are then separated into 4 groups of $\frac{3}{4}$. This method reflects a *part-to-whole concept* of $\frac{3}{4}$.



Students may be interested in knowing that the early development of fractions was influenced by the need to solve problems involving division of whole numbers. A problem found in writings from 1650 B.C. requires that 4 loaves of bread be divided equally among 10 people. According to the *division concept* each person would receive $4 \div 10 = \frac{4}{10}$ of a loaf of bread, as shown at the left.

10 According to the division concept, $\frac{2}{6}$ can be modeled by subdividing 2 cartons into 6 equal parts; 1 part is $2 \div 6 = \frac{2}{6}$ of a carton, or $\frac{1}{6}$ of 2 cartons, as shown here:



Keep in mind the intent here is to introduce students to the division concept as another way to view the meaning of a fraction and this concept is explored in more depth in Lesson 28. (The division concept of a fraction is the primary way fractions are viewed in most algebra courses.)

11 $\frac{2}{3}$, for example, could be viewed as the value of 2 parts of a whole that is divided into 3 equal parts, or as the value of 1 part of a group of 2 wholes that has been divided into 3 equal parts.

Focus Teacher Activity (cont.)

ACTIONS

12 Distribute 2 sheets of 1-cm grid paper to each student. Place a transparency of Focus Master A on the overhead and select 1 or more problems for "eggsploration" by the groups. Discuss.

Fraction Concepts With Egg Cartons Lesson 8

Focus Master A

Eggsplorations

a) Sketch some differently-sized cartons on which you could show sevenths. What other fractions could you show on these cartons?

b) Sketch a 16-egg carton and determine some fractions that can be shown on that carton.

c) Sketch a carton on which you can show both $\frac{1}{3}$ and $\frac{3}{5}$. What are some equivalent fractions on that carton?

d) On a 12-egg carton $\frac{1}{6}$ is 2 eggs. Sketch a carton on which $\frac{1}{6}$ is 3 eggs; $\frac{1}{6}$ is 4 eggs; $\frac{1}{6}$ is 6 eggs; $\frac{5}{6}$ is 5 eggs.

e) Sketch egg carton diagrams (you decide the sizes of the cartons) to show $\frac{14}{5}$ cartons; $\frac{9}{7}$ cartons; $2\frac{7}{9}$ cartons; $\frac{8}{8}$ carton.

f) Sketch a carton that can be divided into both fourths and ninths. What are equivalent fractions that you can show on this carton?

g) Sketch a carton on which you can show eighths. Fill $1\frac{1}{2}$ fourths of this carton.

h) Show two different interpretations of the meaning of $\frac{4}{5}$ and explain a situation that involves each interpretation.

COMMENTS

12 These explorations are intended to bring up some important fraction ideas that are extended in Lesson 9 and later *Visual Mathematics* lessons. It isn't necessary to explore every problem. The intent is to plant seeds of new ideas that are expanded later.

a) Two possibilities are 7-egg and 14-egg cartons (or any number of eggs that is a multiple of 7).

b) A few examples are: $\frac{4}{16} = \frac{1}{4}$; $\frac{3}{8} = \frac{6}{16}$; $\frac{1}{2} = \frac{2}{4} = \frac{8}{16}$.

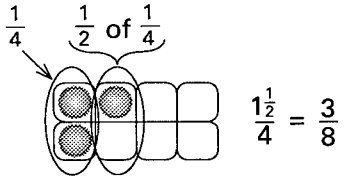
c) One possibility is a 15-egg carton.

d) Three eggs is $\frac{1}{6}$ of an 18-egg carton.

e) This problem involves *mixed numbers* (i.e., a whole number and a fraction) and *improper fractions* (i.e., the numerator is greater than or equal to the denominator). These are explored again in Lesson 9.

f) It is surprising to some students that, for example, $\frac{4}{6} = \frac{2}{3}$ even when the carton size changes.

g) $1\frac{1}{2}$ fourths could be written $1\frac{1}{2}\frac{}{4}$. If the carton has 8 eggs, $\frac{1}{4}$ is 2 eggs and $1\frac{1}{2}\frac{}{4}$ looks like this:



h) $\frac{4}{5}$ could be viewed as the value of 1 part after a group of 4 wholes is divided into 5 equal parts. For example, if 4 boxes of peanuts are shared equally by 5 people, each person gets $4 \div 5 = \frac{4}{5}$ box (i.e., $\frac{1}{5}$ of 4 boxes).

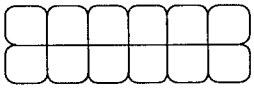
Or, since $\frac{1}{5}$ can be viewed as the value of 1 part when 1 whole is divided into 5 equal parts, $\frac{4}{5}$ is the value of 4 of those parts. For example, if a brownie is cut into 5 equal parts and distributed, $\frac{4}{5}$ is the amount in 4 of those parts.

Eggs are frequently also packed to hold 6, 18, 24, and 36 eggs. If asked, school cafeterias and restaurants may collect odd-sized cartons for you.

Follow-up Student Activity 8.1

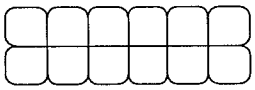
NAME _____ DATE _____

1 Subdivide this carton to show sixths and fill 4/6 of the carton.

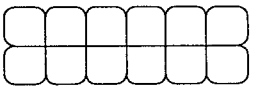


2 On these cartons, show 4/6 of a dozen using twelfths and thirds.

a) use twelfths

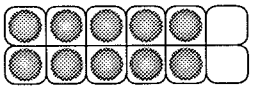


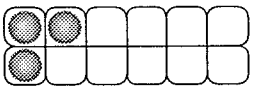
b) use thirds

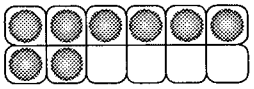


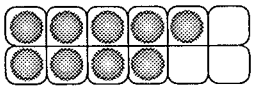
3 List the equivalent fractions shown by your diagrams in Problems 1 and 2.

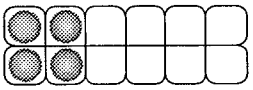
4 For each of Problems a)-i) below, write at least two different fraction names for the part of a dozen that is shown in the diagrams.

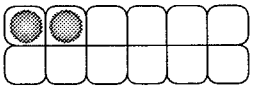
a) _____ 

b) _____ 

c) _____ 

d) _____ 

e) _____ 

f) _____ 

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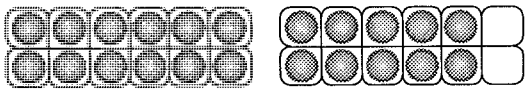
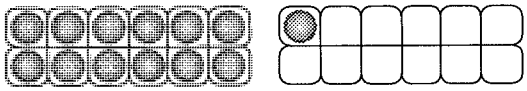
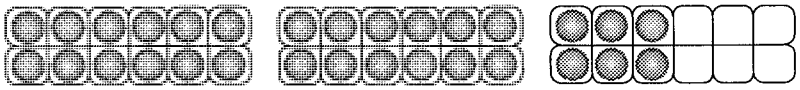
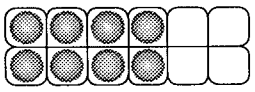
Follow-up Student Activity (cont.)

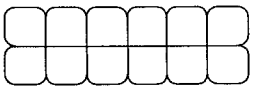
g) _____ 

h) _____ 

i) _____ 

5 Mackenzie gave $2\,2/3/4$ (two and two-thirds fourths) as one fraction name for the part of a dozen shown here: 

Explain how you think she decided this.

6 Mackenzie named another fraction as $4\,1/2/6$. Fill in this egg carton to show what part of a dozen you think she was naming: 

7 On a sheet of grid paper sketch the following:

a) Two differently-sized cartons which each can be subdivided to show tenths. Fill and label $4/10$ of each carton.

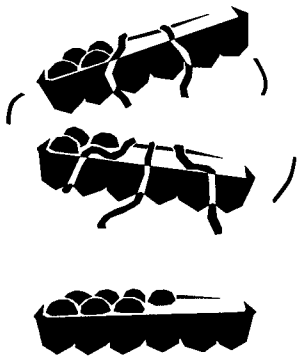
b) A carton which you can subdivide to show both eighths and sixths. Fill and label $3/8$ of the carton and $1/6$ of the carton.

c) Two different ways of viewing the *meaning* of $3/5$. Next to your sketches, write an explanation of how each diagram shows a different meaning of $3/5$.

Building Intuitions About Fraction Operations Lesson 9

THE BIG IDEA

Students' understanding of fraction concepts and intuitions about fraction operations develop when they create models of fraction situations and discuss their observations about mathematical relationships they can see in their models.



CONNECTOR

OVERVIEW

Students combine egg carton fractions and determine their sum.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Egg cartons, 1 per student.
- ✔ "Eggs" (colored tile, pom-poms, etc.), 12 per student.
- ✔ Yarn or string, six 10" lengths per student.
- ✔ Egg Carton Recording Paper (optional, see *Blackline Masters*), 1 transparency.

FOCUS

OVERVIEW

Students make mathematical observations about fraction relationships in a variety of situations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 4-5 copies per student.
- ✔ Butcher paper (optional), 3 sheets.
- ✔ Marking pens (optional), 1 or more.

FOLLOW-UP

OVERVIEW

Students use egg carton diagrams to solve puzzles and model situations involving fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 9.1, 1 copy per student.
- ✔ 1/4" grid paper (see *Blackline Masters*), 2-3 sheets per student.

LESSON IDEAS

FOLLOW-UP

To encourage a variety of observations about each situation on Follow-up Problem 1 have the students work on 2-4 situations as homework each day of the lesson. Then give the students about 5 minutes to share observations in their groups at the beginning of each class period.

It isn't necessary to assign every situation in Problem 1. This gives students an opportunity to "go beyond the requirements" by completing unassigned situations.

ASSESSMENT

Follow-up Problem 2 could be recorded in student journals to provide insights regarding what students view as challenging at this time. Problem 3 is also a useful journal entry.

TIMING

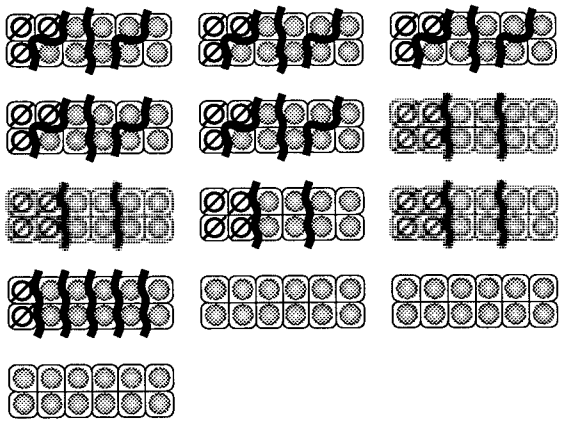
It is helpful to consider the following factors when deciding the amount of time to spend on large group discussion: Are some groups "stuck" and in need of the impetus that other groups' ideas provide? Are certain groups using one method over and over, and therefore, in need of ideas from others? Are there groups that need the chance to clarify their ideas by describing them to and getting feedback from the large group? Is the quality of discussion in the small groups such that sharing at the overhead isn't needed? There are times when small group discussions are rich and student engagement is high and, hence, stopping for large group discussions interrupts rather than facilitates student thinking. It is important to "eavesdrop" on small group discussions with the preceding questions in mind.

SELECTED ANSWERS

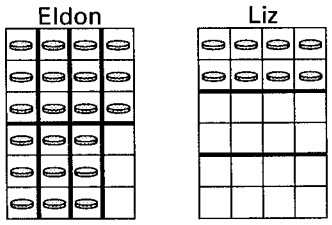
1. Some models and observations:

a)

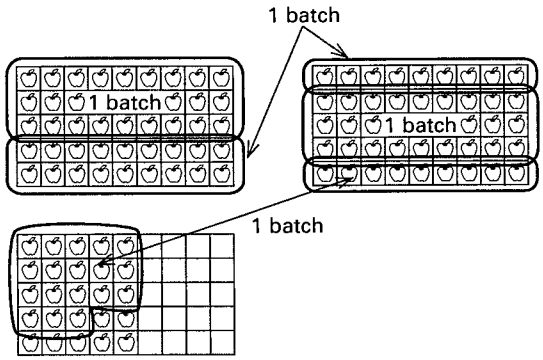
Gene has enough eggs to make $4\frac{1}{2}$ gallons of ice cream.

b) 

The shipment of eggs contained 33 broken eggs.

c) 

The boxes could contain 24 pieces of candy or any multiple of 24. Eldon has $2\frac{5}{8}$ times as much candy left as Liz. Liz has $\frac{8}{21}$ as much as Eldon. Eldon has $\frac{13}{24}$ box more than Liz. Together they have $1\frac{5}{24}$ box.

d) 

Mark has enough apples for 4 batches of applesauce, with $\frac{7}{45}$ box left-over. If he had $\frac{20}{45}$ box more, he could make another batch.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students combine egg carton fractions and determine their sum.

MATERIALS

- ✔ Egg cartons, 1 per student.
- ✔ "Eggs" (colored tile, pom-poms, etc.), 12 per student.
- ✔ Yarn or string, six 10" lengths per student.
- ✔ Egg Carton Recording Paper (optional, see *Blackline Masters*), 1 transparency.

ACTIONS

1 Arrange the students in groups and distribute an egg carton, "eggs," and yarn lengths to each student. Ask the groups to fill 1/3 of one carton and 1/4 of another carton. Have them determine the part of a carton filled when they combine the eggs in both cartons. Ask for volunteers to show their methods and results.

2 Repeat Action 1 for pairs of cartons filled with the following fractional amounts:

a) 1/6 and 3/4;

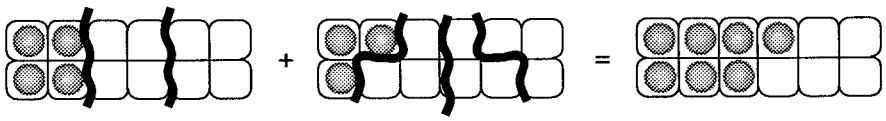
b) 3/12 and 3/6;

c) 1/3 and 1/2.

3 Ask the groups to determine different combinations of two or more fractions of cartons that fill exactly 1 whole carton. Discuss.

COMMENTS

1 Students could demonstrate their methods on egg cartons or on a transparency of Egg Carton Recording Paper. Combining cartons that are 1/3 (4 eggs) and 1/4 (3 eggs) filled produces a carton with 7 eggs or 7/12 of a carton. This is illustrated below:



2 The total amounts are 11/12; 9/12 or 3/4; 5/6 or 10/12. Encourage students to verbalize their methods as they demonstrate them. Acknowledge any equivalent forms of the totals as correct. It isn't necessary to emphasize simplifying the fractions. Rather, encourage students to express their answers according to how they view the fractions (e.g., as sixths or twelfths).

Some students may describe their actions as adding the two fractions, based on the *put-together* method for addition. Rather than discussing rules at this time, keep the emphasis on concepts by focusing on students' actions with the cartons. If the use of symbols comes up emphasize that, for example, 1/6 + 3/4 = 11/12 is a way of recording the action of combining a carton that is 1/6 full with one that is 3/4 full and naming the total amount.

3 Encourage students to predict by picturing combinations in their mind's eye and then use the egg cartons to test their ideas. There are many possibilities, such as: 1/3 + 1/2 + 1/6 = 1; 1/12 + 1/6 + 3/4 = 1; etc. See *Opening Eyes to Mathematics* for background activities involving making wholes.

Connector Teacher Activity (cont.)

ACTIONS

4 (Optional) Pose the following situation for the students to consider:

"In another class the students were asked to combine a carton that was $\frac{2}{3}$ full with one that was $\frac{3}{4}$ full. Chris incorrectly gave $\frac{17}{24}$ of a carton as the total."

Ask the groups to formulate a "visual proof" and clear explanation to show why Chris' statement is incorrect.

COMMENTS

4 You may wish to change the name of the student given in this situation in order to avoid inadvertently embarrassing anyone.

There are several points students may bring up, such as:

- $\frac{17}{24}$ of a carton suggests dividing the carton into 24 parts and filling 17 of them, which is a little more than $\frac{2}{3}$ of a carton. In the given situation, more than a whole carton is filled.

- $\frac{17}{24}$ of *two* cartons are filled. Chris has switched to a new "whole."

- If the "whole" were two cartons connected together, (i.e., 24 eggs) then $\frac{2}{3}$ of a carton would be 16 eggs and $\frac{3}{4}$ of a carton would be 18 eggs, for a total of 34 eggs. There are only 17 eggs in this situation. Chris used one carton as the "whole" when showing $\frac{2}{3}$ and $\frac{3}{4}$, but switched to two cartons as the "whole" when naming the total.

This situation may be unsettling for some students, because they agree with Chris. Rather than expecting that all students "see" Chris' misunderstanding now, it may be instructive to continue the lesson and raise this question again later.

Focus Teacher Activity

OVERVIEW & PURPOSE

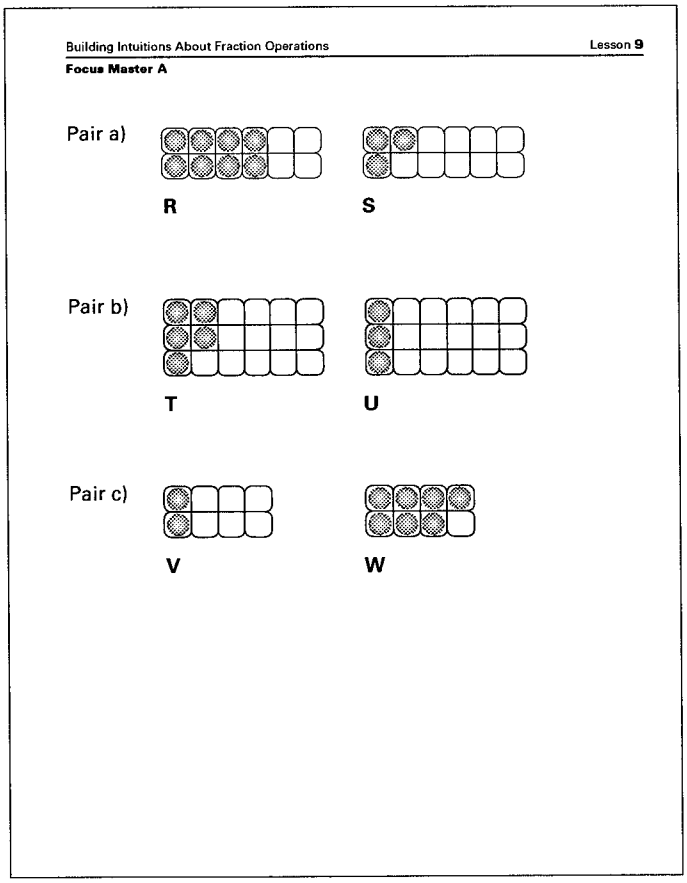
Students make mathematical observations about fraction relationships in a variety of situations. This develops readiness for and intuitions about fraction operations, which are explored in more depth in Lessons 40-44 and in later Visual Mathematics courses.

MATERIALS

- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 4-5 copies per student.
- ✔ Butcher paper (optional), 3 sheets.
- ✔ Marking pens (optional), 1 or more.

ACTIONS

1 Arrange the students in groups and give each student a sheet of 1-cm grid paper. Place a transparency of Focus Master A on the overhead revealing only the diagrams in Pair a). Tell the students that each egg carton diagram in Pair a) represents a fraction of a whole, and the carton shown is the whole (i.e., the unit). Ask the groups to write as many observations as they can about mathematical relationships they can "see" in the pair. Have volunteers share their observations and reasoning.



COMMENTS

1 To help students get started you might offer an observation or two of your own, such as: "I notice carton R is $\frac{4}{6}$ full," or "Carton S has $\frac{3}{8}$ as many eggs as carton R," or "Since carton R is $\frac{4}{6}$ full and carton S is $\frac{1}{4}$ full and carton R has a larger part filled, I can see that $\frac{4}{6}$ is greater than $\frac{1}{4}$."

To maximize student involvement, encourage students to sketch these cartons and then spend 2 or 3 minutes privately writing observations in their journals. Then have individuals share with their groupmates and develop a group list of statements.

Some groups may write their statements in words while others may use numbers and math symbols (or a combination of words and symbols). Encourage students to use whichever method is most comfortable for them. Regardless of the recording method, statements should be reflections of fractional relationships students see in the models.

To collect their responses, you could sketch Pair a) at the top of a sheet of butcher paper and record statements (using words and/or numbers) as they are shared, or have a representative from each group write several statements on the poster (if you do this, ask them to record statements that another group hasn't recorded yet). As these are studied by the large group, encourage students to add other observations to the list.

Keep in mind the purpose here is to draw out students' ideas and, through sharing, to stimulate new ideas. It isn't expected that all four operations come up now. But they may and, if so, keep discussion informal, conceptual, and non-algorithmic. Some groups may only make 2 or 3 observations now and some may begin with observations as simple as, "there are 8 + 3 = 11 eggs in

(Continued next page.)

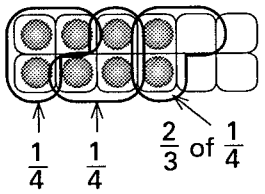
Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)
the 2 cartons. Here are some examples of other observations that are based on the fact that R contains $\frac{8}{12} = \frac{4}{6} = \frac{2}{3}$ carton and S contains $\frac{3}{12} = \frac{1}{4}$ carton.

Two $\frac{1}{4}$'s and two-thirds of $\frac{1}{4}$ (or $2\frac{2}{3}$ of $\frac{1}{4}$) total $\frac{8}{12}$, as shown here:



"$\frac{3}{12}$ carton combined with $\frac{8}{12}$ carton is $1\frac{1}{12}$ carton."

"The difference between $\frac{2}{3}$ carton and $\frac{1}{4}$ carton is $\frac{5}{12}$ carton."

"$\frac{8}{12}$ is equivalent to $\frac{4}{6}$ and $\frac{3}{12}$ is equivalent to $1\frac{1}{2}\frac{}{6}$ (i.e., $1\frac{1}{2}$ sixths)"

"If you double the amount in both cartons their difference is doubled."

Keep in mind the intent of this lesson is to develop intuitions about fraction operations. The student who, for example, describes $\frac{3}{12}$ as $1\frac{1}{2}$ sixths has developed intuitions about fraction multiplication and division.

2 Repeat Action 1 for Pair b) and Pair c) on Master A.

2 These involve "nonstandard" size cartons (i.e., different units) and hence, involve some fractions with denominators different from those possible with a 12-part carton.

3 Give each group another sheet of 1-cm grid paper and write the following situation on the overhead:

> **Carla and Shantal received identical boxes of candy. Carla has $\frac{5}{6}$ of a box left and Shantal has $\frac{3}{4}$ of a box left.**

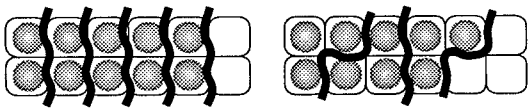
Ask the groups to:

a) on the grid paper draw an "egg carton model" of the situation;

b) write several (as many as possible!) conclusions about mathematical relationships they can "see" in their model;

c) draw two more "egg carton models" of the situation using differently-sized cartons;

3 You may find it helpful to explore and discuss a)-d) one part at a time. Most students will probably make a model for a) similar to the following:



Keep emphasis on verbalizing relationships and using numbers and symbols only as an abbreviated way of recording verbalizations. Students statements, whether verbal or symbolic, can reveal much about their understanding of the meanings of the operations. Following are a few observations that *may* come up:

Carla has $\frac{5}{6} - \frac{3}{4} = \frac{1}{12}$ of a box more than Shantal.

Focus Teacher Activity (cont.)

ACTIONS

d) compare their observations and conclusions in b) to their models for c), i.e., can they make the same conclusions about their models for c) and why or why not.

4 Give each student 2 more sheets of 1-cm grid paper. Give each group of students 1 copy of Focus Master B (a reduced version appears on the next page). Identify which situations groups are to explore and ask them to do the following:

a) draw "egg carton models" of each situation; and

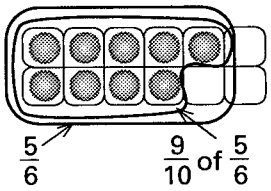
b) write conclusions that are based on mathematical relationships they can "see" about each model.

COMMENTS

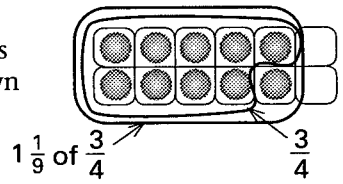
Carla ate $\frac{6}{6} - \frac{5}{6} = \frac{1}{6}$ box and Shantal ate $\frac{4}{4} - \frac{3}{4} = \frac{1}{4}$ box.

Shantal ate $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ box more than Carla.

Shantal has $\frac{9}{10}$ as much candy left as Carla as shown here.



Carla has $1\frac{1}{9}$ as many pieces of candy as Shantal, as shown here.



Together the girls have $\frac{5}{6} + \frac{3}{4} = 1\frac{7}{12}$ boxes of candy.

Some students may draw a 12-part "egg carton model" of this situation and then use that model as a basis for drawing conclusions about the number of pieces of candy each girl ate or has leftover. Since the size of the original boxes is not specified, their conclusions may not necessarily be correct. However, unless a student brings up the issue of differently-sized boxes sooner, you may wish to leave such assumptions "on the floor" until students have explored d). This allows the opportunity for students to discover on their own that the number of pieces in the box could be any number that has both 6 and 4 as factors. Hence, the number of pieces remaining would differ depending on the size of the box, but the fractional parts would not change. For example, there could be 12, 24, 36, 48, ... pieces of candy in each box and Carla could have 1, 2, 3, 4, ..., respectively, more pieces left than Shantal.

4 It isn't necessary to have students explore every situation. Make choices based on the needs and interests of your class, the operations you wish to emphasize, and/or the level of student engagement. There are several ways to organize this: have groups explore and discuss one situation at a time; pick two or three situations for groups to explore, discuss those with the large group, and then have groups pick two or three more to explore; select several for the groups to explore and discuss independently with little or no large group

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson 9 Building Intuitions About Fraction Operations

Focus Master B

Situations

a) Georgina had $\frac{7}{8}$ of a carton of eggs. Jamal borrowed $\frac{2}{8}$ of a carton from her.

b) Rumiana's box of marbles is $\frac{5}{8}$ full. Cal's marbles are in the same sized box and it is $\frac{3}{4}$ empty.

c) At the beginning of the school year, Dylan and Jeff each bought the same-sized package of pencils. Dylan has $\frac{2}{3}$ of his box left and Jeff has $\frac{4}{5}$ of his box left.

d) Mazan bought 3 of the same-sized cartons of soda for the reunion. People drank $\frac{4}{9}$ of each carton.

e) Gary's dad has $1\frac{1}{3}$ dozen eggs. He needs 1 more than double that amount of eggs.

f) Carey had $\frac{1}{2}$ box of candy left after the movie. Then she ate $\frac{1}{5}$ of that on the way home and had 8 pieces left to give her little brother.

g) Judah has $\frac{8}{12}$ of a dozen eggs. A recipe calls for $\frac{1}{4}$ of that amount.

h) Hans bought a bag of apples. After he baked pies, he had $\frac{2}{3}$ of a bag left. Then he gave Bill $\frac{3}{4}$ of these, which was 9 apples.

i) Maria and Justin each bought the same-sized box of candy. Maria ate $\frac{3}{7}$ of $\frac{1}{3}$ of her box and Justin ate $\frac{1}{3}$ of $\frac{3}{7}$ of his box.

j) Fu Li bought a carton of 36 eggs and Jill bought a carton of 24 eggs. Two days later Fu Li had $\frac{5}{6}$ of $\frac{2}{3}$ of his carton left and Jill had $\frac{2}{3}$ of $\frac{5}{6}$ of her carton left.

k) Kelsey and Charlie each bought the same-sized crate of oranges. Last week Kelsey's family ate $\frac{1}{3}$ of a crate and Charlie's family ate $\frac{10}{15}$ of a crate. Charlie's family had 20 oranges left.

Building Intuitions About Fraction Operations Lesson 9

Focus Master B (cont.)

l) Christopher said, "I have 7 pieces of candy left which is $\frac{1}{6}$ of what I had originally."

m) Sonia and her 2 friends had 4 boxes of taffy which they shared equally. Jimmy and his 3 friends had 3 boxes that they shared equally. All of the boxes contained the same amount of taffy.

n) Bryana said, "I have 5 pieces of candy left and that is $\frac{1}{4}$ of $\frac{1}{2}$ of what I had originally."

o) Ted found part of a box of eggs in his refrigerator. He then used $\frac{1}{3}$ of a dozen for baking a cake, and $\frac{1}{6}$ of a dozen for baking brownies. Then he had 2 eggs left.

p) Chieko used $\frac{1}{4}$ of her eggs for eggnog and then used $\frac{1}{2}$ of the remaining eggs for a pie. She had 3 eggs left over.

q) Gretchen had her choice of receiving 1 part of 4 boxes shared equally by 3 people of 1 part of 7 boxes shared equally by 5 people. All of the boxes were identical.

COMMENTS

4 (continued.)
discussion. Some situations could be assigned as homework, or as journal entries; or, individual students could produce Eyewitness Books about selected situations (see Extended Projects in *Starting Points*).

Students may record their observations in words, as number statements, or use a combination of these. Some students may make observations about fraction relationships in the situations without recognizing connections to the operations. This is okay since the purpose here is to develop intuitions.

Number statements should reflect what students can "see" in their model and be based on their understanding of the meanings of the operations rather than on the rote application of algorithms. Methods that students find work in general and are personally meaningful will eventually become their algorithms. Symbolic ways of recording students' methods need not be emphasized now since they will be developed later in *Visual Mathematics*, after students have developed comfort working with materials and diagrams and are able to clearly verbalize relationships and methods.

Focus Teacher Activity (cont.)

ACTIONS

5 (Optional) Distribute 2 additional sheets of 1-cm grid paper to each group and have groups of students design egg cartons on which they can carry out one or more of the following fraction computations.

a) $1/3 + 1/5$

b) $5/8 + 1/2$

c) $1/7 \times 2/5$

d) $3/4 \div 1/5$

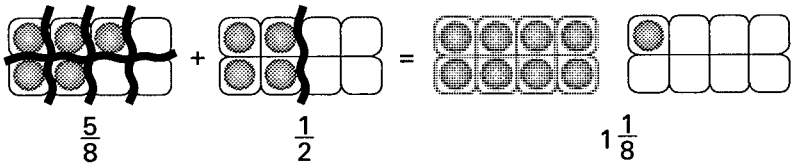
e) $3/5 - 1/6$

COMMENTS

5 Choose from these computations based on students' comfort level with each operation. For example, some students may not be ready for c) and d) to be posed in this format. Alternatively, c) could be posed as, "Represent $1/7$ of $2/5$ of a carton," and d) as, "How many $1/5$ cartons does it take to fill $3/4$ of a carton?"

a) This requires an egg carton that holds 15 eggs or a multiple of 15.

b) This requires an 8-egg carton (or a multiple of 8). Here is an 8-egg carton solution:



c) This needs a 35-egg (or a multiple of 35) carton.

d) This requires a 20-egg (or a multiple of 20) carton.

e) This requires a 30-egg (or a multiple of 30) carton.

TEACHER NOTES:

Follow-up Student Activity 9.1

NAME _____ DATE _____

1 On grid paper make "egg carton diagrams" to model each of these situations (use whatever-sized cartons are needed). Then write as many observations as you can about mathematical relationships you can "see" in each of your diagrams.

a) It takes $\frac{1}{6}$ of a dozen eggs to make a gallon of ice cream. Gene has $\frac{3}{4}$ of a dozen eggs.

b) A store manager found a damaged shipment of eggs. The eggs were packed in cartons that each hold 1 dozen eggs. Five cartons each had $\frac{1}{4}$ of their eggs broken. Four cartons each had $\frac{1}{3}$ of their eggs broken. One carton had $\frac{1}{6}$ of its eggs broken. Three cartons had no eggs broken.

c) Eldon and Liz bought identical boxes of candy. Eldon has $\frac{7}{8}$ of a box left and Liz has $\frac{1}{3}$ of a box left.

d) Mark needs $\frac{3}{5}$ box of apples for 1 batch of his special applesauce recipe. He has $2\frac{5}{9}$ boxes of apples.

e) Katrina brought 6 cartons of donuts to share equally among the 5 groups of students.

f) $\frac{2}{3}$ of the earth's surface is covered by oceans and $\frac{1}{10}$ is covered by glaciers.

g) Ted had no money left over after he spent $\frac{1}{2}$ of his year's income on food and rent, $\frac{1}{3}$ of his income on clothing, $\frac{1}{12}$ on entertainment, and saved $1,400.

h) $\frac{2}{3}$ of Ms. Quan's 5th grade class were boys. She sent 4 boys to another class and replaced them with 4 girls. Now $\frac{1}{2}$ of Ms. Quan's class are boys.

i) The 7 girls on the team each received $\frac{4}{7}$ package of socks.

(Continued on back.)

Follow-up Student Activity (cont.)

2 For each of the following, fill in the blanks with fractions or mixed numbers to make a challenging (to you!) fraction computation that you think you could solve by drawing egg carton models. Then show how to solve each problem using an egg carton model (on grid paper).

a) _____ + _____ = _____

b) _____ − _____ = _____

c) (challenge) _____ × _____ = _____

d) (challenge) _____ ÷ _____ = _____

3 Write a word problem that would require each computation you wrote in Problem 2.

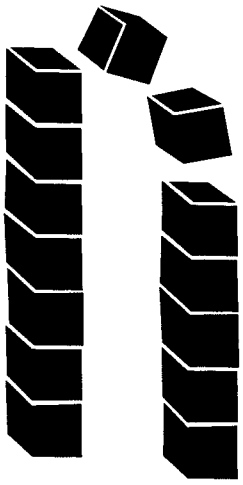
4 A question I still have about fractions is...

Averaging Two Numbers

Lesson 10

THE BIG IDEA

Insights into the concept of average of 2 numbers are developed through the process of leveling-off 2 columns of cubes. This process leads to methods for finding the average of 2 numbers and extends to finding averages of several numbers.



CONNECTOR

OVERVIEW

Students record their views of the meaning of average.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).

FOCUS

OVERVIEW

The average of 2 numbers is related to the process of leveling-off 2 columns of cubes. Students use this leveling-off model to solve problems involving averages of 2 numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Cubes, 30 per student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per pair of students and 1 transparency.
- ✔ Demonstration half-cubes (optional), 4 for the teacher.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Blank transparencies and overhead pens (optional), 1 per pair of students.

FOLLOW-UP

OVERVIEW

Students determine the average of 2 numbers and solve word problems involving averages.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity, 10.1, 1 copy per student.

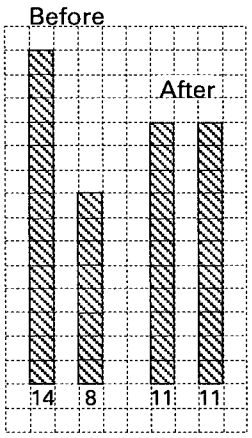
LESSON IDEAS

FOLLOW-UP

Students may find explaining their thinking about Problem 2 a challenge. Remind them that you want them to grow in their comfort and ability to discuss their thinking. As time passes and students have more experience with written explanations, their confidence and the quality of their explanations will improve. The problems students write for Problem 3 could be exchanged with classmates who solve the problems by building or sketching models. Students could write responses to Problems 4 and 5 in their journals.

SELECTED ANSWERS

1. a)



The average of 14 and 8 is 11.

2. a) 29 points
   b) 12 candy bars
   c) 176 and 158 points

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Prior to exploring the concept of average in the Focus activity, students record their views of the meaning of average. This is useful information for instructional planning and later for assessing students' growth in understanding.

MATERIALS

✔ Student journals (optional).

ACTIONS

1 Write the following statement on the overhead and ask students to spend about 3 minutes completing (in writing) the statement. Encourage students to include examples to help explain how they think about the meaning(s) of average.

To me, the word "average" means:

COMMENTS

1 If students are keeping journals, this is an appropriate journal entry. It will provide baseline information regarding their views and understanding of average. Encourage students to write as much as they can and remind them there are no right or wrong answers, rather you are interested in how they think about average at this time.

While it may seem more efficient to have students express their thoughts orally, and without writing, doing so doesn't provide information about each individual's views. This information is useful for both you and the students to use when tracking their needs and growth.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

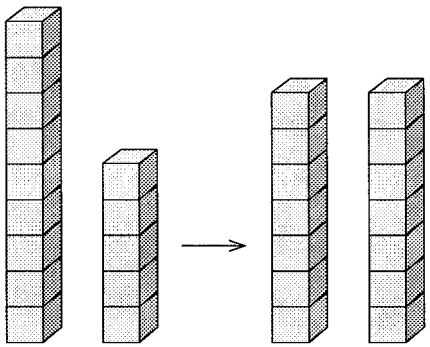
The average of 2 numbers is related to the process of leveling-off 2 columns of cubes. Students use this leveling-off model to solve problems involving averages of 2 numbers. This model is extended in Lesson 11 to include finding the average of several numbers.

MATERIALS

- ✔ Cubes, 30 per student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per pair of students and 1 transparency.
- ✔ Demonstration half-cubes (optional), 4 for the teacher.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Blank transparencies and overhead pens (optional), 1 per pair of students.

ACTIONS

1 Arrange the students in pairs and distribute cubes to each student. Ask the students to each form 2 columns of cubes, one of height 9 and the other of height 5. After everyone has their columns formed, ask the students to level-off their columns so that both have the same height. Ask for volunteers to describe a variety of methods.



2 Repeat Action 1 for 2 columns of cubes with heights of 2 and 8. If it hasn't come up, point out that *averaging* is the mathematical name for the leveling-off process.

COMMENTS

1 You may need to remind students to use all the cubes in their 2 columns, add no additional cubes, and keep 2 columns.

Students may use a variety of methods for leveling-off the 2 columns, such as: move cubes one-by-one from the taller to the shorter column until their heights are equal; knock the columns down into a large pile and rebuild them into 2 equal columns; determine the difference between the heights of the 2 columns and then add half the difference to the shorter column; or, combine the 2 columns into one tall column and then divide that into 2 equal columns. Regardless of the method, the leveled-off height is 7.

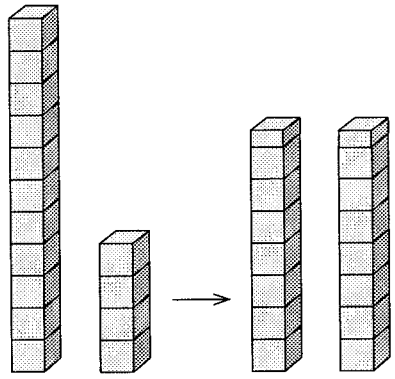
2 The leveled-off height of columns of heights 2 and 8 is 5. This leveled-off height is the *average* of the heights of the 2 original columns. Thus, the average of 8 and 2 is 5. Similarly, in Action 1 the average of 9 and 5 is 7.

Experience with leveling off columns of cubes, describing their methods, and listening to others' methods helps students to develop a strong visual model for the meaning of average. To keep students focused on the concept of average, algorithms are purposely not discussed here.

Focus Teacher Activity (cont.)

ACTIONS

3 Ask the students to form 2 columns of cubes with heights of 11 and 4. Have them level-off these 2 columns to determine the average of 11 and 4. Discuss their methods. Repeat for other pairs of numbers.



The average of 11 and 4 is $7\frac{1}{2}$.

4 Place a transparency of Focus Master A on the overhead and ask pairs of students to *use cubes to model the situations* and to use their models as a basis for making mathematical conclusions about the situations. Give each pair of students a sheet of 1-cm grid paper and ask them to record their models and methods of using the models to make mathematical conclusions. Have volunteers illustrate their observations at the overhead.

Averaging Two Numbers Lesson **10**

Focus Master A

Situations

a) Two columns of cubes are leveled off at a height of 12. One of the original columns had a height of 4.

b) Two numbers have an average of 15. Each number is increased by 8.

c) The average of 2 numbers is 9.

d) The difference between 2 numbers is 5. The average of the numbers is 11.

e) Two columns of cubes have a total of 32 cubes. One of the columns has 10 cubes.

COMMENTS

3 The students may wonder what to do with the extra cube. Suggestions may be sought from other students. You may wish to cut a few half cubes to use for demonstration purposes. They can be made by splitting a straight-grained wooden cube with a sharp instrument.

4 A master for 1-cm grid paper is in *Blackline Masters*. A transparency of the grid paper is useful for demonstrations at the overhead.

One way to get them started is to have students suggest questions they could answer by using their model. For example, a question to accompany Situation A could be, "What is the height of the other column?"

The purpose of this Action is to reinforce the leveling-off model for averaging and to engage students in using that model to represent situations. Using cubes to model the situations prior to making diagrams, reinforces the idea that diagrams are representations of concrete models and/or images of models that students carry in their minds.

Although standing cubes in vertical columns has more visual effect, when numbers become too large for columns to stand, it is helpful to lay columns on a side.

Focus Teacher Activity (cont.)

ACTIONS

5 Have the students imagine leveling-off columns of 23 and 13 cubes. Ask for several volunteers to describe the methods they used.

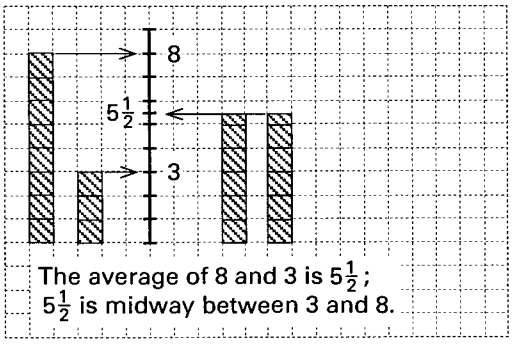
6 Ask students to each work with a partner to list general observations they can make about the leveled-off (average) value of any 2 numbers. Ask them to illustrate their observations with examples. Discuss.

COMMENTS

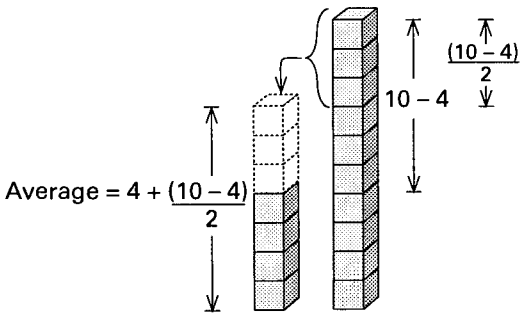
5 Students may use a variety of methods. Any method that yields a correct answer is acceptable.

6 Students may wish to use grid paper to make illustrations of their observations. Following are a few observations that students could make:

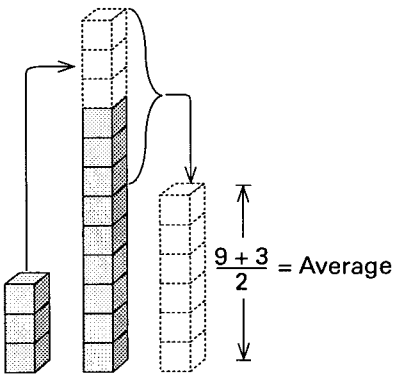
The average of any 2 numbers is midway between them, as illustrated here:



The average height of 2 columns is always one-half the difference between their height added to the height of the shorter column, as shown below:



The average height of 2 columns is one-half the total height of the 2 columns, as shown in the example below:



Focus Teacher Activity (cont.)

ACTIONS

7 Ask the students to imagine and/or sketch more than one leveling-off method of finding the average of 42 and 26. Repeat for other pairs of numbers, encouraging students to try out each other's methods.

8 Place a transparency of Focus Master B on the overhead and ask pairs of students *to build or sketch a model* of each situation. Ask them to use their models as a basis for writing mathematical conclusions about each situation. Ask for volunteers to demonstrate their conclusions and reasoning about each situation.

Lesson **10** Averaging Two Numbers

Focus Master B

More Situations

a) The high school football team scored 16 and 28 points during their first 2 games this year. Last year their average score after the first 2 games was 23 points per game.

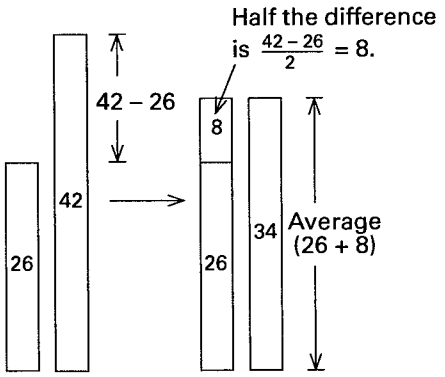
b) Wylie earned 14 points on his first assignment. He wants to earn enough points on his 2nd assignment to have an average of 21 points.

c) The Lions scored an average of 87 points for their first 2 basketball games. One of their scores was 92.

d) The difference between Joshua's weight and Paul's weight is 9 pounds. The average of their weights is 93 pounds.

COMMENTS

7 If necessary, point out to students that sketches don't need to show every cube and the purpose of the sketches is to facilitate and communicate the students' thought processes. For example:



Some other possible pairs of numbers to pose include: 92 and 88, 54 and 56, 39 and 40, 103 and 106, 62 and 78. Emphasize using the leveling-off model to make conclusions. It is intended that sketches are representations of students' actual actions with cubes or of their mental images of actions with cubes.

8 One way to facilitate discussion is to provide pairs of students with a blank transparency and an overhead pen, and ask the pairs to record their models and observations of selected situations prior to coming to the overhead.

Some students may use numbers and math symbols to help communicate mathematical conclusions about their models. These are important to discuss. It is important, however, to keep emphasis on the use of the leveling-off model. We prefer not to discuss algorithms here, unless a student brings one up. Students have a tendency to stop thinking about meanings and relationships when an algorithm emerges too quickly.

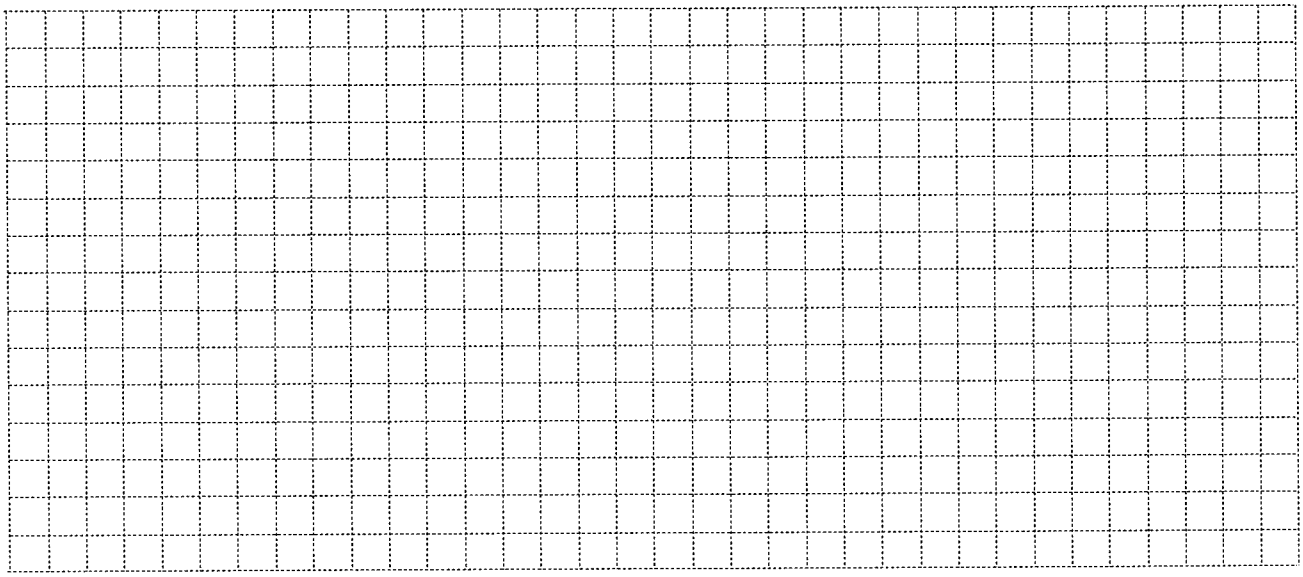
Follow-up Student Activity 10.1

NAME _____ **DATE** _____

1 The pairs of numbers below represent the heights of stacks of cubes to be averaged (leveled-off). On the grid sketch the front views of columns of cubes with these heights before and after they are leveled-off. Label the heights of all columns.

a) 14 and 8

b) 7 and 12



2 Draw sketches that show how to *use a model or diagram of cubes* to solve each of these puzzle problems. Then write an explanation of your methods and reasoning. (Remember a sketch doesn't need to show each cube.)

a) Andrew scored 21 points during his last basketball game. What must he score during his next game in order to have a 25 point average for the 2 games?

(Continued on back.)

Follow-up Student Activity (cont.)

b) During the first week of a fund raising project Maria sold 18 candy bars. After the second week her average sales for the 2 weeks was 15 candy bars per week. How many candy bars did she sell during the second week?

c) Tyson bowled 2 games last night. The difference between his 2 scores was 18 points. His average score for the 2 games was 167 points. What did he score on each of the 2 games?

3 Write two interesting word problems that involve averaging.

4 Describe one or two ideas related to averaging that you learned or understand better after our class explorations of averaging.

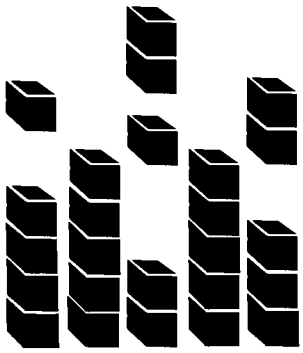
5 What question(s) do you have about averaging?

Averaging Several Numbers

Lesson 11

THE BIG IDEA

The leveling-off process provides intuitive understanding for the concept of average of several numbers. This visual model provides a conceptual basis for solving problems involving the average of several numbers.



CONNECTOR

OVERVIEW

Students use cubes and the leveling-off process to solve problems involving averaging.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Cubes, 30 per student.
- ✔ Connector Master A, 1 transparency.

FOCUS

OVERVIEW

The leveling-off process is extended to develop methods for finding the average of several numbers and to solve word problems.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Cubes, 30 per student.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Transparency of 1-cm grid paper (see *Blackline Masters*).
- ✔ Blank transparencies and overhead pens (optional), 1 per pair of students.

FOLLOW-UP

OVERVIEW

Students use the leveling-off process for sketching columns on grids and solving word problems involving the average of several numbers.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 11.1, 1 copy per student.

LESSON IDEAS

FOLLOW-UP

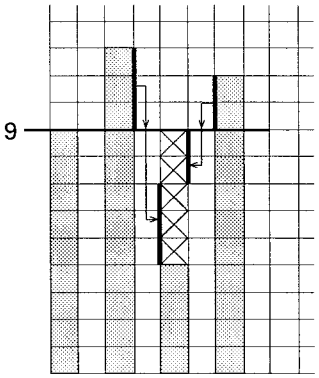
For Problem 3 you might let students pick 4 problems as homework and make the others optional. Students will benefit from opportunities to discuss and refine their thinking about these problems. This could mean giving students 5-10 minutes at the beginning or end of class for small group sharing.

If students need a "boost" on problem 4, you could pose questions like: Could all the numbers be even or odd? Why or why not? Is there a smallest (largest) possible number? What if no numbers are the same? What if...?

TEACHER NOTES:

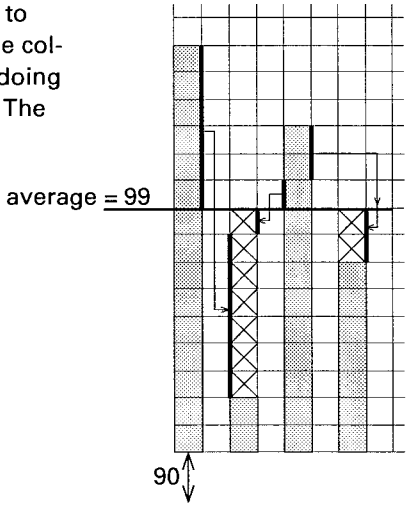
SELECTED ANSWERS

1. a) The average is 9, as shown at the right.



b) The average is 10.

2. It is only necessary to show the tops of the columns. One way of doing this is shown here. The average is 99.



3. a) $8.50
   b) 54
   c) $18\frac{1}{3}$
   d) 184
   e) 90
   f) It would increase to 93.
   g) 15

Connector Teacher Activity

OVERVIEW & PURPOSE

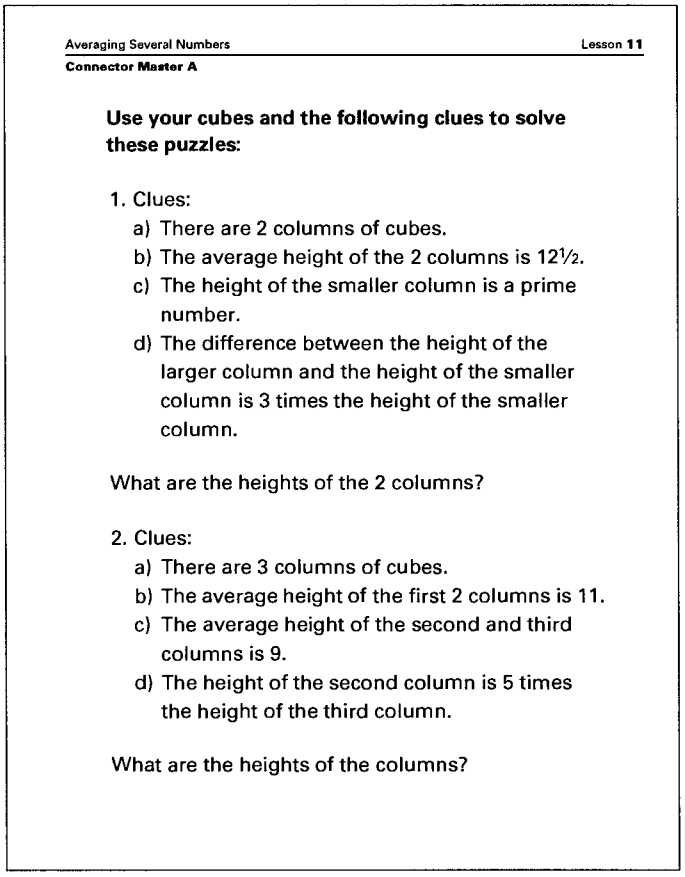
Students use cubes and the leveling-off process to solve problems involving averaging.

MATERIALS

- ✔ Cubes, 30 per student.
- ✔ Connector Master A, 1 transparency.

ACTIONS

1 Arrange the students in groups and distribute cubes to each student. Place a transparency of Connector Master A on the overhead and ask the groups to use their cubes, together with the clues given on the transparency, to determine the heights of the columns of cubes in Puzzle 1. Discuss their methods.



Averaging Several Numbers Lesson 11

Connector Master A

Use your cubes and the following clues to solve these puzzles:

1. Clues:
   a) There are 2 columns of cubes.
   b) The average height of the 2 columns is 12½.
   c) The height of the smaller column is a prime number.
   d) The difference between the height of the larger column and the height of the smaller column is 3 times the height of the smaller column.

What are the heights of the 2 columns?

2. Clues:
   a) There are 3 columns of cubes.
   b) The average height of the first 2 columns is 11.
   c) The average height of the second and third columns is 9.
   d) The height of the second column is 5 times the height of the third column.

What are the heights of the columns?

2 (Optional) Repeat Action 1 for Puzzle 2.

COMMENTS

1 The heights of the columns are 20 and 5. One method of finding these is to begin with 2 columns of height 12½ or one column of height 13 and one of height 12. Then shift the cubes until the height of the shorter column is a prime number and the difference between the height of the larger column and the height of the smaller column is 3 times the height of the smaller column.

2 Some students may find this problem a challenge. The heights of the columns are 7, 15, and 3.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use cubes and the leveling-off process to develop methods for finding the average of several numbers and to solve word problems.

MATERIALS

- ✔ Cubes, 30 per student.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Transparency of 1-cm grid paper (see *Blackline Masters*).
- ✔ Blank transparencies and overhead pens (optional), 1 per pair of students.

ACTIONS

1 Distribute cubes to each student. Form 5 columns of cubes with heights 4, 7, 2, 9, and 3 and ask each student to form 5 columns that match yours. When everyone has formed their columns, ask the students to level-off these columns (without changing the number of columns or the total number of cubes). Discuss their methods and results.

2 Ask the students to form 4 columns of heights 8, 5, 4, and 7. Ask them to determine the average height of the columns. Discuss.

3 Repeat Action 2 for a) and/or b) below:

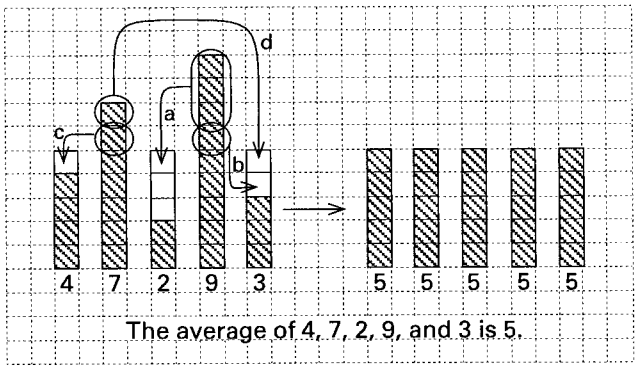
a) 6 columns of height 3, 7, 9, 4, 6, and 1.

b) 4 columns of heights 9, 3, 6, and 8.

COMMENTS

1 The leveled-off height of the columns is the *average* of the heights of the original columns. Students will find the leveled-off (average) height of these columns using a variety of methods. All methods that work are acceptable.

A transparency of 1-cm grid paper is useful for student demonstrations at the overhead. A sketch might look as follows, where the letters near the arrows indicate the order in which a student moved the cubes.



2 It is important that the number of columns and the total number of cubes remain unchanged in the leveling-off process. The average height of the columns is 6.

3 The purpose here is to give students additional experience with leveling-off actual cubes, so that later mental images of the model will be stronger.

The heights of the leveled-off columns are a) 5 and b) $6\frac{1}{2}$ (if half-cubes are not available for students, encourage students to imagine how the 2 leftover cubes could be distributed among the 4 columns).

Focus Teacher Activity (cont.)

ACTIONS

4 Place a transparency of Focus Master A on the overhead and ask the students to use their cubes to model the situations and make mathematical conclusions about each situation (or selected situations). Ask for volunteers to illustrate their reasoning and conclusions.

Averaging Several Numbers Lesson **11**
Focus Master A

Situations

a) Five columns of cubes can be leveled off to a height of 8. The height of the smallest column is 4 and the height of the largest column is 18.

b) Four numbers have an average of 9. Three of the numbers are 7, 11, and 5.

c) The average of 3 numbers is 14. One of the numbers is 6.

d) Six numbers have an average of 8, but none of the numbers are equal to 8.

e) Five columns of cubes are arranged in order from smallest to largest. Each column is 3 cubes higher than the column that comes before it. The average height of the columns is 14.

5 Form 5 columns of blocks with heights of 6, 11, 3, 7, and 5. Ask the students to work with a partner to form columns that match yours and then to determine the average of these numbers.

COMMENTS

4 You may wish to only reveal one situation at a time for exploration and discussion by the students. This allows students to better benefit from others' ideas. Encourage students to use the leveling-off model of averaging as a basis for making conclusions about the situations.

Following are some possibilities for selected situations:

a) Begin with 5 columns of height 8 and then move cubes around to obtain one of height 4 and one of height 18. Here are some possibilities for the missing heights:

4, 5, 5, 8, 18; 4, 5, 6, 7, 18;
4, 6, 6, 6, 18; 4, 4½, 6½, 7, 18.

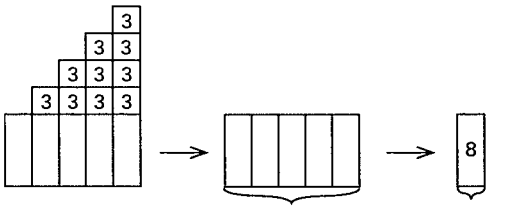
b) The fourth number is 13.

c) The other numbers could be: 0, 36; 1, 35; 2, 34; 3, 33; 4, 32;... 18, 18. There are many other possibilities using mixed numbers, such as 1½ and 34½.

d) Here are some possibilities for the numbers:

5, 6, 7, 9, 10, 11; 4, 5, 6, 10, 11, 12;
1, 1, 1, 15, 15, 15; 2, 3, 4, 12, 13, 14.

e) Since the average of the 5 numbers is 14, there are $14 \times 5 = 70$ cubes. The following diagram illustrates a method of determining the shortest column:



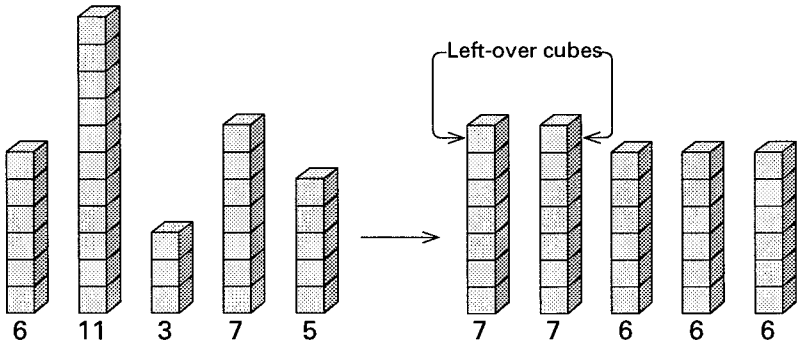
Hence, the smallest number is 8, and by adding 3 successively, the other numbers are 11, 14, 17, and 20.

5 It is not be possible to level-off the columns without cutting cubes. Since the columns, if leveled-off, would be more than 6 but less than 7 cubes high, the average is between 6 and 7.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS



If cubes could be cut, one could cut each of the 2 left-over cubes into 5 equal parts and distribute these parts among the columns. Based on the part-to-whole concept of a fraction introduced in Lesson 8, each column would receive 2 parts of ⅕ each (see Figure 1). Thus, each leveled-off column would be 6⅖ cubes high.

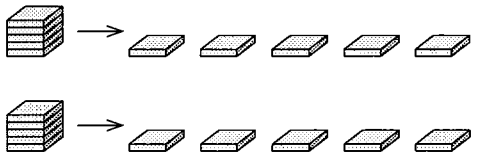


Figure 1

Or, the 2 leftover cubes could be glued together and then cut into 5 equal parts, which would then be distributed among the columns. Each of these parts would be ⅕ of 2 cubes or 2 ÷ 5 = ⅖ of a cube and each column would receive one of them (see Figure 2). Again, the height of the leveled-off columns would be 6⅖ cubes. This method involves the *division concept of a fraction* (see Lessons 8 and 28).

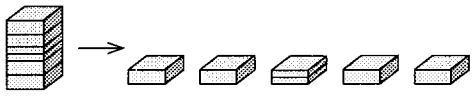


Figure 2

6 Ask students to consider the following:

Suppose columns of 23, 30, 21, 34, and 22 are leveled off.

Have them make a "quick and reasonable" prediction of the height of these leveled-off columns. Discuss their ideas about methods that might be used to find this height. Then have the groups apply one or more of these methods to determine the average of 23, 30, 21, 34 and 22.

6 Many students will probably observe that the average must be between 21 and 34, the lowest and highest columns, and refine their predictions from there.

Students might wish to discuss ideas about methods in their groups. The actual leveled-off height is 26.

One method of finding the average of 23, 30, 21, 34, and 22 is to imagine leveling-off columns of cubes with these heights by taking cubes from one column and

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

6 (continued.)
adding to another. This process can be represented by leveling-off the numbers in much the same manner. For example:

The original numbers:	23	30 ▽	21 ▽	34	22
Taking 5 from 2nd number and adding to 3rd:	23	25	26	34 ▽	22 ▽
Taking 4 from 4th number and adding to 5th:	23 ▽	25	26	30 ▽	26
Taking 3 from 4th number and adding to 1st:	26	25 ▽	26	27 ▽	26
Taking 1 from 4th number and adding to 2nd:	26	26	26	26	26

The average of 23, 30, 21, 34, and 22 is 26.

Another way of viewing this problem is to imagine the 5 original columns stacked on top of one another to form a single column. Dividing this single column in 5 equal parts will produce the leveled-off columns. Thus, the height of the leveled-off columns will be the sum of the heights of the original columns divided by 5:

$$(23 + 30 + 21 + 34 + 22)/5 = 130/5 = 26.$$

This computation is readily performed on a calculator.

7 Write the numbers from a) (see below) on the overhead and ask the students to make a reasonable prediction of the average of the numbers. Discuss their predictions and reasoning. Repeat for b)-d).

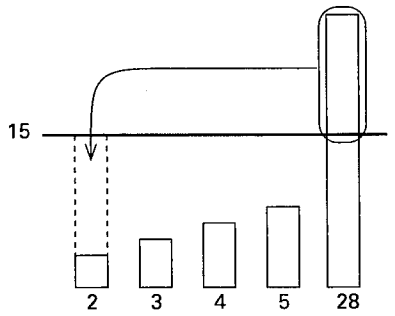
a) 2, 3, 4, 5, 28

b) 26, 42, 40, 24

c) 31, 26, 41, 20, 30

d) 92, 78, 4, 96, 87

7 An important role of visual models is to give students a "sense" for an answer. For example, students may use reasoning similar to the following to predict the average of the numbers in a): "Imagine each number as a column of cubes. The leveled-off height of the tallest and shortest column is 15. The average height of the 5 columns must be less than 15 because each of the 3 "middle" columns is less than 15 cubes high. Hence, a reasonable prediction of the average of 2, 3, 4, 5, and 28 would be about 10." A mental picture of this relationship might look like the following:

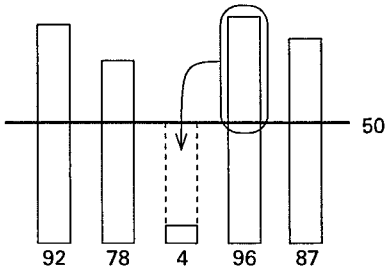


Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

Similarly, students might imagine the numbers from d) like the following:



and use that image as a basis for predicting an average in the range of 65-80.

8 Ask students to find the actual average of the numbers listed in b) and c) of Action 7. Discuss their methods.

8 The averages are 33 and 29⅗. Acknowledge all methods that work, while encouraging students to show visually why their methods work.

9 Place a transparency of Focus Master B on the overhead and have students work in pairs using cubes or diagrams to model each situation. Then ask the pairs to use each model as a basis for creating and answering mathematical questions involving averaging. If students use cubes, ask them to record their models and methods in a sketch. Have volunteers show their models and questions and describe their reasoning at the overhead.

Lesson **11** Averaging Several Numbers
Focus Master B

More Situations

a) The heights of 3 students are 60 inches, 68 inches, and 70 inches.

b) In 5 soccer games, the Troubleshooters scored 3, 2, 0, 1, and 4 goals.

c) In 10 basketball games, Kathy scored 9, 12, 11, 8, 11, 15, 15, 16, 15, and 19 points.

d) Yolinda has an average of 80 points on 3 assignments. On her fourth assignment she needs to bring her average up to 84 points.

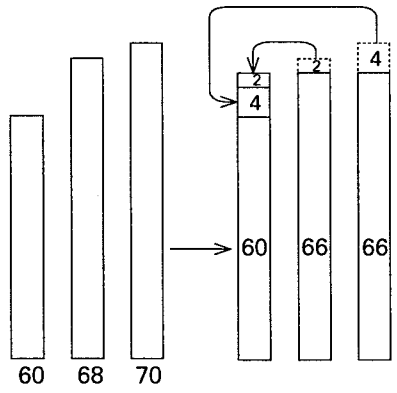
e) After 4 games the basketball team's average score was 95. They want to have a 5-game average of at least 90 points.

f) Sara averaged 20 points per game in her first 12 games and 15 points per game in her last 3 games.

9 Encourage groups to create and explore at least one question about each situation. Sharing can be facilitated by giving students blank transparencies and overhead pens and having them prepare their diagrams before coming to the overhead.

To reinforce the development of students' conceptual understanding of average, encourage them to rely on the leveling-off model to help them answer their questions about the situations (e.g., rather than to rely on "guess and check" methods). Following are possible diagrams of situations a)-f) together with one possible conclusion that can be determined using each diagram.

a) The average of 60, 68, and 70 is 66.



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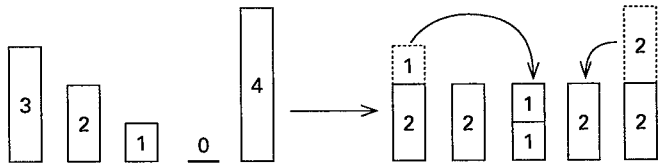
Focus Teacher Activity (cont.)

ACTIONS

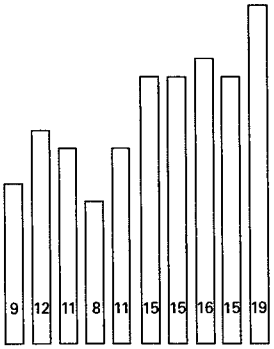
COMMENTS

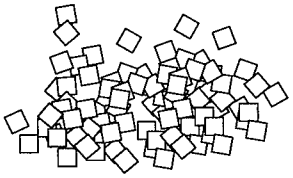
9 (continued.)

b) The average of 3, 2, 1, 0, and 4 is 2.

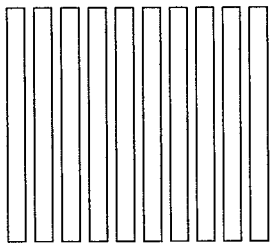


c)



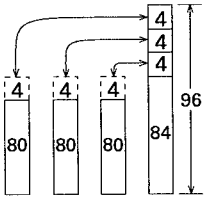


9+12+11+8+11+15+15+16+15+19 = 131

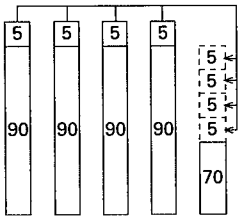


131 ÷ 10 = $13\frac{1}{10}$ is her average.

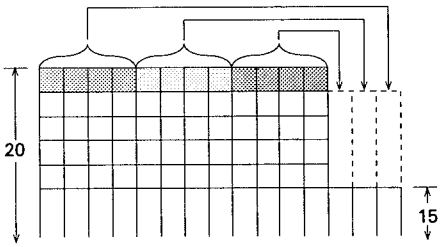
d) A score of 96 is needed. To raise the average to 84, the fourth test must have a score of 84 + (3 × 4):



e) The lowest score they can get is 70. Five can be removed from each of the four 95's and added to 70 to show the average is 90:



f) Sara's overall average for 15 games is 19.



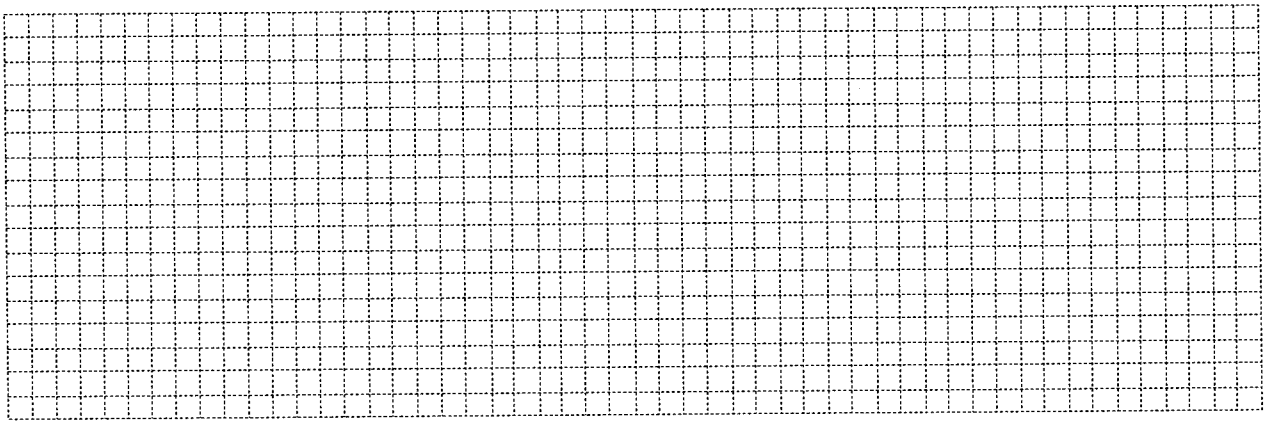
Follow-up Student Activity 11.1

NAME _____ DATE _____

1 For each of the sets of numbers listed below, draw the front views of columns of cubes whose heights are the same as the numbers. Show how to level-off each set of stacks to find the average of that set of numbers. Label the averages on the grid.

a) 9, 12, 4, and 11

b) 5, 15, 8, 9, and 13



2 Make a *sketch* to show the leveling-off of stacks of cubes of heights 105, 92, 102, and 97.

What is the average of 105, 92, 102, and 97?

(Continued on back.)

Follow-up Student Activity (cont.)

3 On a separate sheet make sketches that show how to use a model to solve each of the following puzzle problems.

a) In 4 days of baby-sitting Dan earned \$6, \$7, \$12, and \$9. What was the average amount he earned each day?

b) The average of 6 numbers is 9. What is the sum of the numbers?

c) In 3 games Rachelle made 17, 23, and 15 points. What was the average number of points she made per game?

d) Ramon's average bowling score after 3 games was 152. What must he score on the last game to raise his average to 160?

e) After 5 assignments, Marcia's average was 88 points. Her average for the next 2 assignments was 95. What was her overall average for the 7 assignments?

f) Suppose 3 bonus points were added to each of Marcia's assignments in problem e) above. How would that affect her average?

g) Jeremey scored an average of 18 points per game during the first 5 games of the season. During the 6th game he was injured and scored no points. What was his 6-game average?

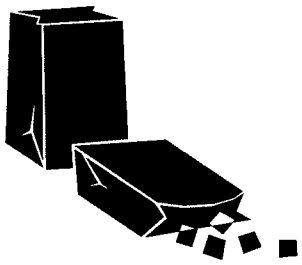
4 The average of 5 numbers is 37. The difference between the largest and smallest number is 12. On another sheet investigate and report all that you can about the 5 numbers.

Probability

Lesson 12

THE BIG IDEA

A rectangular region that is divided into subregions of different colors and areas provides a method of modeling probability. Fractions provide a way of describing these probabilities.



CONNECTOR

OVERVIEW

Students form collections of tile and determine the fractional parts having various colors.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Colored tile, 30 per student.
- ✔ Paper sack, 1 for the teacher.

FOCUS

OVERVIEW

Students are introduced to theoretical and experimental probability through the use of tile and the area model for probability. Probabilities are recorded as fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Colored tile, 30 per student.
- ✔ Paper sacks, 1 per pair of students.
- ✔ Focus Master A, 1 copy per pair of students and 1 transparency.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 1 copy per pair of students and 1 transparency.

FOLLOW-UP

OVERVIEW

Students use fractions to answer probability questions involving selections of colored tile and outcomes of spinners.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 12.1, 1 copy per student.

LESSON IDEAS

PURPOSE
The intent of this lesson is only to introduce the concept of probability and engage students in using fractions to describe chance. There will be several opportunities to discuss probability in this course and this topic is explored in depth in Courses II-IV.

Whenever possible use the term "probability" in informal discussions with students about the chance or likelihood of an event.

JOURNALS
At the end of this lesson you might ask students to complete the following in their journals:

"To me the meanings of theoretical probability and experimental probability are..."

"Sometimes theoretical and experimental probability are close in value and sometimes they are not. I think this is because..."

FOLLOW-UP
Assign parts of this Follow-up as the lesson proceeds. However, encourage students to investigate problems you haven't assigned yet.

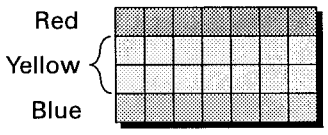
An alternative to having students cut out the rectangle in Problem 1 is to give them colored paper squares.

SELECTED ANSWERS

1. a) Red. Note: Some students may say you can't be sure. It is true that you can't be sure red will be selected but it is the most likely color to be selected. This may need to be discussed by the class.

   b) $\frac{5}{12}$, $\frac{3}{12}$ (or $\frac{1}{4}$), $\frac{4}{12}$ (or $\frac{1}{3}$), $\frac{9}{12}$ (or $\frac{3}{4}$), $\frac{0}{12}$ (or 0), $\frac{8}{12}$ (or $\frac{2}{3}$)

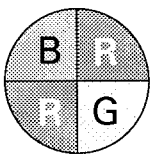
5. The collection of tile is shown in this rectangle (any rectangle with 28 tile and this distribution of colors is acceptable).



   The probabilities are: $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{2}$.

6. Reasonable predictions would be similar (but not necessarily identical) to the following, which are theoretical probabilities: B, 25; P, 12 or 13; R or B, 62 or 63; and 75 times for any letter other than W.

7. The letters B, R, and G represent the colors blue, red, and green, respectively.



TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students form collections of tile and determine the fractional parts having various colors. This area model provides a foundation for introducing probability in the Focus.

MATERIALS

- ✔ Colored square tile, 30 per student.
- ✔ Paper sack, 1 for the teacher.

ACTIONS

1 Place the students in pairs and distribute tile to each student. Ask each pair to use 3 blue, 1 green, and 4 red tile to form a 2 by 4 rectangle. Pose and discuss the following questions: What fraction of the rectangle is red? blue? green? not blue? red or green? yellow?

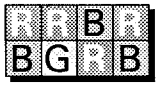
2 Ask each pair of students to build a rectangle of tile that satisfies the following conditions: ⅓ of the tile are green; ¼ of the tile are red; and the remaining tile are blue. Have volunteers share their results and methods at the overhead.

3 Ask each pair to write several questions involving fractions about the rectangle they formed in Action 2. Select some of their questions for solution and discussion by the class.

COMMENTS

1 If colored tile are not available, colored squares can be cut from tagboard. Each pair of students will need at least 6 red, 8 green, and 10 blue to complete this lesson (Connector and Focus).

Here is one possible placement of the tile (B represents blue tile, G green, and R red).

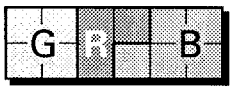


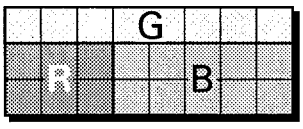
⁴⁄₈ or ½ are red, ³⁄₈ are blue, ⅛ are green, ⁵⁄₈ are not blue, ⁵⁄₈ are red or green, and ⁰⁄₈ or 0 are yellow.

It may help some students to recall the egg carton fraction model and to think of the 2 by 4 rectangle as a carton of 8 eggs.

The use of the word "or" in finding the fraction that are red or green may need to be discussed. Since there are 4 red tile and 1 green tile, there are 5 tile that are red or green. This is the same as the number of tile that are not blue.

2 Here are two rectangles that satisfies these conditions. Any rectangle that contains 12 or a multiple of 12 tile can be subdivided according to these conditions.





3 Here are a few possibilities for questions: What fraction of the total number of tile is blue? What fraction of the total number of tile is red or green? What fraction of the tile is not red? What fraction of the tile is green? What fraction of the total number of tile is the difference between the green and red? Discuss how answers to these questions compare for differently-sized rectangles.

Connector Teacher Activity (cont.)

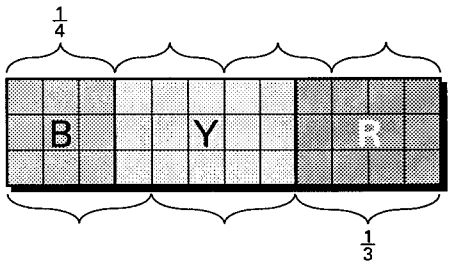
ACTIONS

4 Without letting the students see, place 9 blue tile, 12 red tile, and 15 yellow tile in a sack. Tell the students that you have 36 tile in the sack and that ⅓ are red, ¼ are blue, and the rest are yellow. Ask each pair to form a collection of tile that matches yours and to determine what fraction of the tile are yellow. Discuss their results and methods.

COMMENTS

4 One approach is to count out 36 tile and divide them into 3 equal groups and 4 equal groups to determine that ⅓ of 36 is 12 and ¼ of 36 is 9. Then 36 – (12 + 9) = 15 is the number of yellow tile.

Another approach is to build or sketch a rectangle and group off ⅓ and ¼ of the tile. A 3 by 12 rectangle is shown here with the red tile grouped by thirds and the blue tile grouped by fourths. The remaining 15 tile must be yellow. Hence, $^{15}/_{36}$ (or $^{5}/_{12}$) of the rectangle is yellow.



Focus Teacher Activity

OVERVIEW & PURPOSE

Students are introduced to theoretical and experimental probability through the use of tile and the area model for probability. Probabilities are recorded as fractions.

MATERIALS

- ✔ Colored tile, 30 per student.
- ✔ Paper sacks, 1 per pair of students.
- ✔ Focus Master A, 1 copy per pair of students and 1 transparency.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 1 copy per pair of students and 1 transparency.

ACTIONS

1 Place the students in pairs and distribute the colored tile and a sack to each pair. Ask them to place 3 red, 2 green, and 1 blue tile in the sack. Describe the following game in which a player:

- Selects a tile from the sack, without looking in the sack.
- Records the color of the tile.
- Records 2 points if the tile is red, 1 point if it is green, and 0 points if it is blue.
- Returns the tile to the sack, shakes the sack, and passes the sack to the next player who repeats this process.
- The first player to win 7 or more points wins the game.

After the students have played this game ask them to determine the fewest number of turns to win and the most number of turns a player might take to win. Discuss.

2 Ask the pairs to write a statement predicting which tile they think would be most likely to be selected if 1 tile was drawn randomly from the sack. Discuss their reasoning. Then ask them to write statements describing the chances of drawing:

a) a red tile,

b) a green tile,

c) a blue tile,

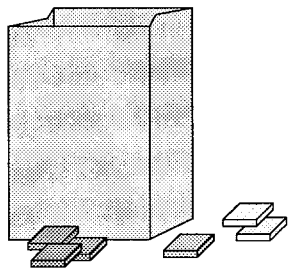
d) a tile that is not green,

e) a tile that is either red or blue or green,

f) a black tile.

COMMENTS

1 A player can win in a minimum of 4 turns by selecting 4 red tile or 3 red and 1 green. There is no maximum number of turns to win, since selecting a green tile does not increase the player's score.



2 The purpose of shaking the sack is to ensure that draws are *random*. If this doesn't come up, you might ask students to speculate about the effect not shaking the bag could have on the results.

The statements for the red tile may be something like: there is a better chance of getting a red tile than a green; there is 3 times more chance of getting a red tile than a blue; there are 3 chances out of 6 of selecting a red tile; the chance of a red tile is $\frac{3}{6}$ or $\frac{1}{2}$. Similarly, the chances of selecting a green tile are 2 out of 6 or $\frac{2}{6}$, the chance of a blue tile is 1 out of 6 or $\frac{1}{6}$, a not green tile is $\frac{4}{6}$, a red or blue or green tile is $\frac{6}{6}$ or 1, and a black tile is $\frac{0}{6}$ or 0.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Discuss their statements and if a student doesn't bring it up, introduce the term *theoretical probability*.

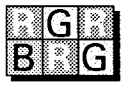
3 Ask the students how they have heard the word probability used. Discuss these uses.

4 Distribute a copy of Focus Master A and a red, blue, and green marker to each pair. Ask each pair of students to place 3 red, 2 blue, and 1 green tile in a sack, shake the sack, and conduct the following experiment: select a tile; record its color on Part a) of Focus Master A; return the tile to the sack and shake the bag; repeat this procedure 6 times.

After students have completed their experiments discuss the meaning of *experimental probability* as compared to theoretical probability and ask the students to use their data and observations to complete Parts b) and c) of Focus Master A.

COMMENTS

2 (continued.)
One way to visually illustrate the meaning of *theoretical probability* is to take the 3 red, 2 green, and 1 blue tile from the bag and arrange them in a rectangle, as shown at the right.



The fraction of the total area which is occupied by each color is the *theoretical probability* that the color will be randomly chosen from the sack.

3 The words *probability* and *chance* are often used interchangeably. Here are a few common uses of these words: The probability (chance) of rain today is very good; in all probability the game will not begin on time, my chances of being able to come to your house today aren't great; the chance that someone will have a pencil I can borrow is good. Statements such as these are usually based on the speakers prior experiences or knowledge about the conditions of the situation.

4 While the fractions that represent the *actual distribution of colors* in the bag are the theoretical probabilities of selecting the colors, the fractions that represent the distribution of colors *in their experiment* are the experimental probabilities of selecting the colors.

The fractions for the experimental probabilities will vary between ⅓ and ⅔ for different groups of students. The experimental probability for a given color of tile may equal the theoretical probability, but this often does not happen, particularly with so few samples. A sample group's data could look like this:

Probability Lesson **12**
Focus Master A

a)

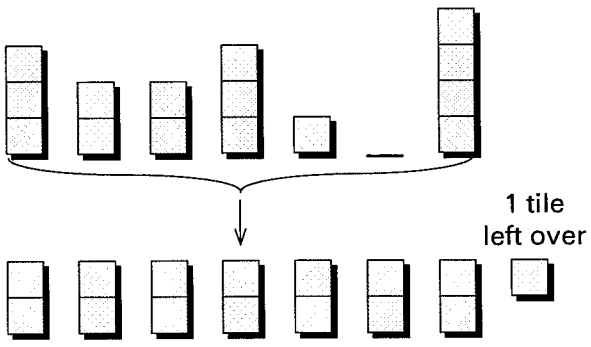
Draw #	Red	Blue	Green
1	I		
2	I		
3		I	
4	I		
5		I	
6			I

Focus Teacher Activity (cont.)

ACTIONS

5 Have each pair of students post their charts from Action 4 on the wall. Ask students to give their opinions about why the experimental probabilities don't always match the theoretical probabilities discussed in Action 2.

6 Ask each pair to devise a method of combining the groups' data posted in Action 5 and to determine the experimental probability of drawing each color, based on this combined data. Discuss the groups' methods and results and their thoughts about the role of *sample size* in an experiment.



COMMENTS

The results of this experiment are recorded on Parts b) and c) of Focus Master A shown below (if available have students use colored markers to fill in area models).

5 The theoretical probability of selecting a green is $\frac{2}{6}$, but this does not guarantee that for every 6 tile selected, 2 will be green. Obtaining a fraction of $\frac{2}{6}$ for the experimental probability is not surprising, but it can not be expected to happen for every selection of 6 tile. Students may suggest that results might be more accurate if they drew more samples or combined all the groups' data.

	(Red)	(Blue)	(Green)
b) Experimental Probability	$\frac{3}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
Theoretical Probability	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{2}{6}$

c)

R	R	B
R	B	G

Area Model of Experiment

R	R	G
R	B	G

Area Model of Actual (Theoretical) Contents of Sack

6 Whereas the results of individual experiments are usually widely varied, the results from combined experimental data are *usually* closer to the theoretical probabilities (unless the class is very small and provided the process is truly random). In general, the larger the sample size the more confident one can be that experimental results more closely resemble theoretical results.

There are several methods which may come up for pooling the results from all pairs of students. For example, some students may form collections of tile to represent each pair's experimental outcomes, combine these collections of tile to form an area model of the whole class' results, and determine the fraction each color is of the total.

Others may arrange the numbers of each color selected by each pair in columns and level-off these columns to find the average number for each color. The example shown at the left shows the number of green tile

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

6 (continued.)
selected by 7 pairs of students (a dash indicates 0 tile). These numbers were then averaged (leveled-off), and based on the results one might predict, "there are approximately 2 green tile per bag."

Issues related to probability, sample size, and confidence are examined in more depth in Lesson 45 and in *Visual Mathematics, Course II.*

7 (Optional) Have each pair of students predict the outcomes of the experiment in Action 4 for 36 draws instead of 6. Discuss their predictions and reasoning.

Give each pair of students a sheet of 1-cm grid paper. Then ask the pairs to conduct an experiment with 36 draws and sketch an area model of their results. Discuss the pairs' results and the class' combined results in comparison to the theoretical probabilities and in comparison to their experimental probabilities from Actions 4-6.

7 Some students may color squares of an area model as they draw the tile, while others may tally their draws and then make an area model of the results.

It may be helpful to discuss the use of tally marks for recording data. For example, a group's data might be recorded like this:

Blue 𝍸
Red 𝍸 𝍸 𝍸 𝍸 ||
Green 𝍸 𝍸

8 Describe the following game for pairs of students to play:

- Place 6 red, 8 green, and 10 blue tile in a sack.
- Take turns selecting a tile from the sack until one student gets a red tile. This student is called the Red Player. The other student is called the Green Player.
- Each player in turn selects a tile from the sack and then returns it to the sack. The Red Player scores 1 point if the tile he/she chooses is red or 1 point if it is blue. The Green Player scores 1 point if the tile he/she chooses is green or 1 point if it is blue.
- The first player to score 10 points wins the game.

After the game is played, have each pair write an explanation that involves probability to prove why the game is or is not fair. Discuss their conclusions and reasoning.

8 Encourage students to use probabilities and visual displays (based on both theoretical and experimental evidence) to support their views. Some students may reason that there is a $\frac{1}{3}$ probability of selecting a green tile, a $\frac{1}{4}$ probability of selecting a red tile, and $\frac{1}{3}$ is greater than $\frac{1}{4}$, so the Green Player has an advantage. Others may say the Green Player has a better chance of scoring a point on each turn since this player can score on more tile than the Red Player. The Green Player has 18 chances out of 24 of scoring a point but the Red Player has only 16 chances out of 24.

This is an example of a game that is *not fair* because, theoretically, both players do not have the same chance of winning. You might ask students to redesign the game so that, theoretically, it is fair. (Fairness is explored in more depth in *Visual Mathematics, Course II.*)

Follow-up Student Activity 12.1

NAME _____ DATE _____

1 Imagine that the tile shown in this rectangle are placed in a sack and that 1 tile is randomly selected.

Red	Blue	Red	Green
Green	Red	Blue	Red
Blue	Green	Red	Blue

a) Which color is most likely to be selected from the sack? How did you decide this?

b) What is the *theoretical probability* that the color of the tile selected is: red?_____ green?_____ blue?_____
red or blue?_____ yellow?_____ not blue?_____

2 Cut out the rectangle in Problem 1 and cut apart the squares. Place all 12 squares in a sack or other container and carry out the following experiment.

• Without looking in the sack, select 1 square and record its color here:

Color	Tally Marks
Red	
Blue	
Green	
Yellow	

• Replace the square in the sack and shake the sack to mix the squares.

• Repeat this process 60 times.

(Continued on back.)

Follow-up Student Activity (cont.)

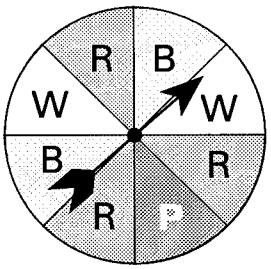
3 Based on the data from your experiment in Problem 2, what is the *experimental* probability that a tile drawn is:
red?_____ green?_____ blue?_____
red or blue?_____ yellow?_____ not blue?_____

4 Using the tile in Problem 2 make up a 2-person game that you think is not fair. Describe the rules of your game on another sheet, and explain why it is not fair and which player has the advantage.

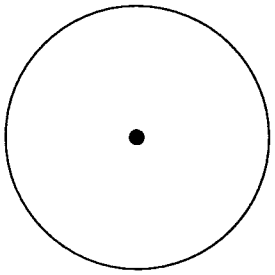
5 A total of 28 red, yellow, and blue tile are placed in a sack. One tile is randomly selected from the sack. The probability of selecting a red tile is ¼. The probability of selecting a yellow tile is twice the probability of selecting a blue tile. Sketch a rectangle at the right showing this collection of tile (mark the color of each tile).

What is the theoretical probability of selecting from your rectangle a tile that is: yellow?_____ blue?_____ not blue?_____ red or blue?_____

6 The spinner at the right has 8 equal parts. Assume the spinner is to be spun 100 times. Predict the approximate number of times you think the spinner will be likely to point to B _____, to P _____, to R *or* B _____, to any letter other than W _____. Tell how you decided each number.



7 Complete the spinner shown at the right so it has 4 parts. Color each part either blue, red, or green so that the probability of spinning a blue is ¼ and the probability of spinning a red is twice the probability of spinning a green.

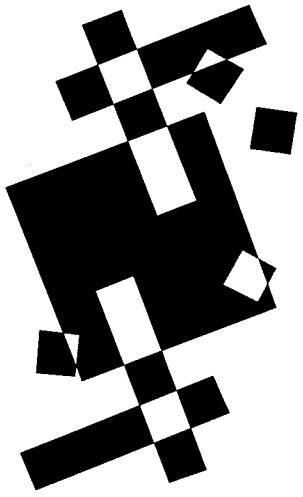


Base Five Grouping & Numeration

Lesson 13

THE BIG IDEA

Explorations with base five area pieces provide insights about the role of grouping and place value in recording numbers. Focusing first on base five keeps attention on conceptual relationships without interference from students' prior knowledge and/or misconceptions about base ten and provides a conceptual basis for later explorations of base ten.



CONNECTOR

OVERVIEW

Students observe the patterns among the base five area pieces and predict the shapes of larger and smaller base five pieces.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area pieces (see *Blackline Masters*); 5 mats, 10 strips, 15 units, and 1 strip-mat for each student.
- ✔ Base five area pieces for the overhead.

FOCUS

OVERVIEW

Base five area pieces are used to examine the role of grouping and place value in recording numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area pieces for each student.
- ✔ Base five area pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students use base five area pieces to form and determine recordings for various minimal collections. Based on relationships they observe about base five pieces, students make predictions about groupings in base four.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 13.1, 1 copy per student.
- ✔ Base five area pieces (see *Blackline Masters*), 1 sheet per student, copied on regular copy paper for use at home.

LESSON IDEAS

ASSESSMENT

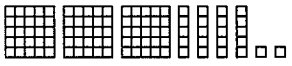
At the end of this lesson, have the students privately reflect on the lesson. Then have them record their thoughts and understandings about base five in their journals. This helps students to clarify their thinking and provides a way for you to assess their understanding at this point in time. By repeating this after students have explored grouping and place value in more depth in future lessons, it will be possible to measure growth in their understanding.

TIMING

Students will explore several lessons involving the concepts of numeration and place value which will provide opportunities for students to clarify ideas that are introduced in this lesson. Hence, it is important not to spend too much time on this lesson. Two to 3 hours is generally adequate.

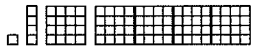
TEACHER NOTES:

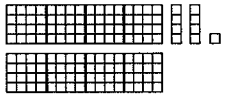
SELECTED ANSWERS

1. 

2. a) 331_{five} b) 1112_{five} c) 1001_{five}
   d) 1310_{five} e) 4043_{five}

3. 1059

4. 

5. 

6. Regrouping can always be done for 4 or more pieces, so the only digits that occur in base four are 0, 1, 2, and 3.

Connector Teacher Activity

OVERVIEW & PURPOSE

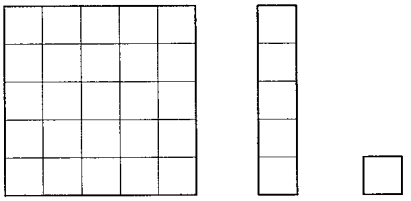
Students observe the patterns among the base five area pieces and predict the shapes of larger and smaller base five pieces.

MATERIALS

- ✔ Base five area pieces (see *Blackline Masters*); 5 mats, 10 strips, 15 units, and 1 strip-mat for each student.
- ✔ Base five area pieces for the overhead.

ACTIONS

1 Arrange the students in groups and distribute the base five mats, strips, and units to each student. Place the following pieces on the overhead projector and tell the students the names of the pieces. Ask each student to form this arrangement of pieces.



2 Tell the students that these pieces represent three figures in a pattern that grows in both directions, to the left *and* to the right. Ask for the groups' observations about this pattern.

COMMENTS

1 Plastic base five area pieces can be purchased or pieces can be made by copying them on cardstock (see *Blackline Masters*) and cutting along the indicated lines. You might wish to have your students do the cutting. This helps students gain familiarity with the pieces.

Each student should have a supply of about 5 mats, 10 strips, and 15 units. This supply is appropriate for use in all future activities that require base five area pieces. The large oblong pieces will be used later in this activity and should be distributed then. To aid in distribution of these materials you may wish to package a supply for a student or group of students in a resealable plastic bag or other container.

Note that there are several introductory lessons involving base four and base five in *Opening Eyes to Mathematics*.

2 It is helpful to have students privately record (e.g., in their journals) observations and then discuss their ideas with groupmates before discussing as a large group.

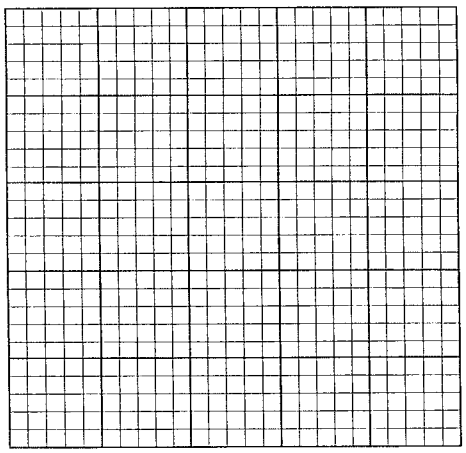
Some possible observations include: each figure seems to be formed by 5 copies of the preceding figure; the pieces are related like pennies, nickels, and quarters; the shapes of the pieces are square, rectangle, square. Some students might observe that these are base five pieces, based on experiences from other classes.

Connector Teacher Activity (cont.)

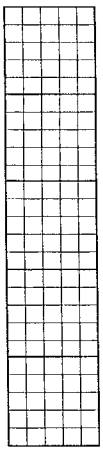
ACTIONS

3 Mention to the students that, because of the groupings of 5, these are called *base five* pieces. If it hasn't come up, ask the students to predict what seems most logical as the piece to the immediate left of the three pieces shown in Action 1 and the most logical piece to the immediate right. Ask for volunteers to show their ideas at the overhead and to explain their reasoning.

4 Distribute one large oblong base piece (called a strip-mat) to each student. Ask the students to predict what seems most logical for the piece to the immediate left of the strip-mat. Discuss their predictions about even larger pieces and about pieces that are smaller than the strip-let described in Comment 3.



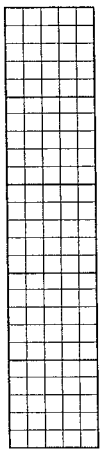
Mat-Mat



Strip-Mat

COMMENTS

3 Acknowledge all logical patterns. If it doesn't come up, point out this possibility for the piece to the immediate left of the mat. It is made by placing 5 mats together to form a 5 by 25 rectangle of 125 units.



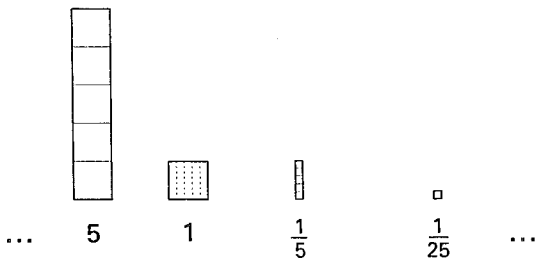
125

Continuing the "square rectangle (non-square)" pattern to the right, the next smaller piece would be a rectangle formed by cutting a unit into 5 congruent "strip-lets," and its area would be 1/5 square unit, as shown below:

1 $\frac{1}{5}$

4 Each large oblong piece is a group of 5 mats arranged in a column. Thus, it is a strip of mats or *strip-mat.* It contains a total of $5 \times (5 \times 5) = 5 \times 25 = 125$ (or 5^3) units. The next larger base five piece is a group of 5 strip-mats. These are arranged to form a square of 25 mats. Hence, this piece is a mat of mats or *mat-mat.* It contains $5 \times (5 \times 5 \times 5) = 5 \times 125 = 625$ (or 5^4) units.

This process of forming base five number pieces in an alternating "square-rectangle (non-square)" pattern in successive groups of 5 of the preceding piece can be continued indefinitely. Thus, 5 mat-mats are grouped to form a strip of mat-mats or strip-mat-mat ($5 \times 625 = 5 \times 5 \times 5 \times 5 \times 5 = 5^5 = 3125$ units). 5 strip-mat-mats are grouped to form a mat-mat-mat ($5^6 = 15{,}625$ units), etc. The piece to the right of the strip-let could be formed by cutting a strip-let into 5 congruent square-lets with area 1/25 square unit as shown here:



... 5 1 $\frac{1}{5}$ $\frac{1}{25}$ ...

Strip Unit Strip-let Square-let

You may wish to display the first several base five pieces on a bulletin board for visual reference during the Focus activity. A mat-mat, strip-mat-mat, and mat-mat-mat can be made by duplicating extra strip-mats and taping them together.

Focus Teacher Activity

OVERVIEW & PURPOSE

Base five area pieces are used to examine the role of grouping and place value in recording numbers.

MATERIALS

- ✔ Base five area pieces for each student.
- ✔ Base five pieces for the overhead.

ACTIONS

1 Place the students in groups and distribute base five area pieces to each student. Ask each student to form a collection of 4 mats, 3 strips, and 7 units. Have the students determine a) the number of pieces and b) the total number of units in this collection.

2 Have the groups find and record different collections of area pieces which total 79 units. Then make a chart on the chalkboard or overhead for listing the various collections the students find. Include a column for the number of pieces in the collection.

COMMENTS

1 Instructions for making and packaging base five area pieces are in the Connector activity.

Each mat, strip, and unit is counted as a single piece. Hence, this collection contains 4 + 3 + 7 or 14 pieces. The total number of units is $(4 \times 25) + (3 \times 5) + 7$ or 122.

2 You may wish to discuss the different strategies groups use for finding collections. Some will be systematic while others will be more random. Many will begin randomly and then develop a system. It isn't necessary that students find all possible collections. However, allow enough time for students to notice the effects of exchanging pieces. Some possible collections are listed below. The asterisk marks the collection with the fewest number of pieces.

M	S	U	No. of Pieces
2	3	14	19
2	5	4	11
1	9	9	19
0	0	79	79
3	0	4	7*
2	2	19	23

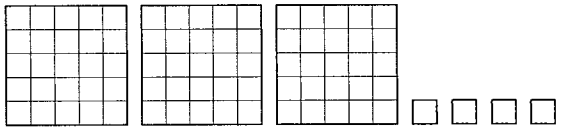
Some students may suggest collections using strip-lets and square-lets (see Connector, Comment 5), resulting in collections like:

M	S	U	Strip-lets	Square-lets	No. of Pieces
2	3	13	5	0	23
1	8	13	4	5	31
0	0	0	395	0	395
2	2	18	0	25	47

Focus Teacher Activity (cont.)

ACTIONS

3 Discuss the base five *representation* of 79.



304_{five} = 79 total units

4 Tell the students that, from now on, all collections are to contain the *minimal*, or fewest, number of pieces. Write the following chart on the chalkboard or overhead. For each line on this chart, ask the students to form the minimal collection of base five area pieces and to use their collections to determine the missing information.

Total Units	M	S	U
113	___	___	___
___	2	1	4
95	___	___	___
21	___	___	___
___	1	0	3
50	___	___	___
___	4	4	4

5 Discuss numerical statements for the first two lines of the chart in Action 4. Then ask each student to write numerical statements for the remaining lines. Discuss their ideas about the role of zero in the recording process.

COMMENTS

3 The collection which totals 79 units and contains *the least number of pieces* is the collection which contains 3 mats, 0 strips, and 4 units. It can be described by the notation 304_{five} and is read as "three, zero, four, base five." This is called the base five representation of 79. Thus $304_{five} = 79$. Note that the representation is written 304_{five} instead of 304_5, because theoretically in base five there is no numeral 5 (see Comment 6).

4 You may wish to reveal and discuss only one line of the chart at a time. It is important to have students use pieces to build each collection. The completed chart is:

Total Units	M	S	U
113	4	2	3
59	2	1	4
95	3	4	0
21	0	4	1
28	1	0	3
50	2	0	0
124	4	4	4

5 Since the minimal collection for 113 total units is 4 mats, 2 strips, and 3 units, the corresponding numerical statement is $113 = 423_{five}$ (read "one hundred thirteen equals four, two, three, base five"). The remaining numerical statements are:

$$59 = 214_{five}$$
$$95 = 340_{five}$$
$$21 = 41_{five}$$
$$28 = 103_{five}$$
$$50 = 200_{five}$$
$$124 = 444_{five}$$

Students may write 041_{five} instead of 41_{five}. This is correct; however, zeroes on the left are usually not recorded since no information is lost if they are omitted.

Focus Teacher Activity (cont.)

ACTIONS

6 Discuss students' ideas about why 0, 1, 2, 3, and 4 are the only digits which occur in base five representations.

7 Ask students to find the base five representations of 200 and 2000 total units. Repeat, as needed, for other numbers.

8 Ask students to sketch the minimal collection for 44⅖ total units and to discuss their ideas about ways to use base five notation to represent this collection.

9 (Optional) Ask the groups to each form the minimal collection of pieces for 423_{five}. Have them do the following:

a) Determine whether this collection is closer in value to 4 mats (400_{five}) or 5 mats (500_{five}) (i.e., determine whether the total number of units in the collection is closer to the number of units in 4 or 5 mats). Discuss.

b) Form the collection of mats and strips only whose value (total units) is closest to 423_{five}.

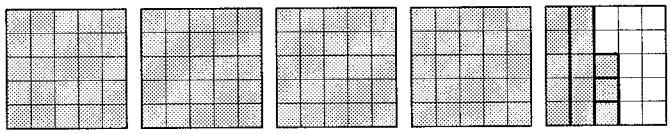
COMMENTS

6 The collection which contains the fewest number of pieces will never contain 5 of the same kind of piece. If it did, the 5 pieces could be exchanged for the next larger piece and the number of pieces in the collection reduced. Therefore, since there is theoretically no numeral 5 in base five, the word "five" is used to indicate the base. However, you may choose to be less formal and record 23_{five}, for example, as 23_5.

7 The collection of 200 units which contains the fewest number of base five area pieces consists of 1 strip-mat, 3 mats, 0 strips, and 0 units. Thus, $200 = 1300_{five}$. The collection for 2000 with the fewest pieces contains 3 mat-mats, 1 strip-mat, 0 mats, 0 strips, and 0 units. Hence, $2000 = 31000_{five}$.

8 This collection contains 1 mat, 3 strips, 4 units, and 2 strip-lets. Since writing 1342_{five} would indicate a collection of 1 strip-mat, 3 mats, 4 strips, and 2 units, or 222 total units, it is necessary to invent a symbol to identify the units place. For example, students may suggest $134{,}2_{five}$ where the comma indicates the 4 is units and the 2 is strip-lets. Some students may suggest a period because of prior experiences with decimals. (Note that decimals are introduced in Lesson 25.)

9 a) One line of reasoning is that 423_{five} is closer in value to 5 mats than 4 mats because 23_{five} covers more than ½ of a mat, as shown here:



This is an example of "rounding" (in base five) to the nearest mat.

b) This is an example of rounding to the nearest strip. One line of reasoning is that since 3 units cover more than ½ of a strip, 423_{five} is closer in value to 4 mats and 3 strips than 4 mats and 2 strips. Hence, 423_{five} "rounds" to 430_{five}.

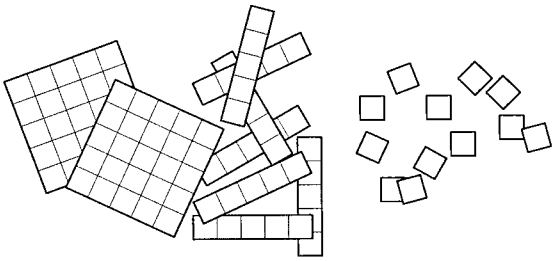
Although this action is intended to only briefly explore the concept of rounding, you may wish to discuss a few other "rounding" problems, such as: round 341_{five} to the nearest mat; round 1421_{five} to the nearest strip-mat and to the nearest mat; etc. Base ten rounding is explored in Lesson 30.

TEACHER NOTES:

Follow-up Student Activity 13.1

NAME _____ DATE _____

1 Use your base five area pieces to form a minimal collection that contains the same total number of units as the collection below. Sketch the minimal collection you formed.



2 For each number of total units listed in the chart below:

- use your base five pieces to form the minimal collection with the same total number of units;
- record on the chart the number of each type of piece in your minimal collection;
- write a numerical statement that shows the base five notation for the collection.

Total Units	Minimal Collection				Numerical Statement
	Strip-Mats	Mats	Strips	Units	
Example) 115	0	4	3	0	$115 = 430_{\text{five}}$
a) 91					
b) 157					
c) 126					
d) 205					
e) 623					

(Continued on back.)

Follow-up Student Activity (cont.)

3 How many total units are in the collection represented by 13214_{five}? Explain your methods for deciding.

4 What do you think a unit, strip, mat, and strip-mat in base four would look like? Sketch pictures of your ideas below.

5 Draw a picture of what you think would be the base four minimal collection for 137 total units.

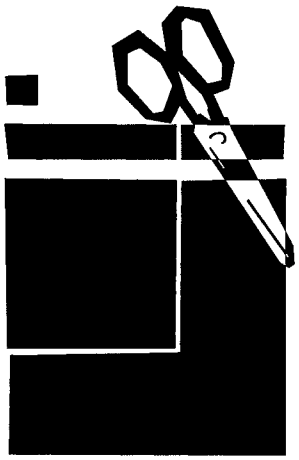
6 What digits do you think are used in base four? Explain your reasoning.

Place Value & Numeration

Lesson 14

THE BIG IDEA

Examining place value and numeration in several bases can lay groundwork for insights about these concepts in base ten. At the same time, students may recognize and adjust misconceptions they have about base ten.



CONNECTOR

OVERVIEW

Students extend their understanding of base five to create base eight area pieces, which are used in the Focus activity.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area pieces for each group.
- ✔ 1-cm grid paper (see *Blackline Masters*), 3 sheets per group of students, and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area pieces for the overhead.

FOCUS

OVERVIEW

Students' experiences with base five grouping and numeration provide the foundation for exploring and generalizing about place value and notation in other bases.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 1-cm grid paper (see *Blackline Masters*), 8-10 sheets per group and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Tape (optional), 1 roll per group.

FOLLOW-UP

OVERVIEW

Students sketch minimal collections of area pieces in various bases and use base notation to record them.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 14.1, 1 copy per student.

LESSON IDEAS

LOOKING AHEAD

Base five activities will continue in Lesson 21 after several lessons that involve area and perimeter. These area and perimeter activities will provide important background for Lessons 23 and 24, where students examine operations with base five. In Lessons 25-31 and 35-37, students explore concepts and operations with base ten.

The sequencing of topics in *Visual Mathematics* is intended to keep student interest high, to enable students to view the topics as closely related to and dependent upon one another, and to provide opportunities for mathematical ideas to resurface regularly in new contexts. This enhances the development of deep and broad mathematical understanding, and reinforces the belief that understanding mathematics is an ongoing process.

SELECTED ANSWERS

1. a) 103_{five}
   b) 1001_{three}
   c) 40_{seven}
   d) 28_{ten}

2. a) $188^{2}/_{7} = 356{,}2_{seven}$
   b) $238^{2}/_{8} = 356{,}2_{eight}$
   c) $498^{2}/_{12} = 356{,}2_{twelve}$
   d) $356^{2}/_{10} = 356{,}2_{ten}$ (or 356.2)

4. Base 6

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students extend their understanding of base five to create base eight area pieces, which are used in the Focus activity.

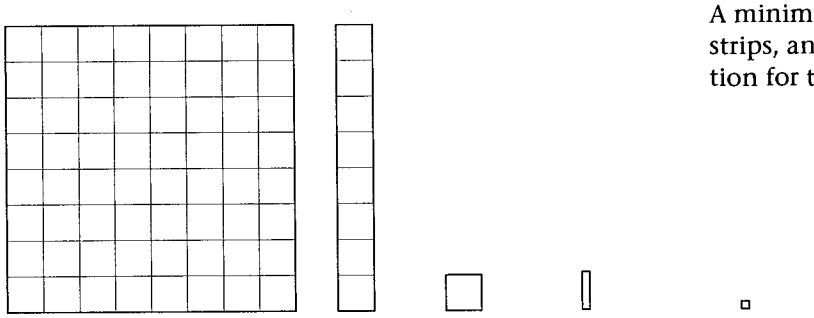
MATERIALS

- ✔ Base five area pieces for each group.
- ✔ 1-cm grid paper (see *Blackline Masters*), 3 sheets per group of students and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area pieces for the overhead.

ACTIONS

1 Place the students in groups of 3-4 and distribute base five area pieces to each group. Ask them to form the minimal collection represented by the symbols 324_{five} and to determine the total number of units in that collection.

2 Give 3 sheets of 1-cm grid paper to each group and a pair of scissors to each student. Ask the groups to discuss and cut out what they believe would be a minimal collection of *base eight* area pieces for 89 total units. Have volunteers share their ideas at the overhead and discuss what larger and smaller base eight area pieces would look like and the number of units such pieces would contain.



Base Eight Pieces

COMMENTS

1 Base five notation was introduced in Lesson 13. 324_{five} (read "three, two, four, base five") is the numerical representation of 3 mats, 2 strips, and 4 units. This collection contains 89 total units. The base five pieces are used in this action in order to give students a model for developing other base pieces in the rest of this lesson.

2 Using the base five pieces as a model, the base eight pieces could be formed as follows: a unit is a small square; a group of 8 units arranged in a row is a strip; and a group of 8 strips arranged in a square is a mat. Each mat contains 64 units. A base eight strip-mat contains 512 units and a mat-mat contains 4096 units.

A base eight strip-let (or other name invented by the students) contains $\frac{1}{8}$ unit and is formed by cutting 1 unit into 8 congruent rectangles. A square-let is formed by cutting a strip-let into 8 congruent squares.

A minimal collection for 89 total units contains 1 mat, 3 strips, and 1 unit. The base eight numerical representation for this collection is 131_{eight}.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students' experiences with base five grouping and numeration provide the foundation for exploring and generalizing about place value and notation in any base.

MATERIALS

- ✔ 1-cm grid paper (see *Blackline Masters*), 8-10 sheets per group, and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Tape (optional), 1 roll per group.

ACTIONS

1 Arrange the students in groups and distribute 8-10 sheets of centimeter grid paper to each group and a pair of scissors to each student. Ask the groups to cut out the minimal collection of base eight area pieces represented by 327_{eight} and to determine the total number of units represented by these pieces. Have volunteers share their results.

2 Ask the groups to determine what would be the minimal base eight area piece collection for each of the following numbers and write the numerical representation for each collection:

a) 180 total units

b) 1000 total units

c) $74\frac{1}{2}$ total units

d) $135\frac{3}{4}$ total units

COMMENTS

1 327_{eight} is the numerical representation for 3 mats, 2 strips, and 7 units. In base eight, this collection contains $(3 \times 64) + (2 \times 8) + (7 \times 1) = 215$ total units.

2 Some groups may wish to cut out these collections while others may make sketches. Some may also wish to use calculators to carry out computations or investigate possibilities. Any of these methods is appropriate.

a) The collection of 180 units which contains the fewest number of base eight pieces consists of 2 mats, 6 strips, and 4 units. Thus, $180 = 264_{eight}$.

b) 1 strip-mat, 7 mats, 5 strips, and 0 units. Thus, $1000 = 1750_{eight}$.

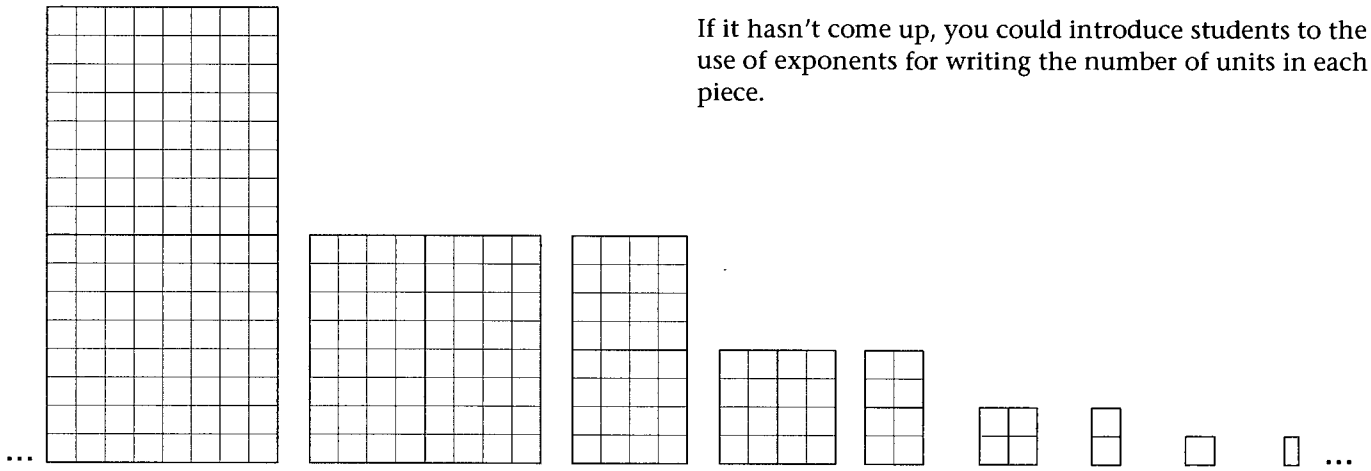
c) The minimal collection for $74\frac{1}{2}$ units would be 1 mat, 1 strip, 2 units, and 4 strip-lets (see Connector, Comment 2), or $112{,}4_{eight}$ (in this case the comma identifies the unit, although students may choose a different symbol).

d) 2 mats, 0 strips, 7 units, and 6 strip-lets, or $207{,}6_{eight}$.

Focus Teacher Activity (cont.)

ACTIONS

3 Ask the groups to cut out the minimal base two area piece collections for 9, 23¼, and 100 total units. Then ask them to determine the base two, or *binary*, numerical representation of these collections.



Base Two Pieces

4 (Optional) Ask the students to form the base two area piece collection for 255¾ total units and to write the binary representation of this collection. Have volunteers share their group's results.

COMMENTS

3 Tape may be useful when groups form the pieces. The first eight base two pieces are shown below.

The collection of 9 units which contains the fewest number of base two pieces consists of 1 strip-mat, 0 mats, 0 strips, and 1 unit. Thus $9 = 1001_{two}$. Also, $23\frac{1}{4} = 10111{,}01_{two}$ and $100 = 1100100_{two}$.

If it hasn't come up, you could introduce students to the use of exponents for writing the number of units in each piece.

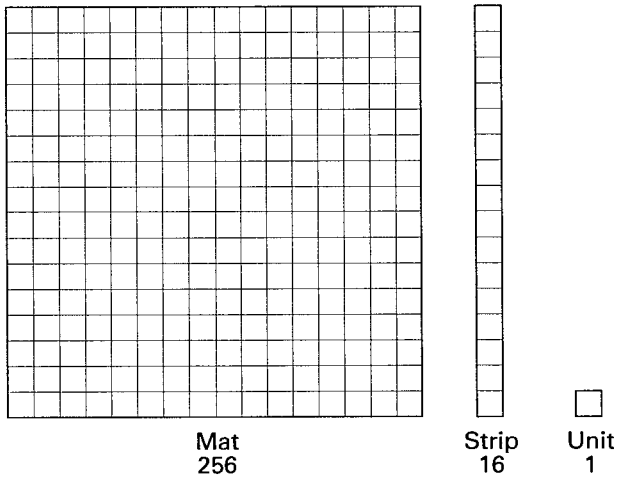
The two digits, 0 and 1, that occur in base two representations can be interpreted as the two positions of an electrical switch, say, 1 is "on" and 0 is "off." Using the binary representation of a whole number allows it to be represented as a sequence of switches in on or off positions. This is analogous to the way computers store numerical information.

4 Since the minimal collection for 255 total units requires 1 of each piece shown in Comment 3 plus 1 square-let, the corresponding numerical statement is $255\frac{3}{4} = 11111111{,}11_{two}$.

Focus Teacher Activity (cont.)

ACTIONS

5 Have the groups cut out the base sixteen minimal collection for a) 100 units and b) 500 units and *invent* a system for representing these collections numerically in base sixteen. Discuss.



Base Sixteen Pieces

6 Have students suggest other numbers of units to model with minimal collections of base sixteen area pieces or diagrams of area pieces. Ask them to write the base sixteen numerical representation of each number.

7 Ask the groups to cut out base ten area pieces that represent the minimal collection for 345 total units. Have them write a numerical representation of this collection. Discuss their results and their ideas about what the minimal collections for 1275 units and other large numbers of units look like.

COMMENTS

5 A base sixteen mat contains $16^2 = 256$ units, a strip-mat contains $16^3 = 4096$ units.

The collection of 100 units which contains the fewest number of base sixteen pieces consists of 6 strips and 4 units; that for 500 units consists of 1 mat, 15 strips, and 4 units.

Base sixteen representations require 16 digits. New digits representing 10, 11, 12, 13, 14, and 15 must be added to the standard collection of digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Students may invent a variety of their own symbols for these additional digits. In machine language computer programming, which uses base sixteen (i.e., *hexadecimal*) representation, it is customary to use the symbols A, B, C, D, E, and F to represent 10 through 15 respectively. Thus, $D6B_{sixteen}$ represents a collection of base sixteen pieces consisting of 13 mats, 6 strips, and 11 units for a total of $(13 \times 256) + (6 \times 16) + 11$ or 3435 units.

6 Have volunteers show or sketch their collections and invite other students to determine the corresponding base sixteen numerical representation.

7 A base ten strip is a row of 10 units; a mat is a square made of 10 strips and totals 100 units; a strip-mat is a row of 10 mats and totals 1000 units.

The minimal collection for 345 total units consists of 3 mats, 4 strips, and 5 units. Because a mat has 100 units and a strip has 10 units, the symbol 345 represents 3 groups of 100, 4 groups of 10, and 5 units.

The collection of 1275 units which contains the fewest number of base ten pieces consists of 1 strip-mat, 2 mats, 7 strips, and 5 units. This collection can be denoted as 1275_{ten}. However, if the base is ten, it is customary to omit the subscript indicating the base. Conversely, if no base is indicated, it is assumed to be ten. You may wish to explore base ten collections involving fractional parts of units. If so, Lesson 25 may provide some ideas.

Focus Teacher Activity (cont.)

ACTIONS

8 Hold up a large handful of unit squares and tell the students you have two hundred thirty-seven altogether. Ask them to imagine what the minimal base ten area piece collection for this number of units would be and to represent this minimal collection with base ten area pieces. Discuss. Repeat with some other numbers.

COMMENTS

8 This Action is intended to help students think of numbers (and represent them) in terms of place value.

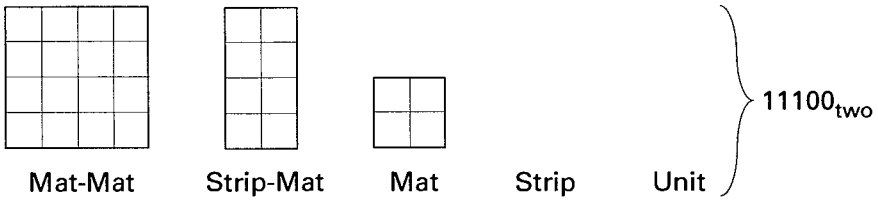
You may wish to extend this action by asking students to imagine what a collection of 237 units would look like if they exchanged as many units as possible for strips. It will be valuable for students to be able to visualize 237 as 2 mats, 3 strips, and 7 units, or 23 strips and 7 units, or 237 units.

After the students have represented a few numbers with their area pieces, you may want to ask them to sketch the area pieces which represent selected numbers. If so, remind students that in a sketch it isn't necessary to show all subdividing lines on the pieces.

Follow-up Student Activity 14.1

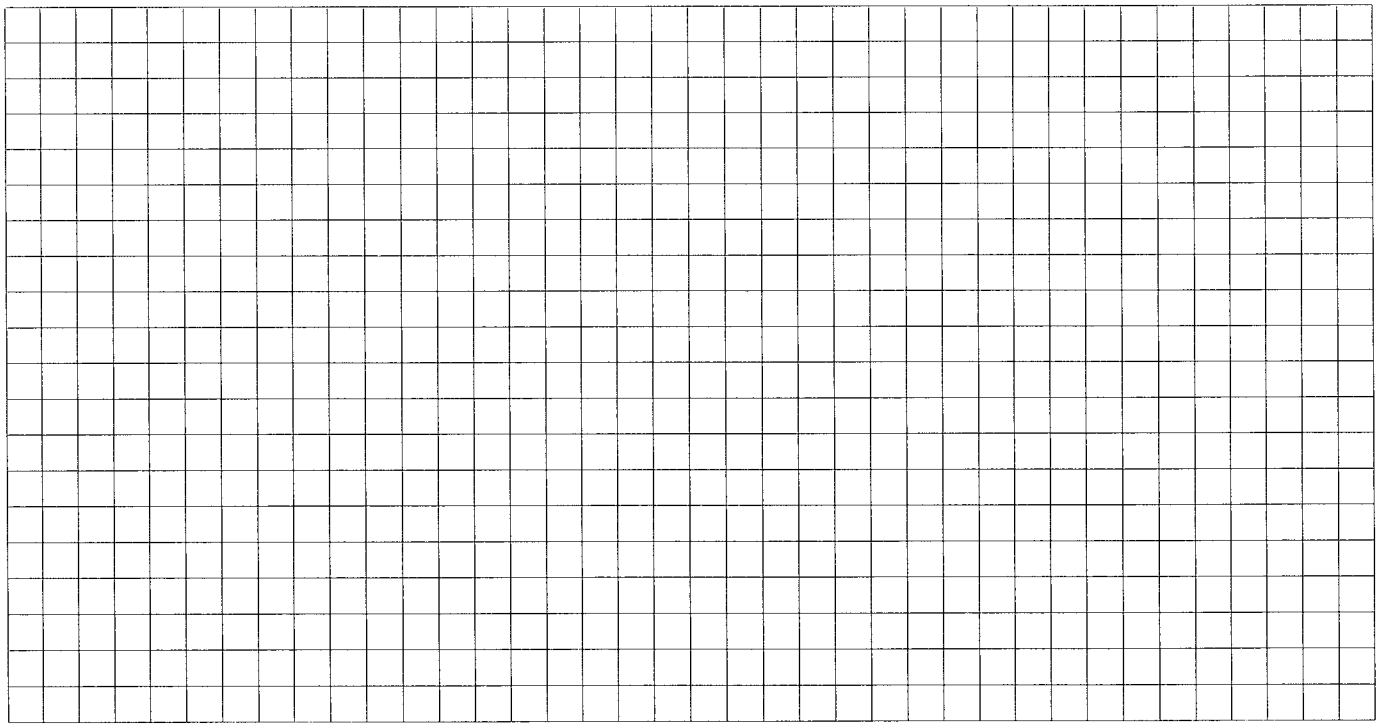
NAME _____ **DATE** _____

1 The base two minimal collection for 28 total units is shown below. It contains 1 mat-mat, 1 strip-mat, 1 mat, 0 strips, and 0 units, and its base two representation is 11100_{two}.



On the grid below sketch the minimal collection of base pieces for 28 total units in each of the following bases. Circle each collection and write its representation in base notation.

a) base five b) base three c) base seven d) base ten



(Continued on back.)

Follow-up Student Activity (cont.)

2 Imagine or draw on the attached grid paper a collection of 3 mats, 5 strips, 6 units, and 2 strip-lets in each of the bases listed below. Then complete the chart below. Remember that for a "sketch" you don't have to show all the grid lines on the pieces.

Sketch of Pieces	Total Units	Base Notation
a) base seven:		
b) base eight:		
c) base twelve:		
d) base ten:		

3 List some reasons why you think a base ten counting system is used around the world.

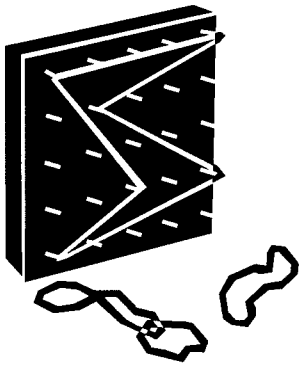
4 In her notebook, Alyssa wrote that a collection of 95 total units was 235_{strip}. The base was not readable. What do you think was the base? Explain your method for finding it.

Geoboard Figures

Lesson 15

THE BIG IDEA

The geoboard provides a means of representing geometric figures and a medium for geometric explorations. Discussion of these explorations provides a meaningful context for developing mathematical vocabulary.



CONNECTOR

OVERVIEW

Students build figures on their geoboards and describe the figures so their classmates can build the same figures.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Geoboard, 1 per student.
- ✔ Geoboard for the overhead.

FOCUS

OVERVIEW

The geoboard is introduced as a means of representing geometric figures and introducing new vocabulary, and as a medium for geometric explorations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Geoboard, 1 per student.
- ✔ Geoboard Recording Paper (see *Blackline Masters*), 1 copy per student.
- ✔ Geoboard for the overhead.

FOLLOW-UP

OVERVIEW

Students find and record geoboard polygons with differing numbers of sides.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 15.1, 1 copy per student.

LESSON IDEAS

ASSESSMENT
Use an observation checklist and informal interviews with groups to collect assessment information while groups are working on the Focus (see *Starting Points*).

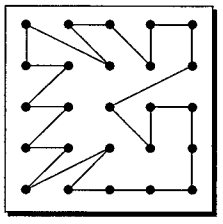
BULLETIN BOARD
The students may want to make a bulletin board display of their polygons from the Follow-up. As students find polygons with greater numbers of sides they can be added to the display.

TIMING
Depending on extensions that come up, this lesson usually takes about 2 hours.

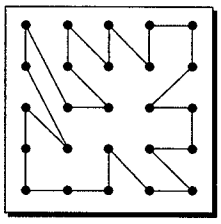
LOOKING AHEAD
Lessons 15 and 16 provide background experiences with the geoboard and introduce several geometry concepts. Lessons 17 and 18 explore the concept of perimeter. Lesson 19 involves geoboard explorations of the concept of area. Together, Lessons 15-19 are a basis for examining area and linear relationships, which, in turn, are the basis for explorations of whole number and decimal operations.

SELECTED ANSWERS

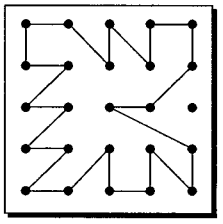
1. Here are examples of polygons with 22, 23, and 24 sides. The maximum number of sides that has been found for a polygon is 24.



22 Sides



23 Sides



24 Sides

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students build figures on their geoboards and describe the figures so their classmates can build the same figures. The purpose of this activity is to sharpen communication skills as well as to prepare students for building geoboard figures in the Focus activity.

MATERIALS

- ✔ Geoboard, 1 per student.
- ✔ Geoboard for the overhead.

ACTIONS

1 Place the students in pairs and have them erect a visual barrier between them. Student 1 forms a triangle on the geoboard and describes it for Student 2 to form. When the first student has completed a description of the geoboard figure, the students compare their figures and discuss any misunderstandings. The students then reverse roles and Student 2 builds a triangle for Student 1. Select an example or two to show at the overhead and ask volunteers to discuss the different ways the figure could be described. Discuss the students' feelings about the activity.

COMMENTS

1 Notebooks or file folders can be used for visual barriers. Students could decide whether the "listener" gets to ask questions about the "describer's" instructions. Students who finish quickly can extend the activity by building a 4- or 5-sided figure.

You might wish to make a list on the overhead of the mathematical vocabulary you overhear the students use in their descriptions. Then have students add to the list.

Some teachers like to distribute geoboards with 8-10 rubber bands already on the boards in a grid design. When students complete their work, they are asked to replace the bands in the grid design. Thus, fewer bands are misplaced. If bands break during the activity, they can be exchanged for new ones from you.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

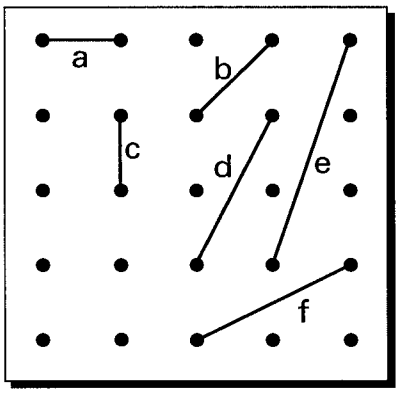
The geoboard is introduced as a means of representing geometric figures and introducing new vocabulary, and as a medium for geometric explorations. One benefit of building figures on the geoboard is the enhancement of spatial visualization skills.

MATERIALS

- ✔ Geoboard, 1 per student.
- ✔ Geoboard Recording Paper (see *Blackline Masters*), 1 copy per student.
- ✔ Geoboard for the overhead.

ACTIONS

1 Put rubber bands on a geoboard as shown below. Show the geoboard to the students and ask them to form these 6 segments on their geoboards.



2 Ask the students to determine the order of the lengths (formed in Action 1) from shortest to longest. Discuss how they arrived at their conclusions.

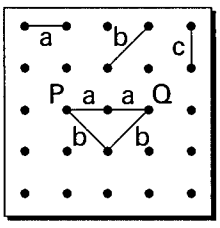
COMMENTS

1 A transparent geoboard on an overhead works well for demonstration purposes. Alternatively, the segments may be drawn on a transparency of Geoboard Recording Paper (see *Blackline Masters*) and displayed on the overhead.

If some students have difficulty reproducing the segments on their geoboards, encourage others to share how they form the segments. For example, someone may suggest segment **e** is formed by coming down 3 spaces and left 1 space (or moving right 1 space and up 3 spaces).

2 Encourage students to find their own methods of comparison. For some, the length relationships may be obvious. Others may need to compare lengths of segments by measuring with rulers or by making marks on paper. Some may recognize that "3 down and 1 over is longer than 2 down and 1 over." Segments **d** ("over 1 and up 2") and **f** ("over 2 and up 1") have the same length. Segments having the same length are called *congruent*. The correct ordering of lengths is **a** = **c**, **b**, **d** = **f**, **e**.

Some students may think segment **b** is the same length as segments **a** and **c**, because "each segment connects two adjacent points." However, as shown in the figure below, there are two paths from **P** to **Q** (a + a and b + b), and a + a is noticeably shorter than b + b. Half the length of the shorter path must be less than half the length of the longer path. So segment **a** is shorter than segment **b**.



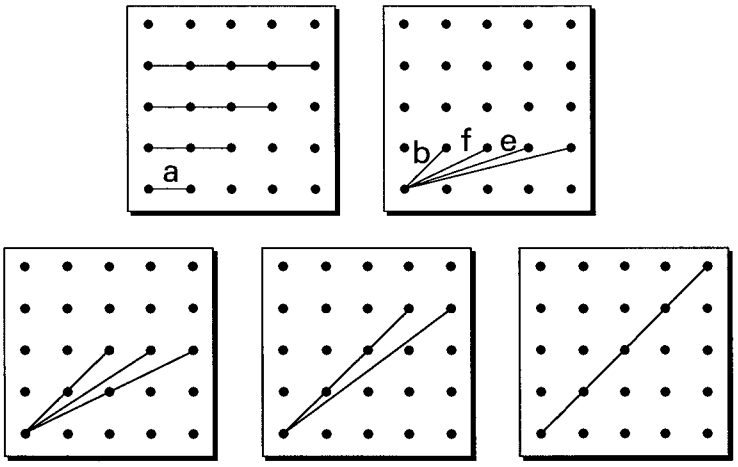
Focus Teacher Activity (cont.)

ACTIONS

3 Ask the students to work in pairs or small groups to find as many geoboard segments, all of different lengths, as they can, and to record the segments on geoboard paper (provide 1 sheet to each student).

COMMENTS

3 There are 14 different lengths that are possible for geoboard segments, including segments **a**, **b**, **e**, and **f** determined in Action 2. One collection of 14 geoboard segments, all with different lengths, is shown below.



To share their results, students can be asked to plot segments on the overhead one at a time. As a segment is plotted, the class can decide if it is a segment of new length.

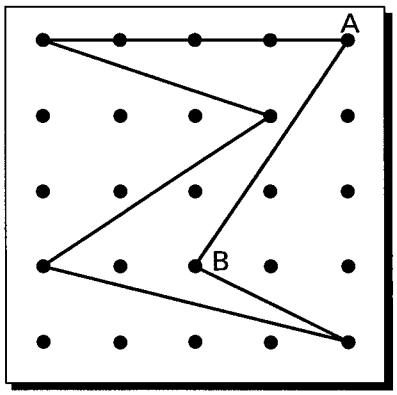
Some students may form all 14 line segments by using the diagonal of the geoboard and only the upper half or lower half of the geoboard (some segments will overlap). This can be done because the geoboard is *symmetric* about this diagonal. Symmetry is explored further in Lesson 16.

Rather than telling students when they have all the segments, you might suggest they develop a convincing argument regarding why they have them all (with no duplicates).

Focus Teacher Activity (cont.)

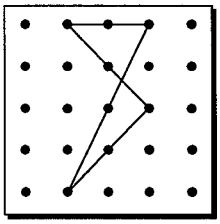
ACTIONS

4 Ask the students to form the polygon shown below. Point out that this polygon has 6 sides. Ask them to make and record polygons with more than 6 sides by altering side AB while leaving the remaining sides fixed. Discuss the results.

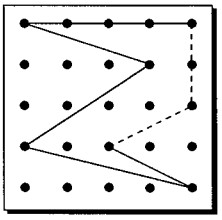


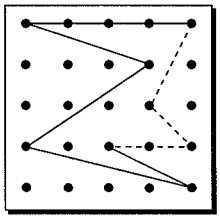
COMMENTS

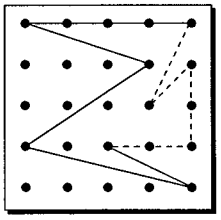
4 In this activity, a figure will be called a *polygon* if its sides are segments and they enclose 1 interior region. Figures of this type are also called simple polygons while figures whose sides are segments, but enclose more than 1 interior region, such as the one shown below, are sometimes called *non-simple polygons*.

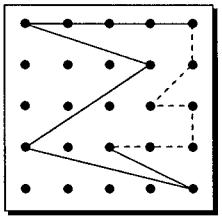


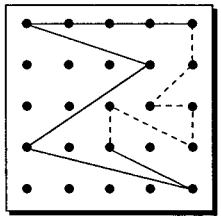
Some students may be surprised to know that the shape shown at the left is a *hexagon* (because it is a 6-sided polygon). Side AB can be altered to form (simple) polygons of 7, 8, 9, 10, 11, 12, and 13 sides. One example of each is shown.

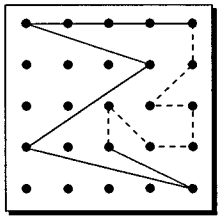

7 Sides

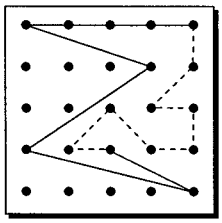

8 Sides


9 Sides


10 Sides


11 Sides


12 Sides

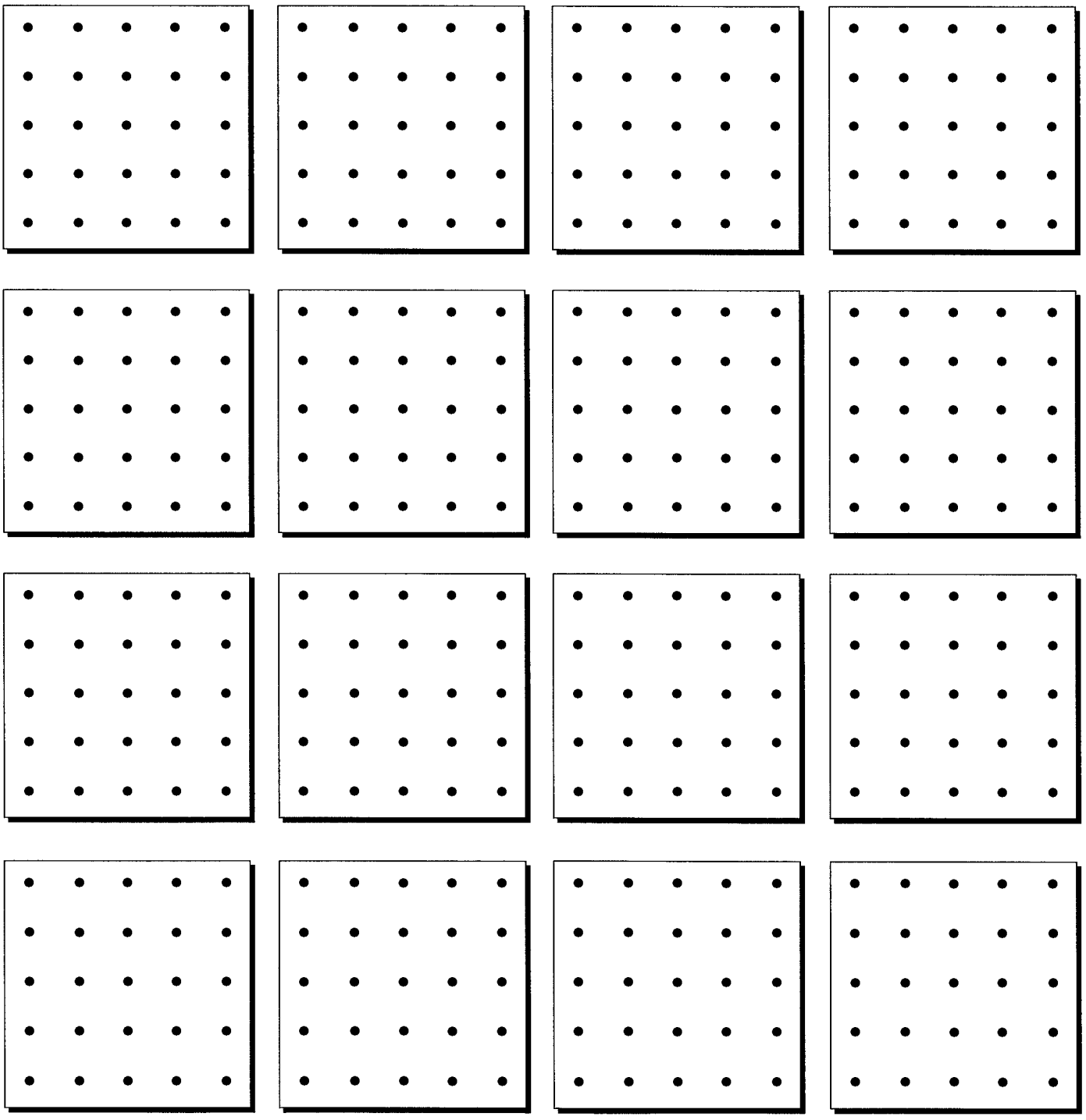

13 Sides

TEACHER NOTES:

Follow-up Student Activity 15.1

NAME _____ **DATE** _____

1 Find and record polygons with differing numbers of sides; 13, 14, 15, 16, etc. What is the greatest number of sides possible on a 25-pin geoboard?

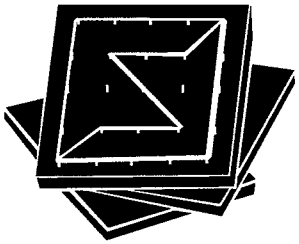


Congruence & Symmetry

Lesson 16

THE BIG IDEA

The geoboard provides a hands-on setting for investigating the meaning of congruence, introducing symmetry, and sorting and classifying polygons. These geoboard explorations help to develop students' visual abilities and later will provide a basis for finding areas of geometric figures.



CONNECTOR

OVERVIEW

Students divide geoboard polygons into congruent subregions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Geoboard, 1 per student.
- ✔ Geoboard recording paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ Geoboard for the overhead.
- ✔ Connector Master A (optional), 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

As they mentally and physically construct, flip, and turn geoboard figures, students have opportunities to develop insights about the concepts of congruence and symmetry and to enhance their spatial visualization abilities.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Scissors, 1 pair per student.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Geoboard for the overhead.
- ✔ Enlarged geoboards (see *Blackline Masters*), 4 sheets per group and 1 transparency.

FOLLOW-UP

OVERVIEW

Students create and collect figures that satisfy certain conditions regarding congruence and symmetry.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 16.1, 1 copy per student.
- ✔ ¼" dot paper (see *Blackline Masters*), 1 sheet per student.
- ✔ 10-cm × 10-cm grids (see *Blackline Masters*), 1-2 sheets per student (see Follow-up comment on the back of this sheet).

LESSON IDEAS

BULLETIN BOARD
Create a bulletin board display of students' pictures collected for Follow-up Problem 4.

LOOKING AHEAD
This lesson gives students opportunities to enhance their visualization skills and to develop intuitions about congruence and symmetry concepts. These are intended to be introductory activities. There will be many opportunities in this course for students to notice and discuss relationships involving symmetry and congruence. These concepts are explored in greater depth in *Visual Mathematics, Course II.*

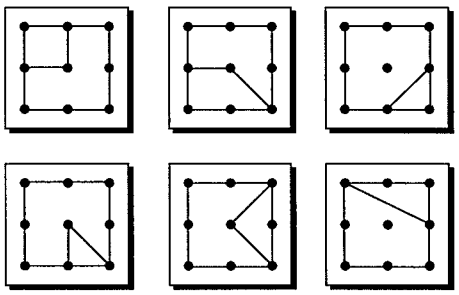
VOCABULARY
There are opportunities for students to use many geometric terms in this lesson. It is important to remember that new language develops through use and when objects and actions are associated with terminology.

If students are building a glossary in their journals (see *Starting Points*) have them accompany all explanations with diagrams to show important relationships.

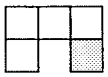
FOLLOW-UP
Create a bulletin board display of students' designs for Follow-up Problem 5. You could save these designs for students to use when they construct a display of base ten number pieces in the Lesson 25 Connector. (This isn't necessary but will make a colorful display.) For that activity, you will need at least 111 10×10 squares. Use that number as a basis for determining the number of grids to give each student.

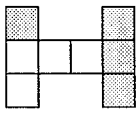
QUOTE
Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing, and testing hypotheses precede the development of more formal summary statements. In the process, definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas to develop informal arguments. The informal exploration of geometry can be exciting and mathematically productive for middle school students. At this level, geometry should focus on investigating and using geometric ideas and relationships rather than on memorizing definitions and formulas.
NCTM Standards

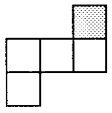
SELECTED ANSWERS

2. 

3. Here is one possibility for each:

a) 

b) 

c) 

Connector Teacher Activity

OVERVIEW & PURPOSE

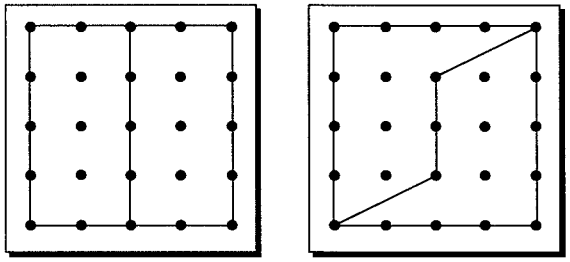
Students divide geoboard polygons into congruent subregions.

MATERIALS

- ✔ Geoboard, 1 per student.
- ✔ Geoboard recording paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ Geoboard for the overhead.
- ✔ Connector Master A (optional), 1 copy per group and 1 transparency.

ACTIONS

1 Place the students in groups of 2-4 and give each student a geoboard. On a transparency of geoboard recording paper, sketch the figures shown below.



a) b)

Ask the groups to form these figures on two of their geoboards and to write down as many observations as they can about the figures, including ways they are alike and ways they are different. Allow several minutes for groups to talk privately and then have them share their ideas with the whole class. Summarize their observations on the chalkboard or a sheet of butcher paper.

COMMENTS

1 Circulate while groups discuss their observations. Their discussions may reveal evidence about students' prior knowledge of geometric concepts such as congruence and symmetry. As groups make observations, encourage them to find ways to justify them. For example, if groups say that both geoboards are divided into 2 regions that have the same size and shape, students could trace the figures or draw the figures on geoboard recording paper and cut out the parts to show that one part can fit exactly on top of the other.

If terminology such as congruence, symmetry, reflection, or rotation come up, discuss students' ideas about their meanings. If they don't come up, notice that congruence is emphasized in the rest of this Connector. Symmetry is explored in the Focus. If questions about area arise, you may wish to examine them now. However, geoboard areas are explored in depth in Lessons 19 and 34. Here are a few observations frequently made by students:

"The border of each figure is the largest possible geoboard square."

"Both figures show the geoboard divided into two regions that have the same size and shape."

"Geoboard a) has line symmetry and geoboard b) does not."

"Each part of geoboard a) and geoboard b) is half of a square with area 16 square units, so each part has area 8 square units."

"The lines that separate the geoboards into two parts each pass through the center pin of the geoboard."

"Both geoboards can be turned half way around and the figures look like they did before being turned."

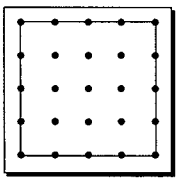
Connector Teacher Activity (cont.)

ACTIONS

2 If it didn't come up during the discussion in Action 1, discuss students' ideas about the meaning of *congruence* and ways of verifying whether regions are congruent.

Ask the groups to each find at least 7 additional ways to divide the 4 by 4 square on the geoboard into 2 congruent regions. Have them make a record of their results on a sheet of geoboard recording paper. Ask volunteers to share several examples at the overhead and to describe their strategies for dividing the square.

3 Form this geoboard figure and ask the groups to do the same.



Pose the following questions for groups to investigate:

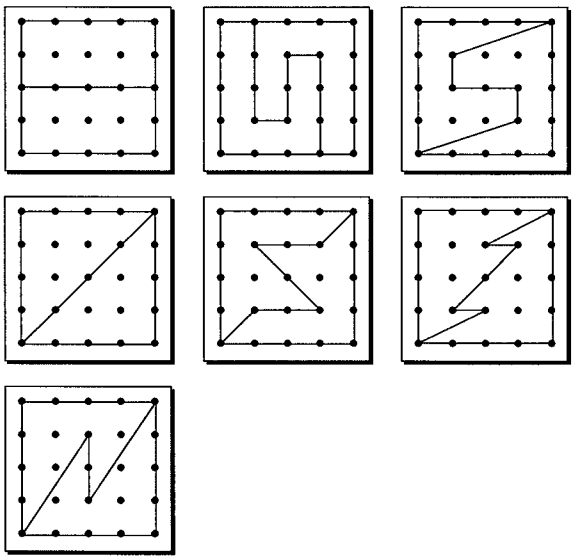
a) Assuming bands are always connected pin-to-pin, in how many ways (if at all) can you subdivide the above figure into 3 congruent parts? 4? 5? 6? 7? 8? etc.? What's the greatest number of congruent parts you can form?

COMMENTS

2 Two figures are congruent if they have the same size and shape. As mentioned in Comment 1, if there is any doubt that the 2 parts of the square are congruent, trace them or record them on geoboard paper, cut them out, and see if one part can be made to fit exactly on top of the other.

Some students may notice that the common border of the congruent regions always passes through the center pin, and so to form congruent parts, they may start at the middle and work in opposite directions; others may begin at opposite corners and work in opposite directions toward the center.

There are many different ways to divide the square into 2 congruent parts. The students may want to make a bulletin board display of different solutions. A few are shown below.



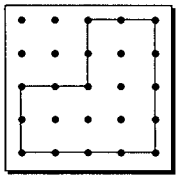
3 This action is suggested as a reminder that congruence isn't limited to forming 2 parts.

a) If bands are restricted to extend only pin-to-pin, the 4 by 4 square can be subdivided into 2, 4, 8, 16, 32, and 64 congruent parts.

Connector Teacher Activity (cont.)

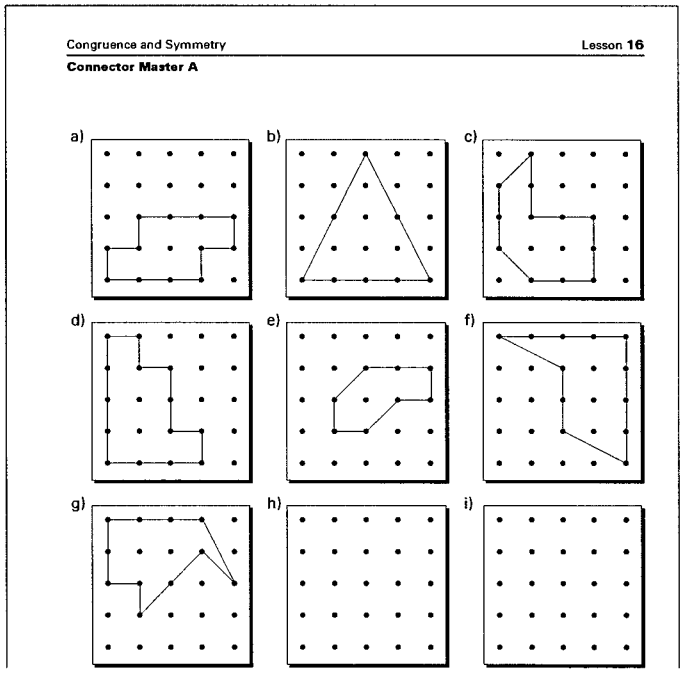
ACTIONS

b) Alter the figure to look like this:



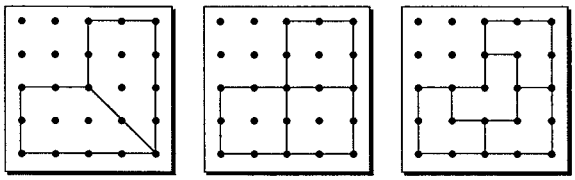
Now can you subdivide this new figure into 2 congruent parts? 3? 4? 5? etc.? If so, in how many ways? What's the greatest number of congruent parts you can form?

4 (Optional) Give each group a copy of Connector Master A and have them decide how it is possible, if at all, to divide each of the shapes on Master A into 2 congruent regions. On the blank geoboards have them create 1 new shape that can be divided into 2 congruent regions and 1 shape that cannot.



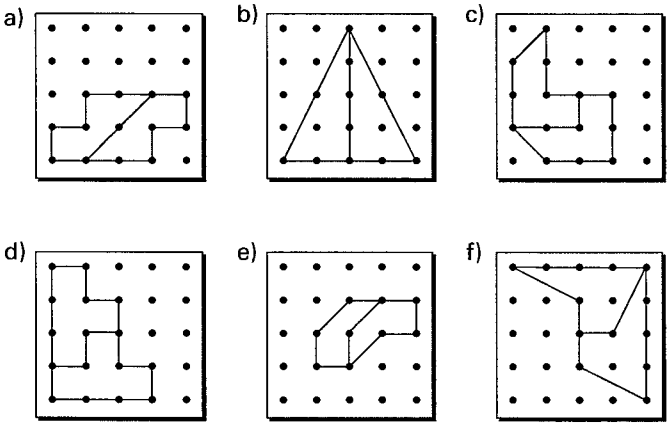
COMMENTS

b) This figure can be subdivided into 2, 3, and 4 congruent regions, as shown here:



Additional subdivisions of these regions can produce 6, 8, 12, 24, and 48 congruent parts. Note that in each of the above cases the subregions are congruent and have equal areas. If it doesn't come up, you may wish to ask students to find ways to divide each figure into subregions that have the same area but are not congruent. You could also have them look for ways to form regions that are congruent, but whose areas differ (this is impossible).

4 To strengthen students' mental visualization abilities, encourage them to imagine ways of subdividing each shape before exploring ideas on the geoboard. If students doubt that 2 regions are congruent, suggest they cut out or trace 1 region to see if it can be turned or flipped to exactly fit on top of the other. It is impossible to subdivide g) into 2 congruent parts. One possibility for each of the other figures is shown here:



TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

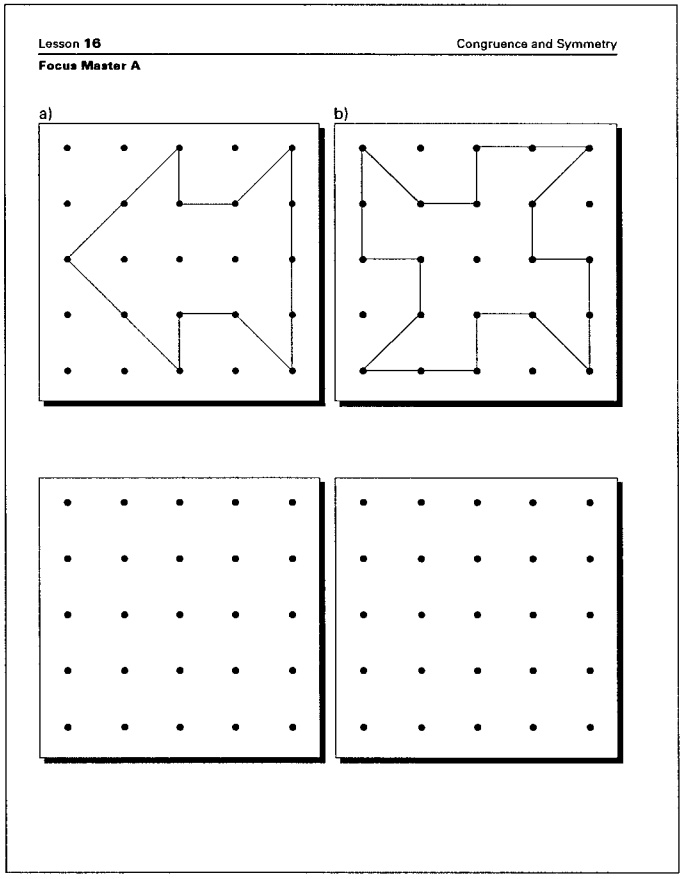
As they mentally and physically construct, flip, and turn geoboard figures, students have opportunities to develop insights about the concepts of congruence and symmetry and to enhance their spatial visualization abilities.

MATERIALS

- ✔ Scissors, 1 pair per student.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Geoboard for the overhead.
- ✔ Enlarged Geoboards (see *Blackline Masters*), 4 sheets per group and 1 transparency.

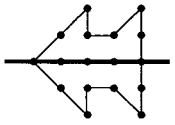
ACTIONS

1 Arrange the students in groups and distribute a copy of Focus Master A and a pair of scissors to each student. Ask the groups to describe any *symmetry* they observe in either of Figures a) or b). If *reflective* and/or *rotational symmetry* don't come up, introduce them to students, along with "frame tests" for symmetry.



COMMENTS

1 As shown below, a line can be drawn through Figure a), dividing it into 2 congruent parts which can each be *reflected* (flipped) across the line to fit exactly on top of the other. This line is called a *line of reflection* or *line of symmetry* and Figure a) is said to have *reflective symmetry*. Students may find it helpful to cut out these figures and try paper folding to find lines of symmetry.



Another method of finding lines of symmetry is to cut out a figure, trace a "frame" about its perimeter, and look for lines the figure can be flipped across so that it fits back into its frame. Notice that if a frame is traced about Figure b), there is no line about which the figure can be flipped and fit into its frame. Hence, Figure b) has no line of symmetry. However, Figure b) can be *rotated* a ¼-turn (90°) so that it fits back into its frame and therefore, it has ¼-turn *rotational symmetry*. The point about which a figure can be rotated clockwise to fit back into its frame is called the *center of rotation* (x marks the center of rotation in the figure below).

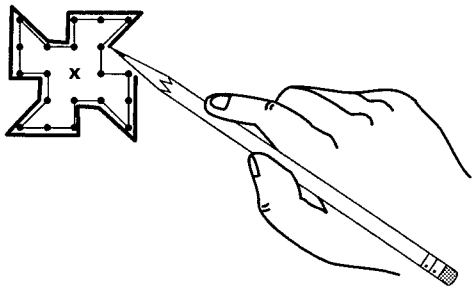
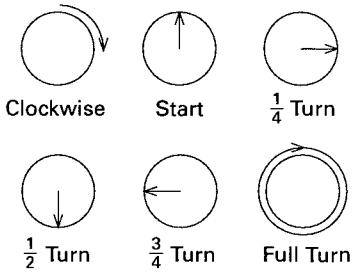


Figure b) can also be rotated clockwise through a ½-turn (180°), ¾-turn (270°), and a full-turn (360°) and fit into

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



2 Give each group a copy of Focus Master B and 4 sheets of Enlarged Geoboards (see *Blackline Masters*). Remind the groups to show lines of symmetry on the figures they create and/or centers of rotation on their drawings and to use the "frame test" described in Comment 1 to verify symmetries. Discuss.

COMMENTS

1 (continued.)
its frame. To determine rotational symmetries, students may find it helpful to cut out a figure, trace its frame, hold a pencil at the center of the figure, and rotate the figure until it fits back into its frame.

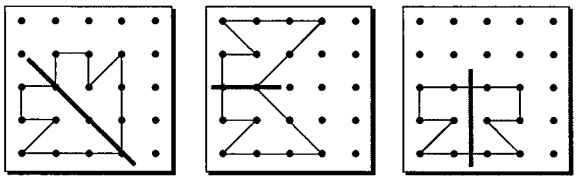
A figure has *reflective symmetry* if it has one or more lines of symmetry, and a figure has *rotational symmetry* if it has a center of rotation about which it can be rotated back into its frame in more than one way (i.e., in ways other than a full turn). Figure b) has 4 rotational symmetries and so we say it has *rotational symmetry of order 4*. Figure a) fits back into its frame only by a full turn, and so it has no rotational symmetry.

To explore the "frame test" for rotational and reflective symmetry, you could have students cut out Figures a) and b) and sketch frames on the blank geoboards from Focus Master A. If students made a bulletin board display of their solutions to the Connector, Action 2, you may wish to have them test the symmetries of some of those figures.

2 To facilitate student sharing and save time, you could have groups record their results on a transparency of Enlarged Geoboards prior to coming to the overhead.

1) Answers will vary; however, f) and g) are not possible.

2) There are many ways to form figures with line symmetry, such as the following (the lines of symmetry are shown in bold):



Notice that a line of symmetry always separates a figure into 2 congruent subregions.

4) In the following diagrams, the center points of rotational symmetry are marked with an "x," and the lines of symmetry are bold.

Focus Teacher Activity (cont.)

ACTIONS

Congruence and Symmetry Lesson **16**

Focus Master B

Use the attached sheets of Enlarged Geoboards to form the following geoboard figures. Draw all lines of symmetry and mark centers of rotations. Use "frame tests" to verify all symmetries.

1. If possible, form a geoboard figure that illustrates:

a) reflective symmetry but not rotational symmetry

b) rotational symmetry of order 2 but not reflective symmetry

c) both rotational and reflective symmetry

d) no symmetry

e) exactly 2 lines of symmetry

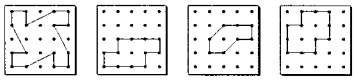
f) exactly 2 lines of symmetry and no rotational symmetry

g) ¼-turn rotational symmetry only

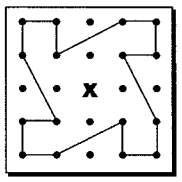
2. Complete this geoboard figure in 2 different ways so the new figures have line symmetry.

3. Form 3 different geoboard figures so this segment is a line of symmetry in each figure.

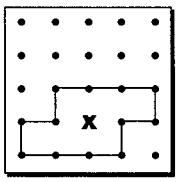
4. Form these figures and determine the symmetry, if any, in each of them.



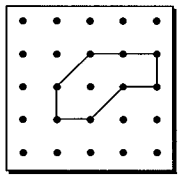
COMMENTS



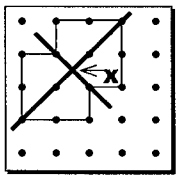
$\frac{1}{4}$-turn, $\frac{1}{2}$-turn, and $\frac{3}{4}$-turn rotation



$\frac{1}{2}$-turn rotation



neither rotation nor reflection



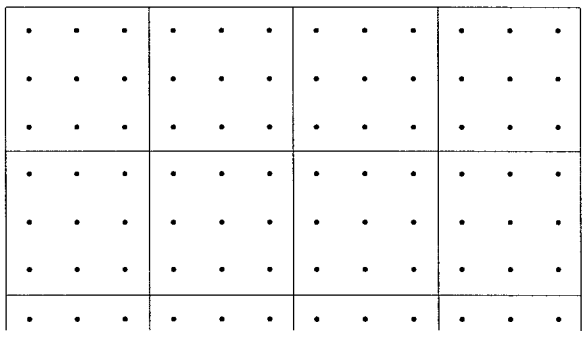
2 lines of reflection and $\frac{1}{2}$-turn rotation

3 Give each group a copy of Focus Master C. Tell them that each square on Master C represents a 3-pin by 3-pin geoboard. Ask the groups to find all the possible *noncongruent* quadrilaterals (both convex and concave) which can be formed on a 3-pin by 3-pin geoboard.

Have volunteers come to the overhead and record possibilities on a transparency of Master C, until the class is in agreement that all possibilities are shown and none are congruent to each other.

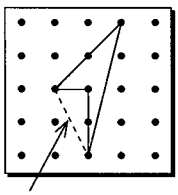
Lesson **16** Congruence and Symmetry

Focus Master C

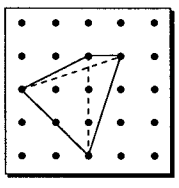


3 You may need to introduce the meaning of *quadrilateral* (a polygon having 4 sides).

One way of telling whether a polygon is *concave* or *convex* is to determine whether at least one *diagonal* (a line segment connecting 2 nonadjacent vertices) lies outside the polygon. For any convex polygon, all diagonals lie inside the polygon. This is not the case for a concave polygon (many students remember concave figures as ones that "cave in").



One diagonal is outside this quadrilateral, which is *concave*.

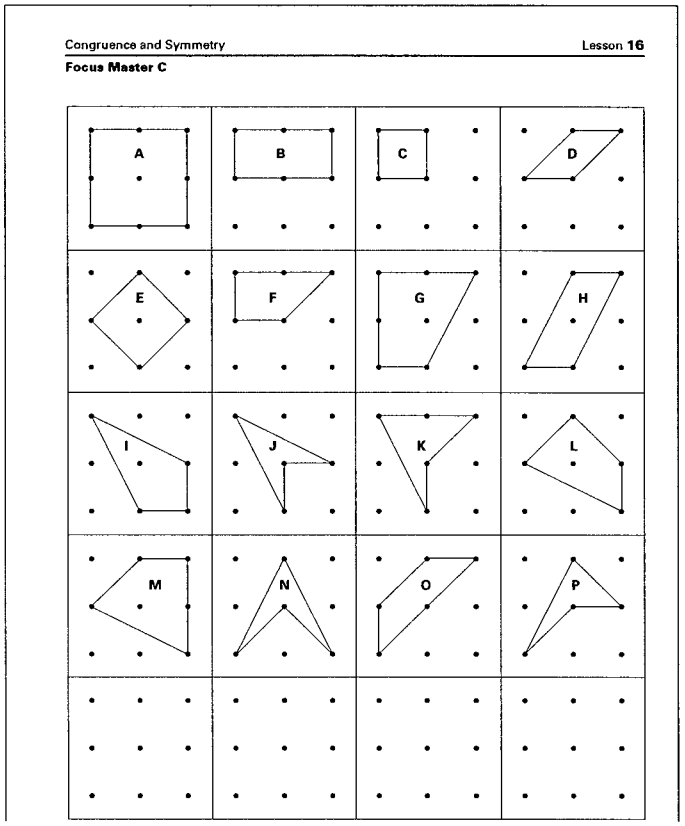


Both diagonals are inside this quadrilateral which is *convex*.

The 16 possible noncongruent quadrilaterals are shown on the next page (letters are shown for reference in Comment 4).

Focus Teacher Activity (cont.)

ACTIONS



4 Give each group a pair of scissors. Write a)-h) below on the overhead and have the students cut out and sort the quadrilaterals from Action 3 into these classifications.

a) No lines of symmetry.

b) Exactly 1 line of symmetry.

c) Exactly 2 lines of symmetry.

d) Exactly 3 lines of symmetry.

e) Exactly 4 lines of symmetry.

f) More than 4 lines of symmetry.

g) Rotational symmetry.

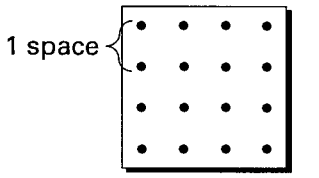
h) Figures that can be divided into 2 congruent regions by 1 diagonal.

Ask the groups to make observations and conjectures about the quadrilaterals that belong in each classification and about relationships between the classifications.

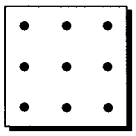
COMMENTS

3 (continued.)
Some groups may wish to explore possibilities on their geoboards. To prevent confusion, suggest they outline a 3-pin by 3-pin geoboard with a band, and then use different colored bands to form quadrilaterals.

It may be helpful to discuss ways of identifying the size of a geoboard. Generally, a 3 by 3 geoboard is interpreted to mean 3 spaces by 3 spaces or 3 linear units by 3 linear units. A 3-pin by 3-pin geoboard is only 2 spaces by 2 spaces. These are illustrated below:



This is a 3-*space* by 3-*space* geoboard.



This is a 3-*pin* by 3-*pin* geoboard.

4 You may wish to let students know that some of the quadrilaterals will fit in more than one category and that some categories may have none of these quadrilaterals. Cutting apart transparent copies of the quadrilaterals will simplify demonstrations at the overhead.

For ease of discussion you might suggest that students label the 16 figures (e.g., as we have in Action 3, with everyone assigning the same letters to the same shapes). Following are the quadrilaterals that fit each classification:

a) D, F, G, H, K, L, M, P

b) I, J, N, O

c) B

d) none

e) A, C, E

f) none

g) A, B, C, D, E, H

h) A, B, C, D, E, H, I, J, N

Some students may use terminology that is unfamiliar to others. If so, have them explain and show examples to their classmates. Such terms might include: *parallel, perpendicular, parallelogram, rhombus, and trapezoid.*

Focus Teacher Activity (cont.)

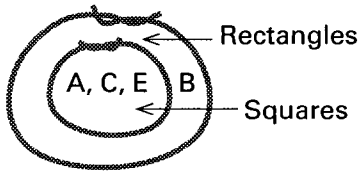
ACTIONS

5 (Optional) Have groups of students investigate and identify other ways to sort and classify the quadrilaterals from Action 3.

COMMENTS

5 There are many possibilities for sorting these quadrilaterals. For example, they might identify concave and convex polygons, trapezoids, squares, parallelograms, polygons with square corners, etc. Students could invent "secret" ways to sort the quadrilaterals and have other students try to guess the classifications based on similarities they observe among the quadrilaterals that are grouped together.

You could also introduce students to *Venn Diagrams* (see Lesson 33) by using yarn loops for grouping the polygons. For example, you might give students 2 loops and ask them to find a way to arrange all the squares in 1 loop and all the rectangles in the other:



TEACHER NOTES:

Follow-up Student Activity 16.1

NAME _____ **DATE** _____

1 On the attached dot paper, draw each of the following figures. Be sure to label each figure.

a) This figure has reflective symmetry across a vertical (↕) line *and* a horizontal (↔) line. (Draw the lines of reflection on your figure.)

b) This figure has rotational symmetry of order 2. (Place an "x" on the center of the rotation.)

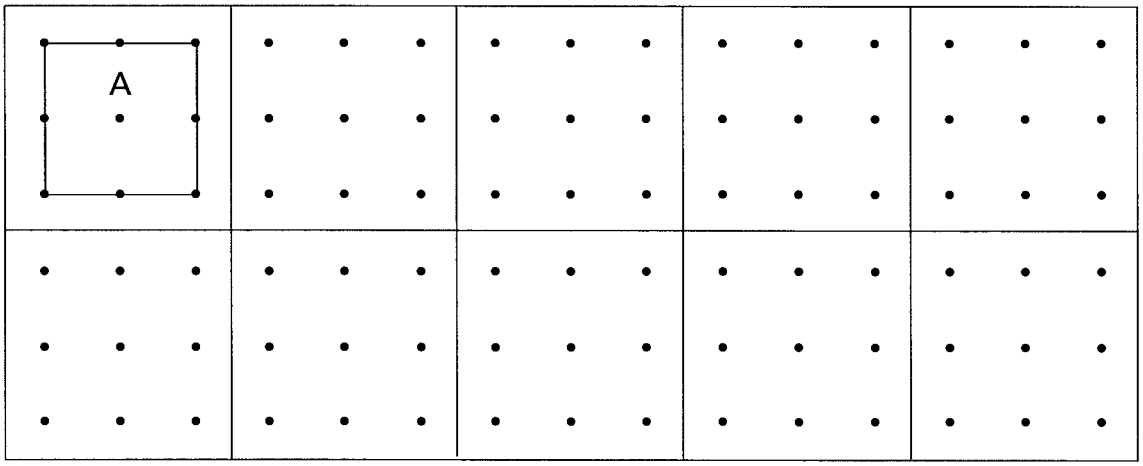
c) This figure is divided into 6 congruent regions.

d) This figure is not a parallelogram. Its area is 12 square units. It has half-turn rotational symmetry. (Place an "x" on the center of the rotation.)

e) This figure does not have reflective symmetry. It is not a simple polygon. Its area is 6 square units. Its perimeter is 22 linear units.

f) This hexagon does not have reflective symmetry. Its area is $4\frac{1}{2}$ square units.

2 Show all the different possible ways to divide Square A into 2 *non*congruent regions if lines have to connect pin to pin.



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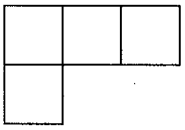
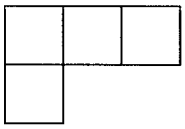
Follow-up Student Activity (cont.)

3 Add 1 or more squares to each figure at the right so the statement is true.

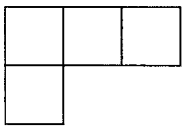
a) This figure has exactly 1 line of symmetry.



b) This figure has exactly 2 lines of symmetry.



c) This figure has half-turn rotational symmetry.



4 Write a letter to students in another school to help them understand the meaning of rotational and reflective symmetry.

5 Color a different design on each of the attached grids so that each 10×10 grid has reflective and/or rotational symmetry. Cut apart the grids and label the back of each colored grid to identify the type of symmetry it has.

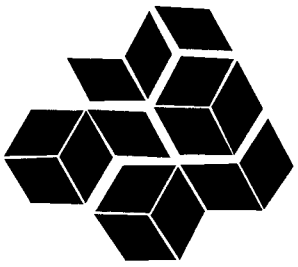
6 Cut out pictures of objects, fabric designs, flags, business logos, etc., that have reflective and/or rotational symmetry. Label each picture according to its type of symmetry. Clip your pictures to this assignment.

Pattern Block Trains

Lesson 17

THE BIG IDEA

By reasoning inductively about observations of visual relationships in geometric patterns, it is possible to make generalizations that describe larger figures in the patterns. Thus, sequences of pattern block arrangements are a meaningful setting for making conjectures and generalizations. Combined with discussions that illustrate the many ways of "seeing" each pattern, these experiences can foster the development of important intuitions about the concept of a variable.



CONNECTOR

OVERVIEW

Students become familiar with the pattern blocks and make observations about their shapes and sizes and relationships to each other.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 package for each group of students (see *Starting Points* for suggestions about packaging pattern blocks).
- ✔ Butcher paper strips (4-6" widths), 8-10 per group of students and some extras.
- ✔ Marking pen, at least 1 per group.
- ✔ Pattern blocks for the overhead.

FOCUS

OVERVIEW

Students examine patterns in sequences of pattern block arrangements.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 package for each group of students.
- ✔ Pattern blocks for the overhead.
- ✔ Focus Master A, 1 transparency.

FOLLOW-UP

OVERVIEW

Students find patterns in sequences of pattern block pieces and write descriptions of these patterns.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 17.1, 1 copy per student.
- ✔ Paper or cardstock sheets of pattern blocks (see *Blackline Masters*), 1 set per student.

LESSON IDEAS

ASSESSMENT
It is important that students not be discouraged by the challenge of writing clearly about their thinking. Rather encourage them to view discomfort as an indicator of a new learning opportunity! In fact, encourage them to express their discomfort in their journals or daily reports. Reading back over such comments later in the year helps students recognize ways they are growing.

FOLLOW-UP
Students may still feel somewhat uncomfortable with making written explanations. This Follow-up could be kept in students' Growth-folios (see *Starting Points*) as a signpost for the development of their abilities to make generalizations and to communicate mathematically.

FOLLOW-UP
Because drawing trains is cumbersome, encourage students to trace their paper pattern blocks (see *Blackline Masters*) if they want to make sketches or to glue cut out pieces in place to show a model. Students could exchange the patterns they create for Follow-up Problem 3 and try to predict each other's 50th train. Or, you could make a bulletin board display of their patterns. Invite students to predict 50th trains and compare their predictions to explanations made by the author of each pattern.

SELECTED ANSWERS

1. b) One way of viewing the 4th train is as 2 sets of double trapezoids and 2 hexagons, so the 10th train will have 5 sets of double trapezoids and 5 hexagons.

2. a) One possibility is that the 3rd train has 3 trapezoids and 4 columns of 3 squares each so, the 20th train will have 20 trapezoids and 21 columns of 3 squares each.

   c) 14 trapezoids

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students become familiar with the pattern blocks and make observations about their shapes and sizes and relationships to each other. These observations may include relationship concepts such as area, length, and symmetry.

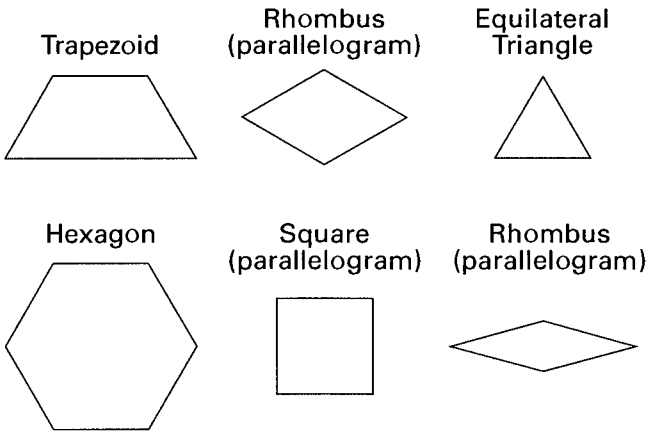
MATERIALS

- ✔ Pattern blocks, 1 package for each group of students (see *Starting Points* for suggestions about packaging pattern blocks).
- ✔ Butcher paper strips (4-6" widths), 8-10 per group of students and some extras.
- ✔ Marking pen, at least 1 per group.
- ✔ Pattern blocks for the overhead (see Comment 1 below and *Blackline Masters*).

ACTIONS

1 Write the word "TRAPEZOID" in large print on an $8\frac{1}{2}$" × 11" sheet of paper, and sketch a large trapezoid on the same sheet. Post this on the classroom wall. Repeat for all the pattern block pieces (see diagrams below).

Arrange the class in 6 groups. Give each group a package of pattern blocks, 8-10 narrow strips of butcher paper, and a marking pen. Assign each group 1 pattern block piece and ask them to record on the butcher paper strips 8-10 interesting mathematical observations about that piece, including its relationships to the other pieces. Have the groups tape their strips under the corresponding name that you posted earlier. Discuss, adding more strips as new observations are made.



COMMENTS

1 You may wish to assign a second piece to groups that finish early. Hence, have additional paper strips available.

It is impossible to predict all the observations students will make, but this does give you a chance to observe students' familiarity with terminology and ways they have integrated concepts like symmetry and area into their way of thinking about shapes.

Here are a few possible observations: 2 *equilateral triangles* cover a blue rhombus (i.e., the area of a triangle is ½ the area of a rhombus); the angles of the triangle are all equal; the angles of the square are all equal; the angles of the hexagon are all equal; the lengths of the sides of the square, triangle, hexagon, and rhombuses are all equal (note that a polygon whose sides and angles are equal is called a *regular polygon*); each piece has at least 1 line of symmetry; the square has 4 lines of symmetry; the hexagon has ⅙-, ⅓-, ½-, ⅔-, ⅚-, and full-turn rotational symmetry; the square has 4 "square corners" (i.e., 4 *right angles*); each shape except the trapezpoid is equilateral (i.e., all sides are equal in length); the perimeter (distance in linear units around the shape) of a hexagon is twice the perimeter of the triangle; etc. Note that perimeter of the pattern blocks is explored in Lesson 18 using linear pieces. If students bring up this topic you may wish to encourage them to explore now with linear pieces.

Pattern blocks for this lesson can be purchased or copied onto colored card stock by using the patterns in *Blackline Masters*. Transparencies of pattern blocks can also be made from these pages.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

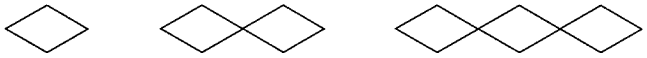
Students examine patterns in sequences of pattern block "trains." Their observations about visual relationships among trains provide a basis for making conjectures about larger numbered trains and generalizations about the structure of any train. These ideas were introduced in Lesson 5, Visual Reasoning.

MATERIALS

- ✔ Pattern blocks, 1 package for each group of students.
- ✔ Pattern blocks for the overhead.
- ✔ Focus Master A, 1 transparency.

ACTIONS

1 Arrange the students in groups and give each group of students a package of pattern blocks. Using blue parallelograms on the overhead, form the following first 3 trains of a sequence of trains. Explain that you have made a 1-car train, a 2-car train, and a 3-car train. Ask the students to use their pattern blocks to form these 3 trains.



2 Tell the students that you see a pattern from the first 3 trains and are picturing the 4th train in your mind. Ask the students to form what they think you have in mind for the 4th train. Have one or more volunteers come to the overhead to show and explain their ideas.

3 Focus attention on this arrangement:

Tell the students that this is the 4th train you had in mind. Ask them to imagine, in their mind's eye, what the 10th train in your pattern looks like. Then ask for a volunteer to come forward to build the 10th train in the sequence without building the intervening trains first. Urge volunteers to discuss their reasons for building that particular train.

COMMENTS

1 Having students actually build the trains helps them to get ideas about relationships among the trains, especially when patterns become more complex. This is useful later when they are asked to make generalizations.

2 Acknowledge a variety of reasonable possibilities for the 4th train. Student verbalizations and demonstrations at the overhead are very important parts of this activity. Emphasis should be placed on how students "see" and think about relationships and on exploring several different ways of viewing each train. This reinforces the ideas that there are many ways to view a problem and there is often more than one "right answer." It also encourages risk taking by students.

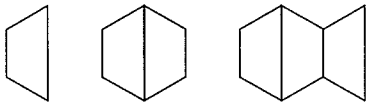
3 Asking students to guess the pattern you are thinking allows you to focus on a pattern that is mathematically rich. It is not intended that other ideas be dismissed as "wrong" answers or less important; rather, they are not how you were thinking. It is important to also extend interesting patterns suggested by students.

This relatively simple first example may not prompt much discussion, but more complicated patterns will. Because of the simplicity of this pattern it is appropriate for introducing the use of a variable to describe any train in the sequence. Many students will say that the number of blue parallelograms is the same as the number of the train. Hence, letting "n" stand for the number of the train, the nth train has n blue parallelograms. (Note that students shouldn't be expected to use variables here, but some students may make verbal statements about relationships between the number of a train and its structure.)

Focus Teacher Activity (cont.)

ACTIONS

4 Place this sequence of red trapezoids on the overhead. Tell the students these are the first 3 trains in a pattern of trains. Ask them to build these 3 trains and to form what seems most logical for trains 4 and 5. Have volunteers share their ideas and reasoning.

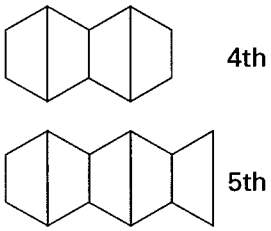


5 Assuming trains 4 and 5 are the ones shown in Comment 4, ask the students to imagine (without building the intervening trains) the 10th train in this sequence. Have a volunteer form or describe this train at the overhead.

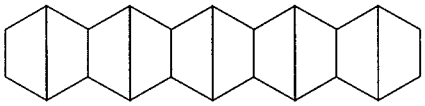
6 Draw attention to train 10 shown in Comment 5 (or another 10th train suggested by a student) and discuss reasons why students believe it looks as it does.

COMMENTS

4 Once again, be receptive to a variety of patterns. These are commonly suggested possibilities for the 4th and 5th trains:

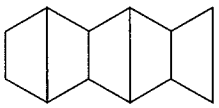


5 To encourage a variety of ways of "seeing" train 10, ask students to think privately about their ideas for train 10 before discussing it with classmates. Most students will suggest the following as the 10th train:



6 It is helpful to have students share their reasons in an open and accepting atmosphere. Students tend to see things in different ways even though the final answer is the same. For example, here are some responses for determining the 10th train shown in Comment 5:

- "There are 10 red trapezoids which make 5 hexagons."
- "The 5th train looks like this:



so the 10th train must be twice as big."

- "The 2nd train is 1 complete hexagon; the 4th train is 2 hexagons; so the 10th must be 5 hexagons."
- "If n is the number of the arrangement, there are $n/2$ whole hexagons in the nth train when n is even, and there are $(n-1)/2$ whole hexagons plus a trapezoid when n is odd."

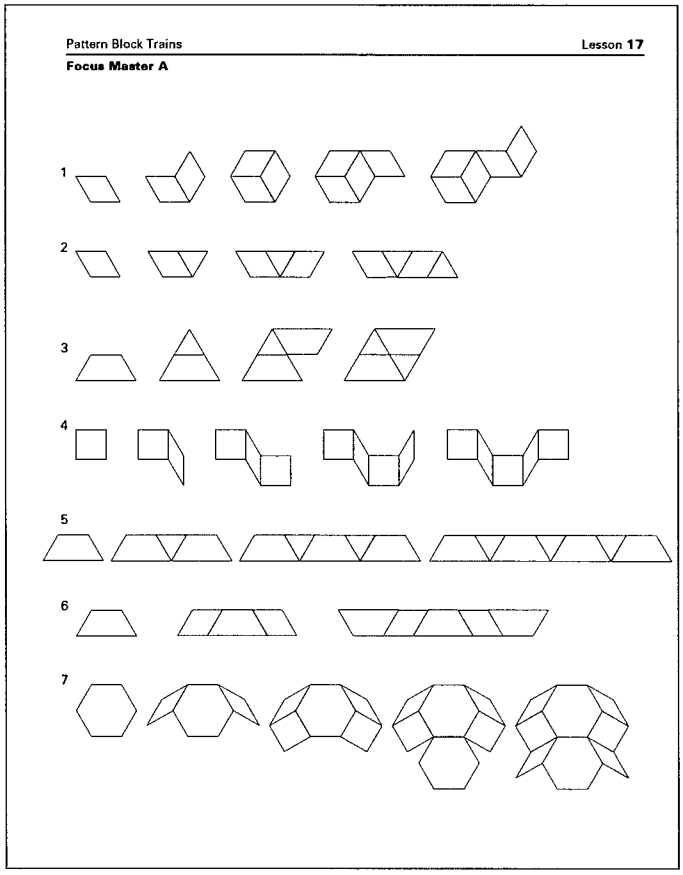
Some students may "see" the answer for reasons they can't verbalize.

For students having difficulty reasoning inductively to determine what larger trains look like, encourage them

Focus Teacher Activity (cont.)

ACTIONS

7 Continue an examination of trains by selecting appropriate patterns from Focus Master A.



8 (Optional) Ask groups to make up their own "interesting" sequence of trains which follow a pattern. Have volunteers share the first 4 trains in their sequence and ask the class to describe the 20th, 100th, etc., trains in the sequence.

COMMENTS

to build a few more trains in the pattern. Also pose questions that encourage students to discuss how the relationships they observe among the first trains could help them to predict what larger trains look like.

7 You may wish to build these train sequences with transparent pattern blocks. If you use a transparency, it is helpful to color the pattern block figures. Allow students to use pattern blocks as needed. Students could explore each pattern individually or in small groups. In any case have students look for a variety of ways of viewing larger numbered trains in each pattern, such as the 100th, 75th, 79th, etc.

If appropriate for your students, ask them to describe, orally or in writing, different ways to "see" *any* train in a sequence. It isn't important that students use algebraic notation. Rather, encourage verbalizations and, as appropriate, model notation as they give descriptions.

Have students describe relationships they see between the number of a train and the quantity and/or position of the pattern blocks. In Sequence 2 on Focus Master A, for example, students might comment that: all odd numbered trains end in a blue parallelogram and all even numbered trains end in a triangle; the total number of pieces in any train is n (the number of the train); even numbered trains have $n/2$ triangles and odd numbered trains have $(n-1)/2$ triangles; even numbered trains have $n/2$ parallelograms and odd numbered trains have $(n+1)/2$ parallelograms; even numbered trains have the same number of parallelograms as triangles; odd numbered trains have one more parallelogram than triangles; etc. (Note that these comments are based on the most common way of viewing the sequence, i.e., each train is formed by adding a triangle/parallelogram (alternately) to the preceding train.)

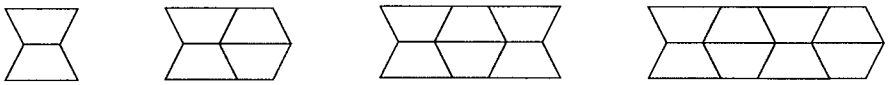
8 This gives you additional information regarding students' understanding of the meaning of patterning. Some students may find it challenging at first to create a pattern.

TEACHER NOTES:

Follow-up Student Activity 17.1

NAME _____ DATE _____

1 The following are the first 4 trains in a sequence.



a) Use your pattern block pieces to form what you think is the 5th train in this sequence. In the space below describe what your 5th train looks like and tell how you decided this.

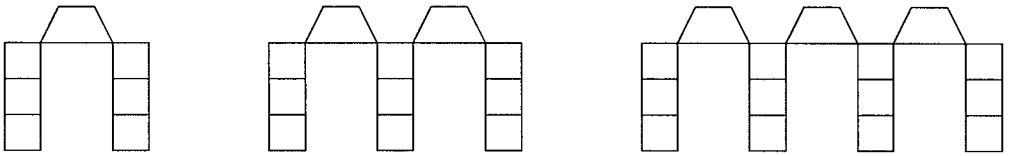
b) Write a description of what you think the 20th train looks like so a person who reads your description could accurately build that train.

c) Which train do you think has 286 trapezoids? Tell how you decided this.

(Continued on back.)

Follow-up Student Activity (cont.)

2 The first 3 trains in a sequence are shown below:



a) Tell how the first 3 trains help you to decide what the 20th train looks like without having to build all of the intervening trains.

b) Write a description of the 100th train in the sequence so a person who reads your description could accurately build that train.

c) If a certain train in this sequence has 45 squares, how many trapezoids are in the train? Explain how you decided this.

d) Show this pattern to an adult. How did they "see" the 100th train and what was their reasoning?

3 Invent an interesting pattern of trains. On another sheet glue or tape in place the first 3 trains in your pattern. On the back of that sheet explain how someone could use your first 3 trains to tell what your 50th train looks like.

Pattern Block Perimeters

Lesson 18

THE BIG IDEA

Sequences of arrangements of pattern blocks provide a context for introducing the concept of perimeter and for developing intuitions about functions and the concept of a variable. Visualizing the 20th and 100th arrangements in a sequence of pattern block arrangements and making generalizations about their perimeters are important background experiences for the study of algebra and solving equations.

CONNECTOR

OVERVIEW

Students build figures with pattern blocks and determine the perimeters of these figures.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 package for each group of students.
- ✔ 1" linear pieces (see Lesson 2), 30 per student.
- ✔ Pattern blocks for the overhead.

FOCUS

OVERVIEW

Students build pattern block trains, compute their perimeters, and use patterns to predict the perimeters of larger trains.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 package for each group of students.
- ✔ 1" linear pieces, 30 per student.
- ✔ Pattern blocks for the overhead.
- ✔ Focus Master A, 1 transparency.
- ✔ Butcher paper strips (4-6" widths), 15-20 per group.
- ✔ Marking pen, 1 or more per group.

FOLLOW-UP

OVERVIEW

Students use patterns to predict the perimeters of larger pattern block trains.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 18.1, 1 copy per student.
- ✔ Paper or cardstock sheets of pattern blocks (see *Blackline Masters*), 1 set per student.

LESSON IDEAS

JOURNALS

Problems 5 and 6 on the Follow-up provide opportunities for students to engage in reflective thinking about their mathematical development. You may wish to have them write their responses to these questions in their journals.

Responses to Problem 1 could also be written in students' journals for comparison to earlier and later statements about the meaning of perimeter.

Another worthwhile journal thought starter to pose after completing this lesson is: "The process I use when I am asked to generalize a pattern is..."

QUOTE

One of the central themes of mathematics is the study of patterns and functions. This study requires students to recognize, describe, and generalize patterns and build mathematical models to predict the behavior of real-world phenomena that exhibit the observed pattern. The wide-spread occurrence of regular and chaotic pattern behavior makes the study of patterns and functions important. Exploring patterns helps students develop mathematical power and instills in them an appreciation for the beauty of mathematics.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

2. The following answers are based on the most common way of viewing this pattern.

   a) A row of 10 parallelograms placed end to end.
   b) 22 linear units.
   c) 202 linear units (e.g., 100 linear units on the top and 100 on the bottom, plus 1 on each end).
   d) Train 22. (Methods for obtaining this answer will vary.)

3. Based on the most common way of viewing this pattern, the 20th train has 20 five-sided polygons (pentagons) placed end to end. The perimeter of the 20th train is 62 linear units. One way to think of this is that the 18 interior polygons each have 3 edges exposed and the 2 polygons on the ends have 4 edges exposed. So the perimeter is $18(3) + 2(4) = 62$.

4. Accept either verbal or symbolic explanations. The nth train (based on the most common way of viewing this pattern) has $n - 1$ triangles and n trapezoids. Each triangle has 1 edge exposed and each trapezoid has 3 edges exposed except for the 2 on the ends which have 4 edges exposed. So the perimeter of the 20th train, for example, is $19 + 18(3) + 2(4) = 81$ linear units.

   Alternately, one could say: There are $2 \times n$ (where n is the number of the train) linear units on the bottom, $n + (n - 1)$ linear units on the top, and 1 linear unit on each end.

Connector Teacher Activity

OVERVIEW & PURPOSE

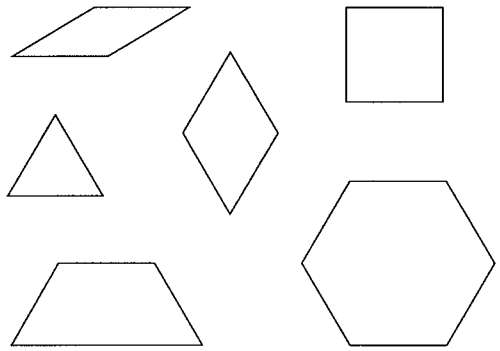
Students build figures with pattern blocks and determine the perimeters of these figures. This activity helps the student see the relationship between linear and area units of measure.

MATERIALS

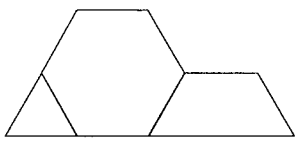
- ✔ Pattern blocks, 1 package for each group of students.
- ✔ 1" linear pieces (see Lesson 2), 30 per student.
- ✔ Pattern blocks for the overhead.

ACTIONS

1 Arrange the students in groups and distribute pattern blocks and linear pieces to each group of students. Discuss students' views of the meaning of *perimeter*. Ask them to provide examples with the pattern blocks to illustrate their explanations. Clarify, as needed, any questions that emerge and then ask the students to find the perimeter of each pattern block.

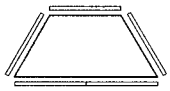


2 Ask each student to form a figure like the following and to determine its perimeter. Discuss their methods.

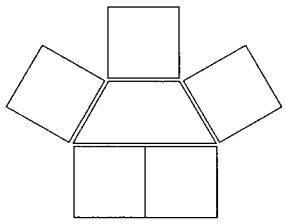


COMMENTS

1 *Perimeter* is the total distance (in linear units) around a shape. Assuming the length of the edge of the square is 1 unit of length, the perimeters of all the pattern blocks can be found by placing linear units end-to-end around the blocks, as shown below for the trapezoid. (The linear pieces introduced in Lesson 2 have the same length as the edge of a square pattern block. If needed, toothpicks could be broken to use as linear pieces.)



Or, again assuming the length of the edge of a square is 1 unit of length, the perimeters of the other pattern blocks can be found by comparing the side of a square to their sides. For example:



Someone may suggest the longest side of the trapezoid has length 1 linear unit. Note that this suggests that a linear piece has length ½ and a square has a perimeter of 2 linear units. This could lead to a discussion of the arbitrariness of the selection of the linear unit. For example, you could ask students to determine the perimeter of the square if the hexagon has perimeter 1 linear unit.

Throughout the remainder of this lesson the length of the side of the square is assumed to be the linear unit.

2 It is important that this is viewed as 1 figure of perimeter 10 linear units rather than 3 distinct figures with total perimeter 14 units. One way to determine the correct perimeter of 10 is to place linear units around

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

3 Ask the students to use their pattern blocks and linear units to form the following figures:

a) A rectangle whose area is 8 square units. Determine its perimeter.

b) A rectangle whose perimeter is 8 linear units. Determine its area.

c) Three different figures whose perimeters are 10 linear units.

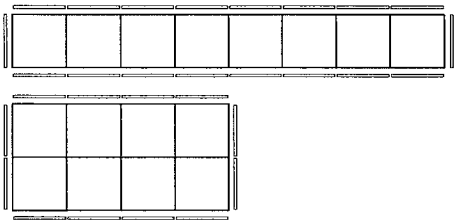
d) A nonrectangular parallelogram whose perimeter is 8 linear units.

COMMENTS

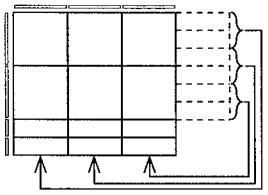
2 (continued.)
the figure. Another method is to determine the perimeter of each pattern block and then to subtract the lengths that are "lost" when the pieces are pushed together. The computation that corresponds to this method would be $(3 + 6 + 5) - (2 \times 2)$.

3 Relationships between the area and dimensions of rectangles were introduced in Lessons 2 and 7, *Basic Operations* and *Factors and Primes*. This and the previous action demonstrate that a figure with a fixed area (or perimeter) does not necessarily have a single corresponding perimeter (or area).

a) Assuming that the square pattern block has an area of 1 square unit, it is possible to build rectangles with perimeters of 18 and 12 linear units, as shown below:

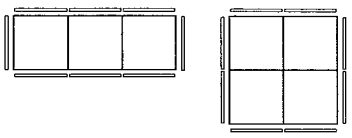


Some students may suggest there are other possibilities if cutting square and linear units is allowed. For example:

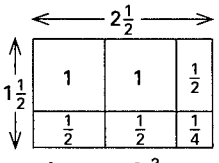


Area = 8
Perimeter = $11\frac{1}{3}$

b) The rectangles below both have perimeter 8, but one has area 3 and the other has area 4, as shown here:



Note that if fractional lengths and parts of squares are allowed, there are other possibilities. For example:



Area = $3\frac{3}{4}$
Perimeter = 8

Focus Teacher Activity

OVERVIEW & PURPOSE

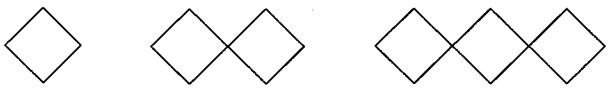
Trains of pattern blocks are built according to a geometric pattern. Students use their observations about patterns to help predict the perimeters of larger trains in the sequence.

MATERIALS

- ✔ Pattern blocks, 1 package for each group of students.
- ✔ 1" linear pieces, 30 per student.
- ✔ Pattern blocks for the overhead.
- ✔ Focus Master A, 1 transparency.
- ✔ Butcher paper strips (4-6" widths), 15-20 per group.
- ✔ Marking pen, 1 or more per group.

ACTIONS

1 Arrange the students in groups and distribute pattern blocks to each group and linear pieces to each student. On the overhead projector or demonstration table form the following sequence of trains. Ask each student or group of students to build these trains.



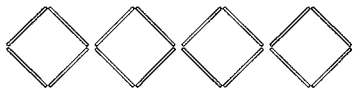
2 Have students imagine the 4th train and predict its perimeter assuming the length of the side of a pattern block square is 1 linear unit. Then have them build the 4th train and confirm their conjecture.

3 Ask students to imagine in their mind's eye what the 10th train in the above sequence looks like and then use that image to determine the perimeter of the 10th train. Discuss their images and methods of determining the perimeter. Repeat for the 20th train.

COMMENTS

1 Many students will say the first arrangement is formed by a diamond. It could be pointed out that this shape is also a square.

2 Encourage students to use linear pieces if needed to help determine the perimeters. For example, a student may suggest that after "removing" the 4th train, the perimeter of the 4th train would look like this:



Assuming each train is formed by adding a square to the previous train, the perimeters of the first 4 trains are 4, 8, 12, and 16 linear units, respectively. Note that the length of one side of a square is assumed to be 1 linear unit throughout this lesson.

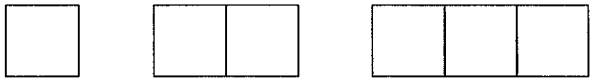
3 To encourage a variety of methods suggest that students think privately about each train and its perimeter. Then have volunteers share their ideas. Discussion is important. Based on the pattern illustrated in Comment 2, many students will say that the 10th train has 10 squares each with a perimeter of 4 units. Others may "see" the perimeter differently. Someone could say, "I see 9 dents on the top and bottom and each dent is 2 units. Then there are 2 units on each end so 18 + 18 + 2 + 2 = 40 linear units." Variety is fun! Based on the pattern described above, the 20th train has perimeter 80 linear units.

Focus Teacher Activity (cont.)

ACTIONS

4 Tell the students that you have a certain train in the preceding sequence in mind and that its perimeter is 100 linear units. Ask them to use their observations about the pattern to determine which train you have in mind. Discuss their ideas and methods.

5 Construct the following sequence of trains on the overhead or demonstration table and ask the students to each build these trains.



6 Have students find the perimeters of the first 4 trains in the sequence. Then ask them to imagine the 20th train (or 50th, 100th, etc.) and predict its perimeter.

COMMENTS

4 The 25th train has a perimeter of 100 linear units. Some students may determine this by finding the number of groups of 4 in 100, or $100 \div 4$. Others may suggest that since the 20th train has perimeter 80 linear units, then the 25th train has perimeter $80 + (5 \times 4) = 100$ linear units.

5 This time the squares share an edge.

6 This problem is interesting because of the different ways it can be approached. For example, if each train is formed by attaching a square to the previous train (the most common way of viewing the sequence):

- Some students may sketch 20 squares and count the perimeter.

- Noticing that the first 4 perimeters are 4, 6, 8, and 10, some students may count by 2's until reaching the 20th perimeter.

- Students who can picture the 20th train may see 20 units along the top, 20 units along the bottom, and 1 unit on each end, or $20 + 20 + 1 + 1 = 42$.

- A few may visualize the 20 squares, having a total perimeter of 80 units, joined together in 19 places. In each of the 19 places 2 sides of the squares come together reducing the 80 units by 38 to leave a perimeter of 42. This could be recorded as $(4 \times 20) - (19 \times 2) = 42$.

Through a discussion of various approaches students can learn from each other. They may also discover that some approaches work more easily than others when finding perimeters of trains too large to build or draw, like the 50th or 100th.

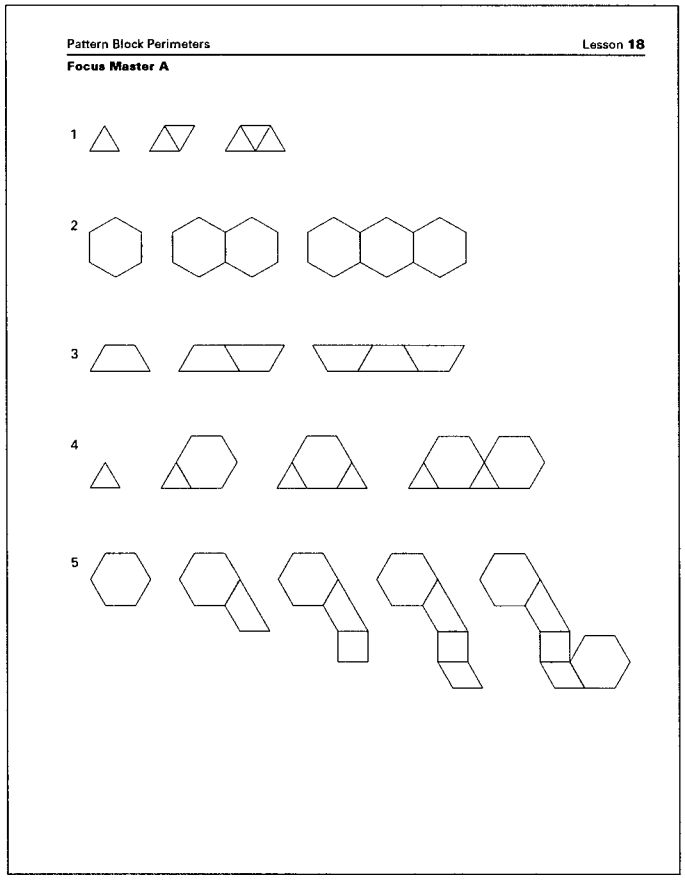
Verbal generalizations of trains provide background and motivation for using variables. So does using number statements to describe their thought processes for determining the perimeter of specific trains. Students who are able to do both may be ready to use variables in their descriptions.

Focus Teacher Activity (cont.)

ACTIONS

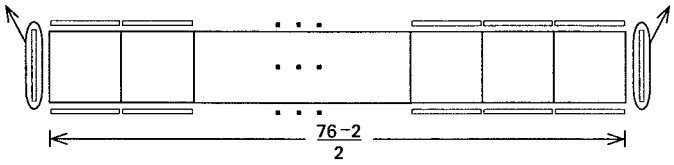
7 Tell the students that you have a certain train in this sequence in mind and that it has a perimeter of 76 linear units. Ask them to use their observations about this pattern to determine which train you have in mind. Discuss their methods.

8 Place a transparency of Focus Master A on the overhead. Ask students to imagine they are constructing the 10th (20th, 100th, etc.) train in Sequence 1 and to look for a variety of ways to "see" its perimeter. Discuss their methods and then have students try out each other's ideas to find the perimeter of other trains in the sequence. Repeat for Sequences 2 and 3 on Master A.



COMMENTS

7 One method of reasoning is based on the fact that each square contributes 2 linear units (a top and a bottom edge), except for the 2 squares on the ends which each contribute 1 additional linear unit. Hence, to determine the number of squares (i.e., the number of the train), subtract the 2 linear units at the ends and divide by 2 as illustrated here:



8 You may wish to build these train sequences with transparent pattern blocks. If you use a transparency, it is helpful to color the pattern block figures. Allow students to use pattern blocks and linear units as needed; some will need to build a few more trains in a series before generalizing. The point here is for students to imagine and verbalize what larger trains look like and to use their mental images to determine the perimeters of the trains.

Whenever possible have students use number statements to describe their thought processes. To motivate discussion you could pose situations about these sequences, such as:

a) Given a particular train in a sequence, say the 10th or 15th, find its perimeter.

b) Given a specific perimeter, find the number of the train with that perimeter.

c) Describe general methods that will work for finding the perimeter of any (i.e., the "*n*th") train in the sequences.

Exploring questions such as these helps to provide background for the study of functions, an important concept in algebra. Functions will be explored informally throughout *Visual Mathematics* courses.

Focus Teacher Activity (cont.)

ACTIONS

9 Sketch an enlarged version of sequence 4 (from Focus Master A) on a strip of butcher paper and post this on the wall. Arrange the students in groups of 2-4 and give each group 8-10 butcher paper strips and a marking pen.

Ask the groups to brainstorm (and record) several observations about patterns and relationships they can see in the sequence. Ask them to include descriptions of ways of seeing the perimeter of the 100th train and the 73rd train in the sequence. Have the groups tape their observations and descriptions under the copy of sequence 4 that you posted. Discuss.

10 Repeat Action 8 or 9 for sequence 5 on Focus Master A.

COMMENTS

9 Encourage groups to make as many observations as possible. If appropriate for your students, ask them to also write a verbal description of how to find the perimeter of any train (i.e., the "*n*th" train, where *n* is the number of the train).

Have the students help you sort the statements they posted, discussing their likenesses and differences. If there are conjectures or generalizations that groups disagree about, encourage them to look for *counterexamples* to illustrate why an idea may work in some but not all cases.

10 An alternative to creating a whole-class list of observations, as in Action 9, is to have each group make a poster of their ideas.

Follow-up Student Activity 18.1

NAME _____ **DATE** _____

1 In your own words, explain the meaning of perimeter.

2 The first 3 trains in a sequence are shown below. Questions a)-d) refer to this sequence:

a) If this pattern continues, describe what the 10th train looks like and how you decided this.

b) What is the perimeter of the 10th train, assuming the length of one edge of a small parallelogram is 1 linear unit? Explain how you "see" the perimeter of train 10:

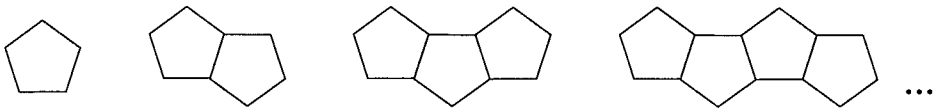
c) Explain how you "see" the perimeter of the 100th train.

d) Which train has a perimeter of 46? Explain how you determined this.

(Continued on back.)

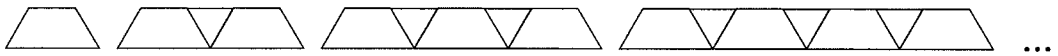
Follow-up Student Activity (cont.)

3 The first 4 trains in a sequence are shown below.



Find the perimeter of the 20th train and explain 2 different ways to "see" this value.

4 The first 4 trains in a sequence are shown below.



Describe a method (other than building the train and counting) that works for finding the perimeter of every train in this sequence.

5 Describe what is easiest and what is hardest now for you about working with visual patterns.

6 Describe one or more ideas about visual patterns that used to be hard for you, but are easier now. Tell what you think helped you to understand better.

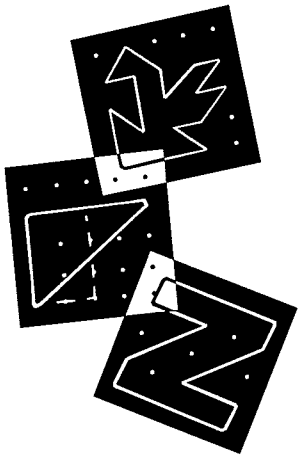
7 Use your pattern blocks to form the pattern in problem 4 and show it to an adult. Find out their ideas about what the 100th train looks like and what its perimeter would be. How do their ideas compare to yours? Explain on another sheet of paper.

Geoboard Area I

Lesson 19

THE BIG IDEA

By exploring area in a formula-free setting, students gain intuitions that will help them later when they invent general methods for finding area. Explorations on the geoboard promote understanding of the concept of area, insights about conservation of area, and an appreciation for a variety of ways to solve problems.



CONNECTOR

OVERVIEW

Students record their views of the meanings of area and perimeter.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).

FOCUS

OVERVIEW

The concept of area is introduced and reinforced as a covering of figures by unit squares and parts of squares. Students form regions on the geoboard and use formula-free methods for finding areas.

MATERIALS FOR TEACHER ACTIVITIES

Teacher Activity 1

- ✔ Geoboards, 1 per student.
- ✔ Geoboard for the overhead or transparency of Geoboard Recording Paper (see *Blackline Masters*).
- ✔ Focus Masters A, B, and C, 1 transparency of each.

Teacher Activity 2 (optional)

- ✔ Area Bingo Cards (see *Blackline Masters*), 1 per student.
- ✔ Markers, 25 for each student (beans, chips, etc.).
- ✔ Geoboard Recording Paper (see *Blackline Masters*), 1 sheet per student.
- ✔ Geoboard Figure Cards (2 pages, see *Blackline Masters*), 1 transparency, cut apart.
- ✔ Geoboards (optional), 1 per student.

FOLLOW-UP

OVERVIEW

Students explain similarities and differences for units of area and length and form rectangles with given perimeters and areas.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 19.1, 1 copy per student.
- ✔ Geoboard Recording Paper, 2 sheets per student.

LESSON IDEAS

ASSESSMENT

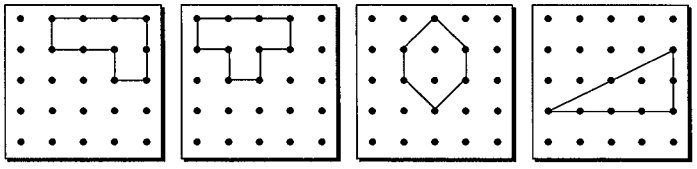
Since the concept of area recurs regularly throughout the course and since it is important to look for growth in understanding over time, be sure to collect written explanations of students' views and understandings for placement in their journals or Work-folios.

Both Focus activities allow time for teacher assessment of student understanding through observation. Anecdotal notes or a checklist such as those described in *Starting Points* may be useful to keep as students work.

TEACHER NOTES:

SELECTED ANSWERS

1. There are many possibilities. Encourage students to be creative! Here are four possibilities that are based on the small square as the area unit.

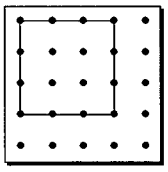


2. The following are based on the small square as the area unit.

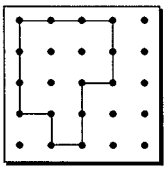
   a) $8\frac{1}{2}$ square units. The figure can be divided into whole squares and half squares.
   b) 10 square units.
   c) 6 square units.

4. There are many possibilities for a) or b).

   a)

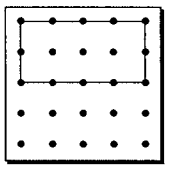


Area: 9 square units
Perimeter: 12 linear units

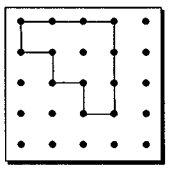


Area: 9 square units
Perimeter: 14 linear units

   b)

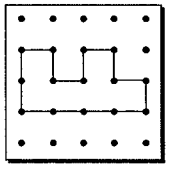


Area: 8 square units
Perimeter: 12 linear units



Area: 6 square units
Perimeter: 12 linear units

   c) A 4 by 4 square is the only rectangle that satisfies this condition.
   d) There are many possibilities including those shapes shown in a).
   e) Not possible.
   f) Not possible.
   g)



5. Students may use complex fractions (such as $4^{1/2}/16$). Acknowledge all correct forms and keep emphasis on using fractions that express how students "see" each part.

A	$4^{1/2}/16$ or $9/32$	B	$3/32$
C	$1/8$	D	same as A
E	$7/32$	F	$2^{1/2}/16$ or $5/32$
G	$3/32$	H	$3/16$
I	$1/16$	J	same as I
K	$1/8$	L	$2/16$
M	$3/16$		

Connector Teacher Activity

OVERVIEW & PURPOSE

Students record their views of the meanings of area and perimeter. This provides important evidence as the teacher and the students monitor the development of their understanding of these concepts.

MATERIALS

✔ Student journals (optional).

ACTIONS

1 Write the following on the overhead and allow about 5 minutes for the students to write their responses. Discuss their ideas.

In my own words, I would describe *area* and *perimeter* as...

Here are some examples and diagrams to illustrate my explanations...

COMMENTS

1 This could be a journal entry that provides useful information about the development of students' views of these concepts. While area has been discussed in relation to rectangular regions (see Lessons 2 and 7), this is the first *Visual Mathematics* lesson in which the concept of area is the focus.

TEACHER NOTES:

Focus Teacher Activity 1

OVERVIEW & PURPOSE

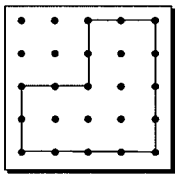
Students form regions on the geoboard and use formula-free methods for finding areas. This activity introduces methods for finding areas such as moving regions (reflecting or translating), enclosing regions in rectangles, and using the complement of a region.

MATERIALS

- ✔ Geoboards, 1 per student.
- ✔ Geoboard for the overhead or transparency of Geoboard Recording Paper (see *Blackline Masters*).
- ✔ Focus Masters A, B, and C, 1 transparency of each.

ACTIONS

1 Form the geoboard region shown below. Show the geoboard to the students and ask them to make the same region on their geoboards.



2 Remind the students that the number of area units needed to *cover* a region is called its *area* and that most often area units are squares. Ask them to use rubber bands to show how they view the area of the region they formed in Action 1. Discuss their ideas.

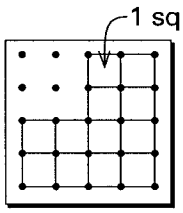


Figure I

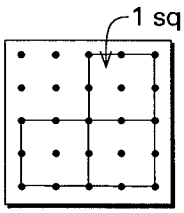


Figure II

3 Tell the students that for now (through Action 8) the small square is the unit of area. Place a transparency of Focus Master A on the overhead (or give each group a copy) and ask the students to form Region a) on their geoboards. Then ask them to determine the area of Region a). Have them place additional rubber bands on the figure to show their thought processes and counting methods. Have several volunteers show their ways of thinking at the overhead.

COMMENTS

1 A transparent overhead geoboard or a transparency of Geoboard Recording Paper works well for demonstrating the formation of regions. Large regions require several rubber bands.

2 Area was introduced in Lessons 2 and 7. Students may ask you to identify the unit of area. If so, suggest they each choose their own. If no one asks, wait to see what comes up.

Many students will probably choose a small square as the unit of area. Thus, the large region has area 12 square units, as shown in Figure I at left.

Others may suggest using a 2 by 2 square as the unit of area. Thus, the area of the large region is 3 square units, as shown in Figure II at left.

Still others may use the large 5-pin by 5-pin square as the unit and show the area is $12/16$ area unit, $3/4$, or $6/8$ area unit, depending on how they subdivide the unit square.

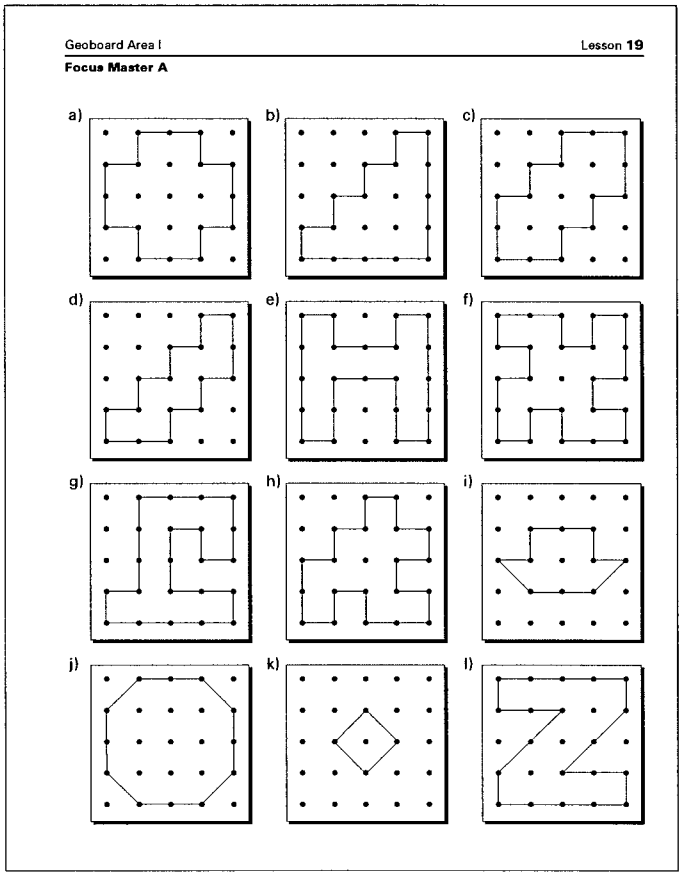
Each of these is a correct way to view the area of the region. In fact, it is possible to use nonsquare units of area (see Action 10).

3 It is valuable to have students see that even though Region a) is a fairly simple figure, there are many ways of "seeing" its area of 12. The examples on the next page show several different students' methods:

(Continued next page.)

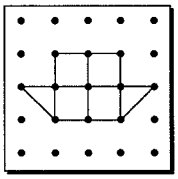
Focus Teacher Activity (cont.)

ACTIONS



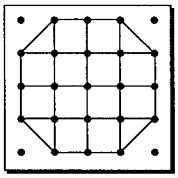
4 Repeat Action 3 for selected other figures on Focus Master A. Discuss students' methods.

i)



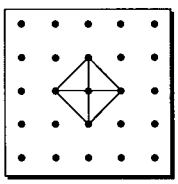
area 5
(4 squares and
2 half-squares)

j)



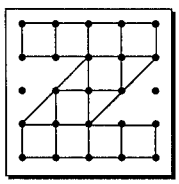
area 14
(12 squares and 4
half-squares) Or, the
4 outside half-squares
have an area of 2, and
16 – 2 = 14.

k)



area 2
(4 half-squares)

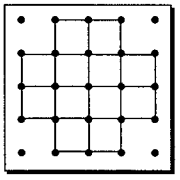
l)



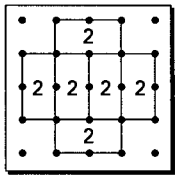
area 12
(10 squares and
4 half-squares)

COMMENTS

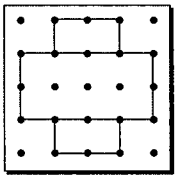
3 (continued.)



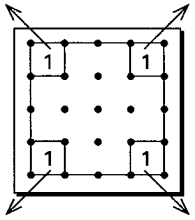
counting 1 by 1



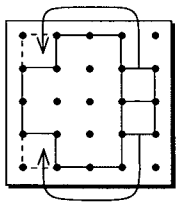
counting by 2's



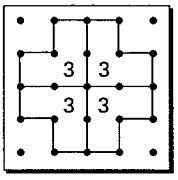
(2 × 4) + 2 + 2



4 × 4 square with
4 missing corners
(i.e., 16 – 4)

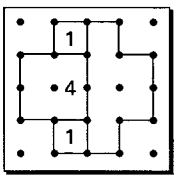


3 × 4 rectangle



4 groups of 3

Some students may point out that lines of symmetry (see Lesson 16) subdivide a shape into congruent regions. Identifying such lines can simplify the process of determining the area of the shape as shown at the right.



2 congruent
regions of
(4 + 1 + 1)

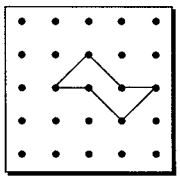
4 It isn't necessary to assign every figure. To keep students together and encourage geoboard use, ask the students to explore one or two figures at a time on the geoboard. To allow for students' varied rates in finding the areas, ask students who finish early to find alternate methods of determining each area. Ask volunteers to demonstrate and explain their approaches at the overhead. This enables students having difficulty to get ideas about other methods they could try.

The areas of the figures on this transparency in square units are: a) 12; b) 10; c) 10; d) 7; e) 10; f) 12; g) 10; h) 10; i) 5; j) 14; k) 2; l) 12. The last four regions involve both squares and half-squares.

Focus Teacher Activity (cont.)

ACTIONS

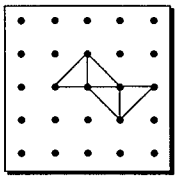
5 Have the students form this region on their geoboard and ask them to determine its area. Discuss.

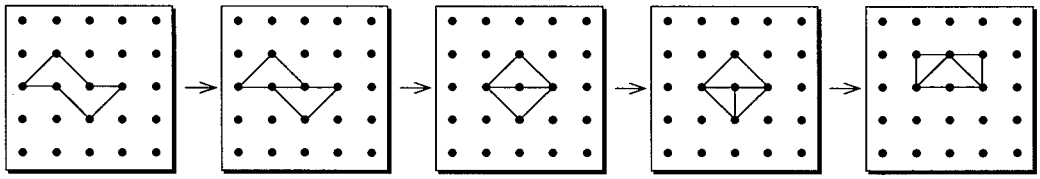


COMMENTS

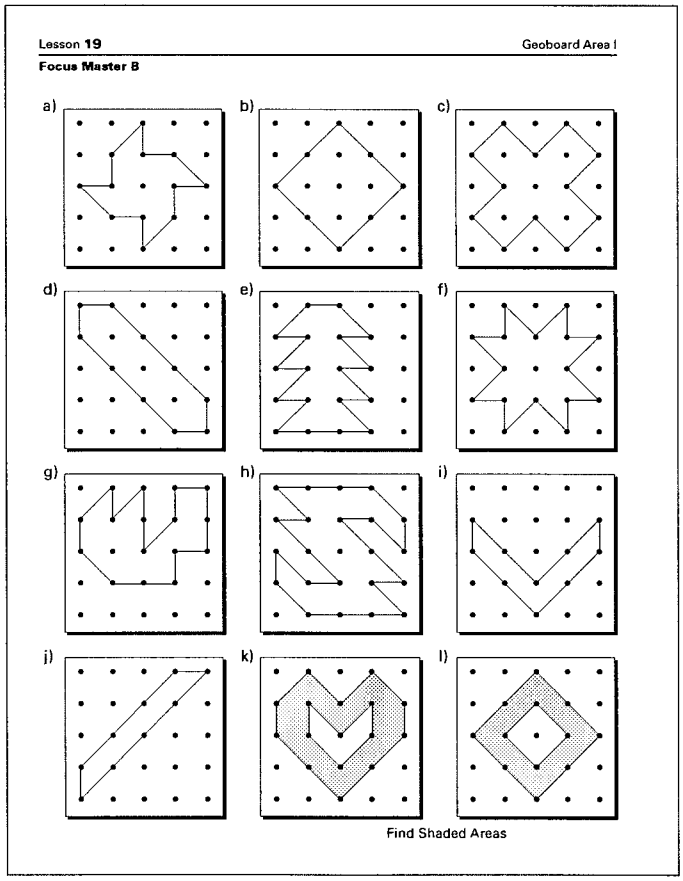
5 The region can be subdivided into 4 half-squares.

Some students are able to obtain the area by mentally manipulating parts of the region into a convenient shape. (One possible sequence of manipulations is given below.) This ability should be acknowledged and encouraged.





6 Place a transparency of Focus Master B on the overhead (or give each group a copy) and ask the students to form Region a) on their geoboards and to determine its area. Have volunteers show a variety of methods at the overhead. Repeat for other selected figures.



6 Base the number of these figures you assign on the success students experience.

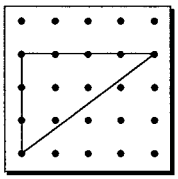
Each of these regions can be partitioned into squares and half-squares.

The areas of the figures on this transparency in square units are: a) 6; b) 8; c) 10; d) 7; e) 8; f) 8; g) 8; h) 11; i) 4; j) 3½; k) 8; l) 6.

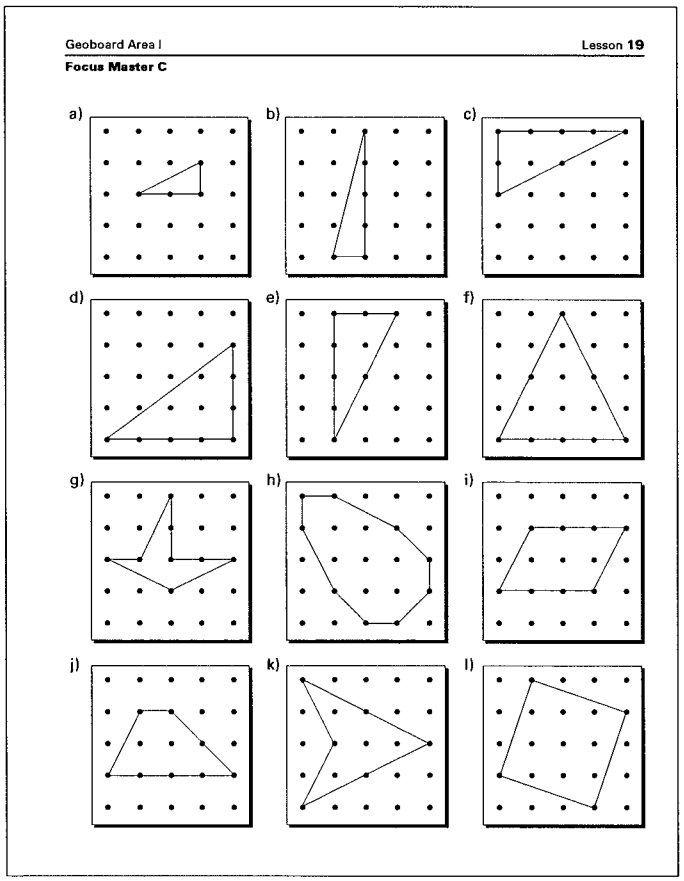
Focus Teacher Activity (cont.)

ACTIONS

7 Tell the students to form this triangle on their geoboards and to find its area. Discuss their methods.

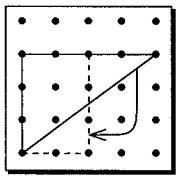


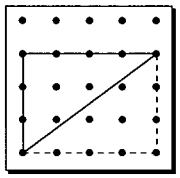
8 Place a transparency of Focus Master C on the overhead (or give each group a copy) and ask the students to build Region a) on their geoboards and to determine its area. Discuss. Repeat for other selected regions on Master C.



COMMENTS

7 This triangle doesn't divide nicely into squares and half-squares, so it is interesting to see how students find its area. A common method is to cut off the right side of the triangle and fit it under the left side to make a rectangle of area 6. Another useful method is to enclose the whole triangle in a rectangle of area 12 and notice that the triangle is half of this rectangle.



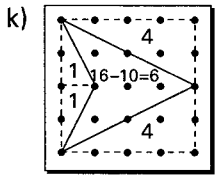


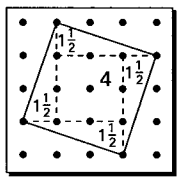
Discuss these approaches and others students devise.

8 It is important to discuss different approaches to finding the area of each figure students explore. Students will be surprised by the variety of methods and will feel more comfortable taking risks with more difficult problems. This will also provide them with a source of new methods to try or to adapt when their own techniques become cumbersome or ineffective.

The areas of the figures on this transparency in square units are: a) 1; b) 2; c) 4; d) 6; e) 4; f) 8; g) 3; h) 10½; i) 6; j) 5; k) 6; l) 10.

Here are some possible methods for finding the areas of figures k) and l).



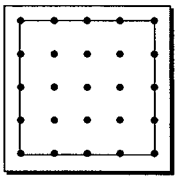


Experiences with enclosing triangles in rectangles to determine their areas lays important groundwork for later development of formulas. To allow for the development of intuitions and conceptual understanding keep emphasis here on formula-free methods. To reinforce concepts explored previously (e.g., congruence and symmetry) draw attention to methods that rely on their use. Geoboard areas are explored more in Lesson 35.

Focus Teacher Activity (cont.)

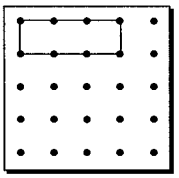
ACTIONS

9 Form the following square at the overhead.

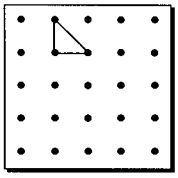


Ask the students to determine the area of Region b) on Focus Master C using the above square as the area unit. Discuss their methods and conclusions. Repeat for other polygons on Master C (or on Masters A and B).

10 (Optional) Repeat Action 9 for the following area units:



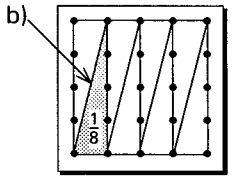
Unit I



Unit II

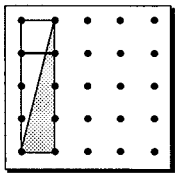
COMMENTS

9 One possibility is to subdivide the unit into 8 regions congruent to Region b). Hence, Region b) has area $\frac{1}{8}$ area unit:

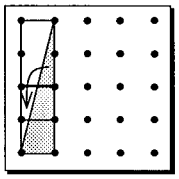


Another method is to reason that since the unit is 16 times as large as in the original problem, then the area of Region b) is $\frac{1}{16}$ of 2 (the original area), or $\frac{2}{16}$.

10 Acknowledge all methods that work. One way to find the area of Region b) using Unit I is to enclose region b) in a rectangle of area $1\frac{1}{3}$ units. Since Region b) covers $\frac{1}{2}$ of this rectangle, the area of Region b) is $\frac{1}{2}$ of $1\frac{1}{3}$ or $\frac{4}{6}$, as shown here:



Or rotate the top most part of Region b) to fill 2 small squares of area $\frac{1}{3}$ each, as shown here:



Using Unit II, Region b) has area 4. Note that in Lesson 35 students will discuss the rationale for the convention of using squares as area units and carry out more in-depth investigations with standard and nonstandard area units.

TEACHER NOTES:

Focus Teacher Activity 2 (optional)

OVERVIEW & PURPOSE

Students determine the areas of geoboard figures and place markers on their Bingo cards for numbers that match these areas. This game reinforces the concept of area as the number of squares and parts of squares needed to cover a figure.

MATERIALS

- ✔ Area Bingo Cards (see *Blackline Masters*), 1 per student.
- ✔ Markers, 25 for each student (beans, chips, etc.).
- ✔ Geoboard Recording Paper (see *Blackline Masters*), 1 sheet per student.
- ✔ Geoboard Figure Cards (2 pages, see *Blackline Masters*), 1 transparency, cut apart.
- ✔ Geoboards (optional), 1 per student.

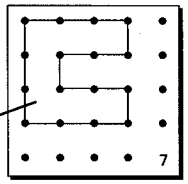
ACTIONS

1 Give each player one or more Area Bingo Cards, some bingo markers, and Geoboard Recording Paper and/or a geoboard with rubber bands. Place a transparent Geoboard Figure Card on the overhead and ask the students to:

- determine the area of the figure (if needed, by forming it on Geoboard Recording Paper or a geoboard);
- check their cards for a number equal to the area;
- use markers to cover any numbers that match the area.

Repeat the above process with additional Geoboard Figure Cards until a player gets 5 markers in a row horizontally, vertically, or diagonally. As an alternative, play until someone has covered all the spaces on a card.

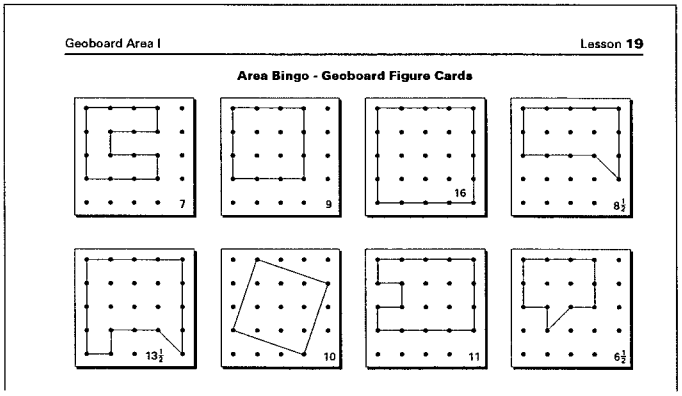
Area Bingo				
$1\frac{1}{2}$	$11\frac{1}{2}$	9	8	$7\frac{1}{2}$
$13\frac{1}{2}$	14	16	15	$9\frac{1}{2}$
7	13	FREE	3	$5\frac{1}{2}$
1	$4\frac{1}{2}$	11	4	$12\frac{1}{2}$
$2\frac{1}{2}$	6	$10\frac{1}{2}$	12	$\frac{1}{2}$



2 (Optional) Make up new Area Bingo Cards that show the areas of the shapes on the Geoboard Figure Cards, but now using the large geoboard square as the area unit. Play as described above.

COMMENTS

1 Prior to class prepare a class set of Area Bingo Cards and 30 transparent Geoboard Figure Cards. To do this, copy the Area Bingo Cards (see *Blackline Masters*) on cardstock and cut them apart. Cut out a transparent copy of each of the Geoboard Figure Cards (see *Blackline Masters*) and place a sticker behind the number on the lower right corner of each geoboard (the sticker conceals the number, which is the area of the figure on the geoboard). Also gather a supply of Area Bingo markers such as beans, colored chips, small pieces of paper, etc.



Note that all players may start with a marker on the FREE space. To encourage discussion among students, you may prefer to have pairs of students play as a team. If you decide to add more Geoboard Figure Cards, be sure the regions have areas that are on the Area Bingo cards or make new cards.

2 A master for blank Area Bingo Cards is in *Blackline Masters*. If the large geoboard square is the area unit, the area of each region on the Geoboard Figure Cards is 1/16 of the area now shown on the cards. Note that this could be repeated for other area units.

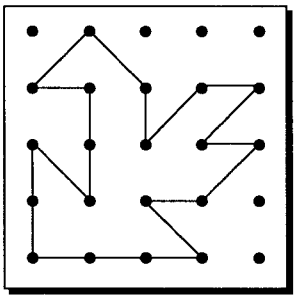
TEACHER NOTES:

Follow-up Student Activity 19.1

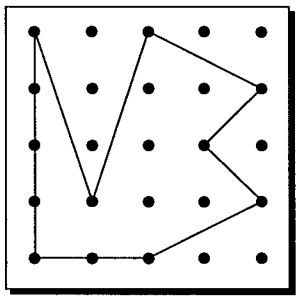
NAME _____ **DATE** _____

1 On the attached sheet of Geoboard Recording Paper, record as many figures as you can that have area 4 square units.

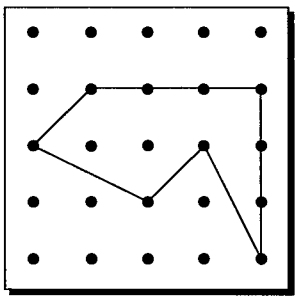
2 Find the area of each of the following figures. Below each figure write an explanation of the methods you used to determine the area and tell what you used as the area unit.

a) 

Methods:

b) 

Methods:

c) 

Methods:

3 Explain in your own words what area means.

(Continued on back.)

Follow-up Student Activity (cont.)

4 If possible, sketch and label these figures on Geoboard Recording Paper. If you think it's not possible write "impossible." (Let 1 small square be the area unit.)

a) Two shapes whose areas are the same, but their perimeters are different.

b) Two figures whose perimeters are the same but whose areas are different.

c) A rectangle whose area and perimeter are the same number.

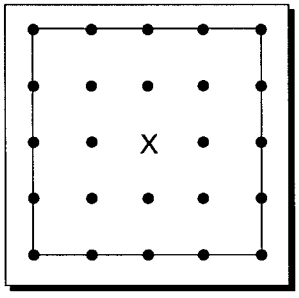
d) Two shapes that are not congruent but have the same area.

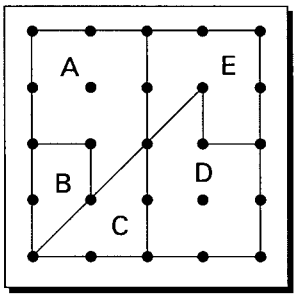
e) Two shapes that are congruent but have different areas.

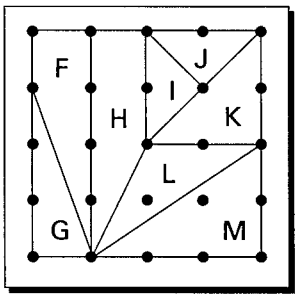
f) A shape that has 1 line of symmetry which divides the shape into 2 non-congruent parts.

g) A shape that has no line of symmetry but can be divided into 2 congruent parts by 1 line.

5 Suppose that the region marked "X" below is the area unit. Determine the area of each of the regions A-M.





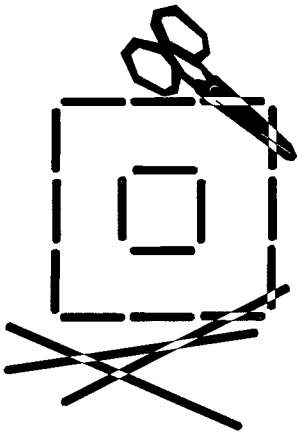


Diagrams & Sketches

Lesson 20

THE BIG IDEA

By first building or sketching models of important mathematical relationships within a situation, it is often possible to "see" solutions to questions about the situation. Involving students in such activities fosters the development of intuitions about the concept of a variable and provides important background to the study of algebra.



CONNECTOR

OVERVIEW

Students are introduced to the use of coffee stirrers as representations of lengths and numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Coffee stirrers, 15 per student.
- ✔ Scissors, 1 pair per student.
- ✔ Connector Master A, 1 transparency.

FOCUS

OVERVIEW

Given situations, students form models that represent relationships in the situations and then create and explore questions about the situations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Coffee stirrers, 15 per student.
- ✔ Tile, 30 per student.
- ✔ Cubes, 30 per student.
- ✔ Scissors, 1 pair per student.
- ✔ Focus Masters A and B, 1 transparency of each.

FOLLOW-UP

OVERVIEW

Students use diagrams and sketches to model relationships and solve puzzle problems.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 20.1, 1 copy per student.

LESSON IDEAS

ASSESSMENT
Make anecdotal notes or use an observation checklist while students work on problems. The degree to which students use models previously explored in class reveals important information about the degree to which they have begun to integrate these ideas into their own way of thinking.

TIMING
The Connector may take longer than usual but it is an important preliminary to the Focus activity.

Rather than assigning every situation on Focus Masters A and B, you could move on after spending 3-5 hours on this lesson. Then as you continue with the next lesson, assign 1 or 2 situations each day (e.g., as part of homework assignments).

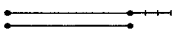
PURPOSE
To develop students' comfort using models and sketches to solve problems, emphasis in this lesson is on having students model relationships and then make mathematical observations and conclusions about their model. When emphasis is on getting an answer, many students use methods such as guessing, and then illustrate the answer with a diagram. While guessing may work to get answers, it generally doesn't lead to as rich or frequent mathematical insights as do visual representations of the relationships.

QUOTE
Teachers foster communication in mathematics by asking questions or posing problem situations that actively engage students, including situations that encourage students to create problems themselves.
NCTM Standards

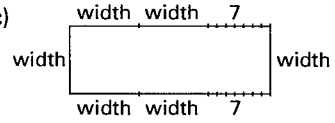
SELECTED ANSWERS

There are many possible visual representations.

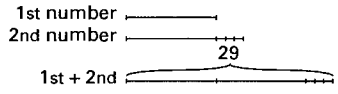
1. a)



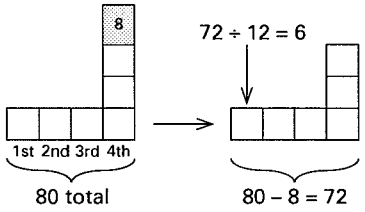
c)



d)

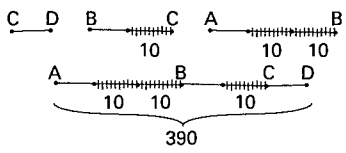


e)



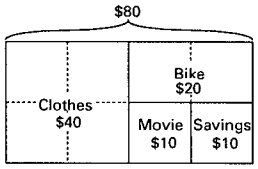
"The 1st, 2nd, 3rd numbers are 6; the 4th number is $6 \times 3 + 8 = 26$."

2. a)

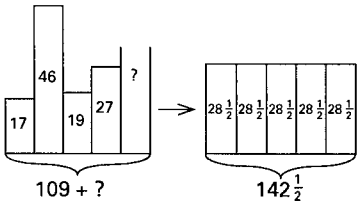


"Together, the three unmarked intervals represent 360 miles. So each unmarked interval represents 120 miles. The distance from A to B is 120 + 10 + 10 = 140 miles."

b)

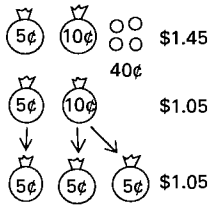


c)



"The difference between 142½ and 109 is 33½, the missing number."

d)



"Since each bag contains the same number of coins, there must be 35¢ or 7 coins in each bag."

Connector Teacher Activity

OVERVIEW & PURPOSE

Students are introduced to the use of coffee stirrers as representations of lengths and numbers. These experiences contribute to the development of students' intuitions about the concept of a variable and about the role of diagrams and models in solving problems.

MATERIALS

- ✔ Coffee stirrers, 15 per student.
- ✔ Scissors, 1 pair per student.
- ✔ Connector Master A, 1 transparency.

ACTIONS

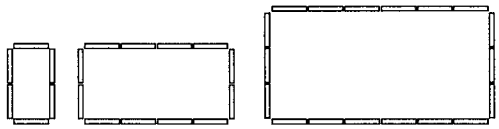
1 Arrange the students in pairs or small groups. Distribute coffee stirrers and scissors to each student. Ask the students to use the stirrers (cutting them if they wish) to form a rectangle that is twice as long as it is wide. Have volunteers form and describe their rectangles at the overhead.

2 Using 6 stirrers (or sections of stirrers) of equal length, form on the overhead the rectangle described in Action 1. Ask the students to determine the length of: the long side of the rectangle if the short side measures 100 linear units; the short side if the long side is 82 linear units; and the long side if the short side is 75 linear units. Discuss their methods. Ask the students to suggest other possibilities for the dimensions of the rectangle.

COMMENTS

1 Use 4-6" plastic coffee stirrers that are flattened on two sides (round stirrers or straws will work but they do roll easily). If stirrers are not available, cut narrow strips of cardstock or tagboard. Cut stirrers into half- or quarter-lengths for overhead demonstrations.

Some students may cut the stirrers into linear units and form specific rectangles such as those shown below. (Note that there may be disagreement about which dimension is width and which is length. There isn't a right or wrong way to view these.)



2 × 1 2 × 4 3 × 6

Other students may form the rectangle shown below without reference to specific dimensions. In this case, a stirrer (or section of one) could have *any* length and represents one dimension of the rectangle, while two stirrers (or sections) represent the other dimension. Hence, this rectangle is a representation of *any* rectangle whose length is double its width.

2 The purpose of this action is to emphasize the difference between representations of specific examples and generalized representations. All rectangles for which one dimension is double the other can be represented by this simple rectangle.

If the short side has length 100 linear units, the long side has length 2 × 100 = 200 linear units; if the long side is 84 units in length, the short side is 84 ÷ 2 = 42 units.

Connector Teacher Activity (cont.)

ACTIONS

3 Ask the groups of students to:

a) use the coffee stirrers to form a representation of a rectangle for which both dimensions are equal;

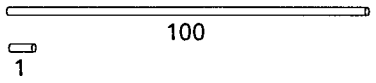
b) sketch a diagram of their representation; and

c) list the dimensions of at least three different rectangles that the diagram could represent.

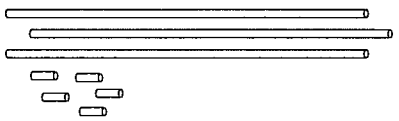
Have a volunteer sketch their results and give possible dimensions. Discuss.

4 Have the pairs of students form generic representations of 2 squares such that the length of a side of 1 square is 3 times the length of a side of the other. Ask them to make diagrams of their representations and then determine 5 different possibilities for the dimensions of the squares. Discuss.

5 Place a coffee stirrer on the overhead and suggest that it represents the number 100. Cut a small section of a stirrer and tell the students that it represents the number 1.



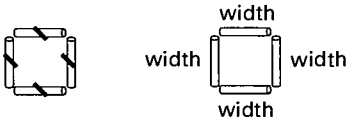
Place the following collection on the overhead and ask the students to tell what number it represents based on the lengths just given.



COMMENTS

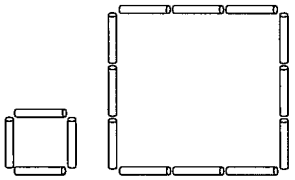
3 Students may need to be reminded not to form specific examples, but rather to form a model that could represent *any* rectangle with these qualities. In this case, the rectangle is a square.

If needed, you might suggest that a clear diagram would make it possible for an observer to build an exact copy of their model without asking questions about the model. Sometimes this requires labeling or marking their diagrams to show the relationships between the dimensions. For example:



The marks on the sides of the first square shown above suggest that the sides all have the same length.

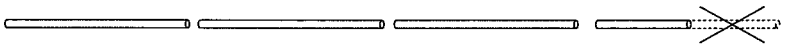
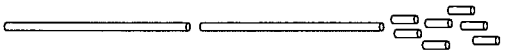
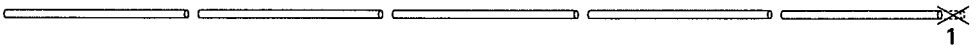
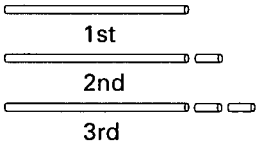
4 Diagrams should look like this:



There are many possibilities for dimensions of these squares: 3×3 and 9×9; 1×1 and 3×3; 200×200 and 600×600; $\frac{1}{3} \times \frac{1}{3}$ and 1×1; etc.

5 In order to represent large numbers like 100, it is impossible to build models that reflect the relative size of the 100 and the 1. Students are generally comfortable with this. If 1 stirrer represents 100, then the collection of 3 stirrers and 5 units represents 305.

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
6 Place a stirrer on the overhead. Suggest that it represents a *secret number*. Ask the students to build models of new numbers according to the conditions that are listed below and then to draw a diagram of each new number.	**6** Secrets are usually intriguing to student. Following are models of each new number.
a) This new number is equal to 3½ of the secret number.	a) 
b) Another number is equal to 2 of the secret number plus 7 units.	b) 
c) A third number is equal to 5 of the secret number minus 1 unit.	c) Encourage students to label their diagrams so it is clear one unit is missing. For example: 
Have volunteers come to the overhead and build or diagram each new number.	
7 Leave the models of a)-c) from Action 6 on the overhead. Tell the students that your secret number is 50. Ask for volunteers to tell the value of each of the new numbers modeled in a)-c) and to describe how they decided each value. Repeat for a secret number of 64, 100, etc.	**7** If the secret number is 50, then the new numbers have the following values: a) $(3 \times 50) + (50 \div 2) = 175$; b) $(2 \times 50) + 7 = 107$; c) $(5 \times 50) - 1 = 249$. Note that each of these numerical statements could be written without parentheses and the statements would still represent the same computations. That is, the computations that represent these models can be carried out using rules for order of operations without parentheses (see Lesson 4).
8 Place a transparency of Connector Master A on the overhead and ask the students to: form a model of the numbers in Situation a), draw a diagram of their model, and then list 5 possibilities for the numbers their model could represent. Have volunteers share their models at the overhead and discuss ways of deciding possibilities for the numbers.	**8** Students may need to be reminded of the meaning of consecutive counting numbers (counting numbers are the numbers 1, 2, 3, ... and consecutive means one number follows directly after the other). Here is one possibility:  These could represent: 8, 9, 10; 1001, 1002, 1003; 575, 576, 577; etc.

Diagrams and Sketches Lesson **20**

Connector Master A

Situations

a) Three consecutive counting numbers.

b) Two numbers whose difference is 8.

c) Three numbers. Two are equal and the third is 6 units more than the sum of the other 2.

d) A rectangle such that 1 dimension is 5 more than 3 times the other dimension.

e) A rectangle whose width is 2 less than its length.

Connector Teacher Activity (cont.)

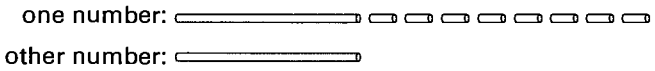
ACTIONS

9 Repeat Action 8 for one or more of situations b)-e).

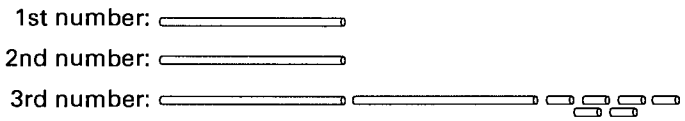
COMMENTS

9 Here is one possible model for each:

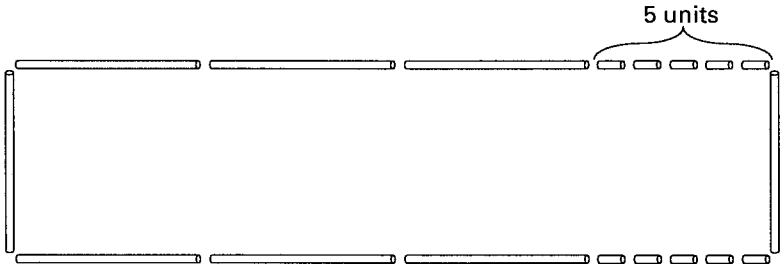
b) The difference model was first introduced in Lesson 2, *Basic Operations*.

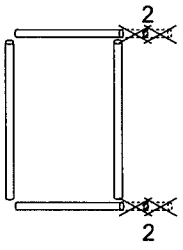


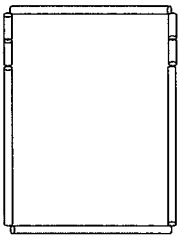
c)



d)



e) 

or 

If students bring up the use of variables, it is important to acknowledge their ideas. However, keep in mind that the intent of this lesson is not to get students to represent statements using algebra. Rather, the purpose is to get students to model information and relationships. In later *Visual Mathematics* courses, symbolic representations will be a natural outgrowth of these experiences.

Focus Teacher Activity

OVERVIEW & PURPOSE

Given situations, students form models that represent relationships in the situations and then create and explore questions about the situations. These experiences can help students to develop strategies and intuitions that will enable them, in subsequent courses, to bring meaning to the use of variables for solving problems.

MATERIALS

- ✔ Coffee stirrers, 15 per student.
- ✔ Tile, 30 per student.
- ✔ Cubes, 30 per student.
- ✔ Scissors, 1 pair per student.
- ✔ Focus Masters A and B, 1 transparency of each.

ACTIONS

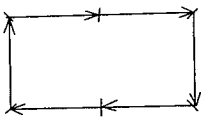
1 Place the students in pairs or groups and distribute cubes, tiles, coffee stirrers, and scissors to each group. Place a transparency of Focus Master A on the overhead and ask each student to build a model that represents the relationships described in Situation 1. Then have them draw a sketch of their model. Ask for volunteers to sketch and explain their models at the overhead.

Diagrams and Sketches I Lesson 20

Focus Master A

Situations

1. Each morning a math teacher runs a total distance of 42 city blocks and always chooses a rectangular running path. This morning her rectangular path was twice as long as it was wide.



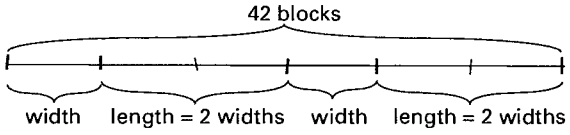
"The total length of the path is 42 blocks, or 6 arrows. The width is 1 arrow and the length is 2 arrows."

COMMENTS

1 To focus students' attention on Situation 1, conceal Situations 2 through 10 on the transparency. Alternatively, copies of Situation 1 can be cut from copies of Focus Master A and distributed to each student (or group of students). Avoid presenting students with the whole page at once.

Notice that no questions are posed, only a situation to model. This keeps emphasis on looking for relationships rather than getting an answer. Many students will, on their own, determine the length and width, perimeter, or area of the rectangle described in Situation 1. Students often on their own pose and explore more complex questions about a situation than we might expect.

Providing an assortment of materials for students to choose allows you to observe their comfort levels with various models and manipulatives and the extent to which they use models that relate to the situation. When students are ready, they will find they don't always need to use manipulatives before sketching. Rather, they will make diagrams of models they picture in their mind's eye. There is no need to rush this process, some students are more comfortable with sketching before others. Following are two examples of ways that students have sketched Situation 1:



"The 42 block path is the same length as 6 'widths' of the rectangle."

Some students may count out 42 tiles (or cubes) and be puzzled when they realize it isn't possible to use them to form a rectangle for which one dimension is double the other. Using 42 tile to represent the perimeter of a

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

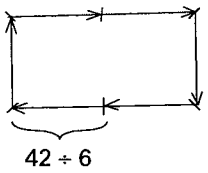
2 Ask the groups to suggest questions they could answer or conclusions they can make based on mathematical relationships they can "see" in their models of Situation 1. Discuss their ideas and pick one or two to explore.

COMMENTS

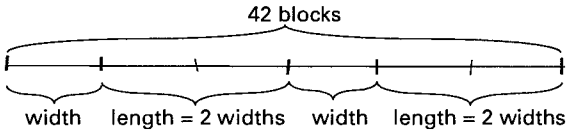
1 (continued.)
rectangle contributes an extra unit of length at each corner. Rather than warning students about this, it is important to let them wrestle with the issues brought about by using a unit of area to represent a unit of length. If a student gets stuck, provide encouragement by asking questions like: "Do you have any ideas about why this model isn't working? What is most difficult for you? What is easiest? If this idea isn't working, could you try another method and come back to this later? What approaches are other students using? Would you be willing to share where you are stuck with the class and ask for their insights?"

2 There are many questions and observations that may come up, although "what are the length and width of today's rectangular path?" is most common. If it doesn't come up, have the students use their models or sketches to determine the length and width of the rectangle in Situation 1.

Using the first example illustrated in Comment 1, the rectangular path is viewed as 6 consecutive arrows (or coffee stirrers) of the same length, so each of the 6 arrows represents a distance of $42 \div 6 = 7$ blocks. The width is 1 arrow or 7 blocks and the length is 2 arrows or 14 blocks, as shown below.



In the second example in Comment 1 (shown again below), the rectangular path is "unfolded" to form a linear path 42 blocks long. This path is the same length as 6 "widths" of the rectangle, so the width is $42 \div 6 = 7$ blocks and the length is 2 widths or 14 blocks.



Explore other questions (e.g., What is the area of the rectangular region enclosed by the path?) that students suggest.

Focus Teacher Activity (cont.)

ACTIONS

3 Repeat Actions 1 and 2 for selected other situations on Focus Master A.

Diagrams and Sketches Lesson **20**

Focus Master A

Situations

1. Each morning a math teacher runs a total distance of 42 city blocks and always chooses a rectangular running path. This morning her rectangular path was twice as long as it was wide.

2. The tall girl looked over the fence into a barnyard containing geese and sheep and announced that there were 5 heads altogether. Her little brother looked under the fence and said he saw 16 legs.

3. When 2 pieces of rope are placed end-to-end they measure 40 meters in length. When the pieces are laid side-by-side one is 10 meters longer than the other.

4. A snail at the bottom of a well 7 feet deep started climbing to the top. Each day it would climb up 3 feet but at night, when sleeping, it would slide down 2 feet.

5. The length of a rectangle is 6 inches longer than the width. The perimeter of the rectangle is 40 inches.

6. Of the girls on the basketball team last fall, $\frac{1}{2}$ also played soccer, $\frac{1}{4}$ also played volleyball, and $\frac{1}{8}$ were also in choir. The remaining 3 girls weren't in other activities. No one was in more than 2 activities.

7. The sum of 2 numbers is 40. Their difference is 14.

8. The average of 4 numbers is 7. The second number is twice the first. The third number is 6 more than the second. The fourth number is 2.

9. Each student in Ms. White's class wrote their favorite color on a slip of paper. Four students wrote green, 8 wrote red, 2 said yellow, 3 said orange, 6 wrote purple, and 1 said blue. To decide what color to paint the classroom door, Ms. White placed these slips in a bag and drew 1 at random.

10. Three consecutive odd numbers have a sum of 51.

COMMENTS

3 Select situations according to the needs and comfort levels of students. Some situations could be saved to include on homework assignments throughout the year.

These sketching activities take time at the beginning. Encourage students who have difficulty to talk their ideas over with groupmates, and to share their "thinking in progress" with the class when they are stuck. It is important that students view ideas that don't work as valuable and productive. Often great insights are prompted by such ideas.

Classroom presentations of drawings that lead to visual solutions will help students generate ideas for other visual solutions. It is beneficial to have several students present visual solutions to the same question. Rarely do two individuals create the same diagram for the same reasons. Even though one student may say that another student has given their solution, the reasoning is likely to be different.

Following are sample sketches of the relationships described in the remaining situations on Focus Master A, along with one sample math question whose solution can be determined by using the diagram. To motivate discussion you may wish to pose specific math questions about some situations. Keep emphasis on *using the models to answer the questions.*

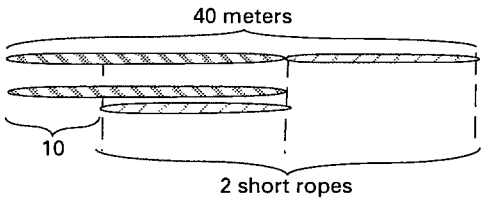
2: How many geese and how many sheep are there?

"If all had two legs:

This is a total of 10 legs, so 6 more are needed:

There are 3 sheep and 2 ducks."

3: How long is each piece of rope?



"Twice the shorter rope is 40 – 10 = 30 meters. Hence the shorter rope is 15 meters. The longer rope is 15 + 10 = 25 meters."

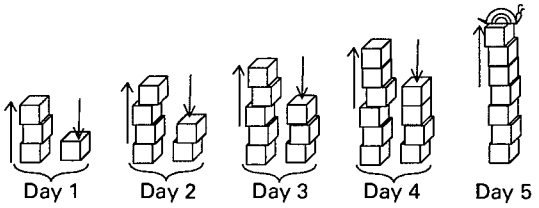
(Continued next page.)

Focus Teacher Activity (cont.)

COMMENTS

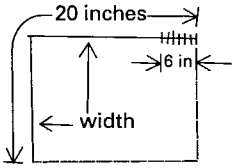
3 (continued.)

4: How many days will it take the snail to reach the top?



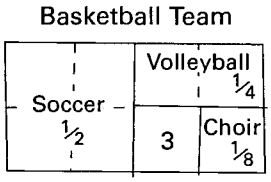
"The snail will reach the top in 5 days."

5: What are the dimensions of the rectangle?



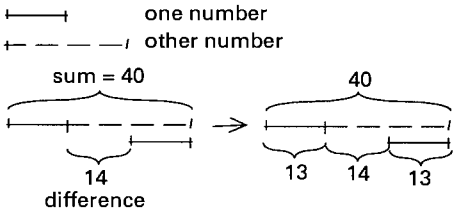
"Twice the width is 14 inches. So width = 7 inches, length = 7 + 6 inches."

6: How many girls were on the basketball team?



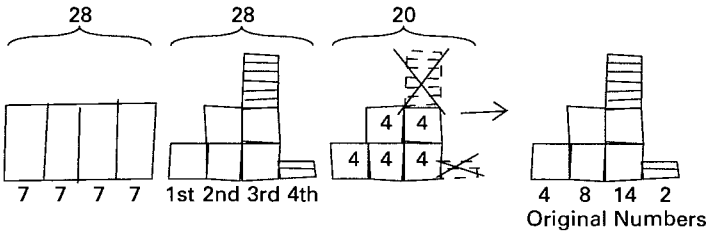
"The total number of basketball players is 3 × 8 = 24."

7: What are the two numbers?



"One number is 13. The other number = 14 + 13 = 27."

8: What are the four numbers?

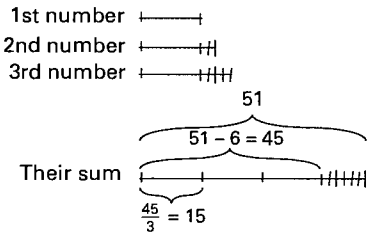


9: What is the probability (theoretical) the color will be either green or red?

G	G	R	R	Y	P	P	P
G	R	R	R	Y	O	P	P
G	R	R	R	O	O	P	B

"This 3 × 8 rectangle represents all possible choices. Since ½ of the area is either green or red, the probability of choosing either red or green is ½."

10: What are the numbers?



"The numbers are 15, 17, and 19."

Focus Teacher Activity (cont.)

ACTIONS

4 Place a transparency of Focus Master B on the overhead. Ask the students to build or sketch a model which represents the relationships in Situation 1. Have several volunteers illustrate their models at the overhead. Have others point out conclusions they can make based on mathematical relationships they can "see" in the models.

Lesson **20** Diagrams and Sketches

Focus Master B

More Situations

1. Max cut a piece of string 43 inches long into two parts so that one part is 4 inches longer than twice the other part.

2. There are 29 more girls than boys in a school. There are 533 students in the school.

3. There are 3 secret numbers. The first number is twice the second number. The third number is twice the first number. Their sum is 112.

4. The perimeter of a rectangle is 216 feet. The length is twice the width.

5. In a school election, Mary received 160 more votes than Jim, and Jim received twice as many as Allen. A total of 440 votes were cast.

6. A certain triangle has perimeter 43 inches. The second side of the triangle is 4 inches longer than the first side, and the third side is 5 inches longer than the second.

7. A rectangle is 3 times as long as it is wide. If the length of the rectangle is decreased by 1 linear unit, the area of the rectangle is decreased by 5.

8. The average of 5 numbers is 56. Four of the numbers are the same. The average of one of the 4 numbers and the fifth number is 80.

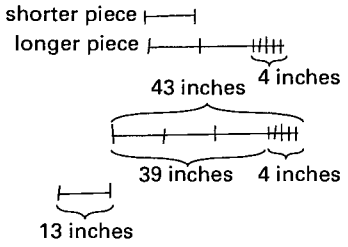
9. The perimeter of Square A is 24 linear units. The area of Square A is 4 times the area of Square B.

10. One student has only nickels in her hand and a second student has exactly the same number of dimes and no other coins. Together they have a total of $1.05.

5 Repeat Action 4 for one or more of the remaining situations on Focus Master B.

COMMENTS

4 Again, an alternative is to give each student or group of students a slip of paper with Situation 1. Some students may want to use coffee stirrers to model the situation and then sketch their model. Encourage this. In the visual model shown here, the shorter piece is represented by a line segment. If the shorter piece is doubled and increased by 4 inches, the longer piece is obtained. If the two pieces are placed end-to-end, the resulting length is 43 inches.



One observation might be that by subtracting 4 from 43, one sees that the total length of the 3 remaining segments is 39 inches long. So the shorter piece is 39 ÷ 3 = 13 inches long and the length of the longer piece is 13 inches + 13 inches + 4 inches = 30 inches.

5 Assigning these one at a time and sharing and discussing solutions in class is a productive and enjoyable method for learning to solve problems visually. Some students may wish to build a concrete representation before sketching. This is appropriate; however, when problems involve large numbers, a sketch is often easier to use. As you circulate while students work, ask for volunteers to record their sketches (or illustrations of their models) on a blank transparency and to share these with the class. To motivate discussion, you may wish to pose some specific math questions about some situations.

If students have difficulty with these, you may wish to simplify numbers or assign these situations one a day over several days while also moving on to other lessons.

(Continued next page.)

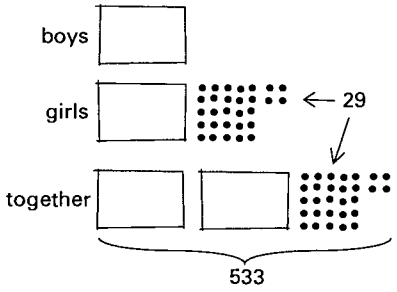
Focus Teacher Activity (cont.)

COMMENTS

5 (continued.)
A natural next step in answering mathematical questions about situations is to use algebraic symbols. However, students should first have extensive experiences at the visual level. A good source of more situations is any traditional first year algebra text, but avoid the use of variables now, unless a student initiates a discussion.

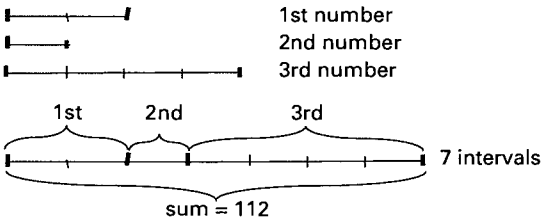
Following are sample sketches leading to visual solutions to questions about the remaining situations on Focus Master B.

2:



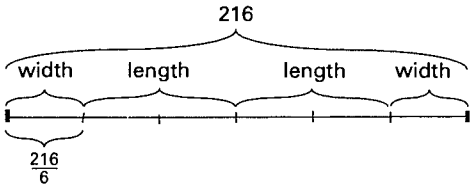
"The two boxes represent 533 – 29 = 504 students, so each represents 252 students. Number of boys = 252; number of girsls = 252 + 29 = 281."

3:



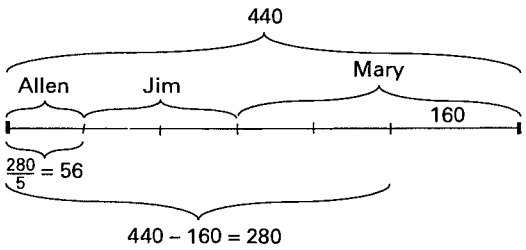
"Each interval represents 112 ÷ 7 = 16. The numbers are 32, 16, and 64."

4:



"Unfold the perimeter into one long piece made up of 6 widths, each measuring $^{216}/_{6}$ = 36 feet. The length is 2 widths or 72 feet."

5:

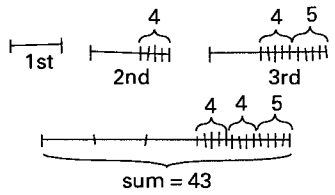


"One of the unmarked intervals represents the total votes for Allen. After subtracting 160 there are 5 equal intervals of $^{280}/_{5}$ = 56. Jim's votes total 2 × 56 = 112, and Mary's votes total (2 × 56) + 160 = 272."

Focus Teacher Activity (cont.)

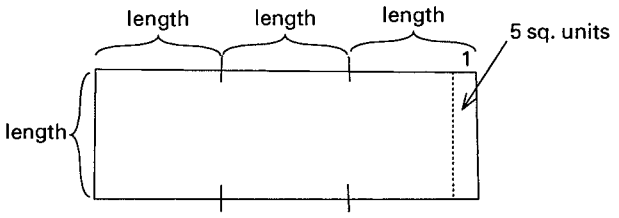
COMMENTS

6:



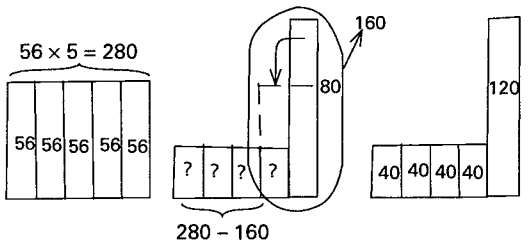
"The length of the first side is (43 – 13) ÷ 3 = 10."

7:



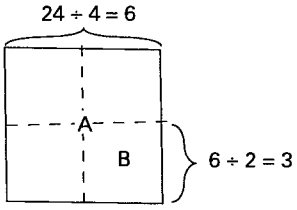
"Since reducing the length by 1 linear unit reduces the area by 5 square units, the 'slice' of area removed is a 1 by 5 rectangle. Hence the original rectangle had width 5 and length 15."

8:



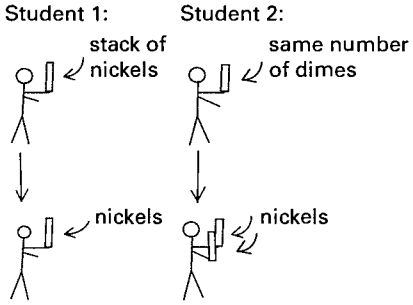
"The 5 numbers average 56, so their total is 5 × 56 = 280. Since the average of one of the 4 equal numbers and the fifth number is 80 those 2 numbers total 160. Removing them leaves 3 equal numbers that total 280 – 160 = 120, so each of these 3 numbers is 120 ÷ 3 = 40. Therefore, the 4 equal numbers are each 40 and the fifth number is 280 – (4 × 40) = 120."

9:



"Since the perimeter of Square A is 24, each side has length 24 ÷ 4 = 6. Subdivide Square A into 4 equal Square B's. Each side of Square B has length 6 ÷ 2 = 3, so the perimeter of Square B is 4 × 3 = 12 linear units."

10:



"The value of each stack of nickels = $1.05 ÷ 3 = 35¢. So, there are 7 coins in each stack."

TEACHER NOTES:

Follow-up Student Activity 20.1

NAME _____ **DATE** _____

1 Draw diagrams that illustrate the relationships described below. Besides the information that is given, what are some mathematical conclusions you can make by studying your diagram?

a) The difference between 2 secret numbers is 3.

b) There are 3 secret numbers. The second number is equal to the sum of 2 of the first number. The third number is 5 more than the second number.

c) The length of a rectangle is 7 units more than 2 times its width.

d) The difference between 2 secret numbers is 3. The sum of the 2 numbers is 29.

e) The average of 4 secret numbers is 20. Three of the numbers are equal to each other and the fourth number is 8 more than the sum of the other 3 numbers.

(Continued on back.)

Follow-up Student Activity (cont.)

2 On a separate sheet of paper, show how to use a diagram or model to solve each of these puzzle problems.

a) To drive from town A to town D, one must drive through town B and then town C. It is 10 miles farther from town A to town B than it is from town B to town C. And, it is 10 miles farther from town B to town C than it is from town C to town D. It is 390 miles from town A to town D. How far is it from town A to town B?

b) Al earned $80 baby-sitting last month. He spent 1/2 of his earnings on clothes, 1/4 on repairs for his bike, and 1/8 on movie tickets. He put the rest in savings. How much did he save last month?

c) The average of a set of 5 numbers is 28 1/2. Four of the numbers are 17, 46, 19, and 27. What is the fifth number?

d) Bob has 4 more dimes than nickels. He has $1.45 in all. How many nickels does he have?

e) Find the 2 secret numbers. The second number is equal to 2 1/2 of the first number. Their sum is 84.

f) A rectangle has perimeter 60 inches. The length of the rectangle is 2 inches more than 3 times its width. What are the dimensions of the rectangle?

g) The average of 5 numbers is 11. The second number is equal to the first. The third number is double the first. The fourth number is equal to the first 3 numbers added together. The fifth number is 7. What are the other 4 numbers?

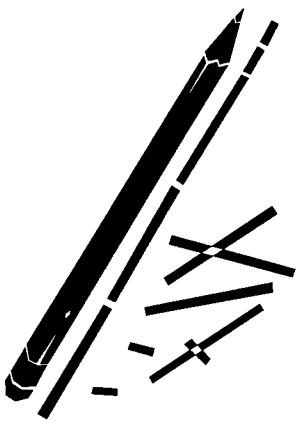
h) A triangle has perimeter 35 inches. The length of side A of the triangle is twice the length of side B. Side C is 5 inches longer than side A. What is the length of each side of the triangle?

Linear Measure

Lesson 21

THE BIG IDEA

Base five provides an unfamiliar setting for examining the concept of linear measure and relating it to area measure. It is important that students explore concepts in settings where it isn't possible to rely on memorized rules, but rather where they are called upon to understand relationships. These activities enable students to recognize misconceptions and resolve uncertainties they have about linear and area measure.



CONNECTOR

OVERVIEW

Students are introduced to base five linear pieces and use them to measure the lengths of objects.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area pieces for each student.
- ✔ Base five linear pieces, 1 chain-chain, 6 chains, and 6 linear units for each student.
- ✔ Base five area and linear pieces for the overhead.

FOCUS

OVERVIEW

Students measure and record the lengths of line segments using base five notation.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area and linear pieces for each student.
- ✔ Focus Student Activity 21.1, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 21.2, 1 copy per student and 1 transparency.
- ✔ Base five area and linear pieces for the overhead.
- ✔ Base five measuring tape (optional, see *Blackline Masters*), 1 per pair of students.
- ✔ Tape (optional), 1 roll per pair of students.
- ✔ Scissors (optional), 1 per pair of students.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 1 sheet per student.

FOLLOW-UP

OVERVIEW

Focus activities with measuring lengths in base five are extended to measuring in base four.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 21.3, 1 copy per student.

LESSON IDEAS

TIMING
This lesson can usually be completed in 1-2 hours.

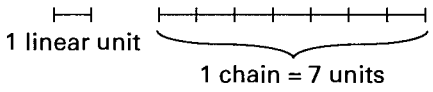
LOOKING AHEAD
In Lesson 22, students measure lengths and areas in base five and examine relationships between the concepts of area and length. Lessons 21 and 22 provide important background for measurement in base ten and for understanding the concepts of multiplication and division.

TEACHER NOTES:

SELECTED ANSWERS

1. a) 111_{four} by 130_{four}

3. A base seven linear unit and chain:



4. Base six.

5. a) One way to describe this length would be 0 linear units, 2 "fourths" linear unit, and 1 "sixteenth" linear unit.

   b) One linear unit, 1 "fourth" linear unit, 2 "sixteenths" linear unit, and 3 "sixty-fourths" linear unit.

Connector Teacher Activity

OVERVIEW & PURPOSE

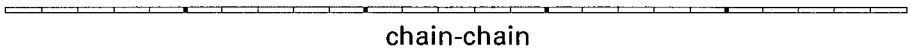
Students are introduced to base five linear pieces and use them to measure the lengths of objects. These linear pieces provide a model for the concept of linear measurement.

MATERIALS

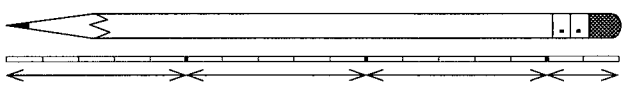
- ✔ Base five area pieces for each student.
- ✔ Base five linear pieces, 1 chain-chain, 6 chains, and 6 linear units for each student.
- ✔ Base five area and linear pieces for the overhead.

ACTIONS

1 Distribute base five area and linear pieces to each student. Ask them to make observations about relationships among the pieces and to predict what larger linear pieces would look like.



2 Ask the students to use their linear pieces to measure and record the lengths of their pens or pencils and the length and width of a standard sheet of paper. Ask volunteers to demonstrate by placing pencils or sheets of paper on the overhead and using transparent base five linear pieces. Discuss.



17 units or 3 chains and 2 units

COMMENTS

1 The linear pieces are introduced to indicate the lengths of edges of base five area pieces, as follows: 1 linear unit is the length of the edge of 1 square unit; 1 *chain* is the length of the long edge of a strip, i.e., 5 linear units; a *chain-chain* (or chain of chains) is the length of a group of 5 chains joined together end-to-end, or 25 linear units.

To make linear units and chains, you can cut base five square units and strips in thirds. It is helpful to use cardstock that is a different color than the area pieces, so the linear pieces are easy to locate (colored plastic base five linear pieces are available from The Math Learning Center). Cut chain-chains from the long edge of a strip-mat. The next longer measure is the length of a group of 5 chain-chains placed end-to-end and is called a *chain-chain-chain*. It equals 125 linear units. This process of grouping by fives can be continued indefinitely.

Base five area pieces are two dimensional and they form either a square or a rectangle. Base five linear pieces are a one dimensional model and they represent lengths. It is important that students view linear pieces as one dimensional. While these pieces do have area (because of how they are cut), they are intended to model only lengths of edges of area pieces.

2 Lengths can be recorded in terms of units only, or in terms of chain-chains, chains, and units. Base five notation for recording lengths is discussed in the Focus. Rounding to the nearest whole unit will come up and should be discussed.

If there is no ambiguity, a linear unit may be referred to simply as a unit. However, in some contexts in this activity, the word "unit" may refer to a square unit. Hence, when the word "unit" is used, it is important to understand whether the word is referring to a square unit or a linear unit.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

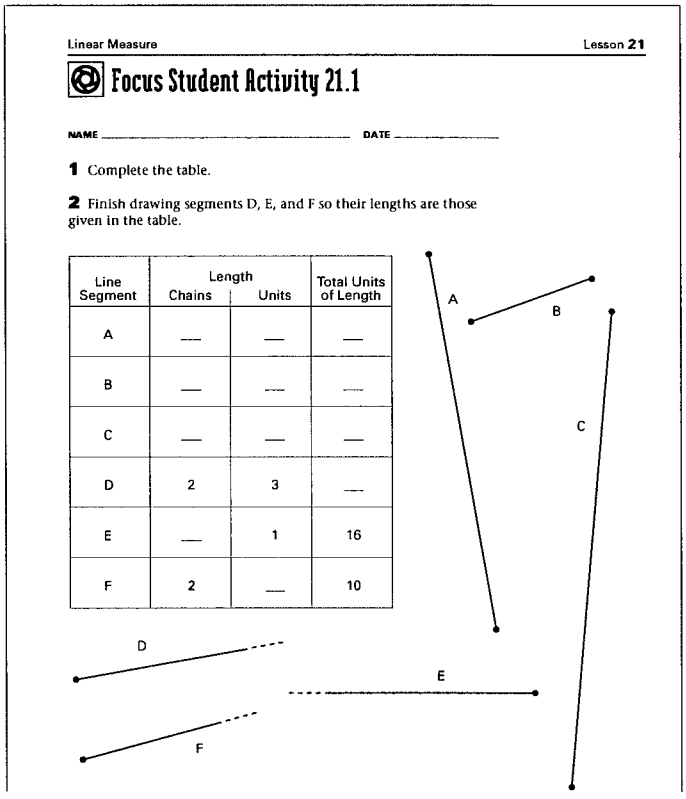
Students measure and record the lengths of line segments using base five notation. This provides an unfamiliar setting for examining the concept of linear measure.

MATERIALS

- ✔ Base five area and linear pieces for each student.
- ✔ Focus Student Activity 21.1, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 21.2, 1 copy per student and 1 transparency.
- ✔ Base five area and linear pieces for the overhead.
- ✔ Base five measuring tape (optional, see *Blackline Masters*), 1 per pair of students.
- ✔ Tape (optional), 1 roll per pair of students.
- ✔ Scissors (optional), 1 per pair of students.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 1 sheet per student.

ACTIONS

1 Arrange the students in groups and distribute area and linear pieces and a copy of Focus Student Activity 21.1 to each student. Ask the students to use their linear pieces to measure segments A-C and to finish drawing segments D-F. Have them complete the table. Discuss the students' results and questions.



Linear Measure Lesson 21

Focus Student Activity 21.1

NAME _____ DATE _____

1 Complete the table.

2 Finish drawing segments D, E, and F so their lengths are those given in the table.

Line Segment	Length: Chains	Length: Units	Total Units of Length
A	—	—	—
B	—	—	—
C	—	—	—
D	2	3	—
E	—	1	16
F	2	—	10

COMMENTS

1 You may wish to work through the first line of the table with the whole class. If this is done on the overhead, a transparency of a chain-chain will allow students to see chain and unit subdivisions.

The chart provides enough information for students to finish drawing segment D, E, and F. Following is the completed chart from Student Activity 21.1:

Line Segment	Length: Chains	Length: Units	Total Units of Length
A	2	2	12
B	0	4	4
C	3	0	15
D	2	3	13
E	3	1	16
F	2	0	10

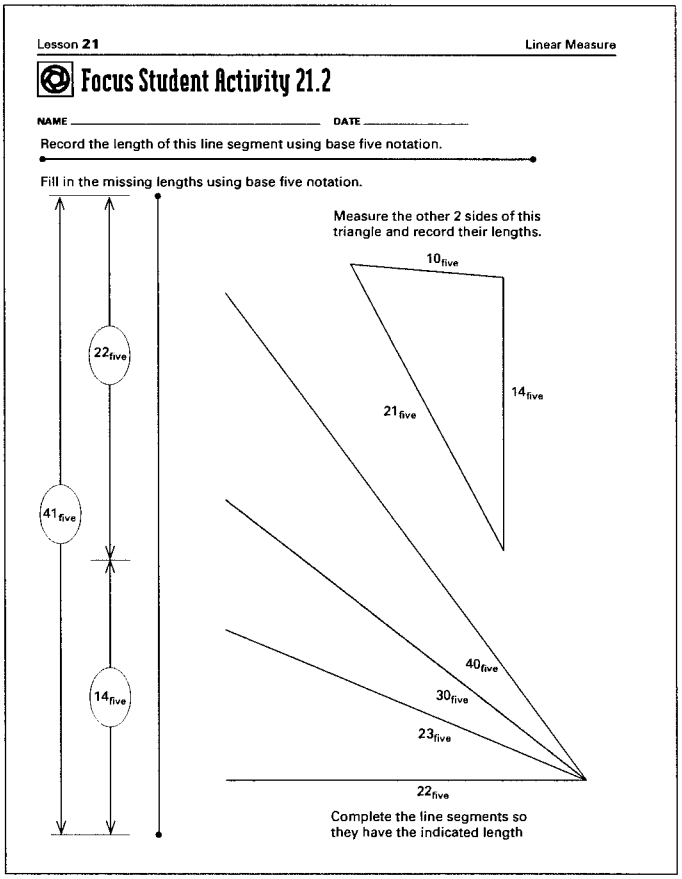
Focus Teacher Activity (cont.)

ACTIONS

2 Have the groups form a length of 2 chain-chains, 7 chains, and 6 units. Ask them to determine the total number of linear units and the minimum number of linear pieces needed to form this length. Discuss with the students a system of measuring lengths using groups of five and discuss how base five notation may be used to record lengths.

3 Ask the students to record the lengths of the line segments on Focus Student Activity 21.1 using base five notation. Discuss.

4 Discuss with the students ways of indicating the length of a line segment. Then distribute one copy of Focus Student Activity 21.2 (a copy with answers is shown below) to each student and ask the students to use their linear pieces to complete the activities on the sheet. Have volunteers share their measurements.


Lesson 21 Linear Measure

Focus Student Activity 21.2

NAME _____ DATE _____

Record the length of this line segment using base five notation.

Fill in the missing lengths using base five notation.

22_{five}

41_{five}

14_{five}

Measure the other 2 sides of this triangle and record their lengths.

10_{five}

14_{five}

21_{five}

40_{five}

30_{five}

23_{five}

22_{five}

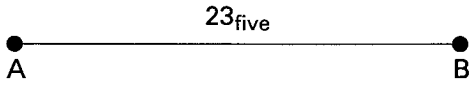
Complete the line segments so they have the indicated length

COMMENTS

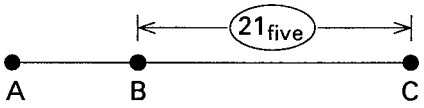
2 The total number of units in 2 chain-chains, 7 chains, and 6 units is 91. The minimum number of linear pieces to represent 91 units is 3 chain-chains, 3 chains, and 1 unit. In base five notation, this is written as 331_{five} units.

3 The length of segment A is 2 chains and 2 units or 22_{five}. The lengths of line segments B through F, respectively, are 4_{five}, 30_{five}, 23_{five}, 31_{five}, and 20_{five}. Notice that it is necessary to use the information in the chart.

4 Sometimes the length of a line segment is simply written alongside the segment.



If there is danger of confusion, the length can be written between two arrows showing the extent of the segment.



In this situation, arrows are used to indicate that 21_{five} is the length of segment BC, and not the length of the entire segment AC.

The completed sheet should resemble the one at left. Note that if the line segments are completed correctly, their endpoints lie on a line.

Focus Teacher Activity (cont.)

ACTIONS

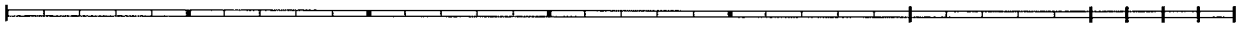
5 Draw several line segments on the chalkboard or on a sheet of butcher paper. Ask for volunteers to use base five linear pieces to determine the total number of linear units in each segment and to determine the base five numerical representation of each.

6 (Optional) Ask students to work with a partner to construct a base five measuring tape and use it to measure and record the lengths of various items in the classroom.

7 Ask the students to predict the number of linear units in a base eight chain and in a base eight chain-chain. Discuss their reasoning. Have students draw (or cut out from grid paper) a line segment whose length is 23_{eight} and determine the number of linear units in the segment. Repeat for base three, base twelve, and base ten.

COMMENTS

5 Draw segments that are at least one chain-chain in length. The length of the line segment shown, written in base five notation, is 114_{five}, indicating the segment is 1 chain-chain, 1 chain, and 4 units long. This has length $(1 \times 25) + (1 \times 5) + 4$ or 34 linear units.



Similarly, a segment of length 302_{five} is 3 chain-chains, 0 chains and 2 units long, which totals $(3 \times 25) + (0 \times 5) + 2 = 77$ linear units. A segment of length 2000_{five} is 2 chain-chain-chains long which equals 2×125 or 250 linear units.

6 A pattern for constructing a base five measuring tape is in *Blackline Masters*. To construct a tape, one needs a copy of this pattern, scissors, and scotch tape. The length of the tape is 1 chain-chain-chain (1000_{five}), which is 125 linear units. To save time prepare measuring tapes prior to class.

Asking students to measure and record lengths of large objects, such as the length of a wall or blackboard, helps to clarify their understanding of place value and notation.

7 A base eight chain contains 8 linear units. A base eight chain-chain contains 8 chains which totals 64 linear units. A segment of length 23_{eight} is 2 (base eight) chains and 3 units or 19 total units long as shown below.

A base ten chain is 10 units long and a base ten chain-chain is 100 units. If the unit length is 1 centimeter, the length of a base ten chain is 1 decimeter and the length of a base ten chain-chain is 1 meter.

TEACHER NOTES:

Follow-up Student Activity 21.3

NAME _____ **DATE** _____

1 Cut out the base four ruler on the bottom of this sheet. Use it to measure the dimensions of the items listed below to the nearest linear unit. Then use base four notation to record the dimensions of each item.

a) this sheet of paper

b) the front cover of a book

c) an envelope

d) another item of your choice: _____

2 Use the base four ruler to help draw line segments whose lengths are given below. Draw the segments in the space provided.

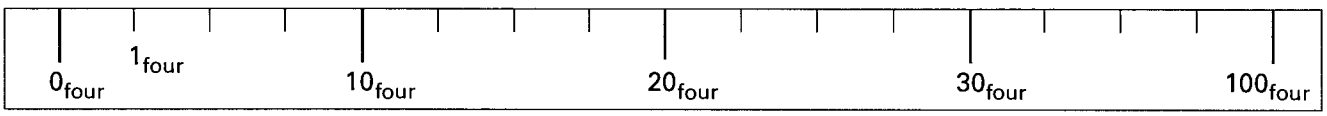
a) 19 linear units

b) 12 linear units

c) 12_{four} linear units

d) 23_{four} linear units

e) 31_{four} linear units



Base four ruler to cut out.

(Continued on back.)

Follow-up Student Activity (cont.)

3 Draw what you think a base seven linear unit and chain look like. Explain how you would draw the length 235_{seven}.

4 For this problem, one linear unit looks like this: ⊢——⊣

The segment below has length $13_{?}$.

⊢————————————————⊣

What do you think is the base in this example? Explain how you can be sure.

5 Jorge changed the linear unit on the base four ruler from Problem 2 to the following.

⊢⟵ 1_{four} linear unit ⟶⊣

a) Jorge drew the following segment and said its length is $0{,}21_{four}$ linear units (based on the above unit). Explain what you think he meant by this measurement.

⊢————————————⊣

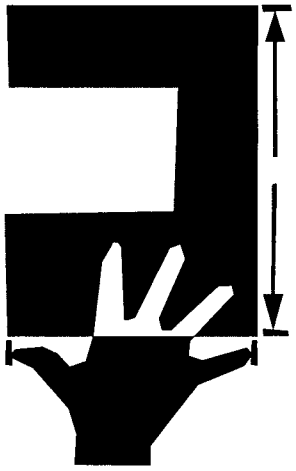
b) Explain how you think Jorge would draw the length $1{,}123_{four}$ linear units (using the above unit).

Dimension & Area I

Lesson 22

THE BIG IDEA

Measuring length and area in base five helps students focus on the similarities and differences of these measures. Such experiences provide important background for understanding the concepts of multiplication and division and relationships between these operations.



CONNECTOR

OVERVIEW

Students recall relationships between the area and dimensions of a rectangle and identify the area and dimensions of one rectangle using several different bases. They have the opportunity to explore historical units of measure.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency
- ✔ Base five area and linear pieces (optional) for each student.
- ✔ Connector Student Activity 22.1 (optional), 1 copy per student and 1 transparency.

FOCUS

OVERVIEW

Base five area and linear pieces are used to measure the dimensions and areas of rectangles.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area and linear pieces for each student.
- ✔ Focus Student Activity 22.2, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 5 sheets per group.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students measure and determine the dimensions and areas of rectangles using several bases.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 22.3, 1 copy per student.
- ✔ Base five area and linear pieces, 1 paper or cardstock set for each student (see *Blackline Masters*).

LESSON IDEAS

LOOKING AHEAD

Because students tend to have difficulties distinguishing between linear measure and area measure, these topics are explored in many contexts in *Visual Mathematics, Course I.* Some of these contexts are: the area model for whole number, fraction, and decimal multiplication and division; area and perimeter of plane figures; and metric measurement.

QUOTE

If students' initial explorations use nonstandard units, they will develop some understandings about units and come to recognize the necessity of standard units in order to communicate.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

1. a-b) 22_{five} by 13_{five}, 11_{five} by 31_{five}, 1_{five} by 341_{five}, 2_{five} by 143_{five}, 3_{five} by 112_{five}, or 4_{five} by 44_{five}

   c) 441_{five}

   d) 303_{five}

   e) 12_{five}

   f) 3 chains, 1 linear unit, and $\frac{2}{5}$ linear unit. Students may record this in several ways. For example: $31\frac{2}{5}_{five}$ (i.e., 3 chains, 1 linear unit, and $\frac{2}{5}$ linear unit) = $31\frac{2}{10}_{five}$, (i.e., $\frac{1}{10}$ is read "one over one, zero base five" and represents one part of a linear unit that is divided into five equal parts, so $\frac{2}{10}$ is 2 of those parts), or $31{,}2_{five}$ (where the comma separates units from "fifths" of units).

2. Both are units for measuring. An area unit is usually square and a linear unit is usually the length of the side of a square area unit. Linear units are used to measure lengths and dimensions of regions, and area units are used to measure the amount covered by a region.

4. a) Dimensions, 12_{five} by 31_{five}
   Area, 422_{five}

   b) Dimensions, $1\frac{2}{5}$ linear units by $3\frac{1}{5}$ linear units. Students may record this in one of several ways, such as: $1\frac{2}{10}_{five}$ by $3\frac{1}{10}_{five}$ (where 10_{five} in the denominator represents "one, zero base five," not "tens"); or $1{,}2_{five}$ by $3{,}1_{five}$ (where the commas separate units from "fifths" of units).

   Area, 4 square units, $\frac{2}{5}$ unit, and $\frac{2}{25}$ unit. (Note that students may record this area in several ways, such as: $4{,}22_{five}$ square units; 4 square units, 2 strip-lets, and 2 square-lets; 4 base five area units and $\frac{12}{25}$ of an area unit; $4\frac{22}{100}_{five}$. Acknowledge all methods that indicate correct areas.)

   c) Dimensions, 13_{four} by 100_{four}
   Area, 1300_{four}

5. a) This segment is two-fourths plus one-sixteenth of 1 linear unit.

Connector Teacher Activity

OVERVIEW & PURPOSE

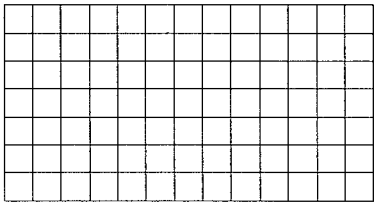
Students recall relationships between the area and dimensions of a rectangle. They identify the area and dimensions of one rectangle using several bases, thus reinforcing distinctions between linear and area measure, while strengthening their understanding of place value and numeration. They have the opportunity to explore historical units of measure.

MATERIALS

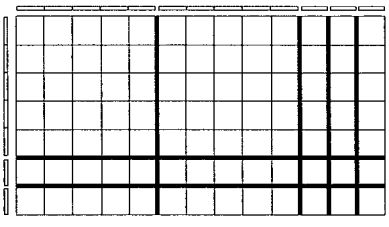
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ Base five area and linear pieces (optional) for each student.
- ✔ Connector Student Activity 22.1 (optional), 1 copy per student and 1 transparency.

ACTIONS

1 Give each group of students one sheet of 1-cm grid paper. Outline the rectangle shown below on a transparency of 1-cm grid paper and ask the groups to do the same on their grids. Ask groups to make several observations about the rectangle. Discuss.



2 If it hasn't come up, make the observation that in base ten the above rectangle has dimensions 7 linear units by 13 linear units and area 91 square units. Ask the groups to select 3 other bases and to determine the area and dimensions of the rectangle in each of those bases. Have volunteers demonstrate their group's methods at the overhead.



COMMENTS

1 Some possible observations include: the rectangle has dimensions 7 linear units by 13 linear units; its area is 91 square units; its perimeter is 40 linear units; it is a model for $7 \times 13 = 91$, $91 \div 7 = 13$, and $91 \div 13 = 7$ (see Lesson 2).

2 The area of the rectangle is the number of square units it contains. Its dimensions are the lengths of its different sides. In base five, the rectangle is comprised of 2 mats, 7 strips, and 6 units. A minimal collection which totals the same number of area units contains 3 mats, 3 strips, and 1 unit. Hence, the area of the rectangle is 331_{five} square units. The length of the longest dimension is 2 chains and 3 units or 23_{five} linear units. The other dimension is 1 chain and 2 units or 12_{five} linear units. This is illustrated at the left. If students have difficulty with this, distribute base five area and linear pieces which can be laid on top of the grid.

Encourage students to cut out other base pieces to cover their rectangles. For example, in base three the dimensions are 21_{three} by 111_{three}, and its area is 10101_{three}. In base twelve its dimensions are 7_{twelve} by 11_{twelve} and its area is 77_{twelve}.

Connector Teacher Activity (cont.)

ACTIONS

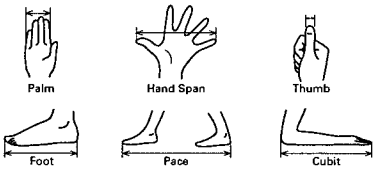
3 (Optional) Discuss with students some historical background of linear measurement. Give each student a copy of Connector Student Activity 22.1 to complete. Discuss.

Dimension and Area I — Lesson **22**

Connector Student Activity 22.1

NAME _____ DATE _____

Shown below are several very old units of linear measure.



1 Select 6 objects to measure using your body measures for the above units. Record your measures on the chart below.

Object	Unit Length	Measurement
	Hand span	
	Palm	
	Thumb	
	Cubit	
	Pace	
	Foot	

2 Tell which of the above historical units seems most reasonable for measuring the length of each of these objects. Tell why you selected each unit.

a) the height of a person

c) the thickness of a book

b) the length of a football field

d) the length of a table

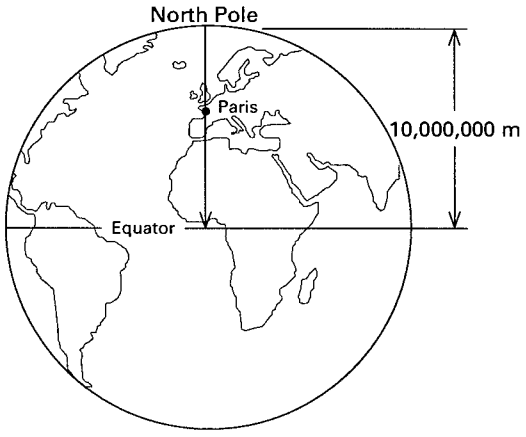
(Continued on back.)

COMMENTS

3 Many of the first units of measure were parts of the body. For example, the hand, foot, span, and cubit (see Student Activity 22.1) were all units of linear measure used by the early Babylonians and Egyptians. Some students may be familiar with the current practice of measuring a horse by the number of hands from the ground to the horse's shoulders (the hand has been standardized as 4 inches).

Throughout history, measurement systems have evolved to meet the needs of society. The English system of linear measurement arose from several nonstandard measures. For example, the inch was the length of any 3 barleycorns placed end to end, the foot was the length of a human foot, and the yard was the distance from a person's nose to the end of their outstretched arm. Gradually, the English system was standardized. For example, in the twelfth century, the yard was established by royal decree of King Henry I as the distance from *his* nose to *his* thumb. Students may be interested to know that today shoe sizes vary by $\frac{1}{3}$ of an inch because they are measured by the unit of length called the barleycorn, where 3 barleycorns = 1 inch. Thus, a size 8 shoe is 1 inch longer than a size 5 shoe.

The metric system was developed by the French Academy of Sciences in 1790. To create a system of "natural standards," scientists divided the length of a meridian from the equator to the North Pole by 10,000,000 to obtain the basic unit of length, the meter.



Connector Teacher Activity (cont.)

ACTIONS

Lesson **22** Dimension and Area I

Connector Student Activity (cont.)

3 Explain what you think are the advantages and the drawbacks of the historical measures mentioned on page 1.

4 Suppose a ream of paper (500 sheets) is 3 thumb widths thick. How thick (in thumb widths) is one sheet? Explain your methods.

5 Each of these segments was drawn using a different linear unit. Without using a ruler, invent a method of locating and labeling a close approximation of 1, 2, and 3 on each segment. (You will need to extend some segments.)

a) 0 — 3

b) 0 — 2

c) 0 — $\frac{1}{2}$

d) 0 — $\frac{3}{4}$

e) 0 — $\frac{4}{3}$

f) On another sheet, explain the methods you used for b), d), and e).

COMMENTS

Smaller and larger linear measures in the metric system are obtained by multiplying and dividing the meter by 10, 100, or 1000. (Note that during the Focus activity, students will continue to explore linear and area measure using nonstandard units. Both the English and metric systems are explored in Lesson 43.)

Following are responses to selected problems on Student Activity 22.1. You might point out that it is okay to use a straightedge (e.g., the edge of a sheet of paper or file folder) but not a ruler which is marked to show standard unit measures.

4. $\frac{3}{500}$ of a thumb width. One method is to use the division model to divide 3 by 500: $3 \div 500 = \frac{3}{500} = .006$. Another method is to divide 500 by 3 to get the number of sheets of paper for each thumb width; $500 \div 3 = 166\frac{2}{3}$. Then each sheet is $\frac{1}{166\frac{2}{3}}$ thumb widths.

5b) Bisect the given line segment to get the point for 1, and then use the unit length from 0 to 1 to obtain the point for 3.

5d) One method is to divide the given line segment into 3 equal parts to get $\frac{1}{4}$ and then use the length from 0 to $\frac{1}{4}$ to obtain the point for 1. Another method is to think of $\frac{3}{4}$ as $3 \div 4$ and mark off 4 segments equal to the given segment to obtain the point for 3.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

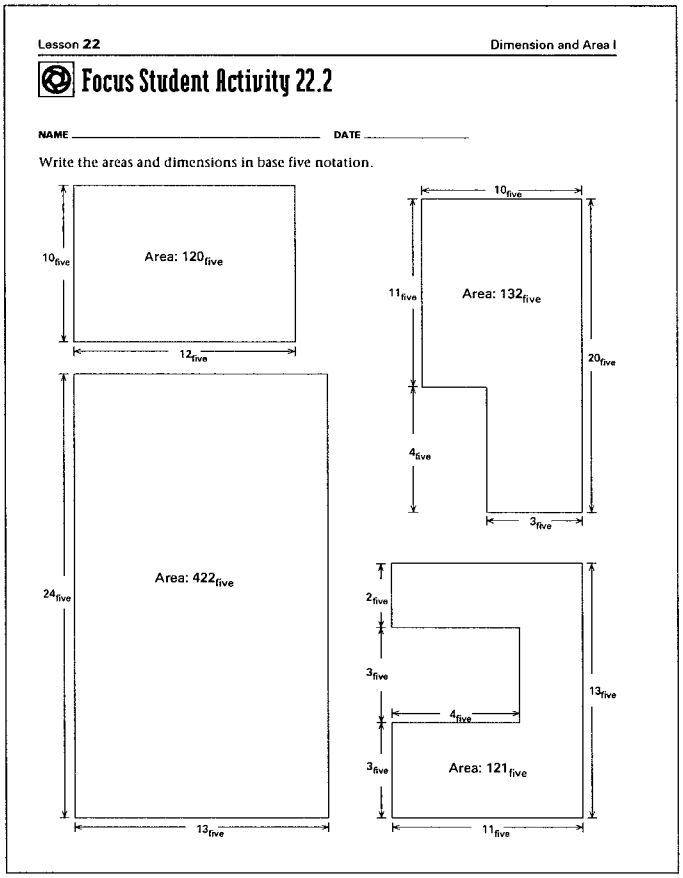
Students measure and record the dimensions and areas of rectangles using base five. This activity provides important background for explorations of the four basic operations and, in particular, for showing relationships between multiplication and division.

MATERIALS

- ✔ Base five area and linear pieces for each student.
- ✔ Focus Student Activity 22.2, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 5 sheets per group.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area and linear pieces for the overhead.

ACTIONS

1 Distribute base five area and linear pieces and a copy of Focus Student Activity 22.2 to each student. Ask them to determine the missing dimensions and areas and record them using base five notation. Discuss with the students how they arrived at their answers.



COMMENTS

1 The copy of Student Activity 22.2 shown at the left contains the answers.

Students may use various methods to arrive at their answers. Some may fill in the areas with base five area pieces and line the edges with linear pieces, and then make exchanges to arrive at minimal collections. Others may mark off regions or edges of the figures which are equivalent to different area and linear pieces.

Students can be asked to demonstrate their methods on the overhead using a transparency of Student Activity 22.2 and transparent base five pieces.

Focus Teacher Activity (cont.)

ACTIONS

2 Place a transparency of Focus Master A on the overhead. Ask the students to use base five area and linear pieces to construct the rectangles described and to provide the information requested using base five notation. Discuss with the students how they arrived at their answers.

Dimension and Area I — Lesson **22**

Focus Master A

a) A rectangle whose area is 242_{five}. Record its dimensions.

b) A square whose side has length 13_{five}. Record its area.

c) A rectangle with dimensions 32_{five} and 13_{five}. Record its area.

d) A rectangle with area 134_{five} and one dimension 4_{five}. Record its other dimension.

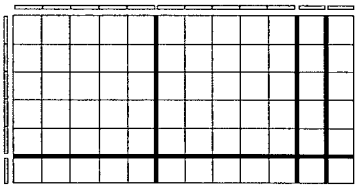
e) A rectangle with area 1341_{five} and one dimension 23_{five}. Record its other dimension.

f) A rectangle with area 222_{five} and one dimension 20_{five}. Record its other dimension.

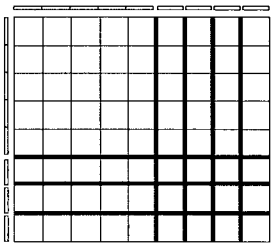
COMMENTS

2 Students may demonstrate their construction of the rectangles on the overhead using transparent base five pieces.

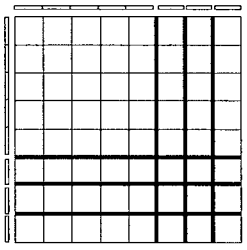
a) Here is one rectangle. Its dimensions are 22_{five} and 11_{five}.



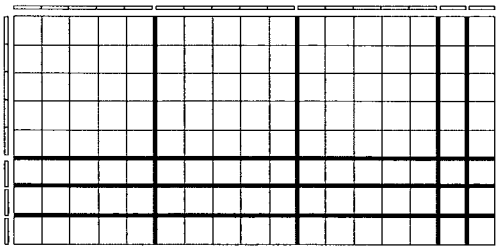
Other rectangles of this area are also possible. By trading 1 mat for 5 strips and 2 strips for 10 units, the following rectangle can be formed. Its dimensions are 13_{five} and 14_{five}.



b) A square whose side has length 13_{five} is shown below. The pieces forming this square are equivalent to 2 mats, 2 strips, and 4 units. Hence, its area is 224_{five}.

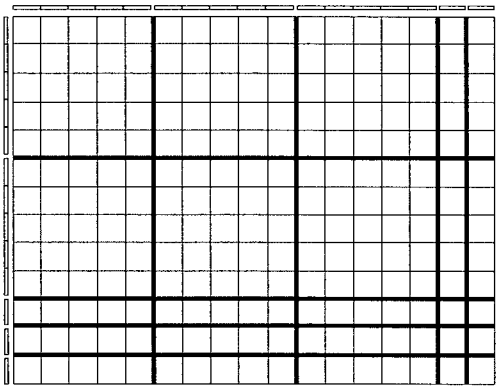


c) This rectangle has area 1021_{five}.



Focus Teacher Activity (cont.)

ACTIONS



3 Distribute 5 sheets of 1-cm grid paper to each group and a pair of scissors to each student. Have the students cut out area and linear pieces to form one or more of the following:

a) Three different rectangles whose areas are each 500_{six}. Record their dimensions.

b) A rectangle with area 10212_{three} and one dimension 11_{three}. Record the other dimension.

c) A rectangle with area 320_{ten} and one dimension 16_{ten}. Record the other dimension.

Ask for volunteers to illustrate their methods at the overhead.

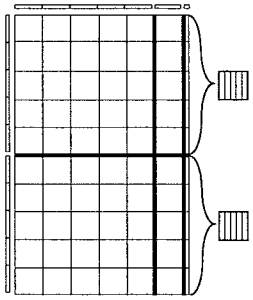
COMMENTS

d) As shown at the left, the other dimension is 21_{five}. In order to construct this rectangle, one must trade the mat for 5 strips.

e) One must make a number of trades to construct this rectangle (shown at the left). Its missing dimension is 32_{five}.

In parts d) and e) it is helpful to first place linear pieces to indicate the dimensions of the rectangle and then use area pieces to fill in the rectangle.

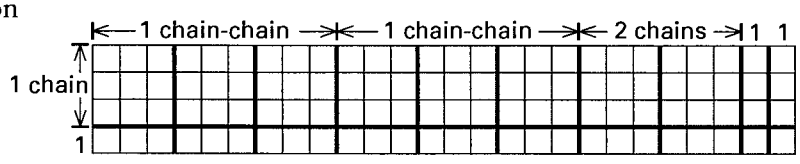
f) For this situation it is possible to form a 20_{five} by 11_{five} rectangle with 2 units left over. Alternatively, one can subdivide the 2 units into strip-lets in order to form a rectangle with dimensions 20_{five} by $11,1_{five}$, as shown below (note that $11,1_{five}$ represents a length of 1 chain, 1 linear unit, and $\frac{1}{5}$ linear unit).



3 Students may demonstrate methods by making sketches, or by laying cut out pieces (when they fit) on the overhead.

a) Here are a few possibilities: 10_{six} by 50_{six}, 30_{six} by 14_{six}, 20_{six} by 23_{six}, 1_{six} by 500_{six}.

b) The other dimension is 2 chain-chains, 2 chains, and 2 units (i.e., 222_{three}), as shown below. Several trades are necessary to form this rectangle.



c) The other dimension is 20_{ten}.

TEACHER NOTES:

Follow-up Student Activity 22.3

NAME _____ **DATE** _____

1 Use base five number pieces to construct each of the following rectangles and then record the information requested using base five notation.

a) A rectangle whose area is 341_{five}. Record its dimensions.

b) A different rectangle whose area is 341_{five}. Record its dimensions.

c) A square whose side has length 21_{five}. Record its area.

d) A rectangle with dimensions 11_{five} and 23_{five}. Record its area.

e) A rectangle with area 223_{five} and one dimension 14_{five}. Record its other dimension.

f) A rectangle with area 312_{five} and one dimension 10_{five}. Record its other dimension.

2 Explain how a linear unit and an area unit are alike and how they are different.

3 In problems a)-d) that follow, ⊢—⊣ is 1 linear unit. On each of the line segments mark off the indicated length and record the total number of linear units it contains.

a) 13_{five}

b) 111_{three}

c) 14_{twelve}

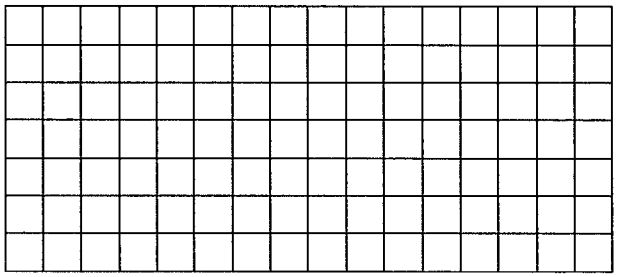
d) 14_{ten}

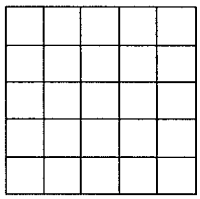
(Continued on back.)

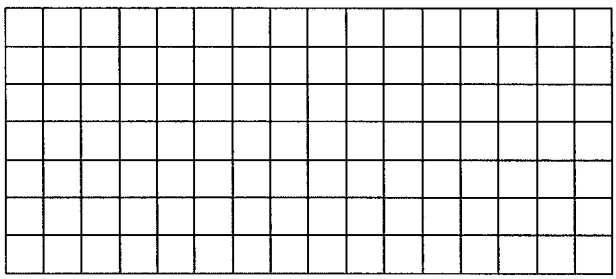
Follow-up Student Activity (cont.)

4 Mark each rectangle below to show how you determined its area and dimensions.

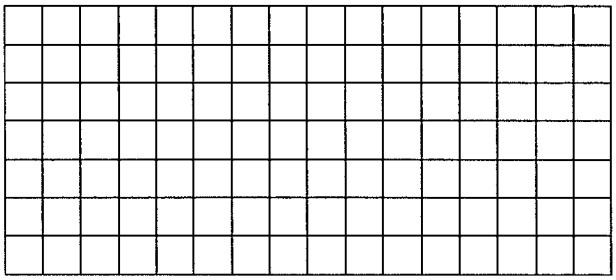
a) Suppose that ☐ is 1 area unit and ⊢⊣ is 1 linear unit, what are the base five dimensions and area of this rectangle?



b) Suppose that  is 1 area unit and ⊢⊣⊢⊣⊢⊣ is 1 linear unit. Now what are the base five dimensions and area of this rectangle? Explain how you determined these measurements.



c) Suppose that ⊢⊣ is 1 linear unit and ☐ is 1 area unit. What are the base four dimensions and area of this rectangle? Explain how you determined these measurements.

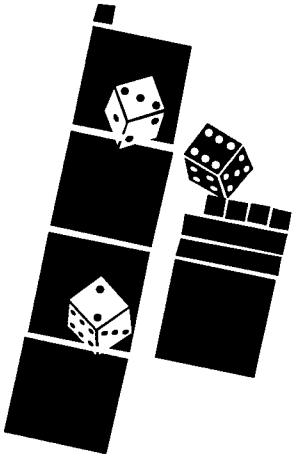


Addition & Subtraction Concepts

Lesson 23

THE BIG IDEA

Using area and linear pieces to explore addition and subtraction in several bases enhances students' understanding of place value and promotes insights about the operations. Such experiences enable students later to invent algorithms that make sense to them.



CONNECTOR

OVERVIEW

Students play several base five "trading games" that draw upon their understanding of place value.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area and linear pieces for each student.
- ✔ Cubical Dice (see pattern in *Blackline Masters*), 3 per group, 2 of one color and one of another color for each group.

FOCUS

OVERVIEW

Area and linear pieces are used to perform addition and subtraction computations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area and linear pieces for each student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 4 sheets per group and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students explain their understanding of the meanings of addition and subtraction and show how to use number pieces to solve computations in several bases.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 23.1, 1 copy per student.
- ✔ Base five area and linear pieces (see *Blackline Masters*), 1 paper or cardstock set per student.

LESSON IDEAS

TIMING
The Connector and Focus activities can usually be completed in a total of about 2 hours of in-class exploration.

LOOKING AHEAD
The intent of this lesson is not to "master" base five addition and subtraction, but rather to explore the operations of addition and subtraction from a conceptual perspective. These activities will be the basis for exploring base ten (whole number and decimal) subtraction and addition in Lesson 26.

FOLLOW-UP
When evaluating the Follow-up, place emphasis on students' diagrams and explanations and the extent to which the diagrams and explanations reveal understanding.

SELECTED ANSWERS

4. a) 1320_{five}
   b) 134_{five}

5. $135_{eight} + 257_{eight} = 414_{eight}$

6. 22_{three}

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students play several base five "trading games" that draw upon their understanding of place value. These activities provide opportunities for students to develop intuitions about addition and subtraction.

MATERIALS

- ✔ Base five area and linear pieces for each student.
- ✔ Cubical Dice (see pattern in *Blackline Masters*), 3 per group, 2 of one color and one of another color for each group.

ACTIONS

1 Place the students in small groups and give each student a set of base five area and linear pieces. Give each group three standard dice, two of one color and one of another color.

2 Ask each group to play *Trading Up* according to the following procedures: One student rolls a die to determine the number of units to place on the table, and makes all necessary trades in order to have a minimal collection. The die is passed to the next group member who repeats these actions, adding the new pieces to those on the table. Play proceeds until the group forms a strip-mat (or whatever goal the teacher establishes).

3 Pose questions like the following for the groups to discuss: What is the fewest number of rolls of the die necessary to form the strip-mat? What is the greatest number of rolls necessary to form the strip-mat? Is it possible to get a strip on the first roll? a mat? etc. What is the probability of getting a strip or more on the first roll?

COMMENTS

1 A pattern for making cardstock dice is in *Blackline Masters*, or numbered stickers can be placed on wooden cubes.

2 To speed up this game have groups roll two dice at a time, and use the sum of the numbers rolled to determine the number of units to add. For base five there can be no more than 4 of any piece; then a student must trade units for strips, strips for mats, and mats for strip-mats. Emphasize the need for keeping minimal collections on the table. This keeps the focus on developing intuitions about place value and its relationship to regrouping. Let the class decide whether it is necessary to reach the exact goal or at least that amount.

To create a more game-like atmosphere, before play begins you might let students determine the criteria for winning the game [e.g., the first group to form a strip-mat, the group that forms a strip-mat in the fewest (or most) rolls, etc.]. Hence, you might have each group tally their rolls as they form a strip-mat.

3 The fewest number of rolls possible with one die to obtain a strip-mat is 21, and the greatest number is 125 (all ones). The probability of rolling a 5 (i.e., of getting a strip) on the first roll is $\frac{1}{6}$, since there are 6 equally likely outcomes on a die and 5 is one of those outcomes. Similarly, the probability of rolling a 6 (hence, getting a strip and 1 unit) on the first roll is $\frac{1}{6}$. Thus, there is a $\frac{1}{6} + \frac{1}{6} = \frac{2}{6}$ chance of getting a strip or more on the first roll. Encourage students to pose and discuss other questions.

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
4 Ask each group to play *Trading Down* according to the following procedures: Each group places a strip-mat (or another piece or combination of pieces, as determined by the teacher) on the table; one student rolls a die (or dice) to determine the number of units to remove from the beginning collection and makes all necessary trades in order to maintain a minimal collection. The die is passed to the next group member who rolls the die, removes the indicated number of units from the collection on the table, and makes sure the remaining collection is minimal. Play proceeds until no pieces are left.	**4** Many students will make alternative trades (e.g., instead of removing 4 units they may remove a strip and place a unit on the board). Using the sum or product of two dice quickens the pace of the activity and allows for the option of beginning with a larger collection. This game develops readiness for subtraction in base five. The winner of this game might be the group that finishes in 25 rolls, the group that takes the fewest rolls, etc.
5 Have the groups play *Trading Up and Down* according to the following procedures: Beginning with 2 mats (or another collection determined by the teacher) on the table, one student rolls three dice, adds to the 2 mats the number of units indicated by the sum (or product) of the numbers on the two dice of matching color, removes the number of units indicated by the third die, and checks to be sure a minimal collection remains on the table. The dice are passed to the next group member who rolls the dice, adds and removes pieces as indicated by the numbers on the dice, and makes sure a minimal collection remains. Play proceeds until the group attains a goal collection established by the teacher. Before play begins, let students decide the criteria for winning this game.	**5** This game develops readiness for base five addition and subtraction while reinforcing place value concepts.
6 Pose questions such as the following regarding *Trading Up and Down*: What is the largest collection that can be added to the board in one roll? the smallest? What is the minimum number of rolls needed to reach the established goal? Encourage students to pose other questions that interest them.	**6** Playing *Trading Up and Down*, the largest collection that can be added to the board in one turn is $6 + 6 - 1 = 11$, or 1 strip and 1 unit. The smallest is a negative amount, as $1 + 1 - 6 = {}^{-}4$ requires that 4 units be removed from the board. The minimum number of rolls to obtain a strip-mat is 12.
7 (Optional) Repeat one or more of Actions 2-5 using base five linear pieces	**7** As with the area pieces, exchanges are necessary each time a group has more than 4 of one type of base five linear pieces.

Focus Teacher Activity

OVERVIEW & PURPOSE

Area and linear pieces are used to perform addition and subtraction computations. Emphasis is placed on viewing these arithmetical operations conceptually rather than on the development of paper-and-pencil algorithms.

MATERIALS

- ✔ Base five area and linear pieces for each student.
- ✔ 1-cm grid paper (see *Blackline Masters*), 4 sheets per group of students and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Base five area and linear pieces for the overhead.

ACTIONS

1 Place the students in groups and distribute base five area and linear pieces to each student. Write the following arithmetical expressions on the chalkboard:

a) $14_{\text{five}} + 23_{\text{five}}$

b) $224_{\text{five}} + 343_{\text{five}}$

c) $301_{\text{five}} - 134_{\text{five}}$

d) $1124_{\text{five}} - 321_{\text{five}}$

Have the groups devise ways of performing the indicated computations by manipulating base five area or linear pieces, using paper and pencil only to record their results. Ask volunteers to show their methods at the overhead.

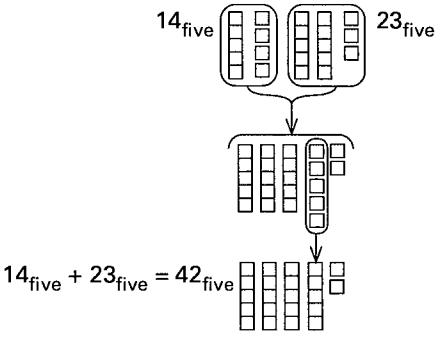
COMMENTS

1 Circulate among the students, offering hints as appropriate (see Lesson 2, Basic Operations, for ways of viewing arithmetical operations). Encourage the students to discuss with each other ways of performing the computations.

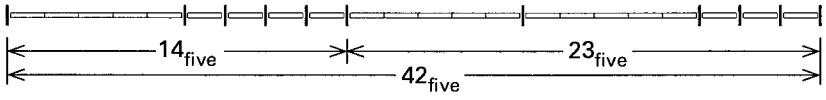
Some students may find rules for computing in base five and not feel a need to use or refer to the base five pieces. Discourage this! The purpose of studying base five is to acquire an understanding of concepts, rather than to learn to compute in base five.

Some possible methods for the computations follow:

a) An addition may be performed by combining collections of base five area pieces, and then converting this combined collection to an equivalent collection containing a minimum number of pieces. See the diagram below:



If students don't bring up methods involving linear pieces you might ask them to do so. Note that a) can also be performed using linear pieces. Lengths of 14_{five} and 23_{five} are placed together end-to-end and their total length is measured to find the sum, as shown here:



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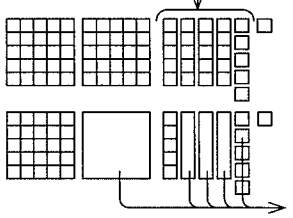
Focus Teacher Activity (cont.)

ACTIONS

301_{five}

Take-Away Method for $301_{five} - 134_{five}$:

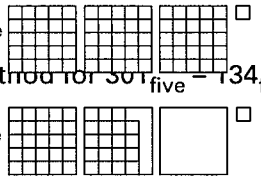


112_{five} take away 134_{five}

$301_{five} - 134_{five} = 112_{five}$

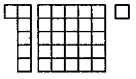
301_{five}

Difference Method for $301_{five} - 134_{five}$:



134_{five}

112_{five}



2 Repeat Action 1, if needed, for other computations, such as:

a) $224_{five} + 303_{five}$

b) $1000_{five} - 4_{five}$

c) $301_{five} - 244_{five}$

COMMENTS

1 (continued.)
b) Combining collections for 224_{five} and 343_{five}, and then making exchanges, results in a collection of 1 strip-mat, 1 mat, 2 strips, and 2 units. This could be recorded symbolically as $224_{five} + 343_{five} = 1122_{five}$. Note that some students may combine mats first, then strips and units, while others may combine units first. It is important that students notice the end result is the same, regardless of the order of combining pieces.

Similarly, lengths of 2 chain-chains, 2 chains, and 4 linear units combine with 3 chain-chains, 4 chains, and 3 units to form (after trading) 1 chain-chain-chain, 1 chain-chain, 2 chains, and 2 units. These actions could also be recorded as $224_{five} + 343_{five} = 1122_{five}$.

c) Shown at the left are two methods of doing this subtraction with area pieces. In the *take-away* method, after making exchanges, a collection for 134_{five} is taken away from a collection for 301_{five}. Although more cumbersome, linear pieces could also be used to carry out the take-away method for this computation: begin with a length of 3 chain-chains and 1 unit; exchange 1 chain-chain for 5 chains; exchange 1 chain for 5 units; remove 4 units, 3 chains, and 1 chain-chain.

In the *difference* method, a collection is found that will make up the difference between collections for 301_{five} and 134_{five}. Lengths of 301_{five} and 134_{five} can be compared similarly.

d) The two methods in Comment c) may also be used to obtain $1124_{five} - 321_{five} = 303_{five}$.

2 a) $224_{five} + 303_{five} = 1032_{five}$

b) $1000_{five} - 4_{five} = 441_{five}$

c) $301_{five} - 244_{five} = 2_{five}$

Focus Teacher Activity (cont.)

ACTIONS

3 Distribute 4 sheets of 1-cm grid paper to each group and a pair of scissors to each student. Ask the groups to cut out and use area and/or linear pieces to perform the following computations and to look for more than one method of performing each computation. Have volunteers demonstrate their methods at the overhead using a transparency of grid paper.

a) $45_{six} + 123_{six}$

b) $247_{ten} + 158_{ten}$

c) $111_{three} - 22_{three}$

d) $304_{ten} - 167_{ten}$

COMMENTS

3 Keep emphasis on conceptual relationships and students' actions with pieces. Base ten operations are explored more in depth in Lessons 26, 27, 31, 37, and 38. Methods for adding and subtracting in any base are like those described in the Action 1 comments, although the number of units in a trade is determined by the value of the base.

a) 212_{six}

b) 405_{ten}

c) 12_{three}

d) 137_{ten}

TEACHER NOTES:

Follow-up Student Activity 23.1

NAME _____ **DATE** _____

1 Explain in your own words the meaning of the *take-away* model for subtraction. Draw a diagram to show an example.

2 Explain in your own words the meaning of the *difference* model for subtraction. Draw a diagram to show an example.

3 Explain and draw a picture to illustrate what addition means to you.

(Continued on back.)

Follow-up Student Activity (cont.)

4 Use your base five pieces to do the following computations. Then make a sketch to show your methods with the pieces. Be sure to show how you traded pieces. Next to each sketch, write the answer using base five notation.

a) $324_{\text{five}} + 441_{\text{five}}$

b) $432_{\text{five}} - 243_{\text{five}}$

5 Draw a sketch of base eight area pieces to show how to use them to find $135_{\text{eight}} + 257_{\text{eight}}$. Show any trades you need to make.

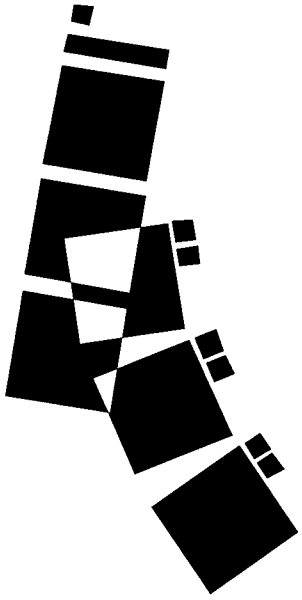
6 Draw a sketch of base three linear pieces and show how to use them to find $121_{\text{three}} - 22_{\text{three}}$. Show any trades you make.

Multiplication & Division Concepts

Lesson 24

THE BIG IDEA

Exploring multiplication and division in base five enhances students' understanding of the meanings of these operations. Base five area and linear pieces are especially useful for deepening students' understanding of the area model for multiplying and dividing.



CONNECTOR

OVERVIEW

Students recall the models for multiplication and division that were introduced in Lesson 2.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces (with lengths equal to the side of a square tile, see Lesson 2), 30 per student.
- ✔ Tile and linear pieces for the overhead.

FOCUS

OVERVIEW

Students use base five area and linear pieces and their understanding of the meanings of multiplication and division to perform computations in base five.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base five area and linear pieces for each student.
- ✔ Base five grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Base five area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students describe their views of the meanings of multiplication and division and apply this knowledge to several computations in base five, base six, and base three.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 24.1, 1 copy per student.
- ✔ Base five grid paper, (see *Blackline Masters*), 2 sheets per student.
- ✔ Base five area and linear pieces, 1 paper or cardstock set per student.

LESSON IDEAS

LOOKING AHEAD

It isn't necessary to spend excessive time getting students to "master" the ideas in this lesson. They will explore conceptual models for the basic operations again in this course when they investigate base ten whole numbers, decimals, and fractions. Often in mathematics deep understanding doesn't occur until students have explored an idea several times in varied contexts.

TIMING

About 2 hours of class time is usually adequate for exploring this lesson.

SELECTED ANSWERS

3. 31_{five}

4. 1103_{five}

5. a) 1442_{five}

   b) 21_{five}

   c) 1022_{five}

   d) 233_{five}

   e) 422_{five}

   f) 12_{five}

6. $24_{six} \times 13_{six} = 400_{six}$

   $2102_{three} \div 12_{three} = 111_{three}$

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students recall the models for multiplication and division that were introduced in Lesson 2. These models are used in the Focus when students explore multiplication and division with base five pieces.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces (with lengths equal to the side of a square tile, see Lesson 2), 30 per student.
- ✔ Tile and linear pieces for the overhead.

ACTIONS

1 Place the students in groups of 3-4. Distribute tile and linear pieces to each student. Ask the groups to use the tile and linear pieces to model at least two *different* interpretations of the meanings of each of the following computations.

a) 5×4 b) $24 \div 6$

Ask volunteers to share their group's models at the overhead.

2 (Optional) Repeat Action 1, as needed, for these computations:

a) 3×8

b) $27 \div 9$

c) $30 \div 6$

d) 2×9

COMMENTS

1 Conceptual models for multiplication and division were introduced in Lesson 2.

a) Some possibilities are: 5 groups of 4 (square units or linear units); 4 groups of 5; or a rectangle with dimensions 4 linear units by 5 linear units and area 20 square units. These illustrate the repeated addition and area models for multiplication.

b) Here are some possibilities that involve the grouping, sharing, and area models for division: 4 groups of 6 (for $24 \div 6$ interpreted as, "how many groups of 6 in 24?"); 6 groups of 4 (for $24 \div 6$ interpreted as, "how many are in each group if 24 is divided into 6 equal groups?"); or a rectangle with area 24 square units, one dimension 6 linear units, and the other dimension equal to the quotient $24 \div 6 = 4$ linear units.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use base five area and linear pieces to perform multiplication and division computations. Emphasis is placed on developing insights about the meanings of the operations rather than on developing paper-and-pencil algorithms.

MATERIALS

- ✔ Base five area and linear pieces for each student.
- ✔ Base five grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Base five area and linear pieces for the overhead.

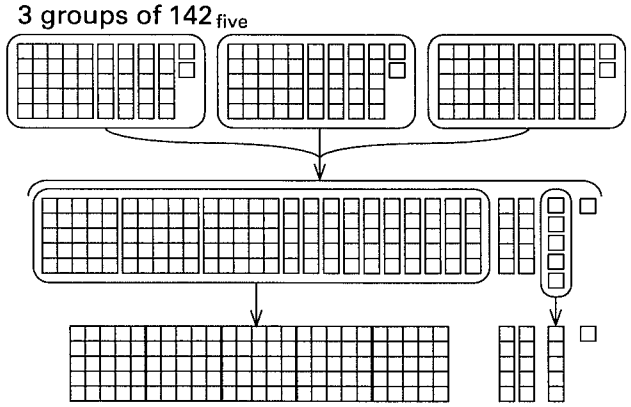
ACTIONS

1 Distribute base five area and linear pieces to each student. Write the following arithmetical expression on the overhead or chalkboard: $3_{five} \times 142_{five}$. Have students devise ways of performing this computation by manipulating base five area and/or linear pieces. Have volunteers demonstrate their methods at the overhead.

COMMENTS

1 Some students may want to convert to base ten, compute, and then convert back to base five. Discourage this! The purpose of this activity is to explore the concept of multiplication rather than converting between bases and performing algorithms. Encourage students to share several ways of viewing this problem.

One method is to use the repeated addition method with number pieces, i.e., combining 3 collections of 142_{five} and then making exchanges, as shown below:



A second method is to use the repeated addition method with three lengths of 142_{five} placed end-to-end, forming a total length of 1031_{five}.

A third method is to use the area model for multiplication: use linear pieces to form the dimensions of a $3_{five} \times 142_{five}$ rectangle; fill in the rectangle with area pieces; determine its area (product) by finding the minimal collection of pieces used.

Focus Teacher Activity (cont.)

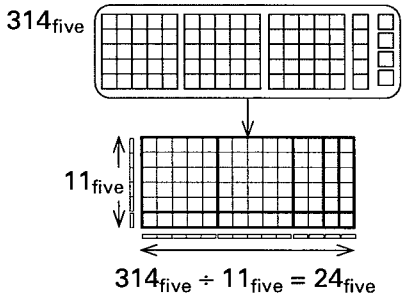
ACTIONS

2 Repeat Action 1 for the following computations:

a) $24_{five} \times 13_{five}$

b) $311_{five} \div 3_{five}$

c) $314_{five} \div 11_{five}$



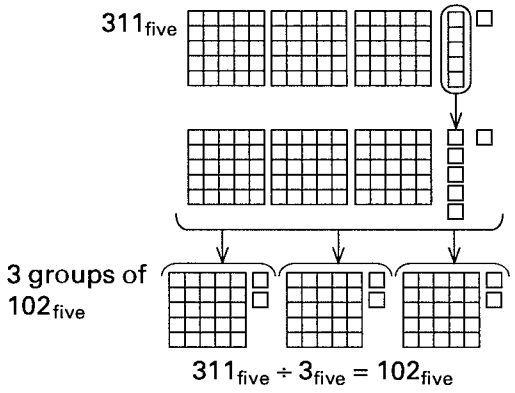
3 Ask the students to perform additional computations, as necessary to become familiar with ways of computing with base five pieces.

COMMENTS

2 a) One way to compute this product is to form a rectangle whose dimensions are the numbers being multiplied. The area of this rectangle is the desired product.

Pieces in the rectangle can be exchanged to obtain 4 mats, 2 strips, and 2 units.

b) To find this quotient, a collection for 311_{five} may be divided into 3 groups, making exchanges as necessary, as shown below.



c) One method of finding this quotient is to form a collection for 314_{five} and arrange its pieces into a rectangle with one dimension 11_{five} (to obtain the rectangle shown at left, 1 mat must be exchanged for 5 strips). The other dimension is the desired quotient. This method may be adapted to any division.

Notice that this divides 314_{five} into 24_{five} groups, each group being a column of the rectangle. The number of objects in each group is the number of rows.

3 Computations may be selected from the following:

a) $1021_{five} + 324_{five}$

b) $1314_{five} - 421_{five}$

c) $33_{five} \times 22_{five}$

d) $20_{five} \times 41_{five}$

e) $1232_{five} \div 22_{five}$

f) $424_{five} \div 14_{five}$

Answers:

a) 1400_{five}

b) 343_{five}

c) 1331_{five}

d) 1320_{five}

e) 31_{five}

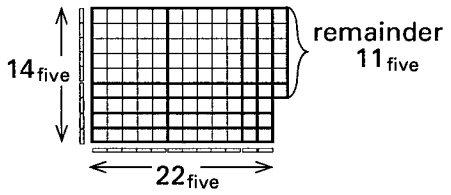
f) 22_{five} w/rem. 11_{five}

Focus Teacher Activity (cont.)

ACTIONS

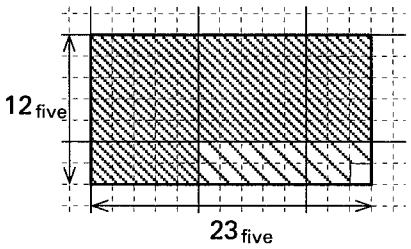
COMMENTS

Notice the remainder in f). If a collection of pieces equivalent to 4 mats, 2 strips, and 4 units is arranged into a rectangular array in which one dimension is 14_{five}, the other dimension is 22_{five} with 1 strip and 1 unit left over. (In forming the rectangle below, 2 mats were exchanged for 10 strips and 1 strip was exchanged for 5 units.) Some students may imagine cutting the leftover units into parts and distributing those parts equally along the end of the rectangle to get the missing dimension $22\frac{1}{14}_{five}$ linear units, or $22\frac{2}{3}_{five}$ linear units.



4 Distribute 2 sheets of base five grid paper to each student. Ask the students to compute $12_{five} \times 23_{five}$ by sketching a rectangle whose dimensions are 12_{five} and 23_{five} and finding its area.

4 If students have difficulty using the grid paper, encourage them to build the models first and then to use the grid paper to make a record of their thinking and actions with the pieces. It is important to encourage the use of manipulatives as long as students need them.



On the above 12_{five} by 23_{five} rectangle, the darker shaded region is equivalent to 3 mats, the lighter shaded region is equivalent to 3 strips, and the unshaded region is 1 unit. Hence, the area is 331_{five}. Since the area of the rectangle is the product of its dimensions, $12_{five} \times 23_{five} = 331_{five}$.

5 Ask each student to enclose a region, on base five grid paper, whose area is 314_{five}. Then ask them to sketch a rectangle which has the same area and has one dimension equal to 12_{five}. Have the students use their completed sketches to compute $314_{five} \div 12_{five}$.

5 Once again, some students will find it helpful to use the pieces first and then to make a recording of their thoughts and actions with sketches on the grid paper. The intent is that students view their grid paper diagrams as representations of actions with pieces. Some students may need more experience with the manipulatives before moving to diagrams.

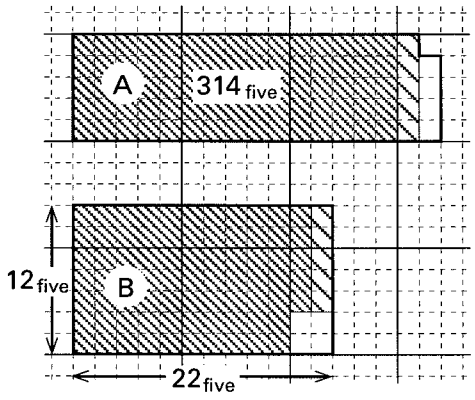
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

5 (continued.)
Region A has area 314_{five} and can be obtained by enclosing 3 mats (the darker shaded portion), 1 strip (the lighter shaded portion), and 4 units (the unshaded portion). This amount of area can be redistributed into a rectangle B with one dimension 12_{five} as shown in the following sketches. The other dimension of the rectangle is 22_{five}. Hence, $314_{five} \div 12_{five} = 22_{five}$.



6 Ask the students to make sketches on base five grid paper to help them do the following computations:

a) $233_{five} + 312_{five}$

b) $321_{five} - 143_{five}$

c) $32_{five} \times 24_{five}$

d) $404_{five} \div 13_{five}$

6 In the following sketches, each dark-shaded region is equivalent to a strip mat, each medium-shaded region can be converted into one or more mats, and each light-shaded region can be converted into strips. The remaining units are not shaded.

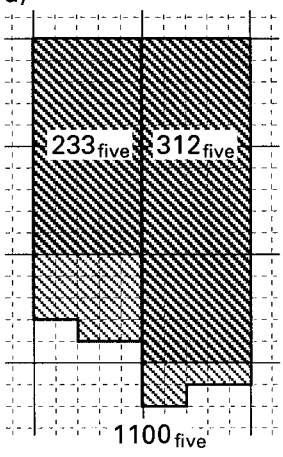
a) $233_{five} + 312_{five} = 1100_{five}$

b) $321_{five} - 143_{five} = 123_{five}$

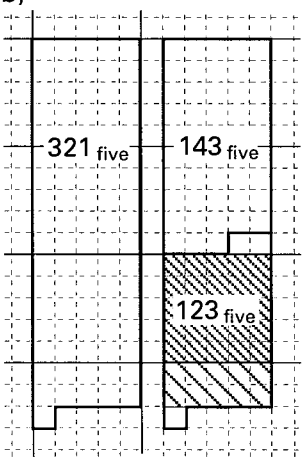
c) $32_{five} \times 24_{five} = 1423_{five}$

d) $404_{five} \div 13_{five} = 23_{five}$

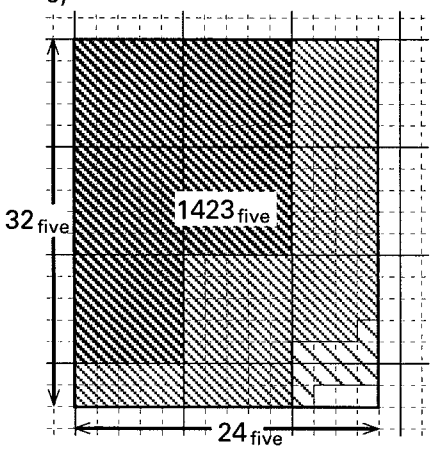
a)



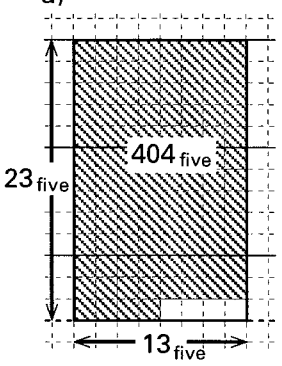
b)



c)



d)



Follow-up Student Activity 24.1

NAME _____ **DATE** _____

1 Explain and draw diagrams to show different ways to view the meaning of multiplication.

2 Explain and draw diagrams to show three different ways to view the meaning of division.

3 Use base five pieces to compute $1313_{\text{five}} \div 23_{\text{five}}$. In the space that follows, explain your methods and then use base five notation to record the quotient.

4 Use your base five pieces to compute $32_{\text{five}} \times 14_{\text{five}}$. In the space below, explain your methods and use base five notation to record the product.

(Continued on back.)

Follow-up Student Activity (cont.)

5 Make sketches on the attached base five grid paper to carry out the following computations. Label your sketches and record the answers using base five notation. (It may be helpful to use base five pieces first and then record your base five piece methods on the grid paper.)

a) $34_{\text{five}} \times 23_{\text{five}}$

b) $231_{\text{five}} \div 11_{\text{five}}$

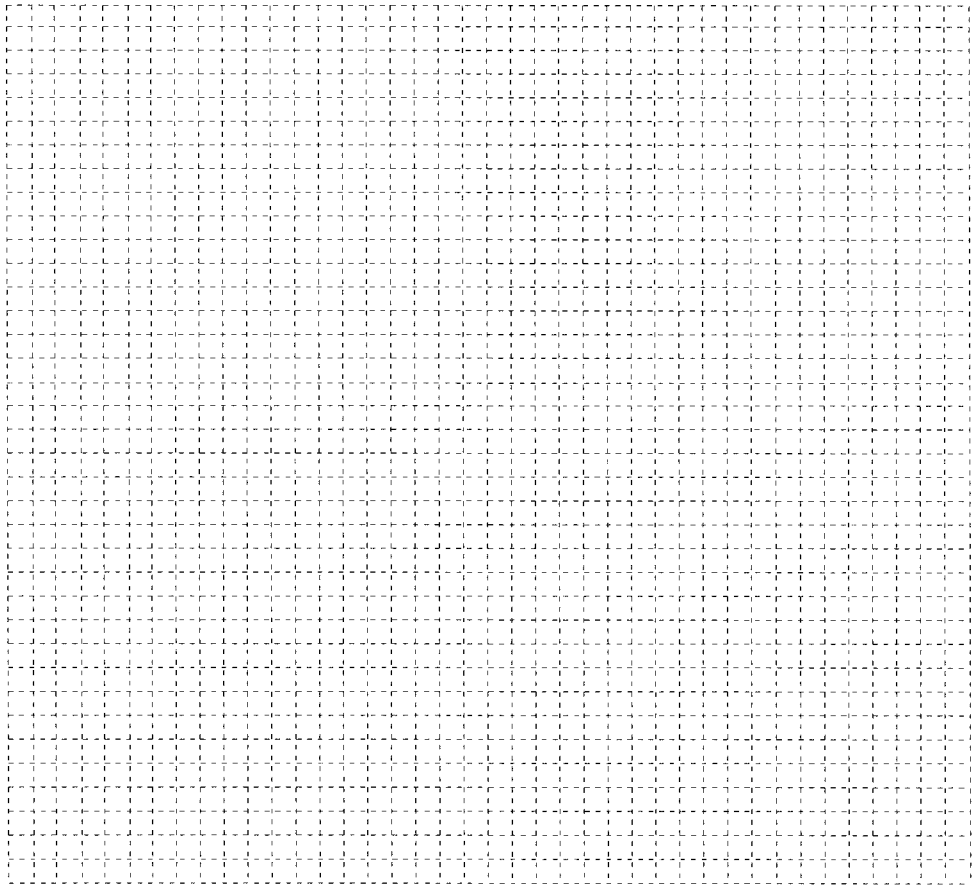
c) $324_{\text{five}} + 143_{\text{five}}$

d) $431_{\text{five}} - 143_{\text{five}}$

e) $24_{\text{five}} \times 13_{\text{five}}$

f) $1013_{\text{five}} \div 34_{\text{five}}$

6 On the grid below sketch diagrams to show how to use the area model to compute: $24_{\text{six}} \times 13_{\text{six}}$ and $2102_{\text{three}} \div 12_{\text{three}}$. Label your sketches.

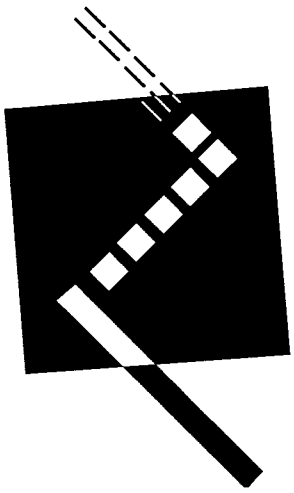


Whole Number & Decimal Numeration

Lesson 25

THE BIG IDEA

Base ten area and linear pieces provide an introduction to the concept of decimal, writing decimals, and reading decimals. Assigning the value of 1 unit to various area and linear pieces, challenges students to expand their understanding of place value. This creates a need for decimal notation and enhances students' number sense.



CONNECTOR

OVERVIEW

Students explore relationships among their base ten area and linear pieces. They use these relationships as a basis for creating larger pieces and for forming minimal collections to represent whole numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces (see *Blackline Masters*), 13 unit squares, 8 strips, and 4 mats for each student.
- ✔ Base ten linear pieces (see Comment 1), 20 linear units, and 10 chains for each student.
- ✔ Index cards or large "Post-It" notes, 10.
- ✔ 10-cm × 10-cm grids (see *Blackline Masters*), at least 28 sheets (i.e., 112 grids) per class.
- ✔ Scissors, 1 pair per student.
- ✔ Tape, 1 roll per group of students.
- ✔ Base ten area and linear pieces for the overhead.

FOCUS

OVERVIEW

Base ten area and linear pieces take on varied decimal values as different pieces are assigned the value one.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Wall displays from the Connector activity.
- ✔ Index cards or large "Post-It" notes, 20.
- ✔ Focus Student Activity 25.1, 1 copy per group.
- ✔ Base ten area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students use sketches of linear and area models, numbers, and words to represent decimals.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 25.2, 1 copy per student.
- ✔ Base ten grid paper Version B (see *Blackline Masters*), 2 sheets per student.

LESSON IDEAS

LOOKING AHEAD
The number sense and mental images that students develop by using area and linear pieces to explore place value and numeration are important in estimation, mental computation, and determining the reasonableness of answers.

TIMING
Depending on students' prior experiences, the Connector activity for this lesson could take 1-2 hours. Although this is longer than the usual Connector, it is important to complete this activity prior to the Focus.

ASSESSMENT
After this lesson is completed, ask students to describe images that come to mind when you say "thirty-two." Have their images changed since you asked that question in the Lesson 1 Connector?

SELECTED ANSWERS

3. a) 413 hundredths
   b) 890 hundredths
   c) 207 hundredths

8. a) 1 hundredth or .01 miles
   b) 2 tenths or .2 miles
   c) 3 ten-thousands or 30,000 miles
   d) 4 hundreds or 400 miles
   e) 6 tens or 60 miles
   f) 7 units or 7 miles

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students explore relationships among their base ten area and linear pieces. They use these relationships as a basis for creating larger pieces and for forming minimal collections to represent whole numbers.

MATERIALS

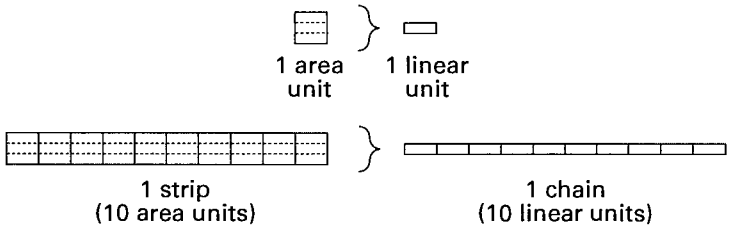
- ✔ Base ten area pieces (see *Blackline Masters*), 13 unit squares, 8 strips, and 4 mats for each student.
- ✔ Base ten linear pieces (see Comment 1), 20 linear units, and 10 chains for each student.
- ✔ Index cards or large "Post-It" notes, 10.
- ✔ 10-cm × 10-cm grids (see *Blackline Masters*), at least 28 sheets (i.e., 112 grids) per class.
- ✔ Scissors, 1 pair per student.
- ✔ Tape, 1 roll per group of students.
- ✔ Base ten area and linear pieces for the overhead.

ACTIONS

1 Place students in groups of 3-4 and distribute base ten area and linear pieces to each student. Discuss students' observations about relationships among the pieces, the value of each piece, and students' predictions about the values and shapes of larger pieces.

COMMENTS

1 Base ten area pieces can be made by copying the pattern in *Blackline Masters* on cardstock and cutting along the indicated lines. Linear pieces can be made by cutting area units and strips in thirds, as shown below. (Note that precut plastic area and linear pieces are available from The Math Learning Center.)



Although the linear pieces must have some width in order to be able to pick them up, it is important that students think of them as representing length only (i.e., theoretically they have no area).

To simplify sorting area and linear pieces, make them in two different colors. For overhead demonstrations make copies of area and linear pieces on transparency film and cut out the individual pieces.

Some observations and relationships that may come up include: 10 square units arranged in a row form a strip; 10 strips side-by-side form a mat; each mat contains 100 units; each linear unit is the length of the edge of 1 square unit; each chain is the length of the long edge of 1 strip; continuing the square-rectangle pattern, a strip-mat is a rectangle that contains 10 mats or 1000 units, a mat-mat is a square that contains 100 mats, etc.; a

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)
chain-chain can be formed by linking 10 chains (or 100 linear units) together end-to-end; a chain-chain-chain is formed by linking 10 chain-chains (or 1000 linear units) end-to-end. Some students may bring up ideas about smaller pieces. If so, you may wish to refer to Focus Comment 1.

2 Give each student a pair of scissors and a supply of base ten mats cut from 10-cm × 10-cm grids (see *Blackline Masters*). Provide each group a roll of tape. Ask the groups to tape together the 10 × 10 grids to form 1 or 2 strip-mats.

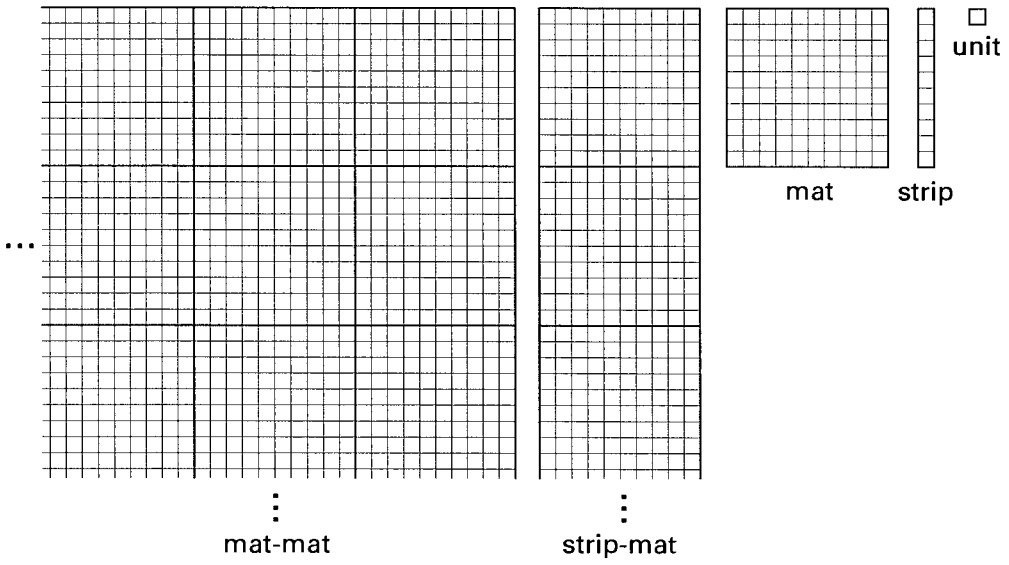
Have the groups combine their strip-mats to create a mat-mat. Form a classroom wall display of the following area pieces: unit square, strip, mat, strip-mat, and mat-mat (see below).

2 If you have saved students' colored grids from the Lesson 16 Follow-up, you could have students use those grids to form their wall display.

Gauge the number of strip-mats for each group to assemble according to the size of your class. The wall display requires a minimum of 112 mats or 28 sheets of 10-cm × 10-cm grids.

Although assembling this display prior to class would save time, it is very valuable for students to have the hands-on experience with forming models of large numbers.

Classroom Wall Display



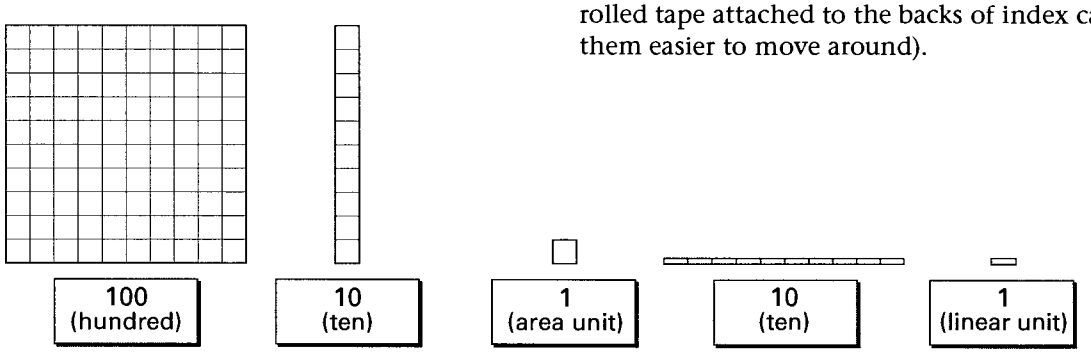
3 Distribute a supply of base ten chains to each group and ask them to each assemble 1 or 2 chain-chains. On another classroom wall (away from the area piece display formed in Action 2) have the groups create a display of the following base ten linear pieces: linear unit, chain, chain-chain, and chain-chain-chain.

3 To save time, cut out the chains prior to class. Each base ten mat can be cut apart to form 30 chains. Gauge the number of chain-chains for groups to assemble according to the size of the class. Forming and stretching out a chain-chain-chain helps students to appreciate the rate at which base ten numbers grow as well as the convenience of working with the area pieces, which are more compact.

Connector Teacher Activity (cont.)

ACTIONS

4 Ask the groups to determine the total number of square units in each area piece displayed and the total linear units in each linear piece. As volunteers share these totals, write them in words and in numbers on index cards or large "Post-It" notes and attach these near their corresponding base ten pieces.



5 (Optional) Ask for volunteers to express the number of units in each base ten piece displayed in Actions 2 and 3 as a product using only ones and/or tens. Introduce students to the use of *exponents* in recording *powers of ten.*

COMMENTS

4 The area pieces in the class display have (in order from largest to smallest) 10000, 1000, 100, 10, and 1 square units. The linear pieces have the same numbers of linear units.

These displays are used again in the Focus activity, where the labels are moved several times (small pieces of rolled tape attached to the backs of index cards make them easier to move around).

5 A strip-mat, for example contains $10 \times 10 \times 10 = 1{,}000$ total units. This can also be written as 10^3, read as *ten to the third power.* In the expression 10^3, the 10 is called the *base* and the 3 is called the *exponent.* The exponent indicates the number of times 10 is a factor of the number.

Powers of ten are explored further in later *Visual Mathematics* courses. It is intended that this action only provide a brief and optional introduction to the topic.

The number of units in each of the pieces can be expressed as follows:

square unit, linear unit: $1 \times 1 = 1 = 10^0$ unit

strip, chain: $10 \times 1 = 10^1 = 10$ units

mat, chain-chain: $10 \times 10 = 10^2 = 100$ units

strip-mat, chain-chain-chain: $10 \times 10 \times 10 = 10^3 = 1000$ units

mat-mat, chain-chain-chain-chain: $10 \times 10 \times 10 \times 10 = 10^4 = 10{,}000$ units

Connector Teacher Activity (cont.)

ACTIONS

6 Write the following statements on the overhead. Have the groups form minimal collections for the numbers represented by each set of pieces.

a) area pieces: 8 tens, 31 units;

b) linear pieces: 153 units;

c) linear pieces: 1 hundred, 9 tens, 19 units;

d) area pieces: 8 hundreds, 32 tens, 30 units.

Collect and display the students' results at the overhead on a chart like the one below.

1000 (thousands)	100 (hundreds)	10 (tens)	1 (units)

7 If it hasn't come up, point out that the base ten numerical representation for the minimal collection in Action 7a) is 111_{ten} and there are 111 total units. So $111_{ten} = 111$. Have the students write the base ten numerical representation for the total number of units in each collection in Action 6. Discuss.

COMMENTS

6 Minimal collections of base pieces were introduced in Lesson 13. To form b) and c) students need to combine chains to form a chain-chain (100 linear units). Hence, you may need to have some extra pieces available.

Here are the minimal collections for these sets of number pieces:

	1000 (thousands)	100 (hundreds)	10 (tens)	1 (units)
a)	0	1	1	1
b)	0	1	5	3
c)	0	2	0	9
d)	1	1	5	0

7 With exchanges, the 8 tens and 31 units in a) have a total of 111 units. The minimal collection consists of 1 mat, 1 strip, and 1 unit. Because a mat is a group of 100 units and a strip is a group of 10 units, the symbol 111 that arises from counting units, can also be viewed as 1 group of 100, 1 group of 10, and 1 unit. Similarly, the numerical representation for the total units in the remaining collections are: b) 153, c) 209, and d) 1150. (Use of numerical representations of collections of base pieces was introduced in Lesson 14.)

Focus Teacher Activity

OVERVIEW & PURPOSE

Base ten area and linear pieces take on various decimal values as different pieces are assigned the value one. The conceptual models for decimal place value and notation that are developed in this lesson are the basis for later explorations of operations with decimals.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Wall displays from the Connector activity.
- ✔ Index cards or large "Post-It" notes, 20.
- ✔ Focus Student Activity 25.1, 1 copy per group.
- ✔ Base ten area and linear pieces for the overhead.

ACTIONS

1 Place the students in small groups and distribute base ten area and linear pieces to each student. Using the display formed during the Connector activity, ask the groups to determine the size, shape, and value of the piece they think belongs at the immediate right of the area unit (i.e., at the right of the small area piece square) and the piece they think belongs at the immediate right of the linear unit (i.e., at the right of the linear piece equal in length to the side of the small square). Then have the groups form and determine the value of the piece at the immediate right of each of these.

Have volunteers describe their group's ideas and reasoning and add the pieces to the wall display, attaching index cards or "Post-It" notes to indicate their values.

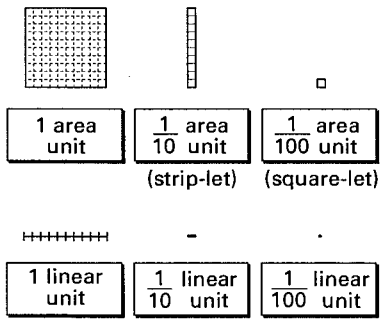
2 Remove all of the labels from the area piece wall display. Attach the "area unit" label to the mat and ask the groups to determine the value of each remaining area piece, based on this new unit. Discuss their conclusions and reasoning. As students identify the new value of a piece, attach (or have volunteers do this) a label to it, making new labels as needed. After all the pieces are labeled, ask for a description of the piece associated with the number 1000.

COMMENTS

1 Continuing the area piece square-rectangle pattern to the right, the next piece would be a small rectangle (strip-let) with area $\frac{1}{10}$ (one-tenth) and dimensions $1 \times \frac{1}{10}$, formed by cutting the square unit into 10 congruent parts. The next smaller piece would be a square-let, formed by cutting a strip-let into 10 congruent squares with dimensions $\frac{1}{10} \times \frac{1}{10}$ and area $\frac{1}{100}$. These smaller area pieces are shown below.

Similarly, the 2 pieces to the right of the linear unit would have length $\frac{1}{10}$ and $\frac{1}{100}$ and be formed by dividing the linear unit into 10 and 100 equal parts, as shown below.

Although these pieces are extremely small to show on the wall, including them helps give students a visual perspective of the relative sizes of numbers.



2 Given this new unit, the value of each piece changes. Some examples are shown on the next page.

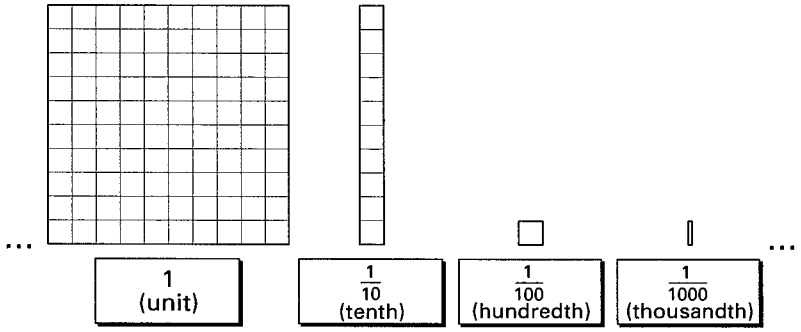
(Continued next page)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

2 (continued.)



Be sure to include the word form of the numbers on the labels. Emphasize the names by reading them aloud to help students distinguish between "hundred" and "hundredth," "ten" and "tenth."

3 Once again remove the labels from the number piece display. Ask for a student volunteer to place the "unit" label under the area piece of their choice. Select another label and ask for a volunteer to attach it in the appropriate position, based on the new unit. Repeat, until all labels are positioned appropriately. Have students determine the numbers associated with the unlabeled pieces and attach new labels to them.

Repeat this Action for several different units, depending on your students' interest and needs.

3 To keep the discussion focused on the pieces in the display, after a student selects the unit, you might temporarily set aside labels that contain numbers for which there are no pieces on the wall display (e.g., if the strip-mat is the unit, then there are no pieces on the display for 100, 1000, or 10000). After all of the pieces on the display have been labeled, ask students to imagine and describe the base ten pieces, (and their location on the wall chart) associated with the labels you set aside.

Having students come forward in pairs is reassuring to them. Before a pair decides where to place a label, have them show it to the class so other students can privately (mentally) predict its placement. You may wish to give pairs the option of drawing a different label or selecting another volunteer to join them if they get stuck.

4 (Optional) Repeat Action 3 for the linear piece wall display.

4 Although the relationships among the linear pieces are not as easy to "see" as pieces get larger, the values of the linear pieces are multiplied or divided by powers of 10 as the unit is moved, just as in the case of the area pieces.

Focus Teacher Activity (cont.)

ACTIONS

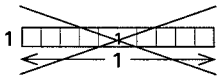
5 Remove the labels from the displays. If it hasn't come up, briefly discuss the convention of using squares as units of area.

Attach the "unit" label to the mat and tell the students that for the remainder of this activity, this represents the area unit and the length of its side is the linear unit. Based on this unit, have groups form a minimal area piece collection for 11 tenths and 13 hundredths. Discuss.

COMMENTS

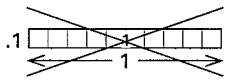
5 Although the choice of an area unit or a linear unit is arbitrary (as seen in Lesson 19), it is customary to use a square as the area unit and the length of its side as the linear unit. This idea is investigated in more detail by students in Lesson 35 and discussion should be limited here.

Notice that if a nonsquare area piece is selected as the unit, its dimensions cannot be 1×1 (since the lengths of the sides are not equal). For example, if the strip is the area unit both dimensions cannot be 1 linear unit:

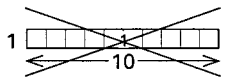


Not possible: both sides are not equal in length.

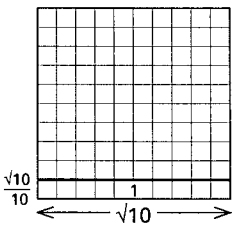
In addition, if the length of one side of a nonsquare area unit is assigned the value of 1 linear unit, then the area of the rectangle is not equal to the product of its dimensions. For example, if the strip is 1 area unit and either side is selected as 1 linear unit:



Not possible: $.1 \times 1 \neq 1$



Not possible: $1 \times 10 \neq 1$



Note that the strip with area 1 unit has dimensions $\sqrt{10} \times {}^{\sqrt{10}}/_{10}$ (this is for teacher information only and is explored by students in later *Visual Mathematics* courses.)

The minimal collection for 11 tenths and 13 hundredths is 1 unit, 2 tenths, and 3 hundredths.

Focus Teacher Activity (cont.)

ACTIONS

6 Give each group a copy of Focus Student Activity 25.1 to complete. Discuss their results and have volunteers complete the chart on a transparency of the activity.

Whole Number and Decimal Numeration Lesson **25**

Focus Student Activity 25.1

NAME _____ DATE _____

1 Use your base ten area pieces to form a minimal collection for each of the following sets of pieces:

a) 9 tenths, 21 hundredths,

b) 1 unit, 4 tenths, 10 hundredths,

c) 15 tenths,

d) 105 hundredths,

e) 10 units, 10 tenths, 10 hundredths,

f) 10 units, 9 tenths, 11 hundredths.

2 Record your minimal collections for a)-f) on this chart.

	100 hundreds	10 tens	1 units	1/10 tenths	1/100 hundredths
a)					
b)					
c)					
d)					
e)					
f)					

7 Point to the entries a), e), and f) on the completed chart from Focus Student Activity 25.1, write the numeral 111, and observe that usually 111 means one hundred eleven. Ask the groups to think of ways to represent entries a), e), and f) so there is no confusion. Then discuss decimal notation.

COMMENTS

6 Here are the minimal collections for these sets of number pieces.

	100 hundreds	10 tens	1 units	$\frac{1}{10}$ tenths	$\frac{1}{100}$ hundredths
a)			1	1	1
b)			1	5	
c)			1	5	
d)			1		5
e)		1	1	1	
f)		1	1		1

7 Some students may suggest a decimal point immediately because of their past experiences. Accept it, but encourage students to invent other systems of their own. Historically, many systems were used. For the collection in Action 5, we write "1.23," but in early times this number was written in forms such as:

$1|23$ $\quad 1(23)$ $\quad 1\overline{|23}$ $\quad 1^{23}$

In England, Europe, and the Scandinavian countries the same decimal is written, respectively, as:

$1 \cdot 23$ $\quad 1{,}23$ $\quad 1,^{23}$

In our system, the period in 1.23 is called a *decimal point* and is always located between the units digit and the tenths digit.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

In entry f) more than a decimal point is needed to resolve all ambiguity. A placeholder is needed to show that there are no tenths. While we normally use a zero, 11.01, students may enjoy creating their own methods during the discussion.

You may wish to introduce more collections of base ten pieces at this point to give the students additional practice in finding minimal collections and writing decimal names.

8 Write the following decimal numbers on the overhead and ask the students to represent the numbers with their base ten area pieces:

1.21 2.07 3.1 3.10 .15 .02 5

8 You may also wish to ask students to imagine or sketch collections that represent a few numbers like 10.01, 121.1 or 101.01. For convenience in representing some larger numbers, some students may suggest "changing the unit." Encourage this. Interesting questions to investigate could be: Imagine trading all base ten pieces for 121.1 into tenths. How many tenths would there be? Or, imagine trading all base ten pieces for 101.01 into hundredths. How many hundredths will there be altogether? etc.

9 Ask the students how they would read the names of the numbers listed in Action 8 if they were talking to someone over the telephone. Discuss ways of reading decimals.

9 There are several ways that a number like 1.21 can be read.

a) One way that is easy to say and easy to understand is, "one point two one."

b) Another possibility is, "one and two tenths and one hundredth."

c) A common way is, "one and twenty-one hundredths."

Method c) is the usual textbook version. The whole-number part of the expression (to the left of the decimal point) is read as a whole number, the word "and" is used for the decimal point, and the decimal part (to the right of the decimal point) is also read like a whole number followed by the name of the smallest base ten piece used. So, for example, 123.45 is read:

"one hundred twenty-three *and* forty-five hundredths."

Visualizing the number piece collection for 123.45 may help one understand this way of reading the number. The collection consists of 1 hundred piece, 2 ten pieces, 3 unit pieces, 4 tenth pieces, and 5 hundredth pieces.

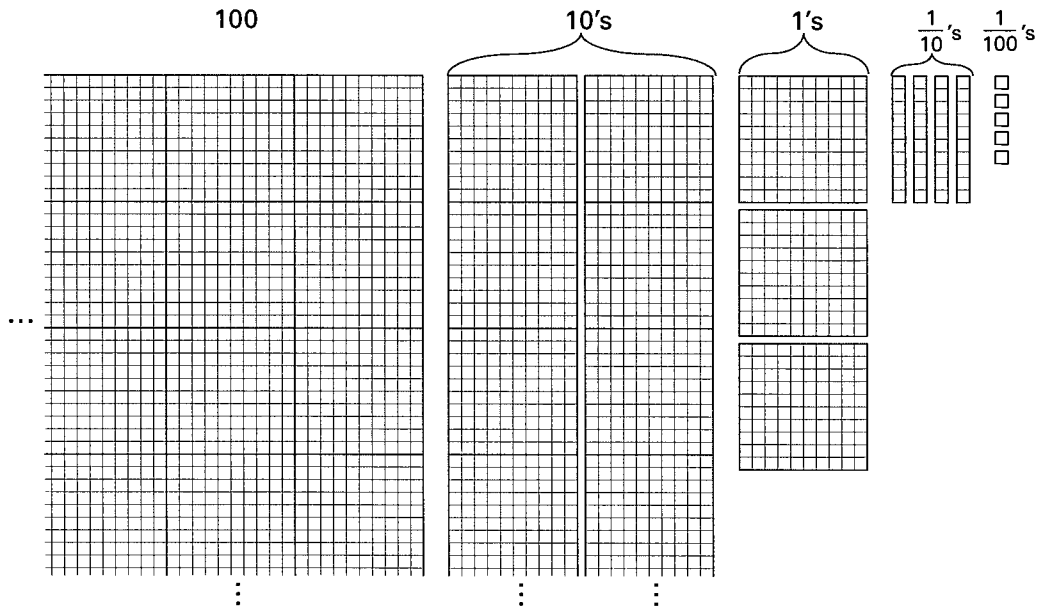
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

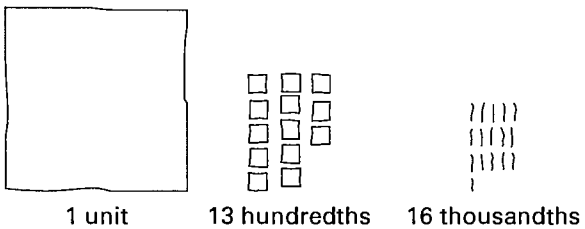
9 (continued.)
"One hundred twenty-three" describes the whole number part of this collection (the word "twenty" means "two groups of ten"). For the fractional part, the 4 tenths can be traded for 40 hundredths. This gives a total of 45 hundredths. A decimal number that has a whole number part and a decimal fractional part is called a *mixed decimal.*



You may also wish to have students represent numbers such as .12, .29, and .8 with their pieces and then exchange the tenths for hundredths to see that they yield 12 hundredths, 29 hundredths, and 80 hundredths, respectively.

10 Ask the students to sketch a collection that has 1 unit, 13 hundredths, and 16 thousandths. Then ask them to sketch the minimal collection that has the same value and write the decimal number which represents it. Discuss ways of reading this number.

10 You may wish to remind students that for a sketch it isn't necessary to show all the subdivisions on an area piece. One unit, 13 hundredths, and 16 thousandths can be represented as follows, where the thousandths are formed by cutting each hundredth into 10 little rectangles:

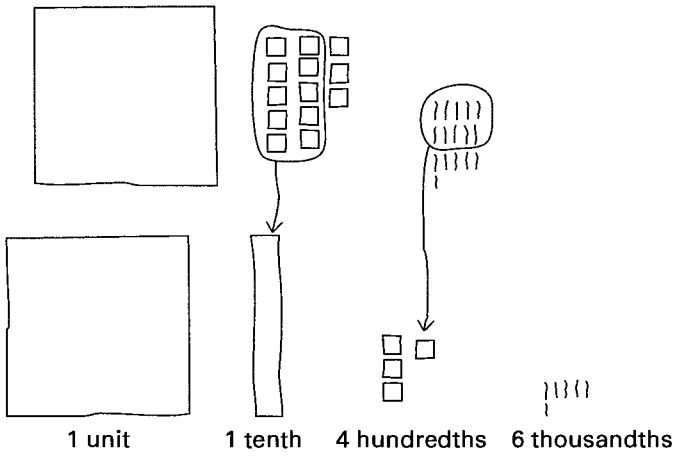


Focus Teacher Activity (cont.)

ACTIONS

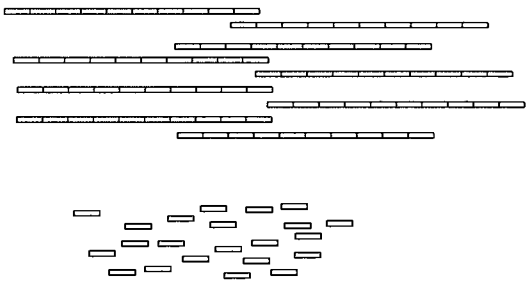
COMMENTS

The trading needed to represent the pieces in base ten can also be done with a diagram:



This collection represents the decimal 1.146 and is read as, "one and one hundred forty-six thousandths."

11 Place or sketch the following linear pieces on the overhead and ask the groups to form the same collection. Then ask them to determine the length of the line segment formed by placing these pieces end-to-end. Have volunteers share their groups' methods and conclusions. If it doesn't come up, ask students to determine a decimal representation of this length.



11 Using the convention that the linear unit is the length of the edge of the square unit, the collection shown contains 9 linear units and 22 tenths linear units (based on the area unit given in Action 5).

Some students may describe the total length as 11 linear units plus $^2/_{10}$ linear unit. Other students may point out that the minimal collection of linear pieces for this collection is 1 chain (10), 1 unit, and 2 tenths, or $(1 \times 10) + (1 \times 1) + (2 \times {}^1/_{10})$. Still others may note that it is 90 + 22 = 112 tenths. On a place value chart these can be recorded as:

100	10	1	$\frac{1}{10}$	$\frac{1}{100}$
		11	2	
	1	1	2	
			112	

Notice that the second collection is the minimal one. Its decimal representation is 11.2.

Focus Teacher Activity (cont.)

ACTIONS

12 (Optional) Repeat Action 11 for 1 or more of the following collections of linear pieces. Ask the students to determine the decimal representation of each length and ways of reading the decimal.

a) 17 linear units, 13 tenths,

b) 38 tenths,

c) 12 linear units, 27 hundredths,

d) 116 hundredths,

e) two hundred three hundredths,

f) $2\frac{1}{2}$ units, $1\frac{1}{2}$ tenths,

g) $34\frac{3}{4}$ units.

COMMENTS

12 Students will need to imagine, cut, or sketch hundredths. Here are the minimal collections and decimal representations for each collection.

	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	decimal
a)		1	8	3		18.3
b)			3	8		3.8
c)		1	2	2	7	12.27
d)			1	1	6	1.16
e)			2	0	3	2.03
f)			2	6	5	2.65
g)		3	4	7	5	34.75

Note that fraction and decimal relationships are explored in depth in Lesson 29. The intent in f) and g) is for students to use their knowledge of fractions to form the decimal collections.

Follow-up Student Activity 25.2

NAME _____ DATE _____

1 Draw each of the following on base ten grid paper. Label each drawing to show the base ten number it represents.

a) □ is one square unit. Draw the minimal collection for 32 square units.

b) ⊢ is one linear unit. Draw the minimal collection for 12 linear units.

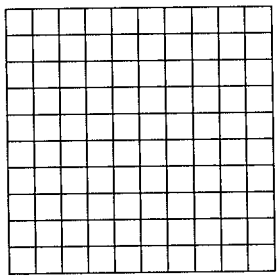
c) □□□□□□□□□□ is one tenth. Draw 2 units.

d) □□□□□□□□□□ is one unit. Draw 2.3 units.

e) ⊢⊣ is 1/10. Draw 1/100.

f) ⊢⊣ is 1. Draw 3.7.

g) ⊢⊣ is 1. Draw a length that is between .2 and .3 linear units (be sure to write the number your drawing represents).

2  is one square unit. Draw and label the following on base ten grid paper:

a) 1.2
b) 1.02
c) 1.20
d) 0.12
e) 10.2
f) 2
g) 2.1
h) 1.002

3 Imagine or sketch the minimal collection for each number below. Then imagine trading all the pieces in each collection for *hundredths only*. Write how many total hundredths there are in each collection.

a) 4.13 b) 8.9 c) 2.07

(Continued on back.)

Follow-up Student Activity (cont.)

d) Explain how you found the total hundredths in b).

4 In this space sketch a minimal number piece collection for 3.247.

5 Imagine exchanging all the number pieces in Problem 4 for thousandths. Describe the results.

6 Imagine exchanging all the pieces in Problem 4 for hundredths. Describe the results.

7 Imagine (or sketch) the number piece collections for the following numbers. Label the unit you use for each collection. Then write in words *two different ways* that you could read each number if you were talking to someone over the telephone.

a) 1.27

b) 72.04

c) 127.654

8 The odometer of a car reads 35467.21. The digit 5 stands for 5000 miles. Tell what each of the other digits stands for:

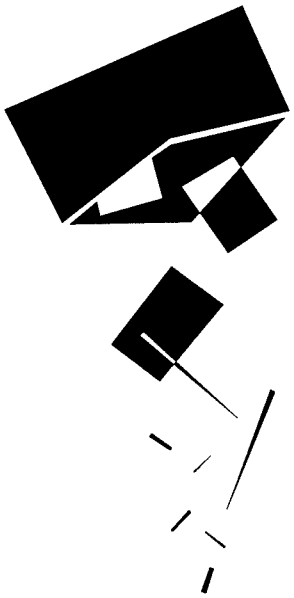
a) 1 c) 3 e) 6

b) 2 d) 4 f) 7

Whole Number & Decimal Addition & Subtraction Lesson 26

THE BIG IDEA

Base ten area and linear pieces provide a conceptual basis for exploring addition and subtraction with whole numbers and decimals. Operations with decimals are an extension of whole number operations—only the unit changes. Mental strategies for adding and subtracting evolve naturally from students' experiences with the base ten number pieces.



CONNECTOR

OVERVIEW

Students use base ten area and linear pieces to relate the models for addition and subtraction explored in Lessons 2 and 23 to sums and differences of multidigit whole numbers. They use their actions with the pieces as a basis for mentally picturing and computing sums and differences of whole numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Connector Student Activity 26.1 (optional), 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

Base ten area and linear pieces provide a model for understanding addition and subtraction of decimals. These models become the basis for mental addition and subtraction strategies.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per group, cut apart and placed in an envelope.
- ✔ Focus Master B, 1 transparency, cut apart and placed in a container.
- ✔ Toothpicks, 6 per student (break each toothpick into 5 approximately equal lengths).
- ✔ Base ten area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students teach an adult about their understanding of decimal concepts and decimal addition and subtraction.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 26.2, 1 copy per student.
- ✔ Base ten area and linear pieces (paper or cardstock), 1 set for each student.

LESSON IDEAS

LOOKING AHEAD

To give students additional practice with decimal sums and differences at this conceptual level, provide base ten grid paper (see *Blackline Masters*) for students to use to find area and linear piece sums and differences. Such experiences help students develop strong visual images and conceptual understanding that makes mental computation simple and natural.

Regularly pose problems for mental computation. Periodically ask students to write explanations of their mental strategies in their journals. Promote the development of several strategies by asking students to describe more than one mental strategy for certain problems. The mental techniques students develop will be the basis for methods of estimating and approximating sums and differences in Lesson 31.

FOLLOW-UP

Rather than waiting until students finish this lesson to assign the Follow-up, allow them to begin drafting and preparing their presentation ideas 2-3 days ahead. The day of their presentation to an adult, you could have them rehearse with and get feedback from a classmate.

JOURNAL

Have students respond to the following thought starter: "What is most difficult for me about mentally computing sums and differences is... What is easiest is..."

SELECTED ANSWERS

Responses will vary.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

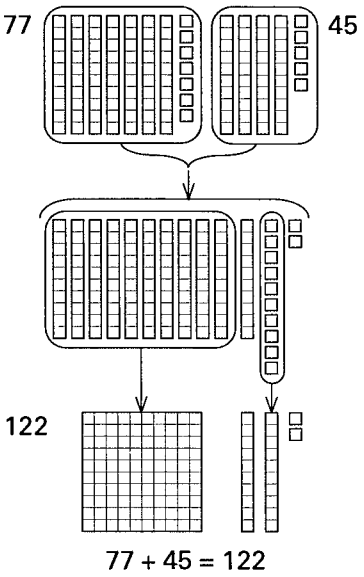
Students use base ten area and linear pieces to relate the models for addition and subtraction explored in Lessons 2 and 23 to sums and differences of multidigit whole numbers. They use their actions with the pieces as a basis for mentally picturing and computing sums and differences of whole numbers.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Connector Student Activity 26.1 (optional), 1 copy per group and 1 transparency.

ACTIONS

1 Arrange the students in groups and distribute base ten area and linear pieces to each student. Ask the students to form a minimal collection to represent 77. Repeat for 45. Next ask students to find a method of using base ten pieces to determine 77 + 45. Discuss.



COMMENTS

1 Watch for evidence that students apply their understanding of the meaning of addition (see Lesson 2, Basic Operations) to their actions with pieces. See also *Opening Eyes to Mathematics* for background explorations of addition and subtraction of whole numbers.

Allow students to choose the unit. Some may select different units to represent 77 and 45, and hence, run into difficulty combining the collections to form 77 + 45. If this happens, allow time for contradictions to emerge and for students to discover the need for using the same unit to represent all numbers within a given problem.

One way to find the sum is to combine area piece collections for 77 and 45 and then make exchanges, as shown at the left. Note that some students may combine the tens pieces, trade for a hundred, and then combine and trade the units pieces. Others may combine and trade units and then combine and trade tens. Either of these methods or another ordering of combinations and trades are acceptable and should be discussed. Some students may feel uncomfortable not "starting at the right" because of rules they have been given previously. Hence, these students may need encouragement to use the method that makes sense to them rather than to feel constrained by rules.

Another method of adding 77 + 45 is to combine linear piece collections for the numbers and make exchanges.

Another interesting possibility for adding 77 + 45 is to form *equal sums* (a useful mental addition strategy, see Action 7) by shifting pieces from one collection to the

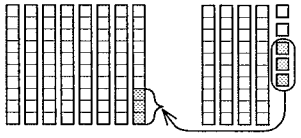
(Continued next page.)

Connector Teacher Activity (cont.)

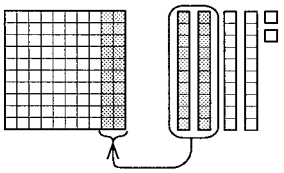
ACTIONS

COMMENTS

1 (continued.)
other before adding. For example, in the following diagram 3 units are shifted from 45 to 77 to get 77 + 45 = 80 + 42 (note that the total number of pieces has not changed):



The above method could be extended further by shifting 2 tens to show 80 + 42 = 100 + 22 = 122, as shown below. Note that both 80 + 42 and 100 + 22 are easier to add than 77 and 45 (they are also easier to mentally picture and combine).

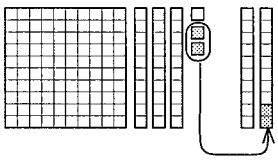


The process of rearranging numbers to *make hundreds*, as illustrated by the last step above, is especially convenient for mental computation.

Throughout this activity many comments emphasize visual methods that are useful for mental computation. If these same methods aren't suggested by students you may wish to share them.

2 Ask the groups to each determine several different ways of using their base ten pieces to compute 133 + 18. Observe the students' methods and discuss several.

2 There are many ways to compute this sum. One method you may wish to share (if a student doesn't suggest it) is to form equal sums by removing 2 units from 133 and adding those to 18, showing that 133 + 18 = 131 + 20 (see below). This new sum, 131 + 20, is convenient to add, whether using the actual pieces or mental pictures of them.



133 + 18 = 131 + 20

3 As needed, repeat Action 2 for these sums:

a) 46 + 27 + 88 b) 156 + 229

3 Some students may find a sum without exchanging pieces (e.g., a student may think of 10 units as 1 "ten" without physically making the exchange). Keep emphasis on having students demonstrate and verbalize their actions. Opportunities to invent and record algorithms occur in Lesson 27.

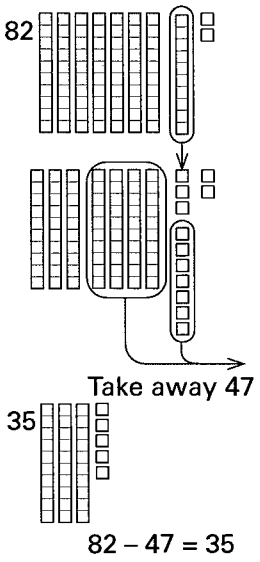
Connector Teacher Activity (cont.)

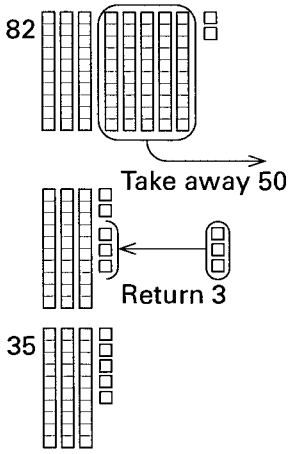
ACTIONS

4 Write the arithmetical expression 82 – 47 on the overhead or chalkboard. Ask the students to devise a way to use their base ten pieces to perform this subtraction. Discuss some of the methods they use.

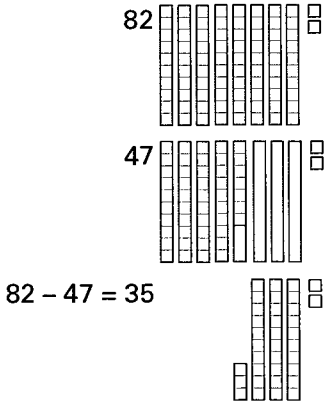
COMMENTS

4 Look for examples of both the *take-away* and *difference* methods. Many students will subtract using the *take-away* method. In this case, they will take 4 tens and 7 units from a collection of 8 tens and 2 units, after exchanging one of the 8 tens for 10 units:





Another way to take 47 units from a collection of 8 tens and 2 units is to take 5 tens from the collection and return 3 units *change,* as shown at the left. This method is often useful for mentally picturing and computing differences.

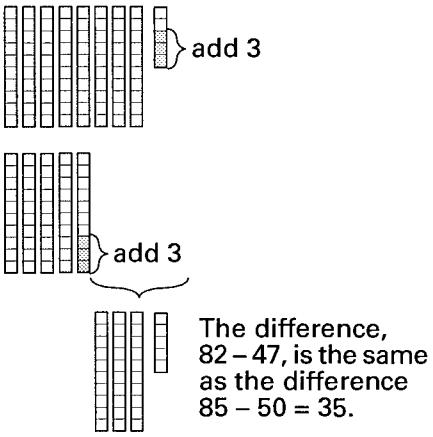


The subtraction can also be done by the *difference* method. In this method, a collection is found that makes up the difference between collections for 82 and 47, as shown at the left. Some students may determine this difference by *adding up* from 47 to 82 (i.e., 47 + 3 + 10 + 10 + 10 + 2, so 82 – 47 = 3 + 10 + 10 + 10 + 2 = 35).

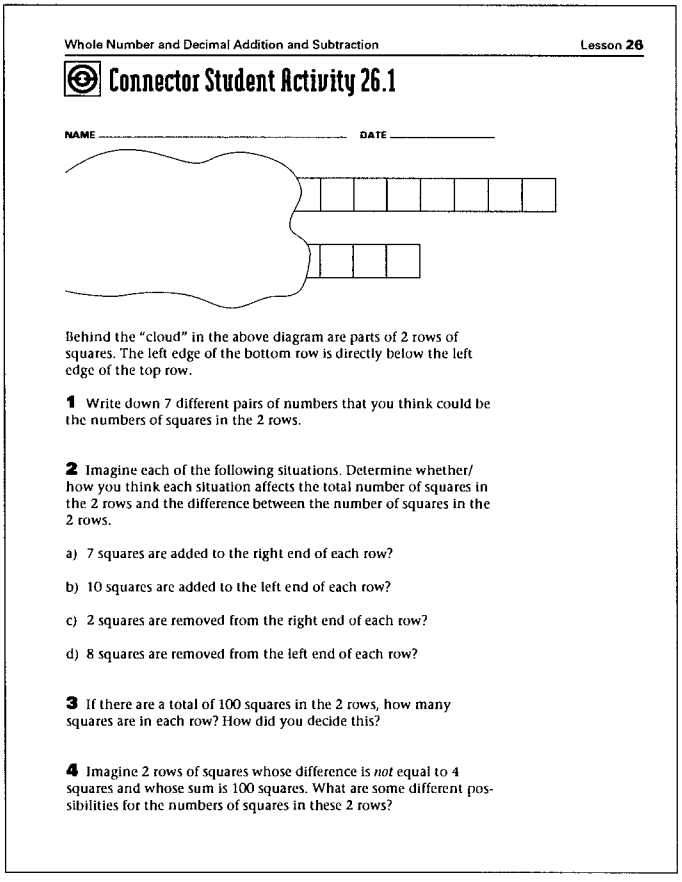
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



5 (Optional) Give each group a copy of Connector Student Activity 26.1 to complete. Discuss their conclusions and observations.



Whole Number and Decimal Addition and Subtraction Lesson **26**

Connector Student Activity 26.1

NAME _____ DATE _____

Behind the "cloud" in the above diagram are parts of 2 rows of squares. The left edge of the bottom row is directly below the left edge of the top row.

1 Write down 7 different pairs of numbers that you think could be the numbers of squares in the 2 rows.

2 Imagine each of the following situations. Determine whether/how you think each situation affects the total number of squares in the 2 rows and the difference between the number of squares in the 2 rows.

a) 7 squares are added to the right end of each row?

b) 10 squares are added to the left end of each row?

c) 2 squares are removed from the right end of each row?

d) 8 squares are removed from the left end of each row?

3 If there are a total of 100 squares in the 2 rows, how many squares are in each row? How did you decide this?

4 Imagine 2 rows of squares whose difference is *not* equal to 4 squares and whose sum is 100 squares. What are some different possibilities for the numbers of squares in these 2 rows?

COMMENTS

4 (continued.)
Yet another method (also convenient for mentally picturing and computing differences) involves finding *equal differences*. For example, the illustration at the left shows that to compute 82 – 47 one could add 3 to *both* collections (without changing the difference) to get 82 – 47 = 85 – 50. The difference is easy to "see" and doesn't require making trades.

All of the preceding methods can be modeled similarly with linear pieces, although linear pieces are not as convenient to use when numbers are large.

5 The purposes of this activity are to strengthen students' mental images of the difference concept of subtraction and to reinforce visual models for equal differences and equal sums (see Comments 1, 2, and 4 and *Opening Eyes to Mathematics*), which are useful methods for mental computation.

A few observations are: the difference between the numbers of squares is always 4; increasing or decreasing both rows by the same number of squares does not change the difference between the numbers of squares (but it does change their sum); equal sums can be formed by adding a certain number of squares to one row and subtracting the same number of squares from the other row (i.e., shifting squares from one row to the other doesn't change the total number of squares); forming equal sums changes the difference between the numbers of squares.

Connector Teacher Activity (cont.)

ACTIONS

6 Repeat Action 4 for one or more of these subtractions:

a) 183 – 65

b) 245 – 78

c) 412 – 267

7 Write the numbers 26 and 59 on the overhead and ask the students to: picture in their mind's eye the area piece collections for these numbers and to mentally determine their sum. Discuss a variety of students mental images and strategies. Repeat as appropriate for other sums and differences. Here are some you could select from:

a) 153 + 89

b) 153 – 89

c) 288 + 199

d) 288 – 199

e) 56 + 71

f) 71 – 56

g) 1005 + 345

h) 1005 – 345

i) 70 + 52 + 30

j) 35 + 52 + 15

k) 318 + 146 + 12 + 3

l) 340 + 17 – 30 + 13

m) 123 – 48

COMMENTS

6 You may want to have the class try specific methods used by students and/or methods described in Comment 4 and discuss which they prefer and why.

7 Some students will benefit from hearing others describe ways their mental images of area piece collections aid each computation. This supports the development of conceptually based mental computation strategies.

One method that several students may use to add 26 + 59 is to shift 1 unit from 26 and add it to 59 to get the equal sum 26 + 59 = 25 + 60, which could be viewed as 20 + 5 + 60 = 20 + 60 + 5 = 85.

Some problems are conveniently solved by adding *compatible* numbers (i.e., numbers whose sum or difference is a "nice" number, such as a power or multiple of 10). For example, in i) 70 and 30 are compatibles as are 35 and 15 in j). As mentioned in earlier actions to promote the development of a range of strategies, periodically ask students to try a specific method or a problem.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Base ten area and linear pieces provide a model for understanding addition and subtraction of decimals. These models become the basis for mental addition and subtraction strategies.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per group, cut apart and placed in an envelope.
- ✔ Focus Master B, 1 transparency, cut apart and placed in a container.
- ✔ Toothpicks, 6 per student (break each toothpick into 5 approximately equal lengths).
- ✔ Base ten area and linear pieces for the overhead.

ACTIONS

1 Place the students in small groups and give each student a set of base ten area pieces and about 30 toothpick pieces. Give each group an envelope of Up – Down Cards (see Focus Master A). Tell the students that, for this Action, the large square in their set of area pieces is the unit, and each toothpick piece represents $\frac{1}{1000}$ unit.

Write the number 1.829 on the overhead and tell the groups to play Trading Up and Down according to the following procedures:

a) Each group begins with 1 area unit and their envelope of Up – Down Cards in the center of the table.

b) Player 1 in each group draws a card from the envelope and places it face up on the table.

c) Depending on the card drawn, Player 1 adds thousandths to the unit on the table or exchanges the unit for an equivalent collection and removes the thousandths. The number on the card indicates the number of thousandths that Player 1 must add or remove. UP indicates to add thousandths, and DOWN indicates to remove them.

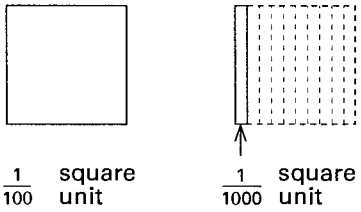
d) Player 1 checks the new collection and makes all necessary trades in order to form a minimal collection, replaces the card in the envelope, and mixes the cards.

e) Play proceeds to the next group member, who repeats procedures b)-d), adding to or subtracting from the previous collection and leaving a new minimal collection for the next person.

f) Repeat the above process.

COMMENTS

1 Forming smaller pieces was introduced in Lesson 25. If the large square is 1 unit, then the strip of 10 small squares is $\frac{1}{10}$ and a small square is $\frac{1}{100}$. The next smaller piece would be formed by cutting a small square into 10 rectangles of value $\frac{1}{1000}$ each.



Since thousandths pieces are difficult to cut from the small squares and are a challenge to handle, toothpick pieces (formed by cutting or breaking a toothpick into 5 approximately equal pieces) can be used in their place. Flat toothpicks are easier to break than round ones and stay in place on desks or tables.

To prepare Up – Down Cards for one group: run a copy of Focus Master A on cardstock; cut apart the cards; place the cards (except the blank one) in an envelope or another container.

To avoid confusion, it may be helpful to model the process of playing Trading Up and Down at the overhead by drawing one Up card and one Down card and having volunteers describe ways to add and remove those amounts.

Since a group's total number of thousandths decimal pieces may be less than some of the numbers in the envelopes, groups will need to devise ways of adding or removing equivalent amounts using other pieces. This

(Continued next page,)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

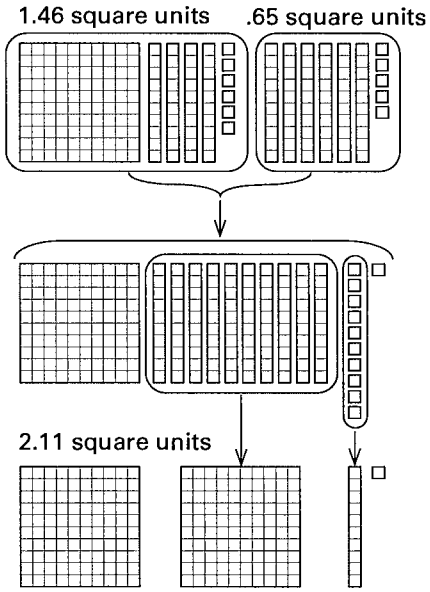
1 (continued.)
activity develops readiness for adding and subtracting decimals and reinforces place value concepts.

An interesting variation of this activity is to have students draw two cards at a time.

2 After groups have played Trading Up and Down for 10-15 minutes (or until they can comfortably make appropriate exchanges), have each group determine the one card they would need to draw or add to their set of cards in order to have exactly 1.829 units. Discuss their methods of deciding the value of the card they need, which group needs the greatest/least amount of thousandths, etc.

2 Acknowledge all methods that work. It isn't expected that students use algorithms to compute these values. Rather, most will use informal strategies based on area piece relationships.

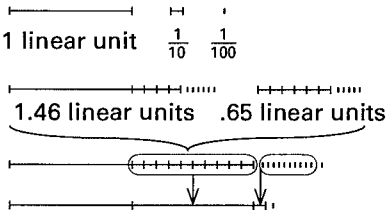
3 Distribute linear pieces to each student. Write the expressions 1.46 + .65 and 2.3 + 1.9 on the overhead and ask the students to use their base ten area and/or linear pieces to find the sums. Have volunteers show their methods at the overhead.



1.46 + .65 = 2.11

3 Using the large area piece square as 1 area unit, combining area piece collections for 1.46 and .65, and making exchanges, results in the minimal collection for 2.11 shown at the left.

Modeling the operation 1.46 + .65 with linear pieces is more difficult to show since $\frac{1}{100}$ of a linear unit is so small. However, relationships among the pieces and exchanges of pieces are similar to those with area pieces. Students may find it easier to sketch (see below) such linear representations than to use linear pieces.



2.11 linear units

1.46 + .65 = 2.11

You may wish to have students show their procedures on the overhead. Different approaches will show that the order in which exchanges of base ten pieces are made does not affect the sum. Encourage students to try ideas explored with whole numbers during the Connector activity (e.g., same sums) to see how they work for decimals.

Focus Teacher Activity (cont.)

ACTIONS

4 Select from the following for students to compute with their base ten pieces (including toothpick thousandths).

a) .467 + .391

b) .099 + .564

c) 1.67 + 3.33 + 2

d) 1.06 + 1.6 + 1.34 + 2.009

e) 207 + 59

Observe methods the students use and discuss some of them.

5 Write the arithmetic expression 1.21 – .54 on the chalkboard or overhead. Ask the students to devise a way of using their base ten pieces to perform this subtraction. Have volunteers explain their methods.

COMMENTS

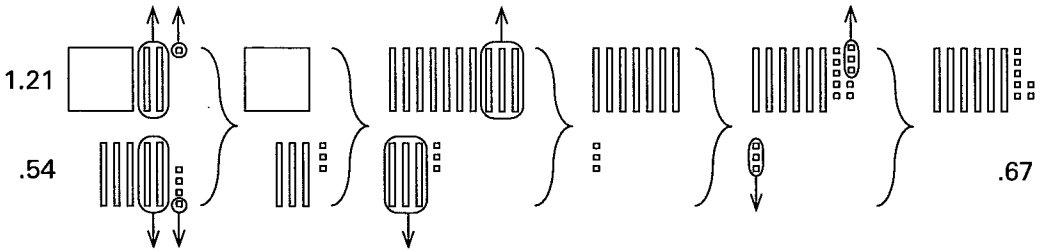
4 Keep emphasis on having students demonstrate and verbalize their actions with the base ten pieces or sketches of the pieces. Opportunities to invent and record algorithms occur in Lesson 27.

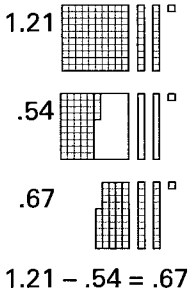
Help build students' comfort with verbalizing decimals by having them express numbers such as 1.34 both as "1 point 34" and as "1 and 34 hundredths." Sums which involve few base ten pieces can be shown on the overhead. Sums involving many base ten pieces are often better demonstrated on a table or the floor or by making a sketch.

Notice that to model e) it is useful to change the unit.

5 Ask students who have used different methods to share them with the class. Following are two common methods. Discuss several that come up.

Method A (shown below): Here the same amount is taken from both sets of area pieces until one set is gone, making exchanges as necessary (*equal differences* method). Note that this method is based on the idea that the difference remains the same as long as equal amounts are taken from both numbers.





Method B (shown at the left): In this method a collection of area pieces is found which makes up the difference between the two sets of area pieces (*add-up* method). Some students will count the difference by adding up from .54 to get 1.21. For example .54 + .06 + .4 + .2 + .01 = 1.21, so the difference is .06 + .6 + .01 = .67.

Again, similar methods will work for linear pieces, but they are not as convenient to handle.

Focus Teacher Activity (cont.)

ACTIONS

6 Select from these subtractions for students to compute with their base ten pieces:

a) 1.024 – 0.831

b) 3.42 – .7

c) 2.3 – 1.39

d) 2 – 1.009

e) 442 – 149

f) 1.25 – 1.09 + .84

Observe the methods the students use and discuss.

7 Explain the following procedures for the activity, Mental Sums and Differences:

a) Without looking, a student volunteer selects a transparent Mental ± Card (see Focus Master B and Comment 7) from a container and places the card on the overhead.

b) Without talking, the student volunteer and all other students in the class use *images* of base ten pieces to mentally compute the sum or difference on the card (i.e., no pencil or paper).

c) The student volunteer describes his or her mental strategies and picks another volunteer classmate to describe a different mental strategy.

c) Or, if stuck, the student volunteer picks two other volunteers to describe their mental strategies.

d) Repeat this process.

Lesson 26 Whole Number and Decimal Addition and Subtraction

Focus Master B

Mental ± Cards

(Copy on a transparency and cut along solid lines.)

4.8 + 5.4	**19 + 63**	**8.7 – 4.9**
.41 + .55	**49.7 + 18.3**	**35 + 52 + 15**

COMMENTS

6 There are many possible approaches to these computations. You may want to suggest that the students try different methods of subtracting with their number pieces. Note that to model e) it is helpful to change the unit.

Here are a few other computations students could explore:

a) 1.98 + 3 + 2.1

b) 12.2 + 4.05

c) 6.7 – 1.14

d) 3.8 – 2

e) 4 – 1.07

7 Prior to class, make a transparency of Focus Master B, cut apart the cards, and place them in a container.

Encourage students to visualize the base ten pieces to help with their mental computations. Discussing several possibilities for each computation will help students develop new techniques. If students are mentally carrying out the traditional algorithm or using one method repeatedly, have them try out specific strategies (e.g., equal sums or differences, a particular method a student used earlier, etc.). Some students may need practice with using *compatible numbers* to "make tens," "make hundreds," etc. To do this you could pose simple combinations like: 3 + 8 + 7; 6 + 12 + 4; 75 + 13 + 25; 2.65 + 1.2 + 2.35; 8.2 + 1.6 + 1.8.

It is helpful to have students discuss mental computation strategies regularly. To do this you could only do a few of these now and/or add more problems to this collection. Have students spend 5-10 minutes repeating this activity weekly. Avoid posing complex sums and differences that are more appropriately computed with a calculator.

To help students understand why each others' methods work, you may wish to have students use base ten pieces to illustrate their mental strategies. It is also helpful to periodically ask students to describe in their journals their mental strategies for solving a particular computation. The strategies developed in this lesson will be useful later for estimation of sums and differences.

Follow-up Student Activity 26.2

NAME _____ **DATE** _____

1 Please read the letter on the back of this sheet. Then fill in the blank after *Dear* to show the name of the adult you would like to teach about your thinking. Sign your name in the space at the bottom of the letter.

a) Have the adult read the letter *before* you do your presentation. Plan to spend about 30 minutes on this assignment.

b) *After* your presentation, complete the following:

What I did was...

I feel very good about the way I presented...

because...

Something that was difficult or challenging for me during this presentation was...

Tomorrow I plan to focus my attention during class discussion on the following ideas about decimals so that I can understand or explain them better:

(Continued on back.)

Follow-up Student Activity (cont.)

Dear _____,

In math right now, we are learning about addition and subtraction of decimals. Instead of memorizing the results of someone else's thinking, we are going to use our understanding of the meaning of addition and subtraction and our understanding of the meaning of decimals to do the work of real mathematicians—beginning tomorrow we will *invent* a variety of systems and rules for operating with decimals! Please keep the rules you know for adding and subtracting decimals a secret for now—it will be fun to compare later. Since explaining my thoughts and ideas helps me to know what I understand and what I have questions about, would you take about 30 minutes to listen while I explain my thinking about *at least four* of the following:

a) how the base ten pieces are related to each other and what even bigger and smaller pieces look like (and why I think this is so);

b) what happens to the value of pieces when I "change the unit" and why this happens (I'll give you some examples);

c) what addition and subtraction mean to me;

d) how I can use the base ten pieces to add 252 + 149 and then to add 3.47 + 1.67 (keep me talking about my thinking while I work!);

e) how I can use the base ten pieces to subtract 232 – 174 and 4.06 – 2.38;

f) how I can use mental pictures of the base ten pieces to mentally compute 3.99 + 4.26 and 5.3 – 2.7.

It is okay if I get stuck. One important reason for sharing my thinking with you is so that I can know where to focus my attention and ask questions during class. After I finish, would you help me by reminding me to write down ideas that were difficult for me to explain? Thanks for listening to my thinking!

_____

Student's signature

Please write down two aspects of my presentation you think I should be pleased about. Then sign your name.

_____

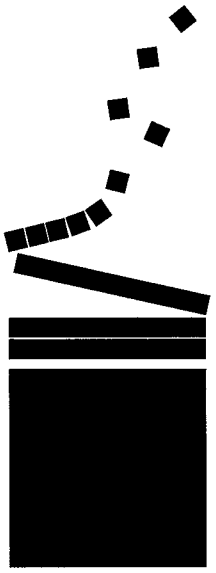
Adult Listener's signature

Inventing Addition & Subtraction Algorithms Lesson 27

THE BIG IDEA

Numbers and math symbols are tools for communicating one's thoughts and actions. Algorithms are generalizations of particular sequences of thoughts and actions.

Students can gain important insights about the mathematical process when they have opportunities to invent recording systems that reflect their own thoughts and actions, to generalize those systems (i.e., create algorithms) and to analyze and compare their algorithms to others.



CONNECTOR

OVERVIEW

Students rely on their understanding of the basic operations and the role of parentheses in mathematical notation to translate mathematical expressions into actions with tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Tile and linear pieces for the overhead.
- ✔ 1-inch linear pieces (see Lesson 2), 30 per student.

FOCUS

OVERVIEW

Students use base ten pieces to find whole number and decimal sums and differences and then invent methods of using numbers and math symbols to record their actions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Poster paper, 1 sheet per group (2 pairs) of students.
- ✔ Marking pens, 1 per group (2 pairs) of students.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Focus Master A (2 pages), 1 copy per group of students.

FOLLOW-UP

OVERVIEW

Students use numbers and math symbols to communicate their area piece methods for finding decimal sums and differences. They describe methods of mentally computing sums and differences of decimals.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 27.1, 1 copy per student.
- ✔ Base ten area and linear pieces, 1 paper or cardstock set per student for use at home.

LESSON IDEAS

ASSESSMENT

Problems on the Follow-up reveal students' views of the relationship between concepts and algorithms. It is important to watch (over time) for evidence that students view algorithms as ways of generalizing and recording mathematical relationships rather than viewing algorithms as *the* mathematical relationships.

QUOTE

As they begin to understand the meaning of operations and develop a concrete basis for validating symbolic processes and situations, students should design their own algorithms and discuss, compare, and evaluate them with their peers and teacher. Students should analyze the way the various algorithms work and how they relate to the meaning of the operation and to the numbers involved.

NCTM Standards

SELECTED ANSWERS

Responses will vary.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students rely on their understanding of the basic operations and the role of parentheses in mathematical notation to translate mathematical expressions into actions with tile.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Tile and linear pieces for the overhead.
- ✔ 1-inch linear pieces (see Lesson 2), 30 per student.

ACTIONS

1 Distribute tile and linear pieces to each student and write the following expressions on the overhead. Ask the students to use the tile and/or linear pieces to model the meanings of the symbols. Have students show their ideas at the overhead.

a) $(4 + 3) - (2 + 1)$

b) $4 - (3 - 2) + 1$

c) $4 + 3 - 2 + 1$

d) $4 - (3 + (2 - 1))$

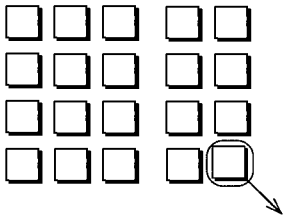
e) $4 \times (3 + 2) - 1$

f) $4 \times (3 + 2 - 1)$

COMMENTS

1 In Lesson 2 students first explored the meanings of the operations. In Lessons 3 and 4 they learned about order of operations and learned that parentheses in a mathematical statement indicate which actions to carry out first. In the Focus students will need to use these ideas to record their actions with base ten pieces.

Here is one possibility for e):



$4 \times (3 + 2) - 1$

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use base ten pieces to find whole number and decimal sums and differences and then invent methods of using numbers and math symbols to record their actions. Various students' methods are examined and generalized.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Poster paper, 1 sheet per group (2 pairs) of students.
- ✔ Marking pens, 1 per group (2 pairs) of students.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Focus Master A (2 pages), 1 copy per group of students.

ACTIONS

1 Arrange the students in pairs and give each student a set of base ten area pieces. Ask the students to set aside their pencils, fold their hands, and watch quietly (no comments or questions) as you carry out the following:

COMMENTS

1 It is important that you not talk or explain your actions, choice of unit, etc., and that students observe without discussing or questioning your procedures.

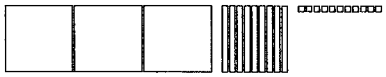
Collection 1 Collection 2

a) *Without talking* place these two collections on the overhead:

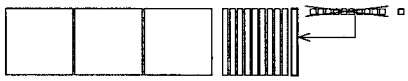
b) Push all of the large squares together:

c) Push all of the strips together:

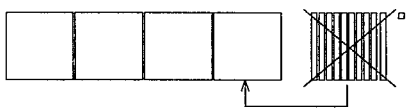
d) Push all of the small squares together:



e) Trade 10 small squares for 1 strip and place this strip with the others:



f) Trade 10 strips for 1 large square and place this large square with the others:

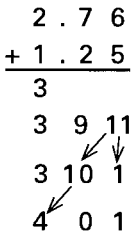


Focus Teacher Activity (cont.)

ACTIONS

2 Repeat Action 1, again without talking, but this time encourage students to make notes, diagrams, or sketches to keep track of your actions in order.

3 Ask the pairs of students to discuss and to invent ways to tell the "story" of your actions without using pictures or words, but rather using *only* numbers and arithmetic symbols. Have volunteers share their recordings at the overhead.



4 Ask the pairs to each:

a) create one decimal addition problem and one decimal subtraction problem that they could solve with the base ten pieces (or sketches of the pieces) and that would each require trading pieces;

b) use the pieces or diagrams of them to solve their problems.

5 Ask the pairs to write, *using words only*, an explanation of their procedures in Action 4 so that someone reading their explanation would be able to repeat exactly their actions with the pieces.

6 Have the pairs each exchange their explanations with another pair of students and take turns trying to duplicate each other's actions with the pieces (with no coaching from the authors!). Next have the pairs discuss differences between how the instructions were interpreted and the actual methods used by the authors. Have them identify ways to improve their explanations.

COMMENTS

2 After completing this, rather than answering questions about your methods, repeat once more steps a)-f), if needed.

3 The intent of this action is to emphasize the role of symbols and numbers in communicating mathematically. Students may suggest you were adding 276 + 125, 2.76 + 1.25, 27.6 + 12.5, etc. depending on their choice of unit. As you circulate give selected pairs of students a blank transparency and overhead pen. Ask them to record their numerical statements on the transparencies prior to sharing them with the class. Having these prepared in advance saves time at the overhead.

Some students may object to recording systems that "start on the left" because of prior experiences with rules for adding. If this happens, remind them their recordings are to tell the story of your actions, in order, rather than what someone feels you *should* have done. At the left is an example of what one pair of students recorded to describe the procedures in Action 1.

4 To encourage a variety of methods, you might remind students that there is no single "best" procedure for computing with base ten pieces. The choice of procedures often depends on the numbers being computed.

Note that all actions in this activity could be carried out with linear pieces. However, area pieces are more convenient.

5 The purpose of this action is to get a *verbalization* of their methods. As students write, circulate and as needed remind them to use words only.

6 It is important that the authors do not orally explain their methods until their partners have decided on their interpretation of the written explanation and have modeled this interpretation with the pieces.

Focus Teacher Activity (cont.)

ACTIONS

7 Give each group of 4 students (formed in Action 6) a sheet of butcher paper and a marking pen. Ask them to invent a way to *use numbers and arithmetic symbols only* to represent each step of their area piece methods for their addition and subtraction problems. Remind them that each recording should make clear all steps of their methods and the order in which they did them.

Post these and discuss a few. Encourage students to decide whether methods generalize to selected other sums/differences.

COMMENTS

7 As you circulate while students work, rather than giving them suggestions about ways to record their methods, tell them what actions their symbols suggest to you. Students may then decide how, or if, they wish to refine their recordings to better reflect *their thinking*.

Leave these posters up for a few days and encourage the students to study each other's posters and speculate about the actions suggested (groups could tape their verbal explanations to the back of their posters).

As examples, following are 3 different ways students might determine and record 2.64 + 3.78 (the comments in parentheses describe the actions represented by the symbols).

Example Recordings of 2.64 + 3.78

Method A: Exchanges are made as individual pieces are combined, starting with the units.

units .	tenths	hundreths	
2 .	6	4	(minimal collection for 2.64)
+3 .	7	8	(minimal collection for 3.78)
5			(result of combining units)
1	3		(result of combining tenths and exchanging 10 for 1 unit)
	1	2	(result of combining hundredths and exchanging 10 for 1 tenth)
6 .	4	2	(minimal collection for sum)

Method B: Combine minimal collections for 2.64 and 3.78 and then make exchanges, beginning with the units.

1's	$\frac{1}{10}$ths	$\frac{1}{100}$ths	
2	6	4	(minimal collection for 2.64)
+3	7	8	(minimal collection for 3.78)
5	13	12	(combined collection)
5	14	2	(combined collection after exchanging 10 hundreths for 1 tenth)
6	4	2	(combined collection after exchanging 10 tenths for 1 unit)

Method C: Make exchanges as individual pieces are combined, starting with the hundredths.

1's	$\frac{1}{10}$ths	$\frac{1}{100}$ths	
2	6	4	(minimal collection for 2.64)
+3	7	8	(minimal collection for 3.78)
	1	2	(combine the hundredths, exchange 10 hundredths for 1 tenth)
1	4		(combine the tenths, exchange 10 tenths for 1 unit)
6			(combine the units)

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Example Recordings of 64.7 – 38.5

Method A: Remove 3 tens from 6 tens, 4 units, and 7 tenths and record the result; exchange 1 ten for 10 units and record the result; remove 8 units and record the result; remove 5 tenths and record the result.

10's	1's	$\frac{1}{10}$ths
6	4	7
–3	0	0
~~2~~ 3	~~14~~ 4	7
–0	8	0
2	6	7
–0	0	5
2	6	2

Method B: Take, 5 tenths from 7 tenths, leaving 2 tenths. Next, take 8 units from the initial collection. In order to do this, 1 ten is exchanged for 10 units so there are now 5 tens and 14 units (recorded above the initial collection). Taking 8 units from 14 leaves 6 units. Finally, taking 3 tens from 5 tens leaves 2 tens. This reflects the procedure generally taught in school mathematics.

~~5~~ 6	~~1~~ 4	. 7
–3	8	. 5
2	6	. 2

Method C: As before, take 5 tenths from 7 tenths, leaving 2 tenths. Now add the same amount to both collections: 10 units to the top collection and 1 ten to the bottom collection. Then take 8 units from 14 leaving 6 units, and take 4 tens from 6 tens leaving 2 tens. This procedure was popular many years ago and depends on the observation that the difference between two collections of number pieces is not changed if the same amount is added to both collections (equal differences).

6	1 4	. 7
~~4~~ ~~5~~	8	. 5
2	6	. 2

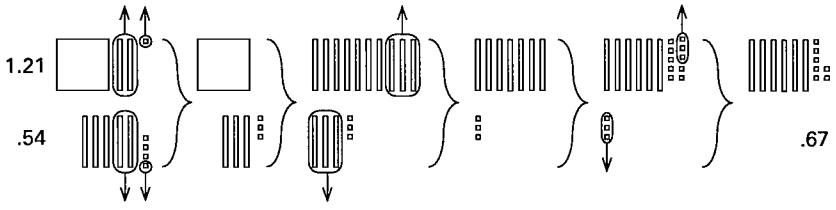
COMMENTS

7 (continued.)
Subtraction is sometimes more difficult for students to record. At the left are three methods of finding and recording 64.7 – 38.5:

Following are 2 possible methods of finding and recording 1.21 – .54.

Example Recordings of 1.21 – .54

Method A:



Remove the same amount from both sets of number pieces until one set is gone, making exchanges as necessary.

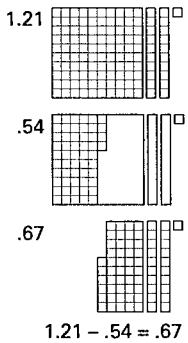
A symbolic recording of this method could be:

1.21 – .21 – .3 – .03 = .67
.54 – .21 – .3 – .03 = 0.00
} The difference is .67

Method B:
Find the collection of area pieces which make up the difference between the two sets of area pieces. Compute the difference by determining the total amount added to .54 to get 1.21.

.54 + .06 + .4 + .2 + .01 = 1.21

so, .67 = 1.21 – .54



1.21 – .54 = .67

Focus Teacher Activity (cont.)

ACTIONS

8 Distribute a copy of Focus Master A (2 pages) to each group and have them carry out the instructions. Then discuss the groups' conclusions and reasoning.

Inventing Addition and Subtraction Algorithms Lesson **27**

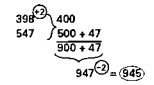
Focus Master A

Students in another class were given several problems to solve using area pieces. Then they used numbers and arithmetic symbols to tell the "story" of their actions with the pieces. Here are five stories about several computations. For each story, do the following on another sheet of paper:

a) Try to use the pieces according to the "story" shown. Then explain what you think were the step-by-step procedures the student used.

b) Create two more addition or subtraction problems that you think would be convenient to solve using this method. Then use the method to solve them. Do you think this method would always work?

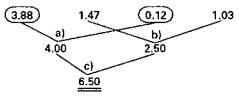
Maria's Story 398 + 547:



Carl's Story for 2.99 + 2.57:

a) 2.99 + .01 = 3.00
b) 2.57 − .01 = 2.56
c) 5.56

Lauren's Story for 3.88 + 1.47 + 0.12 + 1.03:



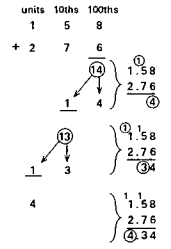
Tania's Story for 7.36 − 5.27:

a) 7.36 + .03 = 7.39
b) 5.27 + .03 = 5.30
c) 2
d) 0
e) 9
f) 2.09

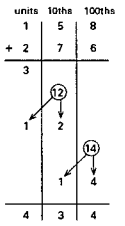
Lesson **27** Inventing Addition and Subtraction Algorithms

Focus Master A (cont.)

Roland's Story for 158 + 276:



Sandra's Story for 158 + 276:



Kieran's Story for 46 + 117:

46 + 117 = 117 + 46
= 110 + 40 + 7 + 6
= 150 + 10 + 3
= 163

Camilla's Story for 5.20 − 3.75:

5.20
− .05
a) 5.15

8.15
− .70
b) 4.45

4.45
−3.00
c) 1.45

Anna's Story for 5.05 − 3.57:

a) ((((3.57 + .03) + .4) + 1.0) + .05) = 5.05
3.6 4.0 5.0 5.05

b) (((.03 + .4) + 1.0) + .05) = 1.48
.43 1.43 1.48

COMMENTS

8 Exploring certain of these "stories" can support the development of mental strategies discussed in Lesson 26; other stories can bring understanding to traditional procedures. To shorten this activity, you could suggest that each group pick 3 or 4 students' stories to investigate or you might pick stories reflecting methods that haven't been suggested by students. Following are possible interpretations of some of the stories.

Maria's procedures might be described as follows: form the minimal collection for 398 + 547; add 2 units to 398 to get 400; combine the collections to get 900 + 47 = 947; remove the 2 units that were added earlier to get the sum, 945. Note that a sum is unchanged if the same amount is both added and subtracted.

Carl appears to have added .01 to 2.99 and removed .01 from 2.57 to get 3.00 + 2.56, (an easier sum to compute than the original problem 2.99 + 2.57).

It looks like Lauren used the method of combining compatible numbers. To add 3.88 + 1.47 + 0.12 + 1.03 she combined 3.88 + 0.12 (.88 and .12 are compatible because they sum to 1). Then she combined compatibles 1.47 and 1.03 to get 2.50. Finally she added the 4.00 + 2.50 to get 6.50.

Tania seems to have added equal amounts to 7.36 and 5.27 to get 7.39 and 5.30 whose difference, 7.39 − 5.30, is the same as 7.36 − 5.27, but easier to compute. By comparing or taking away (it isn't clear which) she found the difference in the units, tenths, and hundredths, in that order, to get 7.36 − 5.27 = 2.09.

Anna appears to have used a counting up method to find 5.05 − 3.57. She started with a collection of 3.57 and added pieces until she had a new collection of 5.05. Then she totalled her add-ons to find the difference, 1.48.

Roland's story is similar to the usual paper-and-pencil procedure for addition, while Camilla's story illustrates the usual procedure for subtraction.

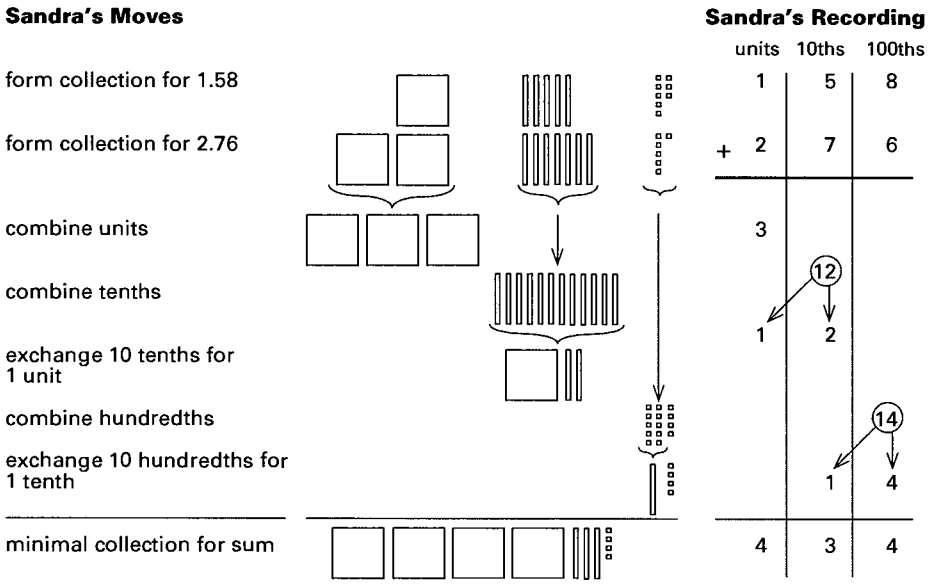
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

8 (continued.)
The following diagrams show the actions Sandra's "story" seems to suggest:



9 Discuss students' ideas about the values and drawbacks of choosing a single algorithm as the one for everyone to use.

9 A method that repeats a process for solving a certain type of problem is called an *algorithm*, or rule. One value of agreeing on one algorithm is that it could simplify communication about actions associated with addition. A drawback could be that in certain situations (e.g., for mental computation) a different order of combining number pieces may be more convenient. Another drawback is that although it may be easy to memorize symbolic procedures without understanding, it is equally easy to forget such procedures when they have no meaning.

Focus Teacher Activity (cont.)

ACTIONS

10 Have the students add and subtract decimals using a calculator. Discuss the students' ideas about relative advantages and disadvantages for adding and subtracting decimals using:

a) base ten pieces,

b) paper-and-pencil algorithms,

c) sketches,

d) calculators,

e) mental strategies, and

f) estimation.

Ask for sample addition and subtraction problems they would prefer to compute using each of the above options for calculating sums and differences.

COMMENTS

10 You may wish to ask questions such as: Which method(s) is/are easiest for you to understand? Which is/are easiest to picture in your mind? Which method(s) would you prefer for explaining decimal addition and subtraction to someone who doesn't understand it? Which method(s) would you prefer when you're in a hurry? Which method(s) would you use if you wanted to be certain the solution was correct? Note that students' may choose different methods for different reasons.

Some students may question the need for developing algorithms with the base ten pieces when they are able to use a calculator. Although it is important that students have ready access to calculators for performing tedious calculations, note that methods explored in this activity promote understanding of decimals and will be helpful in approximating sums and differences and determining the reasonableness of calculator computations. Further, sometimes it is more convenient to compute mentally than with a calculator. The base ten pieces help students to develop insights and mental images that make mental computation easier.

TEACHER NOTES:

Follow-up Student Activity 27.1

NAME _____ DATE _____

1 Use your area pieces to determine 3.24 + 2.79. Then write, *in words only*, a careful explanation of each step of your area piece methods.

2 Invent a way to use *numbers and arithmetic symbols only* to tell the step-by-step story of how you used area pieces to find 3.24 + 2.79.

3 Use area pieces to determine 3.372 – 1.581.

a) Sketch your area piece methods (show all trades):

b) Describe in words each step of your area piece methods:

c) Invent a way to use numbers and arithmetic symbols only to tell the step-by-step story of how you used your area pieces to find 3.372 – 1.581.

(Continued on back.)

Follow-up Student Activity (cont.)

4 Explain three different ways to compute 2.74 + 4.26 mentally. Circle the method you like best.

5 Explain three different ways to compute 15.3 – 9.8 mentally. Circle the method you like best.

6 Explain in your own words what the word *algorithm* means.

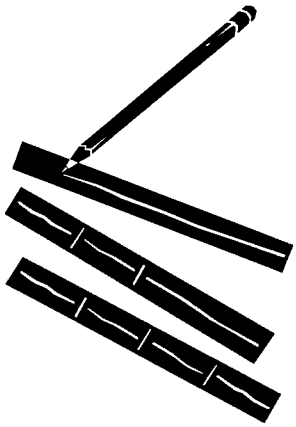
7 When do you think algorithms are helpful and when aren't they helpful?

Division Concept of a Fraction

Lesson 28

THE BIG IDEA

Using the division concept, a fraction is viewed in terms of division of whole numbers. For example, $\frac{3}{5}$ can be visualized as the value of 1 part when 3 units are divided into 5 equal parts. That is, $\frac{3}{5} = 3 \div 5$. This model can be used to visually represent many situations and relationships, including relating fractions and decimals.



CONNECTOR

OVERVIEW

Students recall the division and part-to-whole concepts of a fraction, which were introduced in Lessons 8 and 9.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.

FOCUS

OVERVIEW

Students explore paper folding of areas and lengths as a context for exploring the division concept of a fraction. They also model and make mathematical observations about situations involving the division concept.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 1" by 11" strips of paper, 10 per student.
- ✔ Focus Student Activity 28.1, 1 copy per student, and 1 transparency.
- ✔ Focus Master A, 1 copy per group and 2 transparencies.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Coffee stirrers, 15 per student.
- ✔ Scissors, 1 pair per group.
- ✔ Blank paper, 1 sheet per group.

FOLLOW-UP

OVERVIEW

Students apply the division model to determining fractional areas of figures.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 28.2, 1 copy per student.

LESSON IDEAS

ASSESSMENT

After the lesson ask the students to write about one or more of the following in their journals:

- Ideas or relationships involving fractions they know for the first time or now understand better, and to explain what has helped them most.
- Questions they have about fractions.
- Sketches of two interpretations of the meaning of $\frac{2}{3}$, one using the part-to-whole concept and one using the division concept, and for each diagram, a word problem whose solution is illustrated by the diagram.

FOLLOW-UP

Problem 6b) could be worked on by pairs or groups during class.

LOOKING AHEAD

It is intended that students' conceptions of fractions continue to develop in this lesson. However, note that fractions are the focus of Lessons 40-44.

TEACHER NOTES:

SELECTED ANSWERS

1. 5 square units.

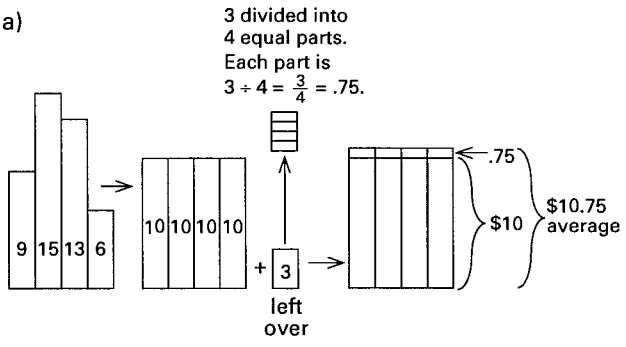
2. One possibility is to divide D into three equal parts to obtain 1 square unit, and then extend D to 5 square units. Another is to extend D to get a region of area 15 square units and then divide this into three equal parts.

3. One quarter of the circle has area $\frac{5}{4}$ square units.

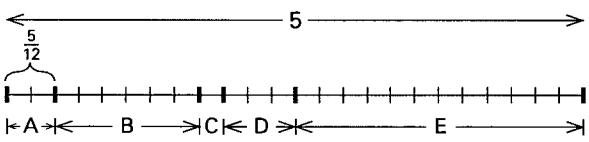
4. If the triangle is split into 2 congruent parts, 1 part has area $\frac{3}{2}$.

5. By the division model, $\frac{4}{3}$ means an area of 4 was divided into 3 equal parts. So, a region that is 3 times bigger than F has an area of 4 square units.

6. a)



6. b)



A = $\frac{5}{12}$, B = $\frac{15}{12}$, C = $\frac{5}{24}$, D = $\frac{15}{24}$, and $\frac{5}{2}$.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students recall the division and part-to-whole concepts of a fraction, which were introduced in Lessons 8 and 9. The division concept is examined more closely in the Focus.

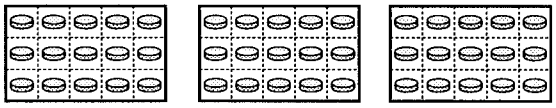
MATERIALS

✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.

ACTIONS

1 Arrange the students in groups and give each group a sheet of 1-cm grid paper. Write the following on the overhead and ask the groups to make diagrams on their grid paper to show ways the candy could be distributed among the 5 students:

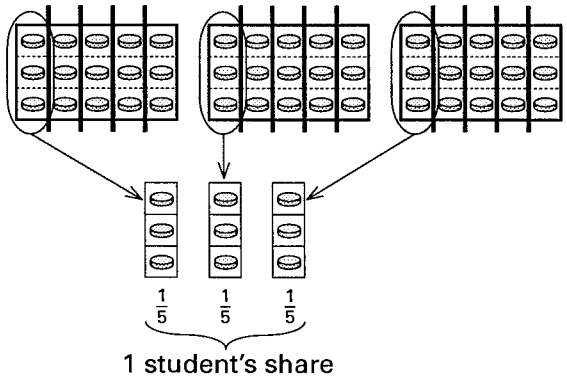
> **Five students wish to share equally 3 identical boxes of candy. Here are diagrams of the boxes:**



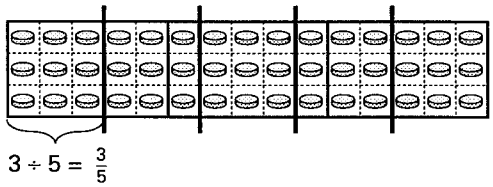
Discuss their ideas about subdividing the boxes and naming the portion each student receives. Point out examples of the *part-to-whole* and *division* concepts of the meaning of a fraction as they come up.

COMMENTS

1 The *part-to-whole* and *division* concepts of a fraction were both introduced in Lesson 8. One example of reasoning that reflects the part-to-whole concept (i.e., a fraction is viewed as a portion of a whole) is to divide each box into 5 equal parts and then give each student 1/5 of each box. Thus, each student receives 3 one-fifths, or 3/5 of a box, as shown here:



An example of reasoning that illustrates the division concept (i.e., a fraction is viewed as a quotient) is to subdivide the total amount into 5 equal parts. Each student receives 1 of these parts which is equivalent to $3 \div 5 = 1/5$ of $3 = 3/5$ of a box:



It is important that students view the part-to-whole and division concepts of fractions as two different *meanings* of fractions. There are certain contexts where one meaning makes more sense than the other. If the division concept isn't suggested by students as a way of viewing this situation, bring it up, since situations involving the division concept are emphasized in the Focus.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students explore paper folding of areas and lengths as a context for exploring the division concept of a fraction. They also model and make mathematical observations about situations involving the division concept.

MATERIALS

- ✔ 1" by 11" strips of paper, 10 per student.
- ✔ Focus Student Activity 28.1, 1 copy per student, and 1 transparency.
- ✔ Focus Master A, 1 copy per group and 2 transparencies.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Coffee Stirrers, 15 per student.
- ✔ Scissors, 1 pair per group.
- ✔ Blank paper, 1 sheet per group.

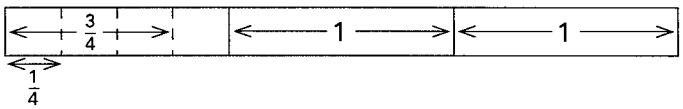
ACTIONS

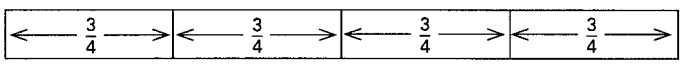
1 Place the students in small groups and give each student ten 1" by 1" strips of paper. Ask each student to label 1 strip "3 linear units" to indicate its length.

Ask the students to each turn their strip over and, keeping in mind its length is 3 linear units, have the groups:

> **Devise and carry out one or more methods (without using a ruler) of finding a length that is $\frac{3}{4}$ linear unit, and then mark this length on the strip.**

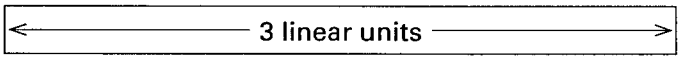
Ask for volunteers to share their methods at the overhead. Ask students to point out examples that illustrate the *part-to-whole* and *division* concepts of a fraction.





COMMENTS

1 Eleven inch strips are convenient to cut from sheets of paper. Any length is appropriate to use, although all strips should be the same length, and it is important not to tell the actual length to students. Here is an example of a labeled strip:



Not allowing rulers encourages students to rely on the meanings and models for fractions. The phrasing of your directions for this action is important, since certain phrasing may predispose students to specific methods. For example, by saying "find a length that is $\frac{3}{4}$ of a unit," you may indirectly suggest Method A shown below. It is helpful to write the instructions on the overhead and to point out that you are not asking for "$\frac{3}{4}$ of the strip."

One way to share ideas so all can see is to tape volunteers' marked strips on the wall so they are lined up directly under one another. Comparing the results of different (valid) methods should demonstrate that each length of $\frac{3}{4}$ linear units is the same. Following are 3 methods to illustrate if they are not suggested by students.

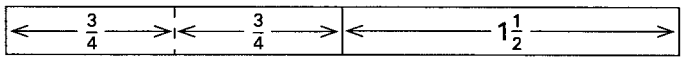
Method A: Fold the strip into 3 equal lengths of 1 linear unit. Then fold one of these lengths into 4 equal parts and mark off 3 lengths of $\frac{1}{4}$. This is an illustration of the *part-to-whole* concept of a fraction, in which a fraction is viewed as a portion of a unit.

Method B: Fold a 3-unit strip into 4 equal parts of length $3 \div 4 = \frac{3}{4}$ linear units. This method illustrates the *division* concept of a fraction in which a fraction is viewed

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



2 Ask the students to label the length of another strip 4 linear units and to paper fold to locate 4/6 by using the following ways of viewing 4/6:

a) as a portion of a whole (i.e., using the part-to-whole concept of fractions),

b) as a quotient (i.e., using the division concept of fractions).

3 Ask the students to each label the length of another strip as 7 linear units, and to devise methods of paper folding to locate 7/3.

4 Ask groups to label the length of another strip as 4/7 linear unit. Ask them to use that strip to help them form a length of 4 linear units. Ask volunteers to demonstrate and explain their methods.

COMMENTS

1 (continued.)
as a quotient. This was introduced in Lesson 9 and again in the Connector of this lesson.

Method C: Fold a strip in half to locate $3 \div 2 = 1\frac{1}{2}$, and then fold the $1\frac{1}{2}$ in half to form of a length of $1\frac{1}{2} \div 2 = \frac{3}{4}$ ($\frac{1}{2}$ of $1\frac{1}{2}$).

2 One way of illustrating 4/6 as a portion of a whole is to fold the strip into fourths to locate 1 unit and then fold 1 unit into 6 equal parts and mark 4 of those parts. Noting that 4/6 is equivalent to 2/3, some students may locate 1 unit and fold it into 3 equal parts and mark 2 of those parts.

A method of illustrating 4/6 as the quotient $4 \div 6$ is to fold the length of 4 into 6 equal parts. The length of one of these parts is 1/6 of 4 or $4 \div 6 = 4/6$ linear units. If a student recognizes $4/6 = 2/3$, they may apply the division model as follows: locate 2 linear units by folding the strip in half; fold a length of 2 into 3 equal parts; one part has length $2 \div 3 = 2/3$ linear units.

To reinforce the different meanings of a fraction you could repeat this action for 4/3 and/or 5/2.

3 Using the part-to-whole concept to locate 7/3 is awkward since it requires folding the strip into 7 equal parts (to locate 1 unit) and then folding the units into thirds. Using the division concept by paperfolding the strip into 3 equal lengths of $7 \div 3 = 7/3$ is more convenient.

Based on experiences in prior classes, some students may suggest it is "wrong" to write a fraction with the numerator larger than the denominator. Although such a fraction is called *improper*, it is appropriate to use this form. Rather than emphasizing a need to convert to mixed numbers, emphasize that in many contexts, such as the study of algebra, this form is preferred.

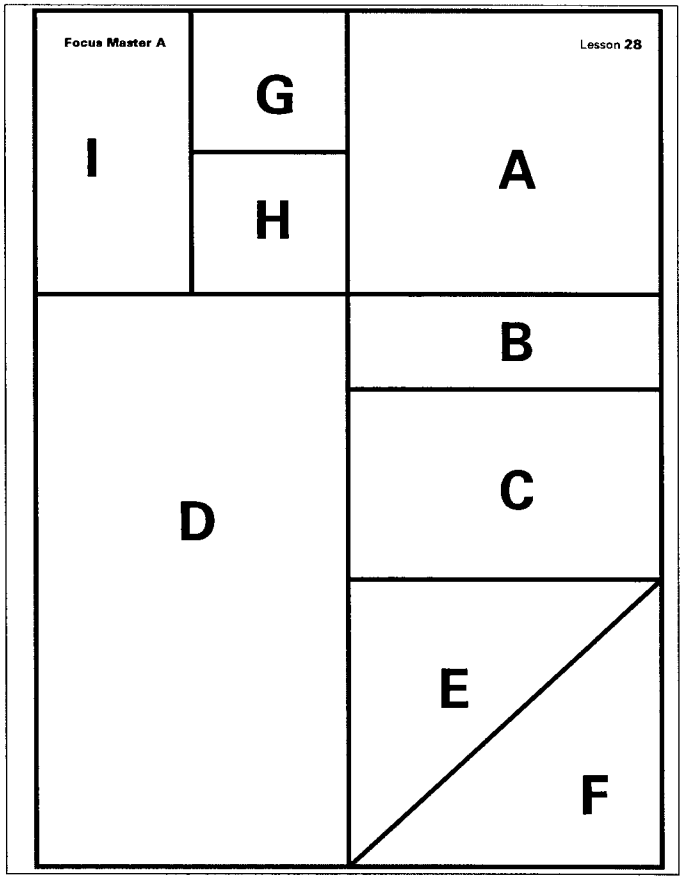
4 Here is one line of reasoning for locating 4 units that is based on the division concept of a fraction: 4/7 represents the length of one piece of a segment of length 4 that has been divided into 7 equal parts, so use 7 lengths of 4/7 to form a length of 4. Another approach is to paper fold the 4/7 strip into 4 equal parts to get 1/7. Seven of these parts equals 1 linear unit, which can be used to obtain 4 units.

Focus Teacher Activity (cont.)

ACTIONS

5 Distribute a blank sheet of paper to each student. Have them determine a method of folding this sheet to divide it into 6 congruent parts. Ask them to assume the area of the whole sheet is 11 and to determine the area of: 1 part, 2 parts (name this in more than one way), 3 parts, 4 parts, etc.

6 Give each group a copy of Focus Master A and a pair of scissors. Ask them to trim the border of the sheet and then determine the area of regions A-I, assuming the area of the whole sheet (without the borders) is 17 area units. Discuss.



COMMENTS

If needed, you could repeat for other lengths. For example, label a strip as $\frac{3}{5}$ linear units and use it to form a length of 3 linear units; use a length of $\frac{7}{3}$ to form a length of 7 linear units.

5 Each part has area $\frac{11}{6}$. The area of 2 of those parts can be viewed as 1 part of an area of 11 that is divided into 3 equal parts of $11 \div 3 = \frac{11}{3}$. Or, 2 parts could be viewed as 2 copies of $\frac{11}{6}$ or $\frac{22}{6}$, which is equivalent to $\frac{11}{3}$.

Similarly, 3 parts can be viewed as $11 \div 2 = \frac{11}{2}$ or 3 of $\frac{11}{6} = \frac{33}{6}$, and 4 parts can be viewed as 4 of $\frac{11}{6}$ and as 2 of $\frac{11}{3}$.

6 To save time, you could trim the borders prior to class. Each of these areas can be determined most readily using the division concept. The area of each region is as follows: A, $\frac{17}{6}$; B, $\frac{17}{18}$; C, $\frac{34}{18}$ or $\frac{17}{9}$; D, $\frac{17}{3}$; E, $\frac{17}{12}$; F, $\frac{17}{12}$; G, $\frac{17}{24}$; H, $\frac{17}{24}$; I, $\frac{17}{12}$.

For student explanations, it is useful to have two transparencies, one that is cut apart and one that is left intact. If groups cut apart their sheet to compare the areas of the parts, it is helpful to have extra copies of Focus Master A available for reference.

Focus Teacher Activity (cont.)

ACTIONS

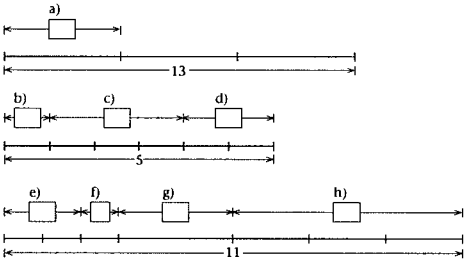
7 Distribute a copy of Focus Student Activity 28.1 to each group to complete. Discuss as needed.

Division Concept of a Fraction Lesson **28**

Focus Student Activity 28.1

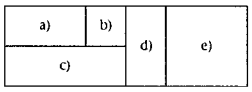
NAME _____ DATE _____

1 Write in the missing lengths in the boxes.



2 This field contains 3 acres. How many acres are in each part?

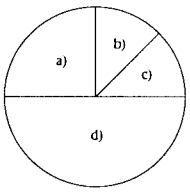
3 This rectangle has area 15. What is the area of each part?



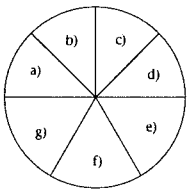
Lesson **28** Division Concept of a Fraction

Focus Student Activity 28.1 (cont.)

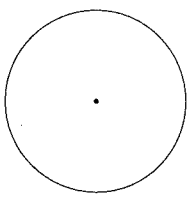
4 The area of this circle is 11. What is the area of each part?



5 This circle has an area of 7. What is the area of each part?



6 This circle has area 12. Subdivide it into parts with these areas: 6, $12 \div 4$, $^{12}/_{12}$, and $12 \div 6$.



COMMENTS

7 As alternatives, you could: have groups work on and discuss 2 or 3 problems at a time; assign only selected problems; or use this as a mid-lesson homework assignment.

It isn't necessary to emphasize simplifying the fractions since the improper forms communicate the relationships between each subdivision and the total value. For example, in 1a) $^{13}/_{3}$ represents the length of one subdivision of a segment of length 13 that has been divided into 3 equal parts. Although this is equivalent to $4\frac{1}{3}$, the symbol $4\frac{1}{3}$ does not as clearly reflect the situation.

Following are answers to the problems:

1. a) $^{13}/_{3}$
   b) $^{5}/_{6}$
   c) $^{15}/_{6}$
   d) $^{10}/_{6}$
   e) $^{11}/_{6}$ or $^{22}/_{12}$
   f) $^{11}/_{12}$
   g) $^{33}/_{12}$ or $^{11}/_{4}$
   h) $^{11}/_{2}$ or $^{66}/_{12}$ or $^{33}/_{6}$
2. $^{3}/_{5}$
3. a) $^{30}/_{12}$
   b) $^{15}/_{12}$
   c) $^{45}/_{12}$
   d) $^{30}/_{12}$
   e) $^{60}/_{12}$
4. a) $^{11}/_{4}$
   b) $^{11}/_{8}$
   c) $^{11}/_{8}$
   d) $^{11}/_{2}$
5. a)-d) $^{7}/_{8}$
   e)-g) $^{7}/_{6}$

Focus Teacher Activity (cont.)

ACTIONS

8 Distribute coffee stirrers, a sheet of blank paper, a pair of scissors, and one copy of Focus Master B to each group. Ask the groups to each build or sketch a model of Situation a) and to make mathematical observations about relationships they notice in their model. Repeat for selected other situations on Focus Master B.

Lesson **28** Division Concept of a Fraction

Focus Master B

Situations

Make a model (or a sketch of a model) of each of the following situations. In addition to the information that is given, what are some mathematical observations you can make about the situation by *studying your model*? Write your observations next to your model.

a) Marie purchased a 5-foot length of gold chain to make identical necklaces for 4 of her friends.

b) The 7-mile hike on the Wilderness Trail is divided into 3 equal parts by rest stations.

c) Three candy bars are shared equally by 8 people.

d) A roll of 300 feet of fishing line was divided equally among a group of people. Each person has $^{300}/_{4}$ feet of line.

e) A small tube of window caulking will caulk 30 feet. Each side of a square window is $^{45}/_{4}$ feet long.

f) A recipe for baked beans calls for $^{3}/_{4}$ of a cup of molasses. Bob used the recipe 4 times.

g) Laura built a rectangular run for her dog. It is 3 times as long as it is wide. She used 35 feet of wire fencing.

h) Courtney bought 3 licorice ropes to share with some of his friends. Each friend got $^{3}/_{7}$ of a rope.

i) Wally built a square rabbit pen that has a perimeter of 13 feet.

j) Willy used 10 feet of fencing to build a square rabbit pen. One side of the pen is the wall of the garage.

k) A problem found in Egyptian writings from 1650 B.C. requires that 4 loaves of bread be divided equally among 10 people.

l) Farmer Bales has a 150 acre farm. $^{150}/_{2}$ acres are planted in hay. $^{150}/_{4}$ acres are planted in corn. $^{150}/_{8}$ acres are covered by trees. $^{150}/_{16}$ acres are taken up by the house and barn. The rest of the farm is covered by a lake.

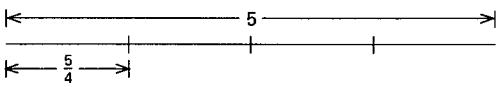
COMMENTS

8 Some students may find the coffee stirrers useful for modeling some of these situations. Other students may cut strips like those used in Actions 1-5. Still others may simply sketch their ideas. Student explanations at the overhead can be facilitated by giving each group a blank transparency and an overhead pen for preparing their demonstrations.

These situations are intended to give students practice with the division concept of a fraction. Select the number of situations you explore according to the needs and comfort level of students. Keep in mind that for most students, other than the brief exposure in Lesson 8 and 9, this is their first experience with the division concept. Do not expect mastery or spend too much time on this lesson. This concept will continue to come up in this course, and it will be explored in more depth in later *Visual Mathematics* courses.

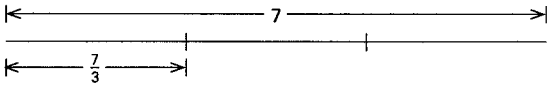
If at first students have difficulty making meaningful conclusions about their models, you might get them started by posing questions for them to answer based on their models. Here is a sample visual model and a sample observation for each of the situations posed:

a)



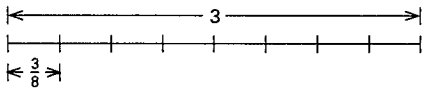
Marie could cut 4 equal lengths of $5 \div 4 = {}^{5}/_{4}$ feet.

b)



Each part of the hike is $7 \div 3 = {}^{7}/_{3}$ miles.

c)



Note that dividing a length that represents 3 candy bars into 8 equal pieces shows that each person receives $3 \div 8 = {}^{3}/_{8}$ of a candy bar. This is based on the division concept of a fraction. Using the part-to-whole concept, each candy bar would be cut into 8 equal pieces, and the 24 pieces distributed equally among the 8 people.

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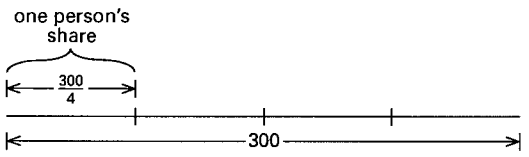
Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

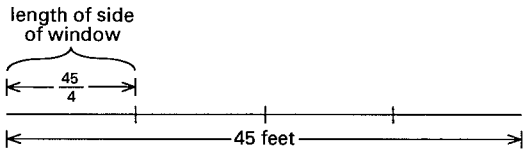
8 (continued.)

d)



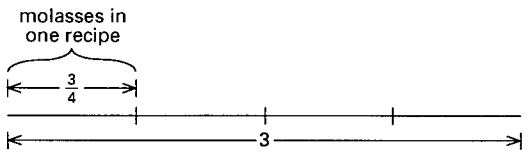
If a length of 300 feet is divided into 4 equal parts, each part has length $^{300}/_{4}$ feet. Therefore, since each person received an equal share of length $^{300}/_{4}$ feet, there are 4 people.

e)



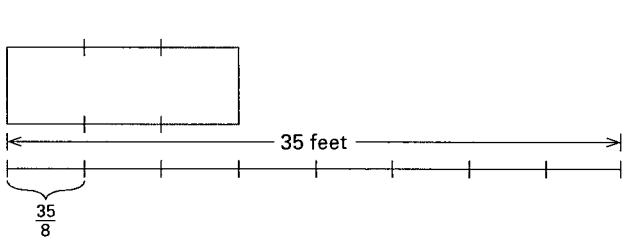
If a length of 45 feet is divided into 4 equal parts, each piece has a length of $^{45}/_{4}$ feet. Therefore, since all of the sides of this square window have an equal length of $^{45}/_{4}$ feet, the total distance around the window must be 45 feet. Since each tube caulks only 30 feet, 2 tubes must be purchased.

f)



$^{3}/_{4}$ cup describes the size of a portion if 3 cups is divided into 4 equal portions; therefore, since Bob used the recipe 4 times, he must have used 3 cups of molasses.

g)



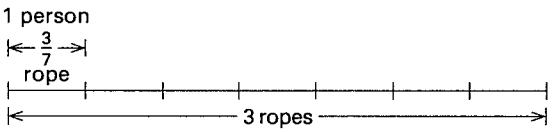
The dimensions are $^{35}/_{8}$ feet by $^{105}/_{8}$ feet.

Focus Teacher Activity (cont.)

ACTIONS

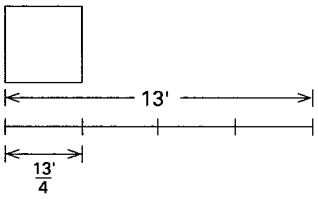
COMMENTS

h)



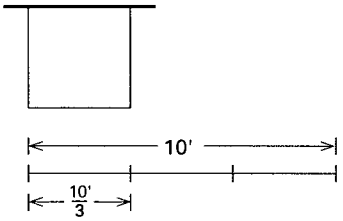
Courtney shared with 7 friends.

i)



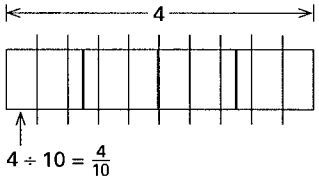
The side of the square is $13/4'$.

j)



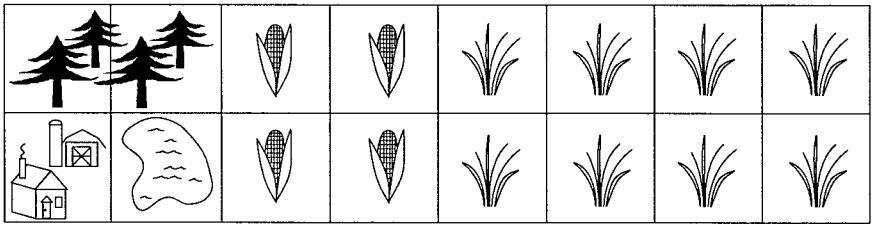
Each side of the pen is $10/3'$.

k)



Each person receives $4/10$ loaf of bread.

l)



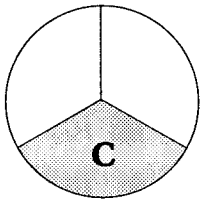
The lake covers $150/16$ acres.

TEACHER NOTES:

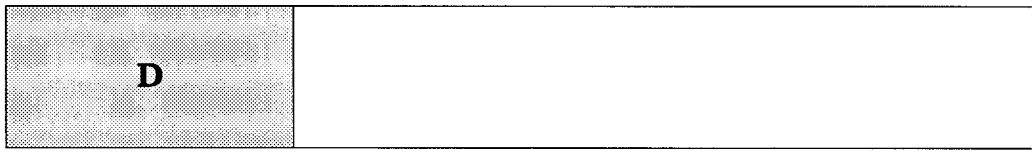
Follow-up Student Activity 28.2

NAME _____ **DATE** _____

1 The area of region C is $\frac{5}{3}$ square units. What is the area of the circle?

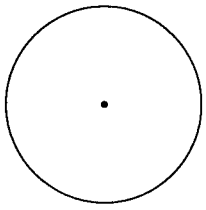


2 The area of regon D is 3 square units. Mark off a rectangle with area 5 square units.



Write an explanation of your method of locating 5 square units.

3 The area of this circle is 5 square units. Shade a part of the circle which has an area of $\frac{5}{4}$ square units. Next to the circle, explain your methods of determining the part of the circle with area $\frac{5}{4}$ square units.



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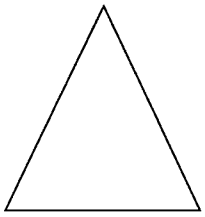
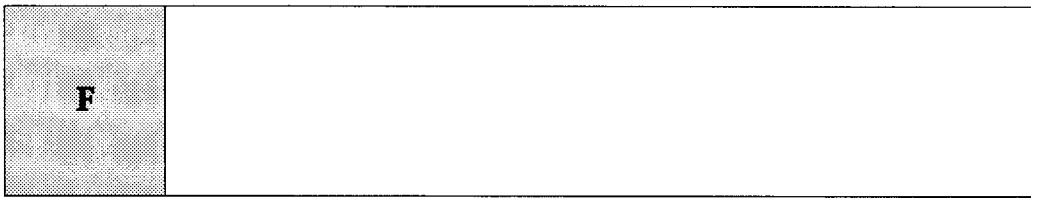
Follow-up Student Activity (cont.)

4 The area of this triangle is 3 square units. Shade a part of the triangle which has an area of $3/2$ square units.



5 The area of F is $4/3$ square units. Mark off a rectangle with an area of 4 square units:



Write an explanation of your method of marking off a rectangle of area 4 square units:

6 On a separate sheet of paper, show how to use a diagram or model to solve these puzzle problems:

a) During the last 4 weeks Michelle has done yard work after school for her neighbor. Her weekly earnings have been \$9, \$15, \$13, and \$6. What was her average weekly earnings for the 4 weeks?

b) Bill cut a 5-foot piece of rope into 5 "mystery" lengths and gave the following clues about the lengths:

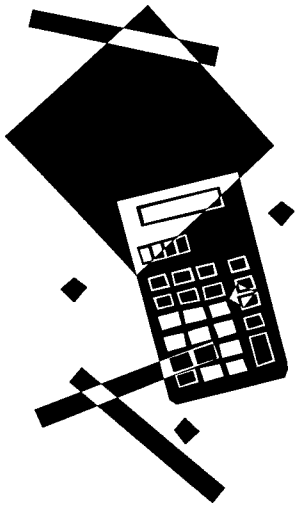
- If the 5-foot piece of rope were cut into 12 equal parts, Mystery Length A has the length of one of those parts.
- Mystery Length B is as long as 3 copies of Mystery Length A placed end-to-end.
- Mystery Length C is half the length of Mystery Length A.
- Mystery Length D is half the length of Mystery Length B.
- Mystery Length E is the remainder of the rope.

How long is each of the mystery lengths?

Relationships Between Decimals & Fractions Lesson 29

THE BIG IDEA

Understanding relationships between fractions and decimals involves integrating conceptual models for both. Knowledge of the part-to-whole and division concepts of fractions, together with an understanding of decimal place value, provide the basis for finding equivalent decimals and fractions.



CONNECTOR

OVERVIEW

Students use base ten area pieces to represent various decimals, and they describe those representations in terms of tenths only, hundredths only, and thousandths only.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces for each student.
- ✔ Toothpicks, 6 per student (each broken into 5 parts to represent thousandths).
- ✔ Base ten area pieces and toothpick pieces for the overhead.

FOCUS

OVERVIEW

Students use decimal grids and calculators to examine relationships between fractions and decimals. Both terminating and infinite repeating decimals are explored.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Decimal Grids Version A (see *Blackline Masters*), 1 per student and 1 transparency (see note).
- ✔ Decimal Grids Version B (see *Blackline Masters*), 3 sheets per student (see note).
- ✔ Overhead pens (optional, see note), 1 per student.
- ✔ Calculators, 1 per student.
- ✔ Scissors, 1 per student.

Note: To make grids reuseable, laminate and use overhead pens.

FOLLOW-UP

OVERVIEW

Students explain methods of using decimal grids and calculators to find equivalent decimals and fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 29.1, 1 copy per student.
- ✔ Base ten area pieces (see *Blackline Masters*), 1 sheet (on standard copy paper) for use at home.

LESSON IDEAS

QUOTE

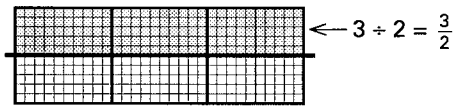
The approach to decimals should be similar to work with fractions, namely, placing a strong and continued emphasis on models and oral language and then connecting this work with symbols. This is necessary if students are to make sense of decimals and use them insightfully...

Decimal instruction should include informal experiences that relate fractions to decimals so that students begin to establish connections between the two systems.

NCTM Standards

SELECTED ANSWERS

1. One possible representation is shown here:



2. $3 \div 2 = 1\frac{1}{2} = 1.5$

5-7 Since the display of a calculator contains a finite number of places, calculator results can at best suggest a decimal may be infinitely repeating (see Comment 10 of the Focus activity). Students could support their conjectures by referring to decimal grid representations of fractions/decimals.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students use base ten area pieces to represent various decimals, and they describe those representations in terms of tenths only, hundredths only, and thousandths only.

MATERIALS

- ✔ Base ten area pieces for each student.
- ✔ Toothpicks, 6 per student (each broken into 5 parts to represent thousandths).
- ✔ Base ten area pieces and toothpick pieces for the overhead.

ACTIONS

1 Arrange the students in groups and give each student a set of base ten area pieces, including toothpick pieces to represent thousandths. Tell them that the large area piece square is the unit. Place 1 unit, 18 tenths, 27 hundredths, and 15 thousandths on the overhead, and ask groups to:

a) form the minimal collection of pieces that has the same value as this collection;

b) determine the decimal that represents this collection;

c) determine the total number of thousandths in the collection if all pieces were traded for thousandths.

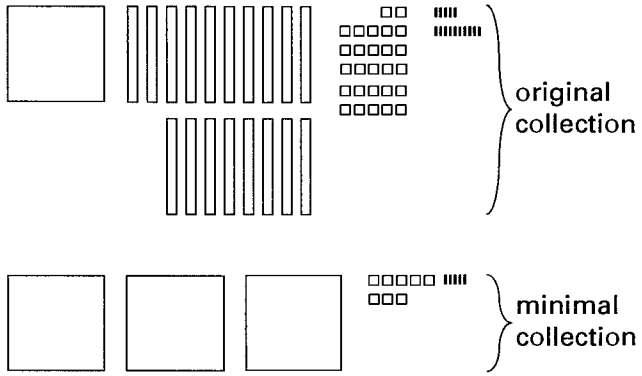
Discuss.

2 Write the following on the overhead: 9 tenths, 7 hundredths, and 354 thousandths. Have the groups repeat parts a)-c) of Action 1 for this collection of area pieces.

3 Write "one hundred fifty thousandths" on the overhead. Ask the groups to form the minimal collection represented by that number, to write its decimal representation, and to describe that collection in terms of tenths only and in terms of hundredths only.

COMMENTS

1 Thousandths pieces can be formed by breaking toothpicks into 5 approximately equal parts. The minimal collection is 3 units, 8 hundredths, and 5 thousandths is formed as follows:



The decimal representation of the minimal collection is 3.085. The total number of thousandths is 3,085.

2 The minimal collection is 1 unit, 3 tenths, 2 hundredths, and 4 thousandths, which is represented by 1.324 and contains 1,324 thousandths.

3 The minimal collection for one hundred fifty thousandths has 1 tenth and 5 hundredths. It represents 1½ tenths, or 15 hundredths (.15), or 150 thousandths (.150).

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
4 Write the decimal .107 on the overhead. Ask the groups to build the minimal collection represented by this decimal and to describe the collection in terms of a) tenths only; b) hundredths only; c) thousandths only.	**4** The decimal .107 represents 107 thousandths which could be described as $1\frac{7}{100}$ tenths and $10\frac{7}{10}$ hundredths.
5 Ask the students to explain why they think 12 dollars and 7 cents is written \$12.07. Have them describe \$12.07 in terms of cents only; dimes only.	**5** Since there are 100 pennies in 1 dollar, 12 dollars and 7 cents contains 12 and $\frac{7}{100}$ dollars or \$12.07. There is a total of 1207 pennies and 120.7 dimes in \$12.07.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use decimal grids and calculators to examine relationships between fractions and decimals. Both terminating and infinite repeating decimals are explored. Decimal and fraction relationships are emphasized again in Lesson 43.

MATERIALS

- ✔ Decimal Grids Version A (see *Blackline Masters*), 1 per student and 1 transparency (see note).
- ✔ Decimal Grids Version B (see *Blackline Masters*), 3 sheets per student (see note).
- ✔ Overhead pens (optional, see note), 1 per student.
- ✔ Calculators, 1 per student.
- ✔ Scissors, 1 per student.

Note: To make grids reuseable, laminate and use overhead pens.

ACTIONS

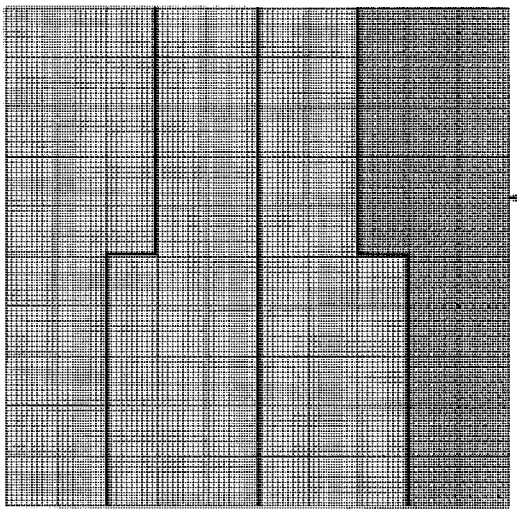
1 Arrange the students in small groups. Give each student 3 sheets of Decimal Grids Version A (see *Blackline Masters*) and a pair of scissors. Tell the students that for this activity 1 decimal grid is 1 area unit.

Ask the groups to each divide 1 unit into 4 equal parts (either by cutting or shading the grid), determine the value of each part, and express that value in a variety of ways. Discuss their conclusions and methods.

COMMENTS

1 To facilitate sharing at the overhead, time is saved by having students record their ideas on transparencies of the decimal grid prior to coming to the overhead or have them hold up their drawings for others to see.

One way to divide the unit is based on the division concept of a fraction (see Lessons 8, 9, and 28). Separate 1 unit square into 4 equal parts; one part is $1 \div 4 = \frac{1}{4}$. Some students may do this by cutting a unit into 4 groups of 2 tenths and 5 hundredths, while others may draw subdivisions on the unit, as shown below.



Another way to view the value of each part is based on the part-to-whole concept of a fraction (see Lesson 8). Since the whole, or unit, has been divided into 4 equal parts, each part represents 1 part out of 4, or $\frac{1}{4}$.

(Continued next page.)

Focus Teacher Activity (cont.)

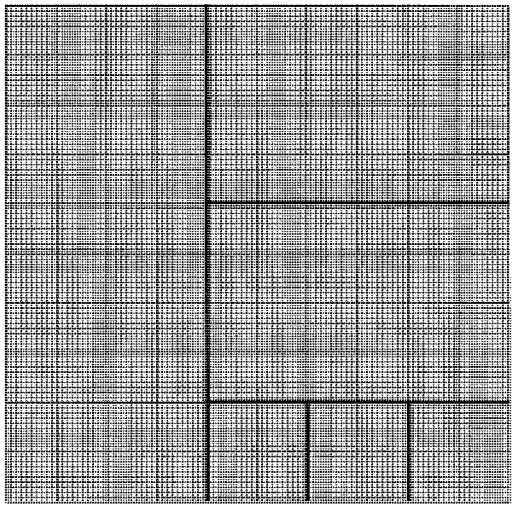
ACTIONS

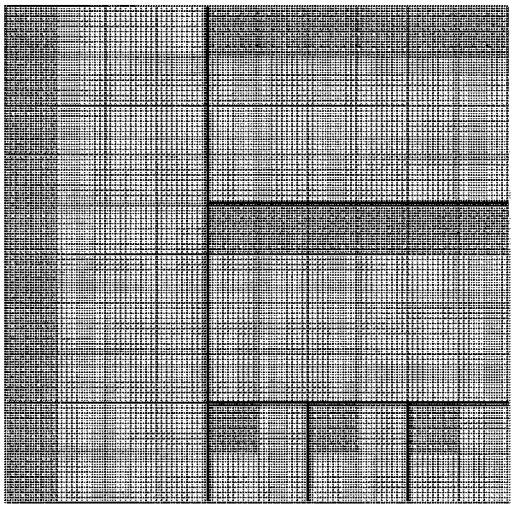
COMMENTS

1 (continued.)
Note that because of the subdivisions on the grid, some students may view each part as 25 of the medium-sized squares which make up $^{25}/_{100}$ of a square unit, while others may see 2 tenths and 5 hundredths, $2\frac{1}{2}$ tenths, $^{250}/_{1000}$, $^{2500}/_{10,000}$, .25, .250, or .2500. Each of these expressions are equivalent to $1 \div 4$, or $\frac{1}{4}$, because they all represent the same part of a unit.

2 Place a transparency of a decimal grid on the overhead and partition it into the following regions. Distribute 3 sheets of Decimal Grids Version B to each student. Ask the groups to partition 1 area unit in the same way and to shade $\frac{1}{4}$ of each region. Have them determine the total amount shaded. Discuss and then have students invent other ways to partition the unit, shade $\frac{1}{4}$ of each part, and determine the total amount shaded.

2 Partitioning the unit into any number of regions of any size or shape and shading $\frac{1}{4}$ of each region produces a total shaded amount equal to $\frac{1}{4}$ unit.





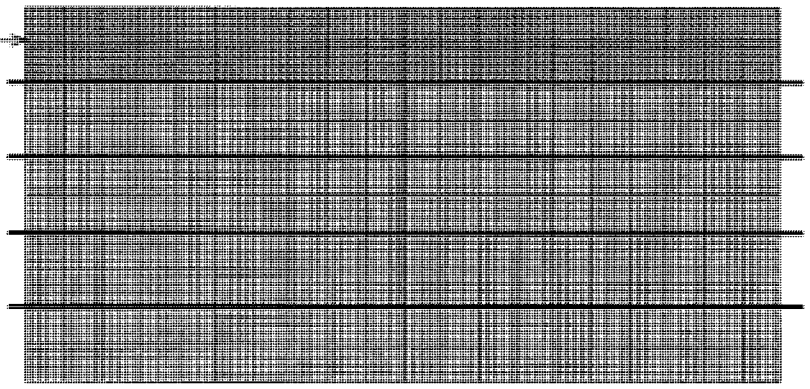
3 Ask the groups to use a variety of ways of modeling $\frac{2}{5}$ with their decimal grids. Have them write several equivalent decimal and fraction representations of $2 \div 5$. Discuss their methods and results.

3 Since students may experiment with cutting and shading the decimal grids, you may need a supply of extra copies. Following is one method of modeling $\frac{2}{5}$ that is based on the division concept of a fraction: place 2 units together; subdivide that 2-unit piece into 5 equal parts; one part represents $\frac{1}{5}$ of 2 units or $2 \div 5 = \frac{2}{5} = 4$ tenths $= {}^{40}/_{100} = {}^{400}/_{1000} = {}^{4000}/_{10,000} = .4 = .40 = .400$ square units (see diagram on next page).

Focus Teacher Activity (cont.)

ACTIONS

$2 \div 5 = \frac{2}{5} = .40$

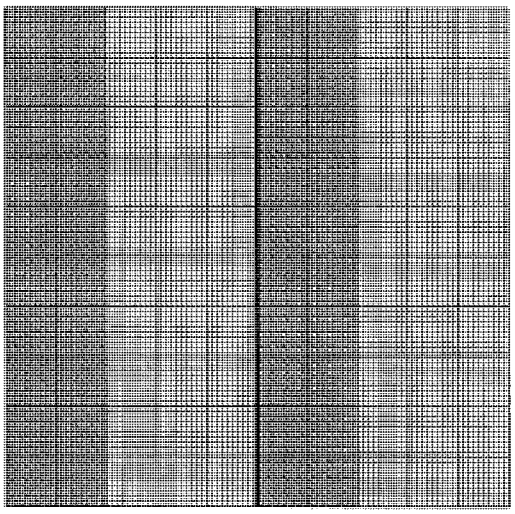


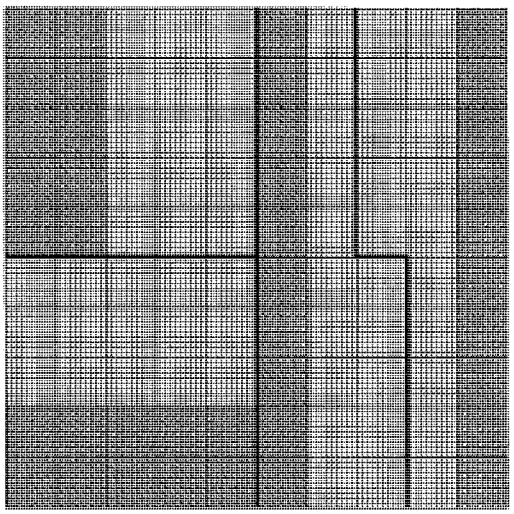
COMMENTS

Another method, based on the part to whole concept, is to subdivide 1 unit into 5 equal parts and shade 2 of those parts. Thus, $2 \div 5$ is viewed as $2/5$ of 1.

An alternate method is to divide each of the 2 units into 5 equal parts, and then distribute these parts into 5 equal groups. Each part represents $1/5$ of a square unit and each group contains $1/5 + 1/5 = 2/5$ square units.

Yet another method is to subdivide 1 unit into regions and shade $2/5$ of each region. The total amount shaded is $2/5$ unit. Here are 2 examples that illustrate this method of determining $2 \div 5 = 2/5 = 4/10 = 4/100 = .40 = .4 = .400 = .4000$, etc.





Focus Teacher Activity (cont.)

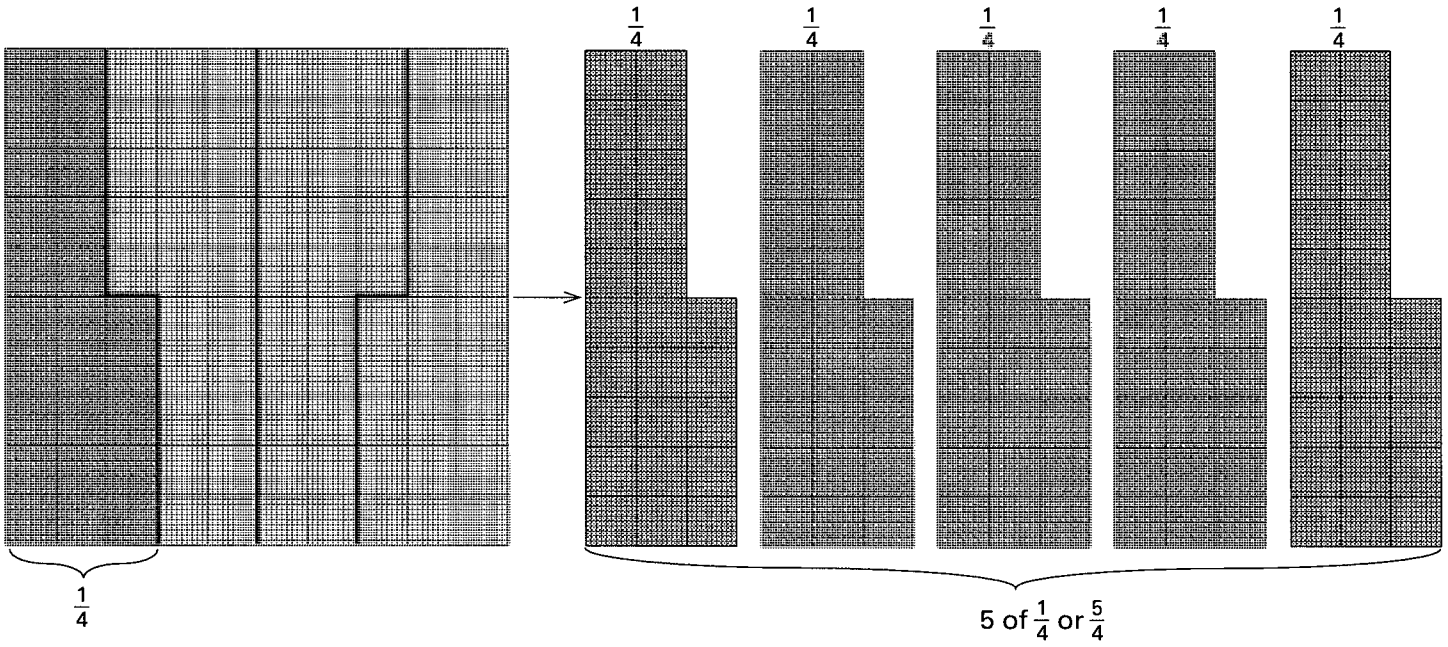
ACTIONS

4 Write the fraction $^{2}/_{50}$ on the overhead and ask the groups to devise methods of using their decimal grids to determine decimal equivalents for it. Have volunteers illustrate their methods at the overhead. As appropriate, repeat for one or more of the following: $^{3}/_{20}$, $^{4}/_{2}$, $^{5}/_{4}$. Encourage students to think about each fraction using both the part-to-whole and the division concepts.

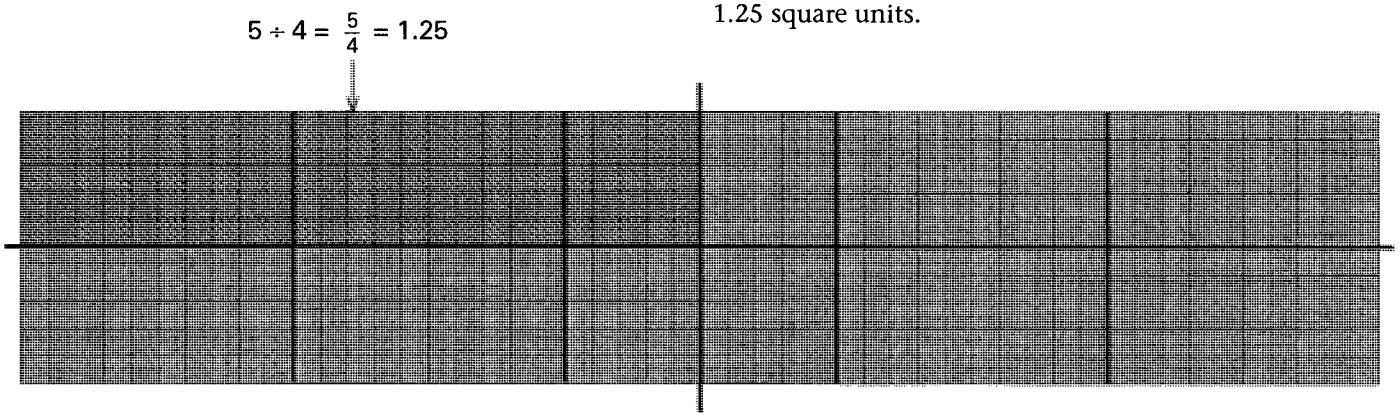
COMMENTS

4 An interesting discussion may emerge from students' efforts to describe $^{2}/_{50}$ as a decimal. The fact that there are 2 groups of 50 hundredths in 1 unit may motivate some students to say $^{2}/_{50} = {}^{1}/_{2}$. If so, rather than explaining why this can't be, invite the class to investigate further by posing a question such as the following to the groups: "When, if ever, is $^{1}/_{2} = {}^{2}/_{50}$. Why or why not?" Be prepared for the possibility of disequilibrium!

Following are two possibilities for determining decimals equivalent to $^{5}/_{4}$. First, using the part-to whole concept, subdivide 1 unit into 4 equal parts (as in Action 1). Each part is $^{1}/_{4} = .25$ square units. Five of the parts are $^{5}/_{4} = {}^{125}/_{100} = {}^{1250}/_{1000} = 1.25 = 1.250$ square units.



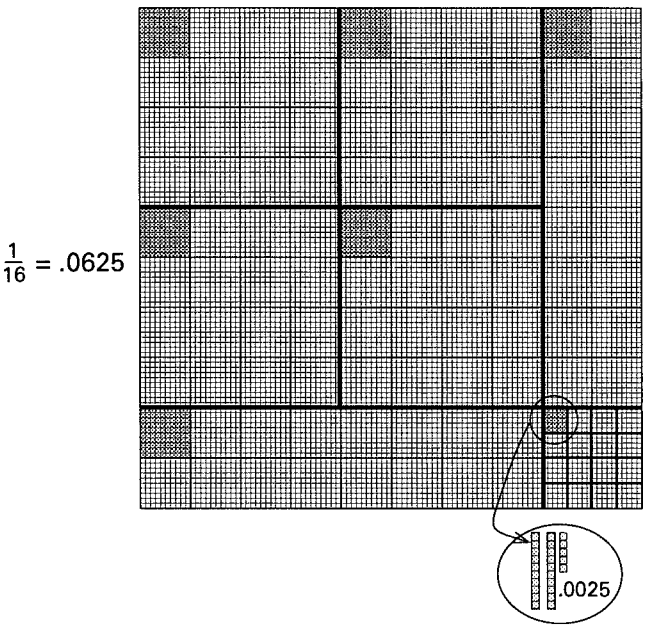
Or, using the division concept, subdivide 5 units into 4 equal parts, as illustrated below. One part is $5 \div 4 = {}^{5}/_{4} = 1.25$ square units.



Focus Teacher Activity (cont.)

ACTIONS

5 Ask the groups to use their decimal grids to find decimal equivalents of 3/8. Discuss and repeat for 1/16.



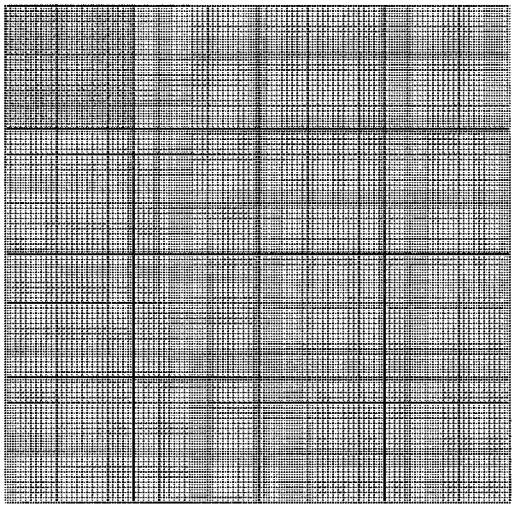
COMMENTS

5 Some students will cut apart grids, while others will shade subdivisions of the grid. One method of finding 3/8 is to divide 3 units into 8 equal parts of .375. Another method is to divide 1 unit into 8 equal parts of .125 and then combine 3 of those parts. Yet another approach is to subdivide the unit into regions and shade 3/8 of each region; the total amount shaded is 3/8 unit.

Note that, if it hasn't come up, a student may suggest using a calculator to carry out the division. This is appropriate and not to be discouraged. However, the emphasis here and in Actions 6-7 is on finding ways to *use the grids* to find decimal equivalents and hence, build understanding of conceptual relationships between fractions and decimals.

One method of forming 1/16 is to subdivide a representation of 1/8 into 2 equal parts of value .0625 each.

Another method of determining decimal equivalents for 1/16 is to subdivide the grid into 16 equal parts and shade 1 of those parts as shown below.

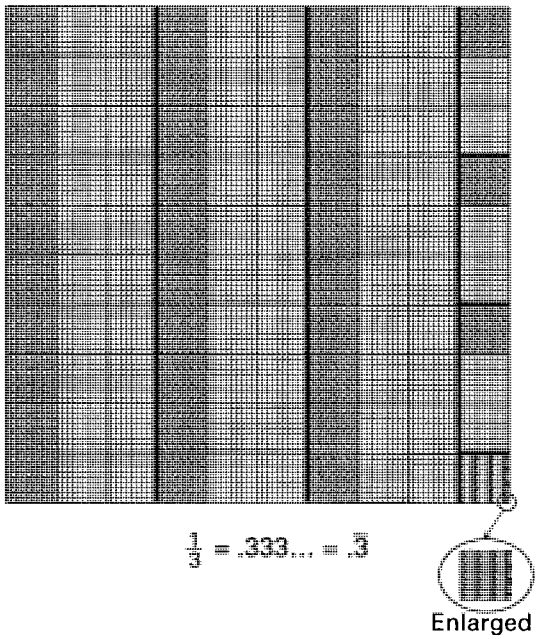


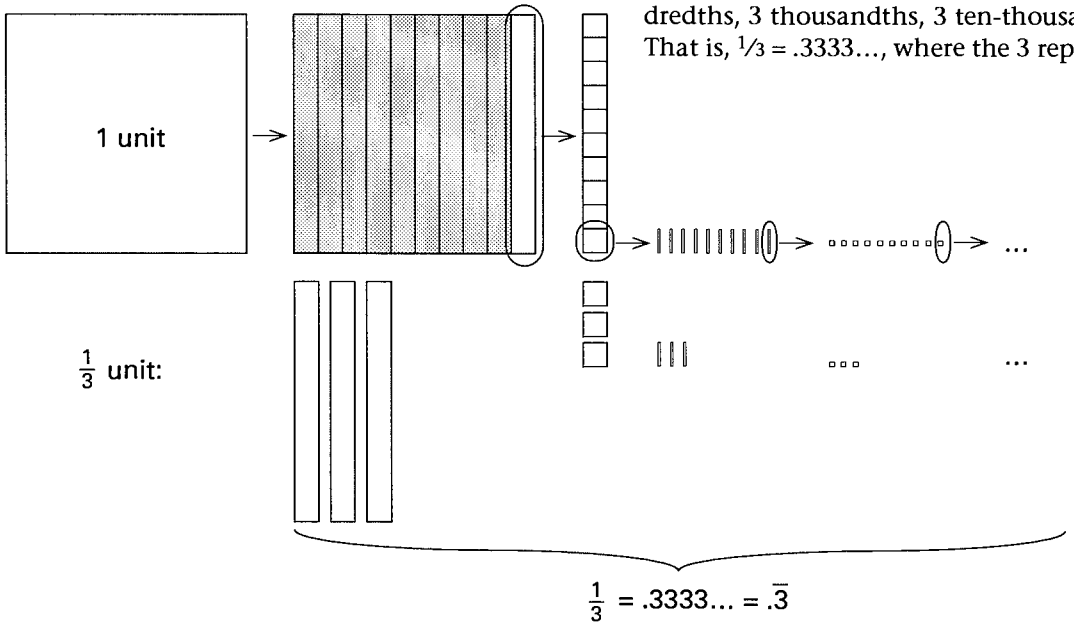
Yet another method is to subdivide 1 unit into regions and shade 1/16 of each region as shown at the left.

Focus Teacher Activity (cont.)

ACTIONS

6 Ask the students to work with their groupmates to devise a way to use the decimal grids to determine the decimal equivalent of ⅓. Have volunteers share their methods at the overhead. If it isn't suggested by a student, point out to students the standard method of writing *infinite repeating decimals.*





COMMENTS

6 Methods of finding ⅓ are similar to those in Actions 1-5. However, in this case the process goes on indefinitely.

One method of shading ⅓ unit is to divide the unit into regions and shade ⅓ of each region. One possibility is illustrated at left. Notice that the remaining hundred-thousandth could be subdivided into 10 parts which again could be divided into 3 groups of 3 with 1 remaining and so on indefinitely. Since ⅓ cannot be expressed as a decimal that terminates after a finite number of places, it is called an *infinite repeating decimal.* The exact decimal equivalent of ⅓ is written .$\overline{3}$. The bar over the 3 indicates that 3 repeats indefinitely. Decimals for fractions such as ½, ¾, ⅕, and ⅜ are called *finite* or *terminating* decimals.

If some students have difficulty seeing ⅓ on the grids you might suggest they use base ten area pieces and trade pieces as needed to partition the unit into 3 equal collections. For example, as illustrated below 1 unit could be traded for 10 tenths. In 10 tenths there are 3 groups of 3 tenths with 1 tenth left over. The left over tenth is then traded for 10 hundredths. In 10 hundredths there are 3 groups of 3 hundredths with 1 hundredth left over. That hundredth is traded for 10 thousandths. In 10 thousandths there are 3 groups of 3 thousandths with 1 thousandth left over. At this stage, it is necessary to imagine the process repeating. That is, at each stage there is 1 piece left over which is traded for 10 of the next smaller piece. Each of those always divides into 3 groups of 3 pieces with 1 piece left over. Hence, a minimal collection for ⅓ is 3 tenths, 3 hundredths, 3 thousandths, 3 ten-thousandths, and so on. That is, ⅓ = .3333..., where the 3 repeats indefinitely.

Focus Teacher Activity (cont.)

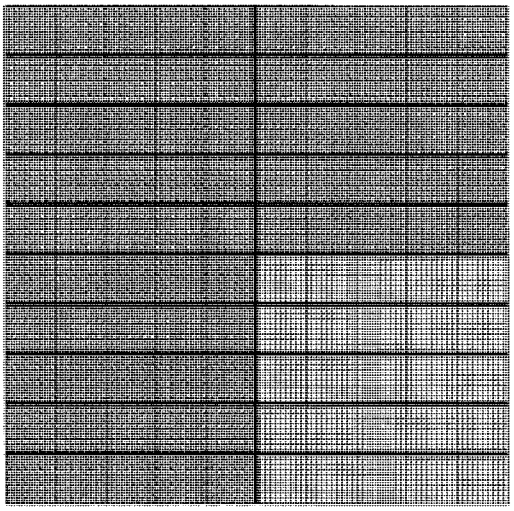
ACTIONS

7 Ask the groups to devise methods of subdividing units to determine the decimal equivalents of $\frac{2}{9}$ and/or $\frac{5}{6}$.

8 Write the following numbers on the overhead:

a) .75 b) .05 c) 1.40 d) .625

Ask the groups to devise methods of using their base ten pieces or decimal grids to determine fractions that are equivalent to each of the decimals. Encourage students to look for more than one way of viewing each decimal as a fraction.

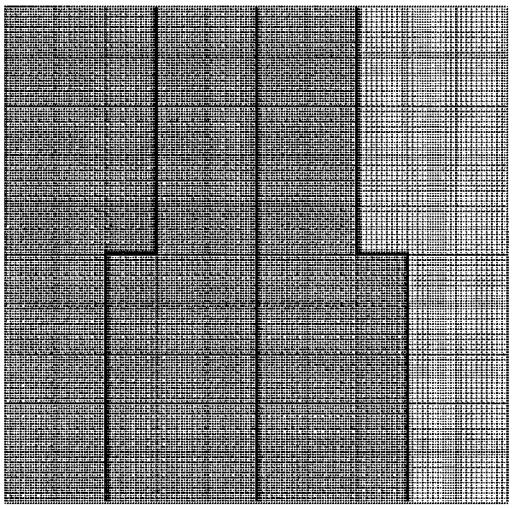


$.75 = \frac{15}{20}$

COMMENTS

7 The decimal equivalent for $\frac{2}{9}$ and $\frac{5}{6}$ are $.\overline{2}$ and $.8\overline{3}$ respectively. Note that the bar is only above the 3 in $.8\overline{3}$ since the 8 does not repeat.

8 The fractional values for these decimals can be found in several informal ways. For problem a), one way is to view the 7 tenths and 5 hundredths as $\frac{75}{100}$, $\frac{750}{1000}$, and $\frac{7500}{10,000}$. Another way is to shade 7 tenths and 5 hundredths and note that it covers $\frac{3}{4}$ of the unit, as shown below.



$.75 = \frac{3}{4}$

Yet another way of viewing .75 is to subdivide the unit into groups of 5 hundredths (i.e., into twentieths) and show that .75 covers $\frac{15}{20}$. This method is illustrated at the left.

One fraction for each of the remaining decimals is listed below:

b) $.05 = \frac{5}{100}$ c) $1.4 = \frac{7}{5}$ d) $.625 = \frac{5}{8}$

Focus Teacher Activity (cont.)

ACTIONS

9 Distribute a calculator to each student and pose the problem of using the calculator to find a decimal approximation for $\frac{2}{3}$. Discuss the groups' results.

10 Ask the groups to use their calculators to find other fractions that they think represent infinite repeating decimals.

COMMENTS

9 Although understanding conceptual relationships can make it easier for students to mentally "see" decimal representations of many fractions, without need for other computations, it is important that students have free access to calculators for carrying out tedious computations. Actions prior to this were aimed at developing understanding of conceptual relationships between decimals and fractions, so students understand why and when to press certain keys to compute the decimal representation of a fraction.

Since $\frac{2}{3}$ is a representation of $2 \div 3$, a decimal approximation can be found by using the calculator to compute $2 \div 3$. The number in the calculator display is an approximation since the bar does not appear (in most calculators) to indicate that the 6 repeats, and some calculators show a rounded last digit (for example, some will show .6667).

10 Note that students have an opportunity to continue exploring this idea on the Follow-up.

If a fraction is in lowest terms and the denominator has factors other than 2 or 5, the fraction has an infinite repeating decimal. Students may not reach this conclusion now.

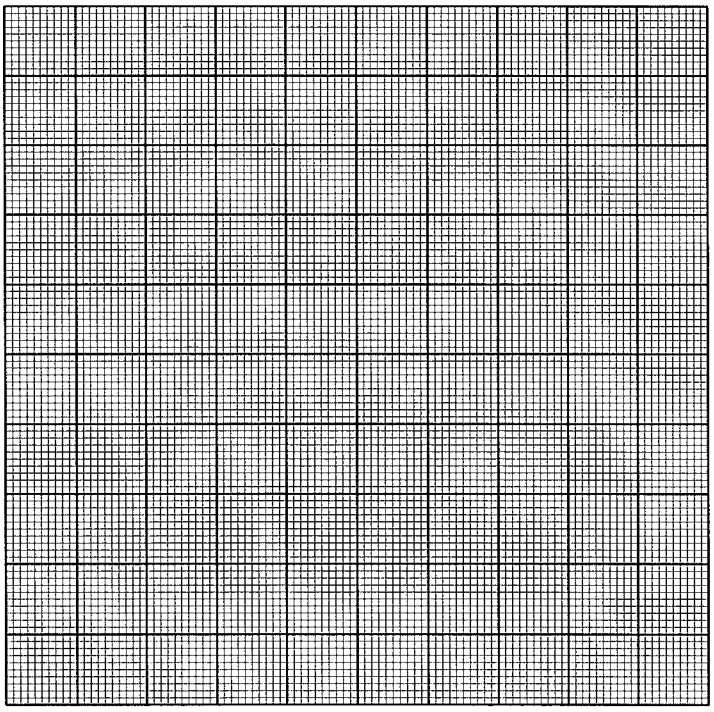
Follow-up Student Activity 29.1

NAME _____ **DATE** _____

1 Form a collection of base ten area pieces that has value $3 \div 2$. Sketch your collection here. Be sure to label your unit.

2 Write a decimal and a fraction that represent your area piece collection in problem 1.

3 Peter says that $\frac{9}{20} = .450$. On the grid shown below, sketch a "proof" to show why you think he is correct or why he is incorrect. Next to your sketch, write a careful explanation of your proof.



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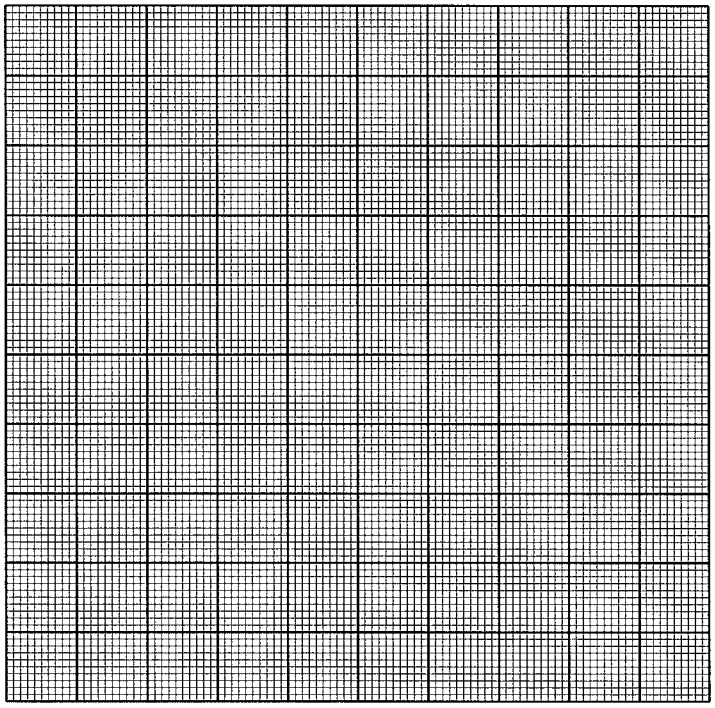
Follow-up Student Activity (cont.)

4 Show how to find the decimal representation for $\frac{1}{6}$ on this grid. Explain your methods.



5 Use a calculator to find 5 fractions that you think represent infinite repeating decimals. Write both the decimal and the fraction form of each number.

6 Use a calculator to find 5 fractions that represent finite terminating decimals. Write both the decimal and the fraction form of each number.

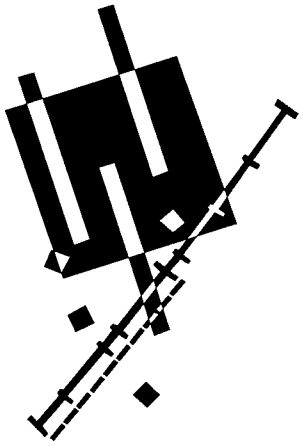
7 Use your calculator to explore decimal representations of a variety of other fractions. On another sheet, explain what you did. What did you notice? Do you have any conjectures? Do you have any questions?

Rounding & Ordering Numbers

Lesson 30

THE BIG IDEA

Base ten area and linear pieces provide a conceptual basis for examining the concept of rounding and for comparing decimal numbers. Such explorations enhance students' number sense and provide important background for mental computation and for making decisions about the reasonableness of answers.



CONNECTOR

OVERVIEW

By varying the unit, students explore ways to represent large and small numbers using the same base ten pieces.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces, including toothpick thousandths, for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Connector Master A, 1 transparency.

FOCUS

OVERVIEW

Students use base ten pieces (and images of base ten pieces) to represent and compare sets of numbers.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces, including toothpick thousandths, for each student.
- ✔ Base ten linear pieces for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Calculators, 1 or more per group.
- ✔ Focus Masters A and B, 1 transparency of each.

FOLLOW-UP

OVERVIEW

Students compare, order, and round decimal numbers. They draw diagrams and make written explanations that are based on their experiences with base ten pieces.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 30.1, 1 copy per student.

LESSON IDEAS

ASSESSMENT
You could have students write the letter for Follow-up Problem 7 in their journals to provide another benchmark in their developing understanding of decimals. Have them add a paragraph describing ways their understanding of decimals has changed or strengthened.

SELECTED ANSWERS

1. a) 5.4 b) $2.35 c) 40. d) 9.2 e) 10.0
   f) 12.0 g) 3000.00

2. a) .03, .30, .32, 3.02
   b) .007, .021, .217, 2.17

4. a) 596 b) 1000 c) 596.5 d) 596.50

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

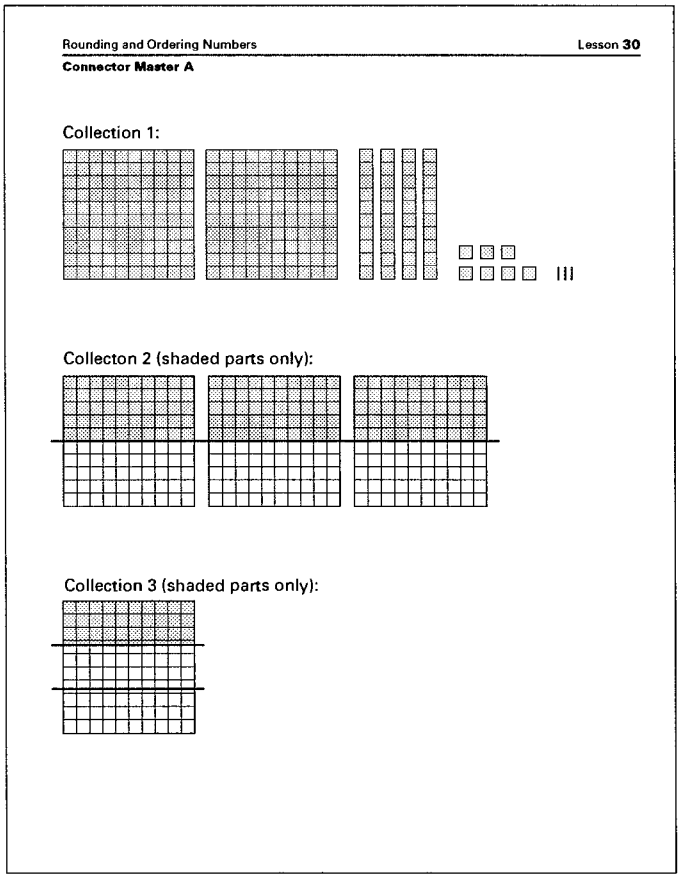
By varying the unit, students explore ways to represent large and small numbers using the same base ten pieces. The relationships between decimals and fractions are reinforced as students assign fraction and decimal values to area piece collections.

MATERIALS

- ✔ Base ten area pieces, including toothpick thousandths, for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Connector Master A, 1 transparency.

ACTIONS

1 Place the students in groups and give each student a set of base ten area pieces and toothpick pieces. Place a transparency of Connector Master A on the overhead and, while covering Collections 2-3, ask the students to form Collection 1 with their area pieces. Ask the groups to determine different possibilities for the value of this collection and record these values using decimals and fractions or mixed numbers. Discuss. Repeat for Collections 2 and 3.



COMMENTS

1 Each of the smallest pieces (i.e., toothpick pieces) on the transparency are intended to represent $1/10$ of a small square.

Here are some possibilities for decimal and fraction representations of Collections 1, 2, and 3 using a) the large area piece square, b) the rectangle (non-square), c) the small square, and d) the small rectangle (non-square), respectively as the unit:

Collection 1: a) 2.473, $2^{473}/_{1000}$, or $^{2473}/_{1000}$; b) 24.73 or $24^{73}/_{100}$; c) 247.3, $247^{3}/_{10}$, or $^{2473}/_{10}$; d) 2473.0 or 2473 (some students may suggest $^{2473}/_{1}$).

Collection 2: a) 1.5, $3/2$, or $1\,1/2$; b) 15.0 or 15; c) 150.0 or 150; d) 1500.0 or 1500.

Collection 3: a) $.\overline{3}$ or $1/3$; b) $3.\overline{3}$ or $3\,1/3$; c) $33.\overline{3}$ or $33\,1/3$; d) $333.\overline{3}$ or $333\,1/3$.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use base ten pieces (and images of base ten pieces) to represent and compare sets of numbers. Comparing, ordering, and rounding visual representations of whole numbers, decimals, and fractions helps students to develop number sense and provides important background for estimation and mental computation.

MATERIALS

- ✔ Base ten area pieces, including toothpick thousandths, for each student.
- ✔ Base ten linear pieces for each student.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Calculators, 1 or more per group.
- ✔ Focus Masters A and B, 1 transparency of each.

ACTIONS

1 Arrange the students in groups and give base ten area pieces and toothpick pieces (thousandths) to each student. Provide each group with at least one calculator to use as needed. Write the following numbers on the overhead. Ask each group to form the minimal collection for each of the numbers, and to arrange the collections in order from the least value to the greatest value. Discuss.

1.320 1.32 1.3200 1 1.0032 1.032 1.302

2 Repeat Action 1 as needed for these sets of numbers:

a) $\frac{3}{4}$, .42, .999, .009, $\frac{1}{5}$

b) .333, .327, $\frac{1}{3}$, .372, .0777, .7

3 Ask the groups to form minimal collections for .3 and .4 and then to form collections for three different numbers that are greater than .3 and less than .4. Ask volunteers to show the new collections formed by their group and to write the decimal number for each. Repeat for numbers that are between:

a) 3.45 and 3.46

b) $\frac{1}{3}$ and .4

COMMENTS

1 The collections for 1.320, 1.32, and 1.3200 are equivalent and have greater value than the other four numbers, whose ordering from smallest to largest is 1, 1.0032, 1.032, and 1.302. Note that students must either imagine or cut $\frac{2}{10000}$ to form 1.0032.

This is an appropriate time to discuss students' observations about the effect of placing zeros in a decimal number.

2 a) The correct ordering is .009, $\frac{1}{5}$, .42, $\frac{3}{4}$, and .999. Be sure to discuss ways of determining the decimal representation of the fractions by subdividing the area piece unit (see Lesson 29).

b) The correct ordering is: .0777, .327, .333, $\frac{1}{3}$, .372, .7. Some students may incorrectly suggest that .333 and $\frac{1}{3}$ are equal. If so, ask for volunteers to explain the differences between them. For example, using a decimal grid, one can show that $\frac{1}{3}$ is equal to $\frac{333\frac{1}{3}}{1000}$ and could be written $.\overline{3}$ or $.333\frac{1}{3}$, while .333 is $\frac{333}{1000}$.

3 Minimal collections for numbers between .3 and .4 include any collections with 3 tenths plus 1 to 9 of any pieces smaller than tenths.

a) One to 9 pieces of any area piece smaller than a hundredth can be added to 3 units, 4 tenths, and 5 hundredths to form a number between 3.45 and 3.46.

b) A minimal collection between $\frac{1}{3}$ and .4 could have 3 tenths and 4 to 9 hundredths plus 0 to 9 of any piece smaller than a hundredth, or it could have 3 tenths, 3 hundredths, and 4 to 9 thousandths plus 0 to 9 of any piece smaller than thousandths, or it could have 3 tenths, 3 hundredths, 3 thousandths and 4 to 9 ten thousandths plus 0 to 9 of any piece smaller than a ten thousandth, and so on.

Focus Teacher Activity (cont.)

ACTIONS

4 Write the following numbers on the overhead. Ask the students to build or imagine the area piece representations of each of the numbers, and to use their models or mental images to help them decide whether the number is less than $\frac{1}{2}$ or greater than $\frac{1}{2}$. Discuss their images and reasoning. Introduce "<" and ">" notation.

a) .5001

b) .250

c) .050

d) $\frac{3}{4}$

e) $\frac{2}{3}$

f) $\frac{2}{5}$

g) .4999

5 Write the numbers shown below on the overhead. For each of the numbers, ask the students to visualize or build area piece representations of the number in order to determine whether the number is closer in value to 0 or to 1. Discuss their reasoning.

a) .49

b) .734

c) .5

d) .099

e) $\frac{3}{7}$

f) .1259

g) $\frac{5}{8}$

6 Write the numbers listed in a) below on the overhead. Ask the students to build, sketch, or imagine area piece representations of each number, and to use their images to determine the correct ordering of the numbers from smallest to largest. Discuss their images and reasoning. Repeat as needed for b)-d).

a) $\frac{1}{3}$, $\frac{1}{4}$, .8, .088, $\frac{9}{10}$

b) .43, .48, .411, .476, $\frac{2}{5}$

c) .309, .3, .199, .364, .30, $\frac{2}{7}$

d) 7.240, 3.009, $2\frac{2}{3}$, 3, 1.999

COMMENTS

4 The notation .5 < .5001 is read ".5 is less than .5001." The symbol > is read "is greater than."

Encourage students to think of a variety of methods of comparing each number to $\frac{1}{2}$. In relation to $\frac{1}{2}$, the numbers listed are: a) greater; b) less; c) less; d) greater; e) greater; f) less; g) less.

Anytime that students have difficulty visualizing collections encourage them to use their area pieces or to make sketches of the pieces, or in the case of fractions, even to make reference to an egg carton. Some students may wish to use calculators to express fractions as decimals and then make comparisons to .5. This is appropriate and calculators should be freely available for that purpose.

5 The answers are: a) 0; b) 1; c) neither; d) 0; e) 0; f) 0; and g) 1. Many students will base their conclusions on whether a number is greater or less than $\frac{1}{2}$.

6 Following are examples of reasoning about ordering each set of numbers:

a) .088, $\frac{1}{3}$, and $\frac{1}{4}$ are each less than $\frac{1}{2}$, and .8 and .9 are greater than $\frac{1}{2}$. .088 is less than $\frac{1}{10}$, which is less than $\frac{1}{3}$ and $\frac{1}{4}$ (a unit split into 10 equal parts has smaller parts than one split into 4 equal parts, which in turn has smaller parts than one split into thirds). Further, 8 tenths are less than 9 tenths, so the correct ordering is .088, $\frac{1}{4}$, $\frac{1}{3}$, .8, $\frac{9}{10}$.

After discussing a) you may wish to write b)-d) all on the overhead for groups to explore at their own pace. Discuss if needed (sometimes the groups' private discussions suffice and large group sharing becomes repetitive).

b) $\frac{2}{5}$ = .4 and the area piece collections for all of these numbers include 4 tenths. Since none of the collections

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

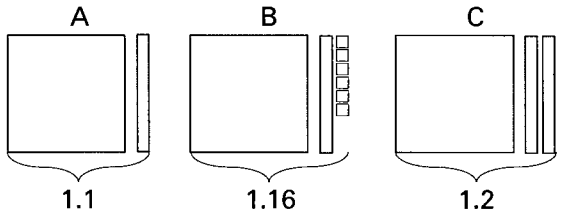
have the same number of hundredths, the ordering can be completed by looking only at the hundredths (the thousandths are not relevant to the ordering process in this example). The order is 2/5, .411, .43, .476, .48.

Another possibility is to compare the total number of thousandths in each collection to get the order .400, .411, .430, .476, and .480. Emphasize comparisons of this type using the area piece representations of numbers rather than stating rules about attaching zeros.

c) In this set of numbers, .3 and .30 are equivalent. Each seventh of a unit is less than 1½ tenths (since 7 groups of 1½ tenths total more than 1 unit), so 2/7 is less than 3 tenths and greater than 2 tenths (and therefore also greater than .199). 3 tenths is the same as 300 thousandths which is less than 309 thousandths. Hence, the ordering is: .199, 2/7, .3 = .30, .309, .364.

d) Since all of these numbers include units, they can first be arranged according to the numbers of units. This leaves only 3.009 and 3 to compare. Since 3.009 is more than 3 units, the final ordering is: 1.999, 2⅔, 3, 3.009, 7.240.

7 Place the three area piece collections shown below on the overhead and ask the groups to form the same collections. Ask the groups to determine whether the value of Collection B is closer to the value of Collection A or the value of Collection C. Discuss their ideas and reasoning.



7 Occasionally, students will translate the expression "is closer to" literally and make reference to physical proximity on the overhead. Hence, it is important to emphasize "closer in *value*."

The collection whose value is closest to 1.16 and uses only units and tenths has 1 unit and 2 tenths (i.e., Collection C). Some students may say this is because it is necessary to add only 4 hundredths to Collection B to form 1.2, and it is necessary to subtract 6 hundredths from Collection B to form 1.1. Others may suggest it is 1.2 because the 6 hundredths in the collection for 1.16 cover more than half of 1 tenth. This illustrates visually why 1.16 *rounds* to the nearest tenth to 1.2. Thus, rounding a given number to a specified place value can be thought of as forming an area piece collection whose value is "closest to" a given number while using no pieces smaller than those representing the specified place value.

Focus Teacher Activity (cont.)

ACTIONS

8 Place a transparency of Focus Master A on the overhead and, covering the other problems, ask the groups to complete a). Discuss. Repeat for selected problems from b)-f).

During discussion ask students to make statements involving rounding about each situation.

Rounding and Ordering Numbers Lesson **30**
Focus Master A

Sketch the following:

a) the minimal collection, using hundreds only, whose value is closest to 631;

b) the minimal collection, using no pieces smaller than a tenth, whose value is closest to 2.35;

c) the minimal collection, using units only, whose value is closest to 2.47;

d) the minimal collection that contains only hundredths and tenths whose value is closest to .635;

e) the minimal collection of tenths only whose value is closest to .635;

f) the minimal collection, using only units, whose value is closest to .635.

COMMENTS

8 Base your selections of problems on students' comfort and prior experience with the concept of rounding.

a) In this case, students are finding the collection that is "rounded to the nearest hundred." To form this collection with their pieces, it is necessary to choose a different unit than in the preceding Actions. A collection of 6 hundreds, or 600, satisfies these conditions since the 31 units (or 3 tens and 1 unit) cover less than half of a hundred.

b) The collection for 2.35 is not closer in value to either 2.4 or 2.3, since 5 hundredths cover exactly half of 1 tenth. That is, rounding to either 2.4 or 2.3 could be justified.

Although in many textbooks students are taught to always round the number 5 (and 6-9) up, whether to round up or down often depends on the context of the numbers. For example, when determining whether one has enough money to buy several items, rounding up is important in order to insure having enough money. On the other hand, it is useful to round down when determining how far one can drive on a tank of gas (to prevent running out of gas). (Students will explore a variety of situations involving rounding in Lesson 31.)

c) Some students may see that 4 tenths and 7 hundredths, or 47 hundredths, cover less than half of a unit, so 2.47 rounds to 2. Others may remember a rule and say that "2.47 rounds to 2 because 4 is less than 5." In this case, ask for a visual explanation. Here is one possibility: 4 tenths and any number of hundredths less than or equal to 9 hundredths covers less than half of a unit.

Still other students may suggest that "since 7 hundredths is more than half of 1 tenth, the 4 rounds to 5 and then the 2.5 rounds to 3." If this comes up, ask why this reasoning does not work.

d) The collection whose value is closest to .635 using only hundredths and tenths has 6 tenths and 3 or 4 hundredths. Using the standard rule of rounding up for 5 or more, the collection rounds to .64.

e) 6 tenths

f) 1 unit

Focus Teacher Activity (cont.)

ACTIONS

9 Ask the students to visualize the area piece collection for a) below. Ask volunteers to describe their images. Repeat for one or more of b)-g), depending on students' needs.

a) 3.501 rounded to the nearest unit (i.e., form the collection, of units only, whose value is closest to 3.501)

b) 7.499 rounded to the nearest unit

c) 312.987 rounded to the nearest ten

d) 312.987 rounded to the nearest whole number

e) 312.987 rounded to the nearest tenth

f) 312.987 rounded to the nearest thousand

g) 238.992 rounded to the nearest hundredth

10 Distribute base ten linear pieces and a blank sheet of paper to each student. Tell the students that for the following actions the linear piece which is the length of the edge of their large area piece square is 1 linear unit. Ask the groups to each use a linear unit to draw a line segment of length 2 linear units and to mark the points 0, 1, and 2 on the segment, as shown here:

11 Write the numbers .2, .5, .7, 1.1, 1.4, and 1.7 on the overhead and ask the groups to use their linear pieces to locate points for these decimals on their line segments, and to mark and label each point. Ask volunteers to locate the numbers on an overhead number line.

12 Write each of the following numbers on the overhead and ask the groups to *approximate* and label a point on their number lines for each number: 1.24, .37, .96, 1.08, 1.79, .04, .62, 1.91, and 1.55. As needed, have volunteers show their methods of locating the points.

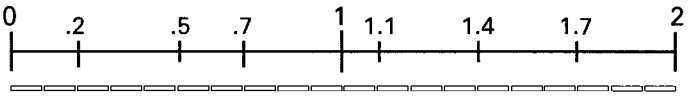
COMMENTS

9 To pose a) for example, you could say, "Imagine in your mind's eye the area piece collection for 3.501." Then allow a few moments for students to create a mental image of this collection before saying, "Now imagine the collection of only units, whose value is closest to 3.501," or "now imagine the collection that represents 3.501 rounded to the nearest unit." Finally, allow a few moments for students to create this new image and then discuss their ideas.

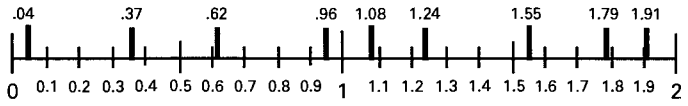
The numbers representing the rounded collections for a)-g) are: 4, 7, 310, 313, 313.0, 0, and 238.99. Models of larger area pieces on display in the classroom (see Lesson 25) make it easier for students to imagine collections for large numbers.

10 It is preferable not to use lined paper for this activity. Having students use the linear pieces to mark lengths, rather than tracing edges of area pieces, helps to reinforce the distinction between linear and area measure.

11 The linear unit has 10 equal parts so each part has length $\frac{1}{10} = .1$. The number line that follows shows the approximate location of the points:



12 Locating hundredths requires dividing the segment between two consecutive tenths into 10 equal parts (just as the tenths *area piece* is divided into 10 equal parts to obtain the hundredths pieces). Showing these divisions is difficult unless the linear unit is enlarged. Approximate locations are shown below:



Focus Teacher Activity (cont.)

ACTIONS

13 Repeat Action 12 for these numbers: .773, .462, 1.204, 1.463, .135, 1.781, .628, 1.908.

14 Place a transparency of Focus Master B on the overhead. Ask the groups to determine which numbers on their number lines (created and labeled in Actions 11-13) satisfy each condition.

Lesson **30** Rounding and Ordering Numbers

Focus Master B

Identify which number(s) on your line is/are:

a) between $\frac{1}{2}$ and $\frac{9}{10}$;

b) closest to but not equal to .7;

c) to the nearest tenth, rounds to 1.0;

d) between .9 and 1.2;

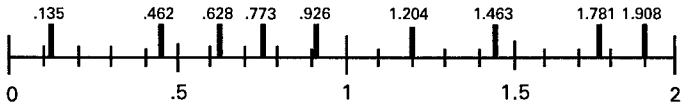
e) less than $\frac{1}{2}$;

f) exactly halfway between .13 and .14;

g) less than $1\frac{3}{4}$ but greater than $1\frac{1}{5}$.

COMMENTS

13 Keep emphasis on making reasonable approximations of the location of each point. Technically, each hundredths segment must be divided into 10 equal parts. Practically, this is not possible without enlarging the linear unit but the process should be discussed. Approximate locations of the decimals are shown here:



14 a) .7, .62, .628, .773

b) .628

c) .96, 1, 1.08

d) .926, 1, 1.08, 1.1, .96

e) 0, .2, .04, .37, .135, .462

f) .135

g) 1.4, 1.7, 1.24, 1.55, 1.204, 1.463

Focus Teacher Activity (cont.)

ACTIONS

15 Ask the groups to name several decimals that could be placed between .773 and .8 on their number lines. Discuss.

16 Ask the students to imagine an extended number line and to picture 3 decimal numbers between 27 and 28. Discuss. Repeat for one or more of the following:

a) 3 decimal numbers between 38.5 and 38.6,

b) 3 decimal numbers between 190.25 and 190.4,

c) 3 decimal numbers that are very close to 500,

d) 3 decimal numbers that round to 53,

e) 2 decimal numbers that are exactly the same distance from 7.0.

COMMENTS

15 Answers will vary. Any numbers with 7 tenths, 7 hundredths, 3 thousandths, and 1 to 9 of any smaller pieces satisfy this condition, as do numbers with 7 tenths, 8 hundredths, and 1 to 9 of any smaller pieces, or numbers with 7 tenths, 9 hundredths, and 1 to 9 of any smaller pieces.

16 c) The numbers will depend on each student's interpretation of "very close." Allow students to decide for themselves and discuss their views. What is meant by "very close" often depends on the context of the numbers. For example, 499 may not be very close if it represents a measurement for some scientific purpose that needs to be accurate to the nearest ten-thousandth. On the other hand, a total earning of $499 is very close to $500.

e) Some examples include 6.7 and 7.3, 6.99 and 7.01, 5.0 and 9.0.

TEACHER NOTES:

Follow-up Student Activity 30.1

NAME _____ **DATE** _____

1 Each of the following statements is incorrect. Place a decimal point or change the position of the decimal point in each number so that the use of the decimal makes sense.

a) Monica's mother is 54 feet tall.
b) Darrell was paid $235 for baby-sitting for one hour.
c) Eric's father is .400 years old.
d) Mrs. Rogers' newborn baby weighed 92 pounds.
e) The official height of a basketball hoop from the floor is .0100 feet.
f) The Smith's bought a turkey which weighed 120 pounds.
g) The distance across the United States is about .300000 miles.

2 In the space below, sketch area piece collections for each set of numbers. Then use your diagram to order the numbers from smallest to largest.

a) .32 .03 3.02 .30

_____ < _____ < _____ < _____

b) 2.17 .217 .021 .007

_____ < _____ < _____ < _____

(Continued on back.)

Follow-up Student Activity (cont.)

3 Draw a sketch of area pieces that represent 3.849.

Now draw the collection closest in value to 3.849 that uses:

a) only units and tenths b) only units c) no pieces smaller than hundredths

4 Round 596.499 to the nearest:

a) unit _____ c) tenth _____
b) thousand _____ d) hundredth _____

5 Imagine a number line that extends from 0 to 160. Name 3 decimals that are:

a) between 154 and 155 _____, _____, _____
b) between 3.7 and 3.8 but closer to 3.7 _____, _____, _____
c) between 14.62 and 14.63 but closer to 14.63 _____, _____, _____
d) greater than $1/4$ and less than $1/2$ _____, _____, _____

6 Arrange these numbers in order from smallest to largest and then explain your methods.

8.932 7.912 $8\frac{9}{10}$ 9.111 8.99 $7\frac{9}{12}$ 8.24

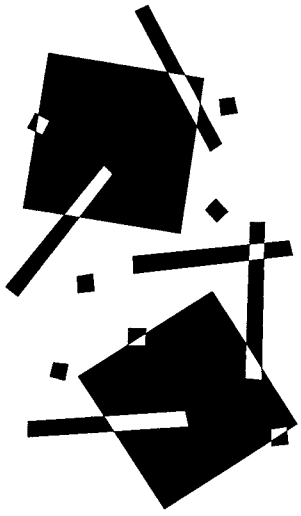
7 On separate paper, write a letter to a younger student (who has base ten pieces) explaining how to use the base ten pieces to understand rounding decimals.

Estimating Sums & Differences

Lesson 31

THE BIG IDEA

Mental computation strategies, together with an understanding of place value, rounding, and the meanings of addition and subtraction, provide a conceptual framework for understanding estimation. Activities that enable students to develop estimation techniques that are conceptually based can enhance students' number sense, build their confidence at computation, and give them tools for determining the reasonableness of calculator results.



CONNECTOR

OVERVIEW

Students describe their views of the meanings of addition and subtraction, and mentally compute several sums and differences.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).
- ✔ Base ten pieces for the overhead.

FOCUS

OVERVIEW

Students combine their understanding of place value concepts with their knowledge of rounding and mental computation to estimate sums and differences.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces for each student.
- ✔ Base ten area pieces for the overhead.
- ✔ Focus Masters A-F, 1 transparency of each.
- ✔ Focus Master G, 1 copy per pair of students and 1 transparency.

FOLLOW-UP

OVERVIEW

Students estimate sums and differences of whole numbers and decimals, describe their estimation strategies, and apply these strategies to determining the reasonableness of calculator computations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 31.1, 1 copy per student.

LESSON IDEAS

ASSESSMENT

Have students write their responses to Follow-up Problem 5 on a separate sheet for inclusion in their journals.

Have students work in small groups to design what they believe would be a meaningful quiz about addition and subtraction estimation. Have the groups also write expected responses to their quiz and then exchange quizzes with another group.

TEACHER NOTES:

SELECTED ANSWERS

3. a) Two possibilities are:
Front-end on leading digits:
30 + 10 + 20 + 60 + 90 + 70 = 280

Rounding to tens digits:
30 + 10 + 20 + 70 + 90 + 80 = 300

b) Use front-end methods to get 3 + 27 + 1 – 16; then form compatibles 32 – 16 = 16, or compatibles 30 – 15 = 15.

c) Form compatibles 300 – 150 = 150 or 275 – 150 = 125.

Or, form "nicer" (compatible) numbers 280 – 160; and then add up: 160 + 20 + 100 = 280

difference of 120

Connector Teacher Activity

OVERVIEW & PURPOSE

Students describe their views of the meanings of addition and subtraction, and mentally compute several sums and differences. This allows students and you to see how their thinking has developed, and to see whether they have integrated models from class into their ways of thinking. Mental computation strategies are applied to estimating sums and differences in the Focus.

MATERIALS

✔ Student journals (optional).

✔ Base ten pieces for the overhead.

ACTIONS

1 Write the following statements on the overhead and ask the students to spend 5 minutes completing them (in writing). Encourage students to include examples and diagrams to help explain their thinking. Discuss their views.

To **me**, **addition means:**

To **me**, **subtraction means:**

2 Write Problem a) below on the overhead and ask the students to mentally compute the sum. Have volunteers describe several different strategies. Repeat for one or more of Problems b)-i).

a) 430 + 510

b) 56 + 8

c) 354 + 430

d) 50 + 25 + 70 + 8 + 30 + 25

e) 200 – 47

f) .357 – .296

g) 6 – 1.3

h) 3.24 – .99

i) 114 + 289

COMMENTS

1 This is an appropriate journal entry. Look for evidence that students view the meanings of the operations from a conceptual perspective (as introduced first in Lesson 2) rather than algorithmically. Understanding these meanings is reinforced in the Focus activity.

One way to discuss students' views is to ask for volunteers to read their explanations aloud and to show their illustrations at the overhead. After several students have shared, invite the class to make additions to their entries based on new ideas shared by classmates.

2 The intent here is to recall mental strategies explored in Lesson 26. To encourage a variety of strategies to come up, suggest that after students mentally solve each problem, they think of another mental method that would also work.

Have overhead base ten pieces available so that students can illustrate visual models they use to mentally compute sums and differences. Emphasize student verbalizations and visual representations of the problems rather than symbolic recordings, which could distract them from the purpose of the activity. However, for some problems you might record students' actions as they describe or model them. There are many possibilities for each of the problems given. Numerical representations of a few follow:

a) Combine the hundreds and then the tens to get $(400 + 500) + (30 + 10) = 900 + 40 = 940$.

b) $(50 + 6) + 8 = 50 + (6 + 8) = 50 + 14 = 50 + 10 + 4 = 64$;
or $56 + (4 + 4) = (56 + 4) + 4 = 60 + 4 = 64$;
or $(56 - 2) + (8 + 2) = 54 + 10 = 64$.

c) $354 + 400 + 30 = 754 + 30 = 784$;
$(300 + 400) + (50 + 30) + 4$.

(Continued next page.)

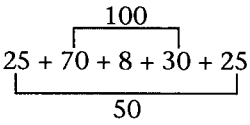
Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

2 (continued.)

d) Use compatibles to get $50 + 25 + 70 + 8 + 30 + 25 = 50 + 50 + 100 + 8 = 208$



e) Form compatibles: $200 - 47 = (200 - 50) + 3$ (since subtracting 50 removes 3 too many) $= 150 + 3 = 153$; or use add-up methods to get

$$47 + \underbrace{3 + 150}_{\text{difference of 153}} = 200;$$

or, adding 3 to each number to get the equal difference $200 - 47 = 203 - 50 = 153$.

f) Add .004 to each to get the equal difference $.361 - .300 = .061$.

g) Add .7 to each to get the equal difference $6.7 - 2 = 4.7$; or since $1.3 = 1 + .3$, compute $6 - 1 - .3 = 5 - .3 = 4.7$.

h) Since subtracting .99 is equivalent to subtracting 1 and adding .01, compute $3.24 - 1 + .01 = 2.24 + .01 = 2.25$; or add .01 to both to get the equal difference of $3.25 - 1 = 2.25$; or add-up to get:

$$\underbrace{.99 + .01}_{\text{difference of 2.25}} + 2 + .24 = 3.24.$$

i) Form the equal sum $(114 - 11) + (189 + 11) = 103 + 200 = 303$.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students combine their understanding of place value concepts with their knowledge of rounding and mental computation to estimate sums and differences. The techniques students develop will help them to determine reasonable approximations of sums and differences that occur in a variety of contexts and to decide whether the results of calculator or pencil-and-paper computations are reasonable.

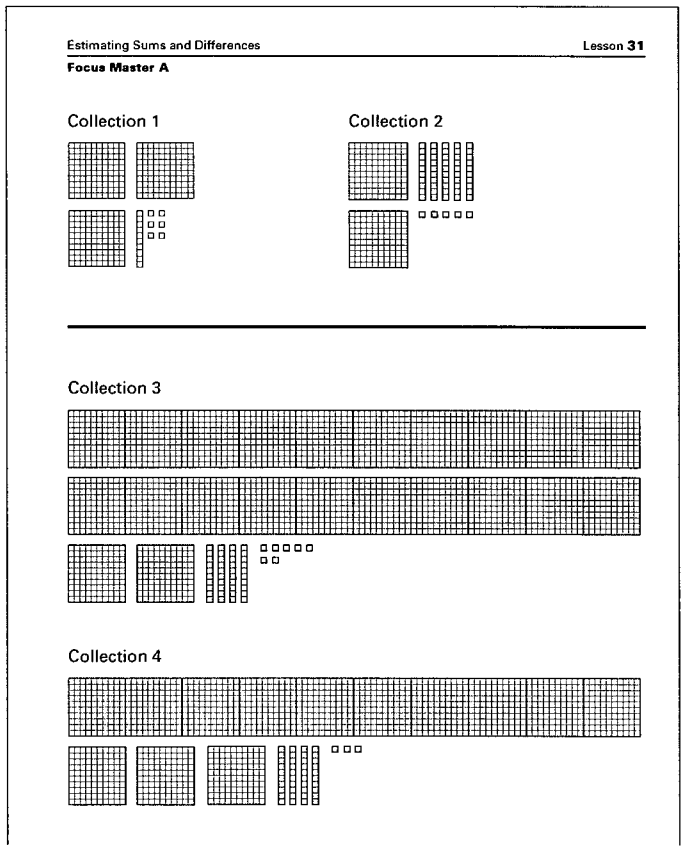
MATERIALS

- ✔ Base ten area pieces for each student.
- ✔ Base ten area pieces for the overhead.
- ✔ Focus Masters A-F, 1 transparency of each.
- ✔ Focus Master G, 1 copy per pair of students and 1 transparency.

ACTIONS

1 Distribute base ten area pieces to each student. Tell the students that for now the *small* base ten area piece square represents one unit.

Place a transparency of Focus Master A on the overhead and show students Collections 1 and 2, covering the other collections on the transparency. Ask the students to form these two collections with their area pieces and then to "quickly and at a glance" make a reasonable approximation of the sum of the values of the two collections (the small square in the diagrams represent a small area piece square). Discuss students' estimates and methods.



COMMENTS

1 The intent of this activity is to develop a variety of visual methods for estimating sums and differences. Listening to other students' verbalizations of their methods helps students acquire a range of estimation strategies, and reinforces the idea that there isn't a best or most reasonable method of estimating.

There are many possible strategies. Here are four methods that may or may not come up:

a) Combine the largest pieces in the two collections to get an estimate of 5 hundreds, or 500. This method of looking only at the largest pieces is called *front-end* estimation.

b) Round each collection to the nearest hundred and then combine to get 6 hundreds.

c) Use the front-end method of looking at hundreds *and* tens to obtain an approximation of 300 + 10 + 200 + 50 = 500 + 60 = 560 (notice that, in this case, the front-end method involves the front *two* place values, and units are ignored).

d) Round both collections to the nearest ten and then combine them to get 5 hundreds and 8 tens (note that this assumes the convention of rounding up a 5).

All of the above are considered reasonable approximations of 316 + 255. That is, based on the above methods, one could write: $316 + 255 \approx 500$; $316 + 255 \approx 600$; $316 + 255 \approx 560$; and $316 + 255 \approx 580$ (where $\approx$ means "is approximately equal to").

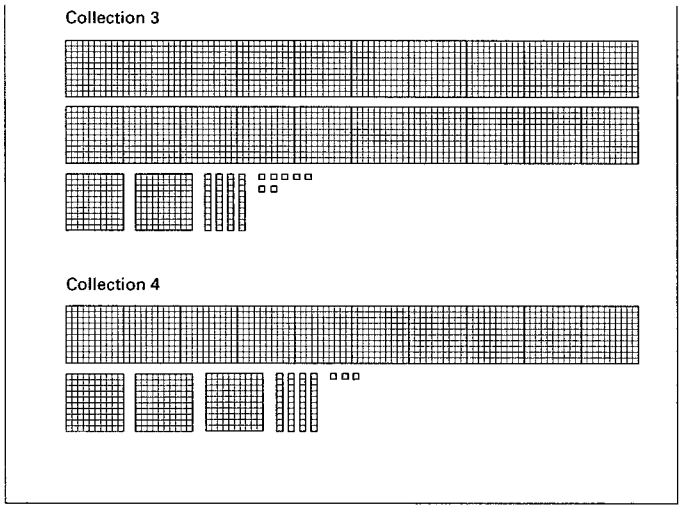
It is important that students not view the estimate that is closest to the exact sum as the "best" estimate. In fact, to prevent this view from developing, avoid discussing exact answers or asking students to compute them.

Focus Teacher Activity (cont.)

ACTIONS

2 Have the students describe a situation for which they feel 500 would be a reasonable estimate of 316 + 255 and one for which 600 would be more reasonable. Discuss.

3 Reveal Collections 3 and 4 on Focus Master A and ask the students to determine a quick and reasonable approximation of the sum of the values of the two collections, so that the sum contains: a) thousands only; b) thousands and hundreds only. Encourage students to base their estimates on what they "see" in the collections and to verbalize why some methods give different approximations than others. Discuss students' ideas about situations in which their approximations would be reasonable/unreasonable.



COMMENTS

2 The intent here is to begin discussion of situations in which underestimating is more appropriate than overestimating, and vice versa.

For example, suppose the goal of a fund raiser is \$500 and one needs to estimate \$316 + \$255 to tell if the goal has been met. Approximating by underestimating to get \$300 + \$200 = \$500 would be reasonable in this case.

However, one might overestimate to get \$316 + \$255 ≈ \$300 + \$300 = \$600 as a method of deciding how much cash to take in order to be sure to have enough money to purchase two items whose prices are \$316 and \$255.

3 Note that the large rectangles in the diagram represent 1000. Since students don't have these pieces, encourage them to imagine combining the collections. If this is difficult for some students, suggest they make quick sketches to determine estimates.

a) Here are three reasonable possibilities: 1) use the front-end method for thousands only to get 3000; 2) round each collection to the nearest thousand to get 1000 + 2000 = 3000; 3) use the front-end method for thousands and hundreds only, to get (1000 + 2000) + (300 + 200) = 3500, and then round to get 4000.

b) Three possibilities are: 1) use the front-end method for thousands and hundreds only to get 3500; 2) round each collection to the nearest hundred and then find the sum to be 1300 + 2200 = 3500; 3) add the front-end collections 2240 + 1340 to get 3580, which rounds to 3600.

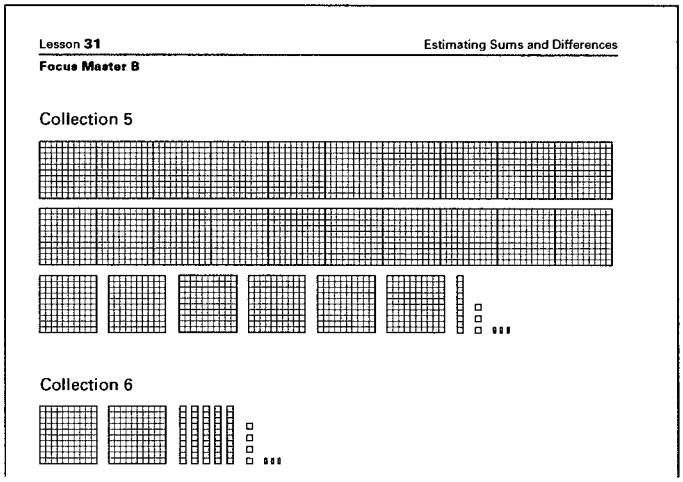
Based on the preceding methods, one could write: 2247 + 1343 ≈ 3000; 2247 + 1343 ≈ 3500; 2247 + 1343 ≈ 4000; and 2247 + 1343 ≈ 3600.

Notice that rounding and then adding does not always produce the same result as adding and then rounding. However, depending on the context, both may give reasonable approximations, as may the front-end method.

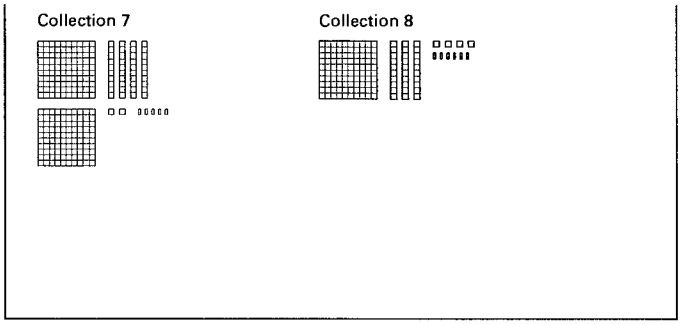
Focus Teacher Activity (cont.)

ACTIONS

4 Place a transparency of Focus Master B on the overhead, revealing only Collections 5 and 6. Tell the students that for now the unit is the *large* square. Ask the students to estimate the total value of the two collections. Discuss their methods and conclusions as well as situations for which they think their estimates would be reasonable/unreasonable.



5 Reveal Collections 7 and 8 on Focus Master B and, using the large square as the unit, have the students estimate the corresponding sum: a) using units only, and b) using units and tenths only.



COMMENTS

4 If the large square is one unit, then the large rectangle is 10 and the smallest rectangle represents 1 thousandth.

Encourage students to look for more than one method of approximating the sum, such as: imagine compatible collections of 28 + 2 = 30; use the front-end method for tens only to get an estimate of 20 + 0 = 20; use the front-end method for tens and units only to get 28; round each collection to the nearest ten and then add to get an estimate of 30; round each collection to the nearest unit and then add to 26 + 3 = 29. Based on the above method, one could write 26.133 + 2.543 ≈ 30, or 20, or 28, or 29.

To give students more practice with forming compatibles, sketch base ten collections that are represented by the following computations and have students approximate the sums: 24 + 75; 23 + 28; 150 + 52; 2.24 + .78; 1.71 + .28; 687 + 243 + 354 + 979 + 847.

5 Here are a few possibilities for each:

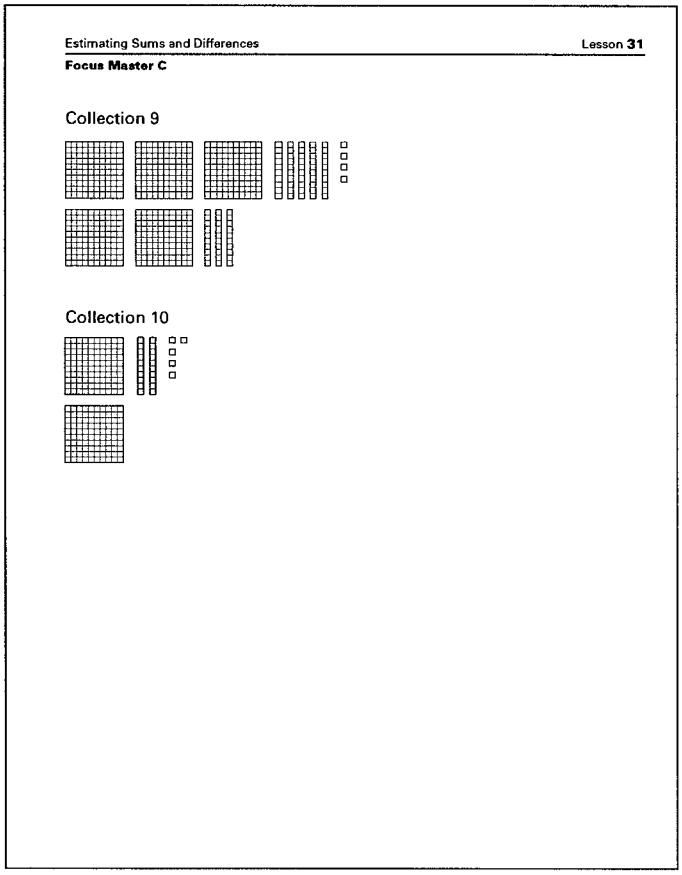
a) Using the front-end method for units only results in an estimate of 3 units. Observing that the total tenths form nearly another unit produces a refined estimate of 4. Rounding each initial collection to the nearest unit and then adding produces an estimate of 3. Adding the initial collections of units and tenths and then rounding to the nearest unit results in an estimated sum of 4. Hence, 2.425 + 1.346 ≈ 3 and 2.425 + 1.346 ≈ 4 are both examples of reasonable statements.

b) Forming and adding compatible collections of 2.5 and 1.5 produces 4.0 as an approximation. The front-end method for units and tenths produces an estimate of 3.7. This adjusts to 3.8 by observing that the hundredths combine to form nearly another tenth. Rounding the initial collections to the nearest tenth and then adding produces 3.7. Hence, 3.7, 3.8, and 4.0 are examples of reasonable approximations for 2.425 + 1.346.

Focus Teacher Activity (cont.)

ACTIONS

6 Tell the students that for now the small square is the unit. Reveal Collections 9 and 10 on Focus Master C and ask pairs of students to form these two collections with their area pieces. Have them also form a third collection that represents what they feel is a reasonable approximation of the *difference* between the numbers that represent Collection 9 and Collection 10. Discuss their methods, and situations for which their estimates would be reasonable/unreasonable.



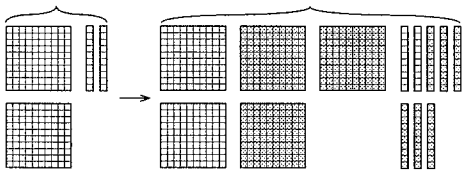
COMMENTS

6 A few of the possibilities are: use the front-end method with hundreds only to determine a difference of 3 hundreds; round the two collections to find a difference of 4 hundreds; compare the front-end hundreds and tens only to get a difference of 3 hundreds and 6 tens or 360.

The add-up method (determining what must be added to the smaller collection to make it equivalent to the larger one) could be applied to rounded collections or front-end parts of collections to approximate differences. For example, observing the front-end hundreds only of Collections 9 and 10, it is necessary to add 3 hundreds to Collection 10 to match the hundreds in Collection 9. Or, looking at the front-end hundreds and tens, 3 hundreds and 6 tens must be added to Collection 10, as illustrated below:

Start with the front-end of Collection 10.

Add pieces until the front-end of Collection 9 is formed.

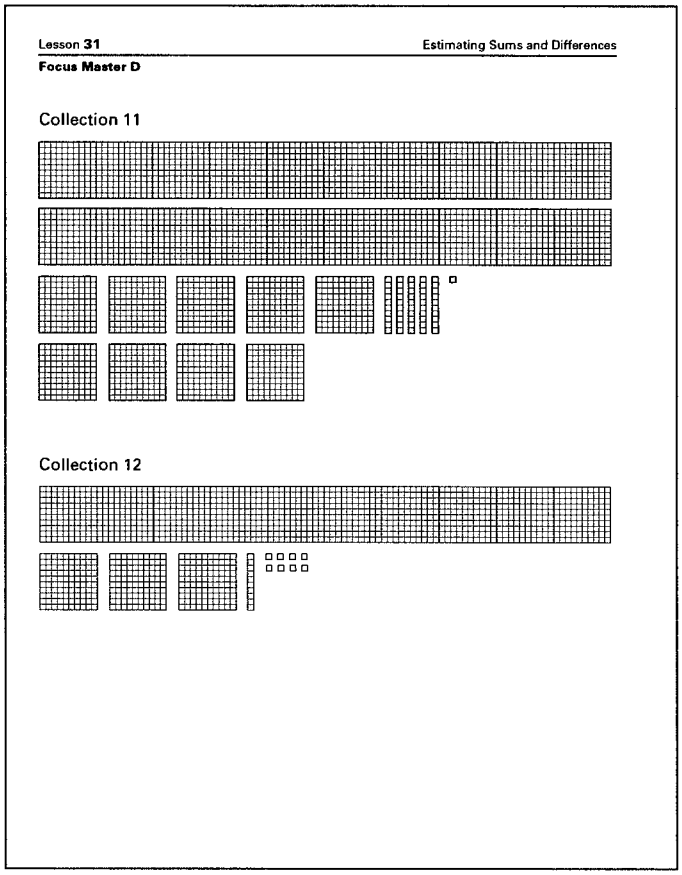


Hence, both 300 and 360 are examples of reasonable approximations of 584 – 225. One could also approximate 584 – 225 using compatibles 575 – 225 to get 350.

Focus Teacher Activity (cont.)

ACTIONS

7 Place a transparency of Focus Master D on the overhead and ask the students to make a reasonable approximation of the value of Collection 11 minus the value of Collection 12, using the small square as the unit. Discuss students' views about the "reasonableness" of various approximations.



COMMENTS

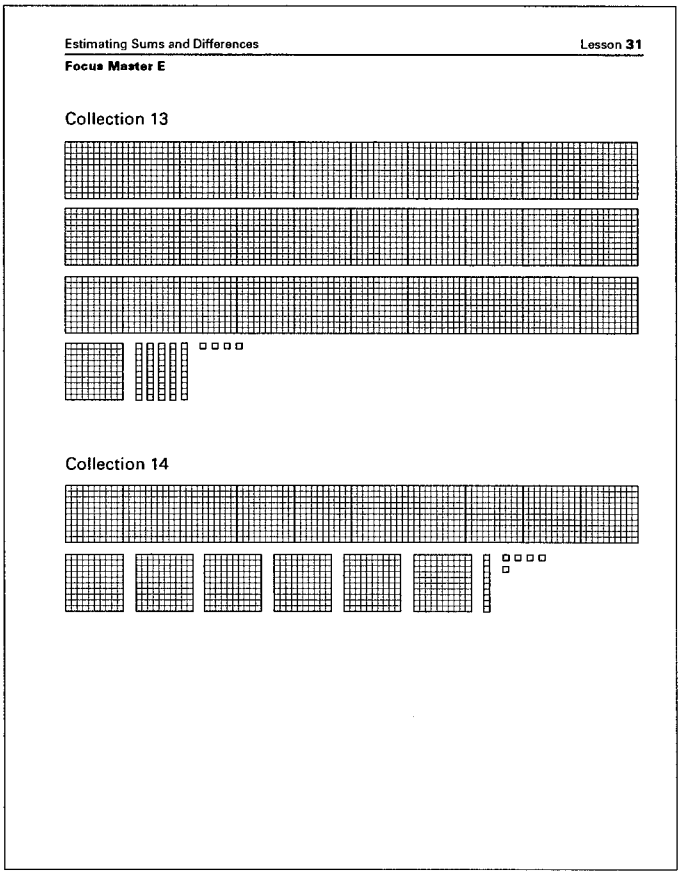
7 Some possible responses, based on the small square as unit, are: use the front-end method of looking at only the largest pieces to determine a difference of 1000; round each collection to the nearest 1000 and compare to find a difference of 2000; compare the front-end thousands and hundreds in the two collections to get a difference of 1000 + 600 = 1600; round Collection 11 to 2 thousands and 10 hundreds and Collection 12 to 1 thousand and 3 hundreds, and then compare to get a difference of 1700.

Some students may suggest that 1000 is not "reasonable" since this approximation does not take into consideration the relatively large difference in hundreds between the two collections. If so, you may wish to discuss contexts in which 1000 would and would not be a reasonable estimate for 2951 – 1318.

Focus Teacher Activity (cont.)

ACTIONS

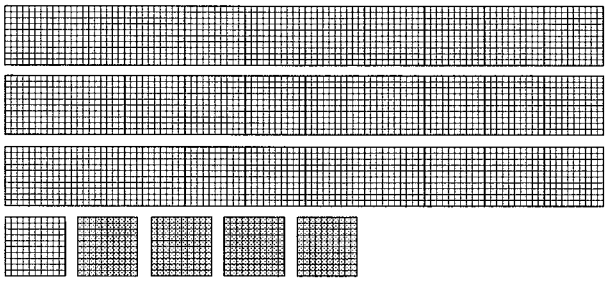
8 Place a transparency of Focus Master E on the overhead. Tell the students that for now the unit is the large square. Ask the groups to each determine at least two different visual methods for approximating the difference between the values of the collections (i.e., for approximating 31.54 – 16.15). Discuss their results and their ideas about situations for which selected estimates would be reasonable/unreasonable.



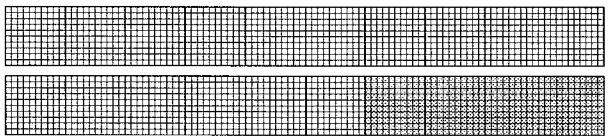
COMMENTS

8 There are many possible methods of estimating 31.54 – 16.15. For example, the strategy of forming same differences introduced in Lesson 26 for mental subtraction can be used to determine differences between rounded collections or between front-end parts of collections. By adding 4 units to each front-end collection of tens and units, an approximate difference is 35 – 20 = 15, as illustrated here:

Front-end Collection 13 with 4 units added:

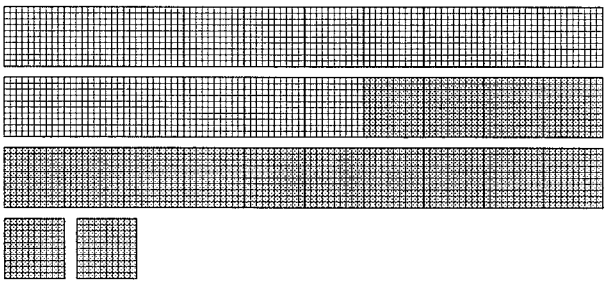


Front-end Collection 14 with 4 units added:



Another possibility is to use the add-up method on rounded collections of 32 and 16. The shaded pieces in the following diagram represent what must be added to the smaller collection to make it equivalent to the larger one (i.e., the shaded pieces represent the difference between the rounded collections).

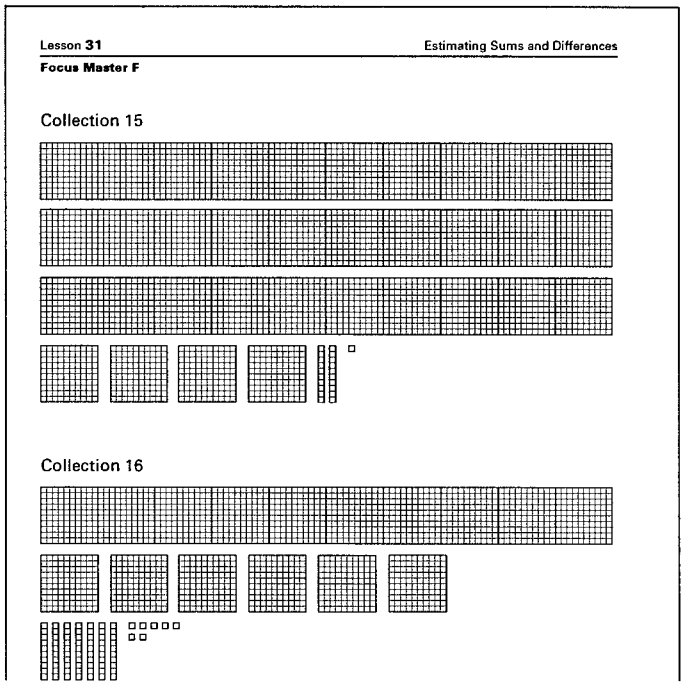
Rounded Collection 14 plus 16 more = Rounded Collection 13



Focus Teacher Activity (cont.)

ACTIONS

9 Place a transparency of Master F on the overhead and ask the students to estimate the difference between the values of Collections 15 and 16, using the small square as the unit. Discuss their methods and situations for which selected estimates would be reasonable/unreasonable.



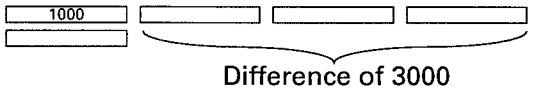
10 On the overhead, write the subtraction problem 4134 – 1525. Suggest that students imagine (or sketch) number piece collections to help them estimate the difference between the numbers. Have students sketch and/or explain their images and methods.

COMMENTS

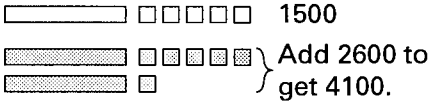
9 Some methods of using diagrams as a basis for approximating 3421 – 1677 include: imagine compatible collections of 3500 and 1500 whose difference is 2000; observe front-end thousands only whose difference is 2000; add-up 1600 + (1000 + 400 + 400) to get a total difference of 1800; add 400 to both front-end collections and find the difference 3800 – 2000 = 1800; subtract 600 from both collections to get 2800 – 1000 = 1800.

The collections on Focus Masters A-C could be used for students to estimate differences using both the large and small area piece squares as the unit. Similarly, the collection on Masters D-F could be used for further practice estimating sums.

10 For example, students may visualize only the front-end thousands and "see" a difference of 3000, as sketched below:



Another possibility is to use the add-up method to find the difference between rounded collections of 4100 and 1500.



Still another reasonable approximation is 4000 – 1500; adding 500 to each number forms the equal difference 4500 – 2000 which is easy to compute mentally.

Yet another approach is to round to get 4000 – 2000 = 2000 as an approximation.

Focus Teacher Activity (cont.)

ACTIONS

11 Repeat Action 10 for the following computation:

$$\begin{array}{r} 4429 \\ 3264 \\ +\ 7056 \\ \hline \end{array}$$

12 Select one or more of the following computations for students to approximate. Have the students use at least two methods of approximating each sum or difference. Then discuss their strategies.

a) .3421 + .6142

b) 1409 + 23 + 382

c) 42,689 + 57,493 + 12,792 + 87,208

d) 36,482 – 9,576

e) 73,891 – 54.915

f) 9.43 + 4.2 + 7.135 + 3.789

g) 13 + 89 + 47 + 72 + 54 + 28 + 17

h) 32,359 – 5,000 + 33,000

i) 590 + 105 – 301

j) 307 – 47 – 52

COMMENTS

11 To encourage students to develop comfort with several methods, suggest they each find three different ways to approximate this sum.

12 To reinforce the development of students mental computation strategies, pose one or two problems for discussion on a regular basis. Encourage students to try out a variety of different strategies.

It is not appropriate to ask students to mentally find exact answers to these computations. In fact, for most of these problems, the most reasonable and efficient method of getting the exact answer is to use a calculator. However, it is appropriate to mentally approximate the answers in order to make a decision about the reasonableness of a calculator result.

Since finding the exact answer may suggest to students that the closest estimate is "best" or that an estimate is not valuable because it is not exact, it is preferable not to compare estimates to exact answers.

Focus Teacher Activity (cont.)

ACTIONS

13 Give each pair of students a copy of Focus Master G. Assign one or more of the Approximation Situations. Encourage students to include diagrams to help explain their reasoning and observations.

Estimating Sums and Differences Lesson 31

Focus Master G

Approximation Situations

Describe what you think are reasonable approximation methods to use for each of the following situations. Write your observations and conclusions about each situation.

a) Jan earned $178 one week and $235 the next week. She needs $400 to buy a guitar. She used approximation to decide if she had enough money.

b) Bill's car gets 30 miles per gallon on the highway and he is going to travel 153 miles. He will make a trip on December 23 and the temperature is expected to be below 15°. To insure that he will have enough gasoline to make the trip, he uses approximation to decide how much gasoline to buy.

c) Baseball cards cost $2.47 each and Sandra wants to buy 3 cards. She uses approximation to determine the amount of money she needs.

d) The distance from the ground to the roof of a building is 44 feet. Parker uses approximation to decide whether to buy a 40 foot ladder or a 50 foot ladder to use while repairing the roof of this building.

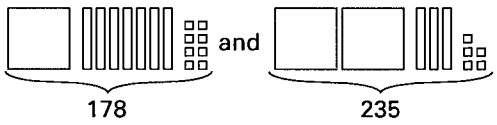
e) Alesea averaged 17.6 points per game for 4 games. The coach used approximation to estimate whether Alesea had scored more than the record of 67 points in 4 games.

f) In 3 weeks of fund raising the class had earned $139, $259, and $196. The students approximated to determine if they had raised more or less money than their goal.

COMMENTS

13 For example, some students might reason about Situation a) as follows:

"I know that 178 = 100 + 70 + 8 and 235 = 200 + 30 + 5 as shown here:



Since 100 + 200 = 300, Jan has at least 300. Since 78 + 35 > 70 + 30 = 100, Jan has more than $300 + $100 = $400. Therefore, Jan has enough money to buy the guitar."

Notice that in Situation a) it is important that approximations be made by underestimating the amount earned, rather than by overestimating. For example, to round up to get 178 + 235 ≈ 180 + 240 would suggest she needs $420 and therefore, does not have enough money. In Situation c), on the other hand, it isn't reasonable to round down since doing so would suggest she needs less money than is the case. It is worthwhile to discuss the rationale for over or underestimating each situation on Focus Master G.

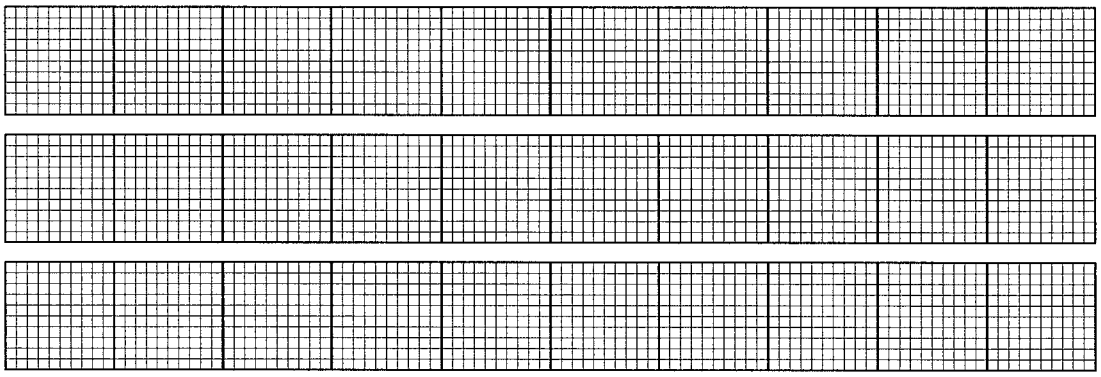
TEACHER NOTES:

Follow-up Student Activity 31.1

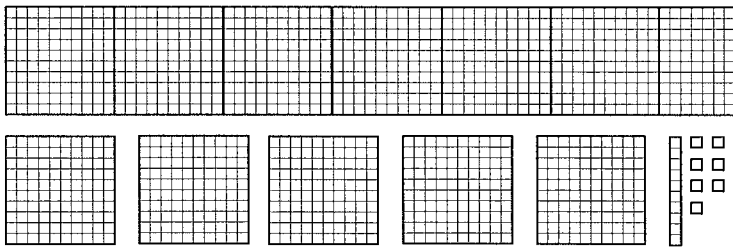
NAME _____ **DATE** _____

1 Describe two different methods of approximating the difference between these two collections of base ten pieces. (The smaller square " ▫ " is the unit.)

Collection A:



Collection B:



Method 1:

Method 2:

2 Describe two different methods of approximating this sum:
5191 + 2365

Method 1:

Method 2:

(Continued on back.)

Follow-up Student Activity (cont.)

3 Estimate each answer. (Note ≈ means "is approximately equal to.") Then explain the methods you used for each problem.

a) $32 + 11 + 23 + 69 + 94 + 76 \approx$

b) $3.4932 + 27.2097 + 1.403 - 16.909 \approx$

c) $286.41 - 158.09 \approx$

4 Use your calculator to solve each computation below.

a) $24.36 + 103 - 47.66 + 75.23 - 149.1 =$ _____

Use estimation strategies to decide whether the calculator answer is reasonable. Then describe your estimation strategies here:

b) $.4689 + .713 + .49 - 1.0099 =$ _____

Use estimation strategies to decide whether the calculator answer is reasonable. Then describe your estimation strategies here:

c) $197.2013 - 9.96 - 73.6217 + 10.3 + 49.98 =$ _____

Use estimation strategies to decide whether the calculator answer is reasonable. Then describe your estimation strategies here:

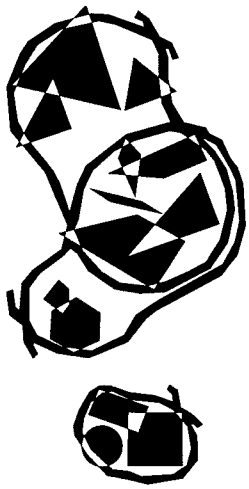
5 On another sheet of paper, explain some techniques for estimating sums and differences that you didn't know before you took this class or that you now understand better.

Angle Measurement

Lesson 32

THE BIG IDEA

Pattern blocks provide an informal introduction to the concept of angle measure. After determining the measures of the pattern block angles, it is possible to use them to approximate other angles and combinations of angles. These experiences develop students' intuitions about angle relationships and readiness for constructing circle graphs.



CONNECTOR

OVERVIEW

Students describe their current perceptions of the concept of angle.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals, (optional).
- ✔ Pattern block squares, 1 per student and 1 for the overhead.

FOCUS

OVERVIEW

Students use pattern blocks and toothpicks to explore the concept of angle measure.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 set per group of students.
- ✔ Focus Student Activity 32.1, 1 copy per group and 1 transparency.
- ✔ Toothpicks (flat are preferable), 30 per student.
- ✔ Focus Masters A, B, and C, 1 transparency of each.
- ✔ Shapes to Sort (see *Blackline Masters*), 1 set per group.
- ✔ Pattern blocks for the overhead.
- ✔ Butcher paper and colored markers for each group.
- ✔ Scissors and straight edges, 1 of each per student.
- ✔ 1-inch grid paper (see *Blackline Masters*), 2-3 sheets per student.
- ✔ Yarn loops, 4 per group.

FOLLOW-UP

OVERVIEW

Students identify and determine the sums of the angles in the pattern block figures. Students combine pattern blocks to form acute, obtuse, right, straight, reflex, and 360° angles. They also use the pattern blocks to help form approximations of other angles.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 32.2, 1 copy per student.
- ✔ Pattern blocks (see *Blackline Masters* for patterns to copy on cardstock), 1/4 sheet (or more) of each shape per student.

LESSON IDEAS

ASSESSMENT/FOLLOW-UP
Have students write responses to Follow-up Problem 4 in their journals and discuss ways their thinking about angles has developed since their entry for the Connector Action 1.

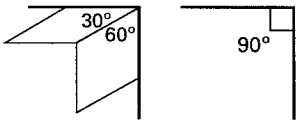
QUOTE
Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing, and testing hypotheses precede the development of more formal summary statements. In the process, definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas to develop informal arguments.
NCTM Standards

TEACHER NOTES:

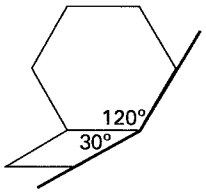
SELECTED ANSWERS

1. Here is one possibility for each:

a) white and blue parallelograms

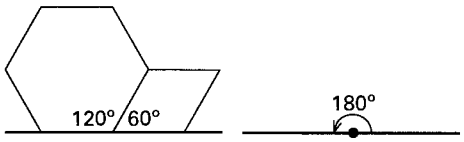


b) a hexagon and white parallelogram

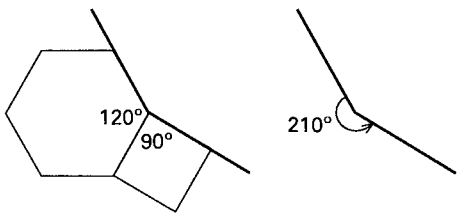


c) It is not possible to form an acute angle using 2 noncongruent pattern blocks since all possible combinations are greater than or equal to 90°. (If using congruent pattern blocks was allowed, then 2 small angles from 2 white parallelograms would form a 60° acute angle.

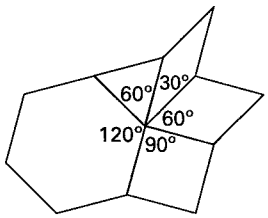
d) hexagon and blue parallelogram



e) hexagon and square

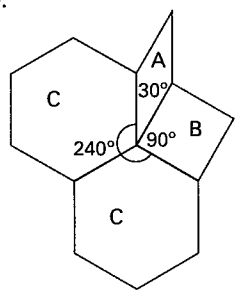


2. a) One possibility is 1 each of a hexagon, triangle, square, white parallelogram, and blue parallelogram, arranged as shown here (note that in this example the pattern blocks used are noncongruent, but 2 of the pattern block *angles* are congruent).

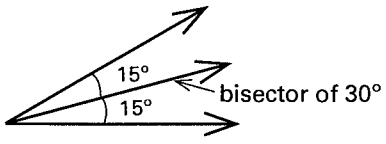


b) One argument why this is not possible is that since 1 right angle has measure 90°, and 1 acute angle has measure less than 90°, their sum will always be less than 180°. Hence, the remaining angle is greater than 180°, which is not obtuse.

c) One possibility is 2 hexagons, 1 square, and 1 white parallelogram.



3. To approximate 15°, for example, many students will trace a 30° angle and either fold it in half or "eyeball" to locate the *bisector*, as illustrated below:



Connector Teacher Activity

OVERVIEW & PURPOSE

Students describe their current perceptions of the concept of angle. This is useful now in planning for the Focus and later for assessing student growth.

MATERIALS

- ✔ Student journals, (optional).
- ✔ Pattern block squares, 1 per student and 1 for the overhead.

ACTIONS

1 Write the following thought starters on the overhead and have the students write responses:

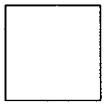
To me the word *angle* means...

Here are some diagrams that help show what I know about angles...

A question or questions I have about angles...

Discuss their ideas.

2 Place the students in small groups and distribute a set of pattern blocks to each group. Have the groups take an orange pattern block from their set and ask them to record as many observations as they can about this pattern block. Discuss. If students don't make observations about the angles, ask them to do so.



COMMENTS

1 If students are keeping journals, have them write their responses in their journals. The point of this Connector is to find out what students know now about angle measure. The Focus activity provides opportunities for students to expand and clarify their understanding. Note that they describe their thoughts about *angle* again in the Follow-up.

2 Students could brainstorm privately in their journals before discussing. Keep discussion of terminology informal and in the context of the activity.

There are many possible observations. For example, the top and bottom faces are squares and the side faces are rectangles; each square has 4 square corners (or 4 right angles or four 90° angles); the lengths of the sides of the square are all equal; the top and bottom faces are congruent; the 4 rectangular faces are congruent; all of the angles are congruent; etc. Encourage students to informally verify their observations.

Depending on students' prior experiences with angle measurement, comments about combinations of angles of the square may come up (e.g., when 2 squares are placed side by side, the angles form a straight line, or 180°; the 4 angles in a square total 360°; the square can be divided into 2 congruent triangles with angle measure 90°, 45°, 45°, etc.). Note if these ideas do not come up here, there are opportunities during the Focus.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use pattern blocks and toothpicks to explore the concept of angle measure. After determining the measures of all of the pattern block angles, they use the pattern blocks to approximate other angles. They also examine the similarities and differences among several polygons and use Venn Diagrams to sort them, with particular emphasis on angle measures.

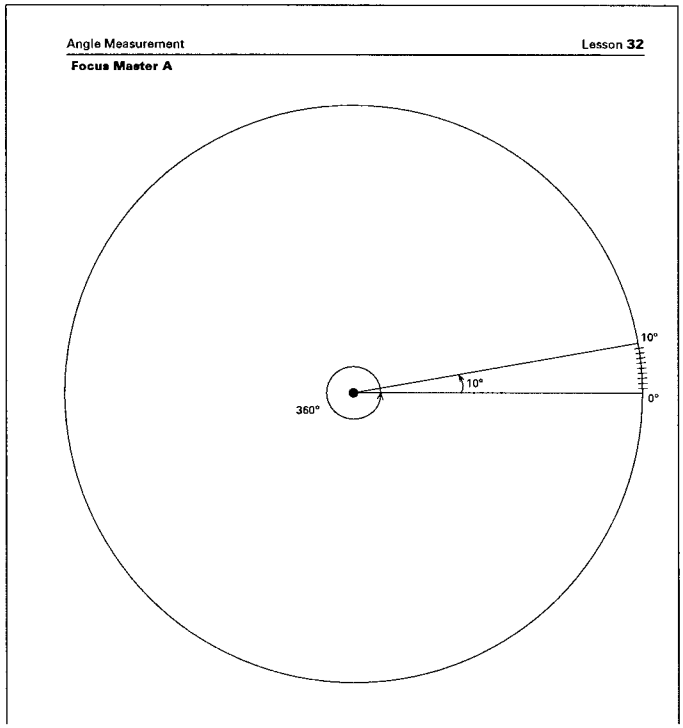
MATERIALS

- ✔ Pattern blocks, 1 set per group of students.
- ✔ Focus Student Activity 32.1, 1 copy per group and 1 transparency.
- ✔ Toothpicks (flat are preferable), 30 per student.
- ✔ Focus Masters A, B, and C, 1 transparency of each.
- ✔ Shapes to Sort (see *Blackline Masters*), 1 set per group.
- ✔ Pattern blocks for the overhead.
- ✔ Butcher paper and colored markers for each group.
- ✔ Scissors and straight edges, 1 of each per student.
- ✔ 1-inch grid paper (see *Blackline Masters*), 2-3 sheets per student.
- ✔ Yarn loops, 4 per group.

ACTIONS

1 Tell the students that for this activity, all discussion of the pattern blocks will be in reference to their *top face only*.

Place a transparency of Focus Master A on the overhead and tell the students that thousands of years ago the Babylonians devised a method for measuring angles by dividing a circle into 360 equal parts called degrees, as indicated on Focus Master A.



COMMENTS

1 Although in the Connector students may have made observations about the rectangular surfaces around the edges of the orange pattern block, for the remainder of this activity the pattern blocks are considered as plane figures.

The Babylonians chose 360° as the measure of a circle because they believed there were 360 days in a year. Although it was later realized that a year has more than 360 days, the use of 360 as the number of degrees in a circle was never changed and is a basis for angle measurement today. (More in depth investigations of angle measurement are in *Visual Mathematics, Course II.*)

Although this activity provides an opportunity to introduce and/or reinforce some vocabulary, its main purpose is to develop intuitions about angle measurement.

Angles are formed by two *line segments* or *rays* called *sides* that intersect at a point, called the *vertex* of the angle. On Master A, for example, the vertex of the 10° angle is the center of the circle, and the sides of the angle extend from the center to points on the circle. A *line segment* (such as a side of a polygon) has 2 endpoints, while a *ray* has 1 endpoint and theoretically, extends indefinitely in one direction (as indicated by an arrowhead). When drawn, angles are usually identified by their vertex, as shown on the next page.

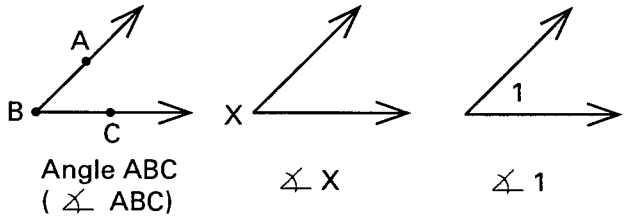
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

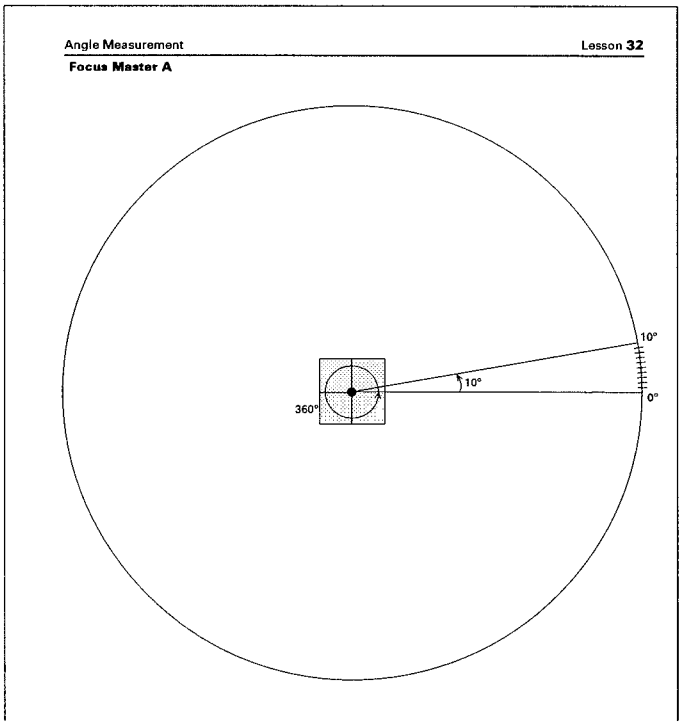
1 (continued.)



Toothpicks are useful for modeling angles on the overhead.

2 Arrange four transparent pattern block squares around the center of the circle on Master A, as shown below, and ask for volunteers to explain how they think your model can be used as a "visual proof" that the measure of a *right angle* (i.e., a square corner) is 90°.

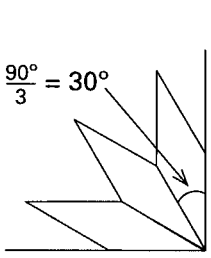
Have the groups determine the measure of each angle in the other pattern blocks. Have volunteers present "visual proofs" of their conclusions.

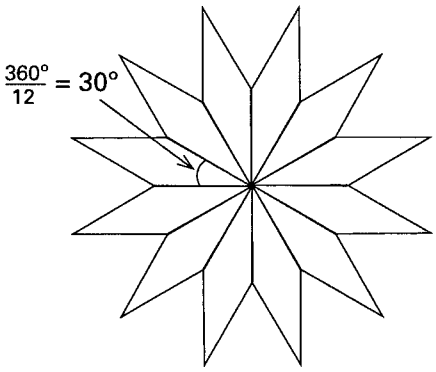


2 One line of reasoning is that since the total degrees in the circle is 360° and the squares subdivide the circle into 4 congruent *central angles* (angles whose vertices are at the center of a circle), then each of the angles is $^{360°}/_{4} = 90°$.

As students discuss the pattern blocks, promote the development of vocabulary by encouraging them to identify the pieces by naming their shapes (parallelogram, rhombus, hexagon, etc.).

There are many ways to find the measures of the angles of the other pattern blocks. For example, if 3 of the smaller angles in the white parallelograms are placed at a point, the angle created is a right angle. This means that each small angle has a measure of $^{90°}/_{3} = 30°$. Some students will determine this by "filling" one corner of a square pattern block with white parallelograms, while other students will trace a right angle and then trace around white pattern blocks one at a time as they fit them into the right angle:





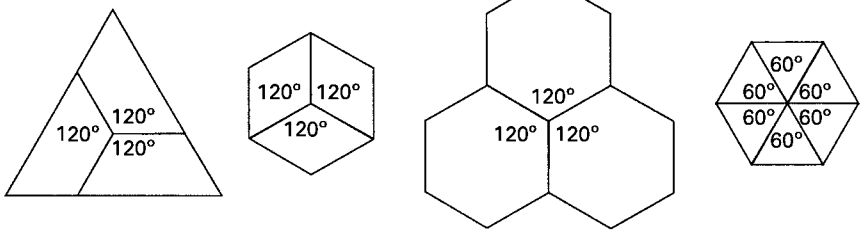
Still others may fit 12 white parallelograms around a point to show that the measure of one of the small angles of a white parallelogram is $^{360°}/_{12} = 30°$ as shown above.

Focus Teacher Activity (cont.)

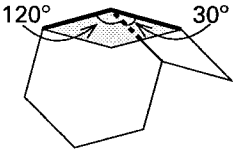
ACTIONS

COMMENTS

One way to find the angle measures of the following pattern block pieces is by placing the pieces around a point, as illustrated below.



Or, once the measure of the angles of one piece are determined, those angles can be used to measure the angles of other pieces. For example, the larger angle in the white parallelogram is 150° since it equals the sum of 1 angle from a hexagon (120°) plus 1 small angle from the white parallelogram (30°), as shown below.

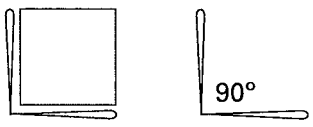


Some students may at first incorrectly predict that angle measures are related in the same way as areas (e.g., since the area of the triangle is ⅙ the area of the hexagon, the measure of 1 angle of the triangle is ⅙ of the measure of 1 angle of the hexagon). Allow time for contradictions to emerge as students explore and discuss their ideas.

3 Give each student about 30 toothpicks. Place a transparency of Focus Master B on the overhead and ask the students to use toothpicks to form the angle described in a), using their pattern blocks as needed. Discuss and repeat for b)-k). Have volunteers form some of these angles on the overhead and for b)-e) have other volunteers estimate the measure of each angle formed.

3 Using toothpicks to form angles helps to clarify the properties of angles, in particular, that the sides are always *straight* lines and these lines intersect at a point (the vertex) to form the angle.

a) A pattern block square can be used to help form a right angle, for example, by placing toothpicks along two adjacent sides of the square, and then removing the square:



b) An acute angle can be formed by placing toothpicks around a 30° or 60° pattern block angle. Or, students may form an angle that, by eye, is less than 90°, compare it to pattern block angles, and use those as a basis for estimating the size of the toothpick angle.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson 32 Angle Measurement
Focus Master B

a) a right angle (90°)

b) 3 different acute angles (an acute angle is less than 90°)

c) 3 different obtuse angles (an obtuse angle is greater than 90° and less than 180°)

d) a straight angle (exactly 180°)

e) 2 different reflex angles (a reflex angle is greater than 180° and less than 360°)

f) a 360° angle

g) ≈ 15°

h) ≈ 1°

i) ≈ 225°

j) 270°

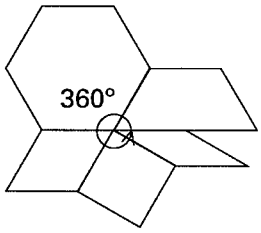
k) ≈ 105°

4 Have the students stand up, face the front of the room, and turn to their *left 90°* (i.e., counterclockwise 90°). Discuss their ideas about other turns or combinations of turns they could make and still end up in the same position.

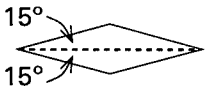
5 Repeat Action 4 for right 180°, left 30°, right 270°, right 390°, and left 745°. Have the students describe situations that involve angles that are greater than 360°.

COMMENTS

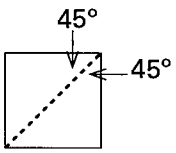
3 (continued.)
f) The 2 sides of a 360° angle lie one on top of the other (to model 360° with toothpicks some students may place 2 toothpicks on top of each other). Many combinations of pattern block angles are possible to form 360°. Some are shown in Comment 2. Another possibility is shown below:

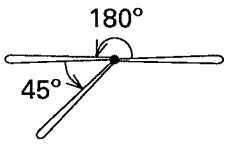


g) Discuss methods students use for approximating 15°. Some may "eyeball" an approximation. Others may make a pattern for 15° by tracing a 30° angle and folding it in half. Still others may note that the long diagonal of a white parallelogram forms two 15° angles as shown here:



i) To form a 225° angle many students will form a straight angle and add to it ½ of a right angle (which they may form by "eyeball," by tracing and folding a right angle into 2 equal angles, or by drawing a diagonal of the square).





4 If you turn with the students, be sure to face the same direction so your left and theirs is the same. A turn of right 270°, or left 30° plus left 60°, or left 180° and right 90°, or left 450° would cause them to end up in the same position as left 90°.

5 There are many examples of situations that involve angles greater than 360°. For example, turning a faucet through 745° may be necessary to turn water on full force. Although the faucet ends in the same position as a turn of 25°, the result of a 745° turn is very different from a 25° turn.

Focus Teacher Activity (cont.)

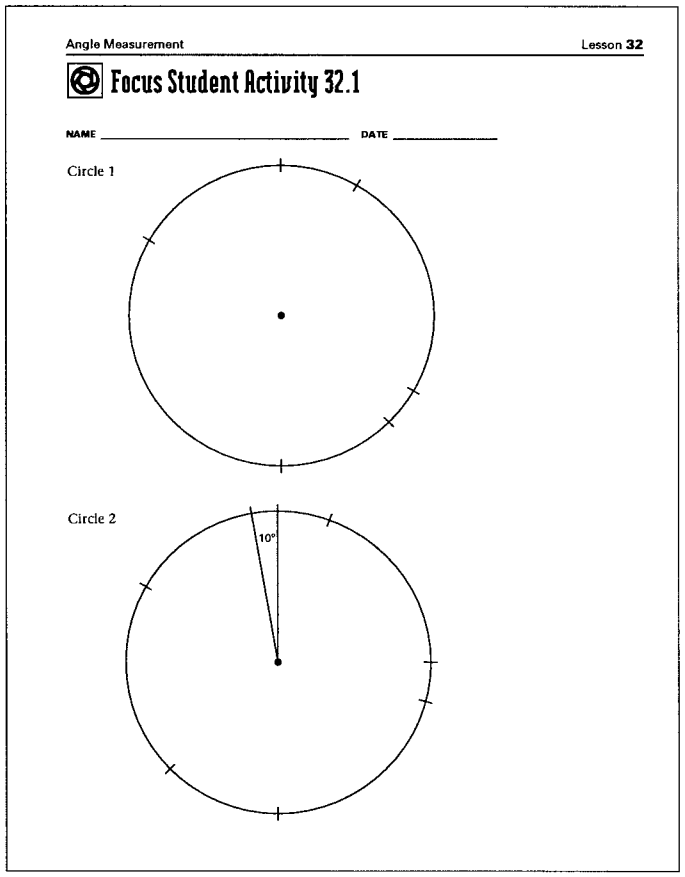
ACTIONS

6 Give each group a copy of Focus Student Activity 32.1. Point out that Circle 1 has been marked in several places. Ask the groups to use a straight edge to draw *central angles* by connecting the center of the circle to each point marked on the circle. Then have the groups determine the following:

a) the measure of each central angle; and

b) (optional) the fractional part covered by each sector of Circle 1.

Have volunteers show their conclusions and methods on a transparency of the activity sheet.


Angle Measurement Lesson 32

Focus Student Activity 32.1

NAME _____ DATE _____

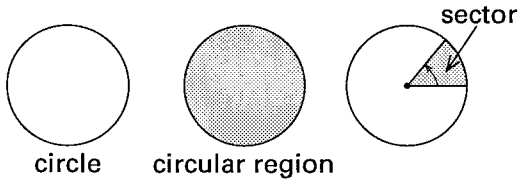
Circle 1

Circle 2

10°

COMMENTS

6 A *central angle* of a circle is an angle whose vertex is the center of the circle and whose sides intersect the circle, cutting off a *sector*.

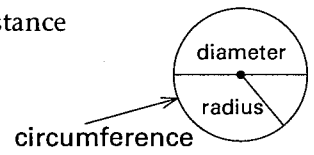


Other terminology related to circles that may come up includes the following:

radius: a line segment that joins the center to a point on the circle; the length of this line segment is also called the radius of the circle.

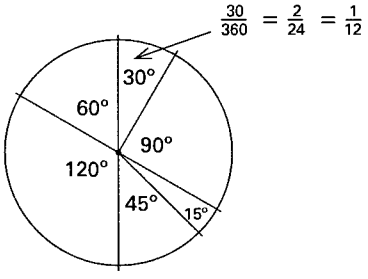
diameter: a line segment that joins two points of a circle and passes through the center; it's length is twice the radius and is also called the diameter of the circle.

circumference: the distance around the circle.



To determine the angle measures on Circle 1 (see below), students may arrange combinations of pattern blocks over the angles, or they may trace angles and look for relationships.

Circle 1:



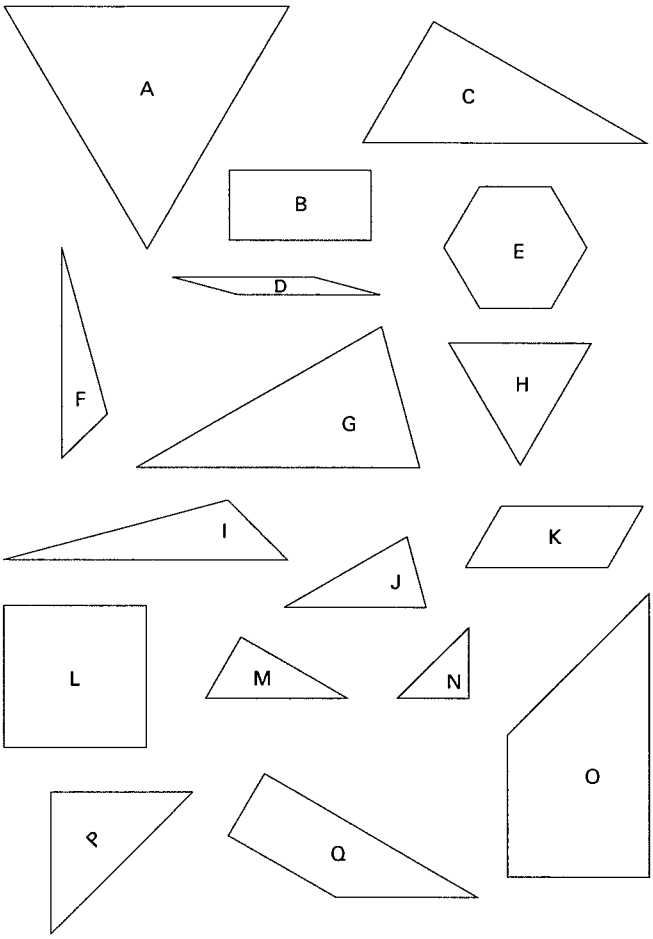
Determining the fractional part of the circle covered by each sector can help to prepare the students for creating and analyzing circle graphs in Lesson 33. These fractions can be determined in many ways. For example, some students may suggest the 30° angle cuts off $^{30}/_{360}$ of the circle. Others may subdivide the entire circle into 24 equal parts (by forming twenty-four 15° angles) to show that a 30° sector covers $^{2}/_{24}$ of the circle. Still others may suggest there are twelve 30° sectors in a circle, so a 30° sector is $^{1}/_{12}$ of the circle.

Focus Teacher Activity (cont.)

ACTIONS

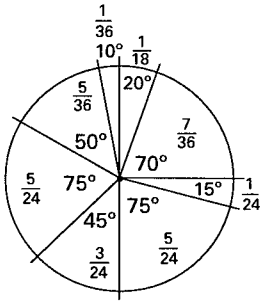
7 (Optional) Repeat Action 6 for Circle 2 on Student Activity 32.1.

8 Distribute a set of Shapes to Sort (see *Blackline Masters*) and a sheet of butcher paper to each group of students. Give each student a pair of scissors. Ask the groups to cut out the shapes and record on the butcher paper all the observations they can about them. Post their observations and discuss selected ones.



COMMENTS

7 Note that some students may find this more challenging than Circle 1. The angles and fractions are shown here:

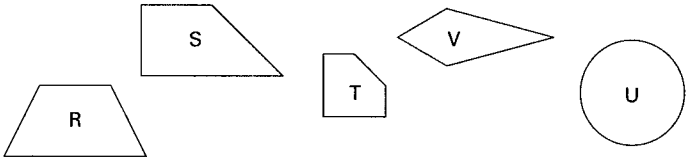


8 Make each group's set of shapes from a different colored cardstock, and laminate the shapes to make them sturdier. To save time cut out the shapes prior to class.

This action provides an opportunity for you to informally assess (by eavesdropping on discussions) students' comfort with geometric terminology and for students to clarify questions they may have about geometric relationships. Discuss new vocabulary as it comes up, but keep emphasis on observations about relationships.

It is impossible to predict what observations students will make. Students may compare side lengths, examine symmetries of shapes, or use pattern blocks to measure angles (all of the angles are multiples of 15°). Some students may investigate which shapes can be combined to form a quadrilateral and in how many ways. They might also be curious about which shapes or combinations of shapes will tile (tessellate) a plane and why.

Many of the students' observations could lead to other conjectures or generalizations. For example, students may be surprised to note that two shapes can have identical angle measures but different areas and perimeters, or to discover that the length of the sides of an angle has no effect on the measure of the angle.



Focus Teacher Activity (cont.)

ACTIONS

9 Distribute 4 large yarn loops, scissors, 2 sheets of 1-inch grid paper and a copy of Focus Master C to each group. Ask the groups to sort their shapes and place them in yarn loops (Venn diagrams) according to the classifications listed in a) from Focus Master C. Pose questions such as the following for the students to investigate and discuss regarding their Venn diagrams:

What other observations or conjectures can you make about the shapes in these classifications?

How would you classify the overlapping region of the Venn diagram?

What are other shapes that would fit in each region of the diagram? Cut out some examples from grid paper.

Suppose the 21 shapes are each placed in a separate envelope. Then the envelopes are shuffled and spread out on a table. If someone randomly selects one of these envelopes from the table, what classifications are most and least likely to be picked? What is the probability of selecting each of these classifications?

Angle Measurement Lesson **32**

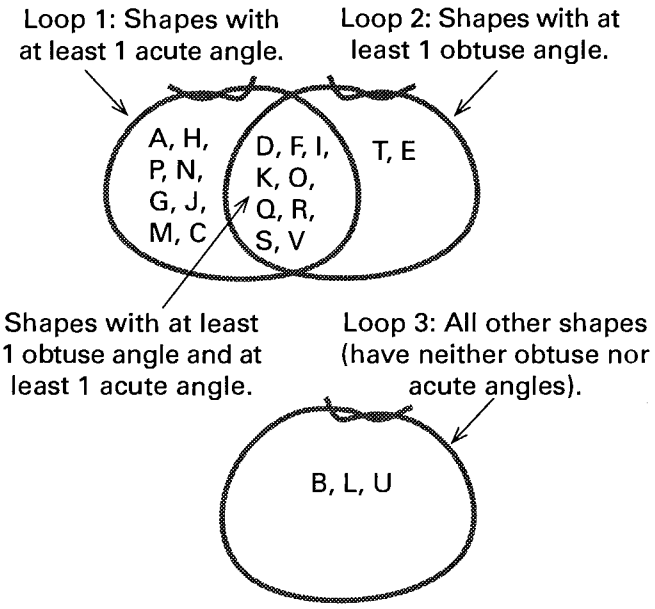
Focus Master C

a) Loop 1: shapes with at least 1 acute angle
Loop 2: shapes with at least 1 obtuse angle
Loop 3: all other shapes

COMMENTS

9 If table or desktop space is limited, students may find it more convenient to arrange the loops and shapes on the floor. Circle diagrams such as those formed by the yarn loops provide a method of visualizing information and drawing conclusions. These diagrams are called *Venn diagrams,* after John Venn who lived from 1834 to 1923 and was a pioneer in the development of logic.

Rather than showing students, allow them time to determine that it is sometimes necessary to overlap the loops. For example, Loops 1 and 2 in a) must overlap (see below) so that certain shapes can be placed in both loops simultaneously.



Pieces that students cut out to add to the regions may reveal misunderstandings or misinterpretations of definitions. For example, some students may form a shape with curved sides and suggest that angles can be formed by intersecting curves. (If such ideas come up, it is important to remind students that definitions are arbitrary and for example, while one could explore another geometry in which angles are formed by intersections of curved lines, that doesn't fit with the standard definition of angle.)

Focus Teacher Activity (cont.)

ACTIONS

10 Repeat Action 9 for one or more of parts b)-f) from Focus Master C.

Angle Measurement Lesson **32**
Focus Master C

a) Loop 1: shapes with at least 1 acute angle
Loop 2: shapes with at least 1 obtuse angle
Loop 3: all other shapes

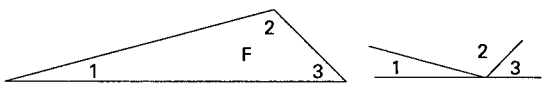
b) Loop 1: quadrilaterals
Loop 2: quadrilaterals whose opposite sides are parallel
Loop 3: all other shapes

c) Loop 1: polygons whose total angle measure is greater than or equal to 360°
Loop 2: polygons whose total angle measure is less than 360°
Loop 3: all other shapes

d) Loop 1: shapes with exactly 1 line of symmetry
Loop 2: shapes with more than 1 line of symmetry
Loop 3: shapes with no line of symmetry

e) Loop 1: quadrilaterals
Loop 2: parallelograms
Loop 3: rectangles
Loop 4: rhombuses

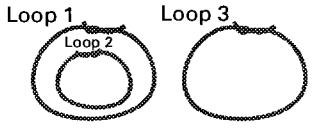
f) Loop 1: you decide
Loop 2: you decide
Loop 3: you decide



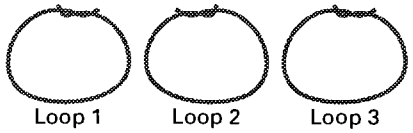
COMMENTS

10 The purpose here is to use Venn diagrams to model relationships. Pick problems to explore based on your time needs and student interest. Rather than using the classifications listed on Focus Master B, you may wish to make up your own classification, based on observations and questions posed by students during Action 8.

b) Students may find it interesting to look for all the similarities they can among the parallelograms (Loop 2 shapes) and make conjectures about parallelograms in general. For example, they might discover that in each of their parallelograms opposite angles are congruent (have equal measures), opposite sides are congruent (have equal lengths), adjacent angles total 180°, and the total angle measure is 360°. Here is one way loops 1, 2, and 3 could be arranged for b):



c) Since none of these classifications overlap, the loops could be arranged like those below:

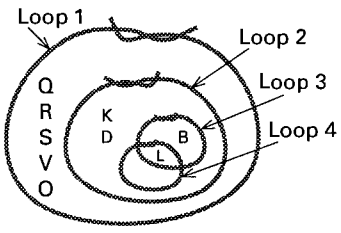


Students may notice that all the triangles have a total angle measure of 180°; all of the quadrilaterals have a total angle measure of 360°; and the hexagon has a total angle measure of 720°. Most students will probably determine the total angle measure in each shape by measuring with the pattern blocks and adding.

Another way of verifying the total angle measure in a polygon is to trace around the corners and place them around a point. This method, which is explored in *Visual Mathematics Course II*, is used in the example at the left to show that the total degrees in shape F is 180°.

d) Line symmetry was introduced in Lesson 16.

e) One way of relating these classifications is shown in the loops at the right.



Follow-up Student Activity 32.2

NAME _____ **DATE** _____

1 If possible, use 2 *non*congruent pattern blocks to form the following angles. If not possible, explain why. On another sheet, trace around each angle and the pattern blocks you use to make the angle. Write the total number of degrees in each angle.

a) a right angle (90°)

b) an obtuse angle (greater than 90° and less than 180°)

c) an acute angle (less than 90°)

d) a straight angle (180°)

e) a reflex angle (greater than 180°)

2 If possible, use pattern blocks to form each of the following angles. If not possible, explain why. On another sheet, trace each angle and the pattern blocks that you use. Write the total number of degrees in each angle.

a) a 360° angle, using 5 noncongruent pattern blocks

b) a 360° angle, using 1 obtuse angle, 1 right angle, and 1 acute angle

c) a 360° angle using 3 different *colors* of pattern blocks and so that the angle formed by color B is 3 times the angle formed by color A, and the angle formed by color C is 8 times the angle covered by color A. (Hint: you can use more than 3 pattern blocks.)

(Continued on back.)

Follow-up Student Activity (cont.)

3 On another sheet of paper, use the pattern blocks to help you draw *approximations* of the following angles. Next to your drawings, explain how you approximated c) and e).

a) 15°

b) 10°

c) 45°

d) 135°

e) 202½°

f) 1°

4 On another sheet, use your pattern blocks to help you draw, if possible, each of the following. If you don't think it's possible explain why. Label the measure of every angle in each shape you draw.

a) A right triangle with a 15° angle.

b) An obtuse triangle (a triangle with an obtuse angle).

c) A right triangle with 1 obtuse angle.

d) A scalene triangle (all sides are different in length) with 2 equal angles.

e) A triangle with angles 45°, 15°, and 135°.

f) A parallelogram with a 60° angle.

g) A parallelogram with a 45° angle.

h) A parallelogram with a 60° and a 45° angle.

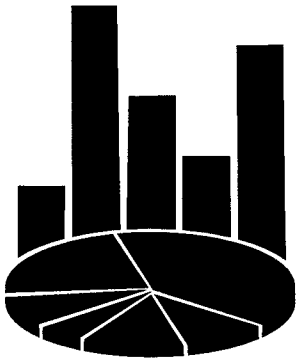
5 On another sheet of paper write as much as you can to show your understanding of the word *angle*. Include diagrams to illustrate your explanation.

Graphing

Lesson 33

THE BIG IDEA

Bar graphs, circle graphs, Venn diagrams, and rectangular area models are meaningful ways to visually display data. Such representations reveal relationships among the data and provide a basis for making predictions and decisions.



CONNECTOR

OVERVIEW

Students complete a survey that will be used during the Focus and write about their views of the meaning of graphing.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals, (optional).
- ✔ Connector Master A, 1 copy per class.
- ✔ Connector Master B, 1 transparency.

FOCUS

OVERVIEW

Students integrate their knowledge of angle measure, area, fractions, and probability to create graphs (bar, circle, rectangle, and Venn diagrams) that represent situations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activities 33.1 and 33.2, 1 copy of each per group and 1 transparency of each.
- ✔ Pattern blocks, 1 set per group.
- ✔ Orange segment strips, 4 strips per group of students.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Connector Master A, 1 copy of the completed survey per group.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Colored markers, several colors for each group.
- ✔ 10/12-cm grid paper (optional), 2 sheets per group.

FOLLOW-UP

OVERVIEW

Students analyze and create graphs (bar, circle, rectangle and Venn Diagrams) of data.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 33.3, 1 copy per student.

LESSON IDEAS

ASSESSMENT
Situation 2 in Action 6 of the Focus activity could be assigned as a project for students to work on independently or in groups. They could be asked to "write-up" their processes and conclusions; this write-up could be assessed with a rubric (provided to students at the time the project is assigned). This would be a meaningful portfolio entry.

GROUPING STUDENTS
Throughout *Visual Mathematics* courses students are asked to work in small groups. Frequently activities call for 1 copy of an activity or 1 set of materials per group, and groups are often asked to each create 1 product (a poster, summary, etc.). The purpose of this is to promote discussion and foster interdependence within groups.

SELECTED ANSWERS

1. a) A, 120°; B, 30°; C, 150°; D, 60°.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students complete a survey that will be used during the Focus and write about their views of the meaning of graphing. This provides baseline information that is useful now for planning and later for making comparisons and determining growth in students' understanding.

MATERIALS

- ✔ Student journals, (optional).
- ✔ Connector Master A, 1 copy per class.
- ✔ Connector Master B, 1 transparency.

ACTIONS

1 Write the following survey questions on the overhead (see Comment 1 regarding other ideas for the subject of this survey):

1. Do you like *rock and roll oldies music*?
2. Do you like *country western music*?
3. Do you like *classical music*?

Have the students answer the questions by writing "yes" or "no" next to the numbers 1, 2, and 3 on a scrap of paper. Collect these.

Graphing Lesson 33

Connector Master A

Question 1: Do you like _____?
Question 2: Do you like _____?
Question 3: Do you like _____?

Student	Question 1	Question 2	Question 3
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			

COMMENTS

1 The results of this survey will provide the basis for Action 7 of the Focus activity. Record the results on a copy of Connector Master A (see partially completed example at the left) and set it aside until it is called for in the Focus.

You may wish to survey students regarding their opinion of some issues or topics (other than music types) that are relevant to your school or classroom or that are based on the students' interests (musical groups, sports, ice cream flavors, school policies, etc.). Select topics for which it is likely some students will respond "yes" to at least two of the questions.

Connector Teacher Activity (cont.)

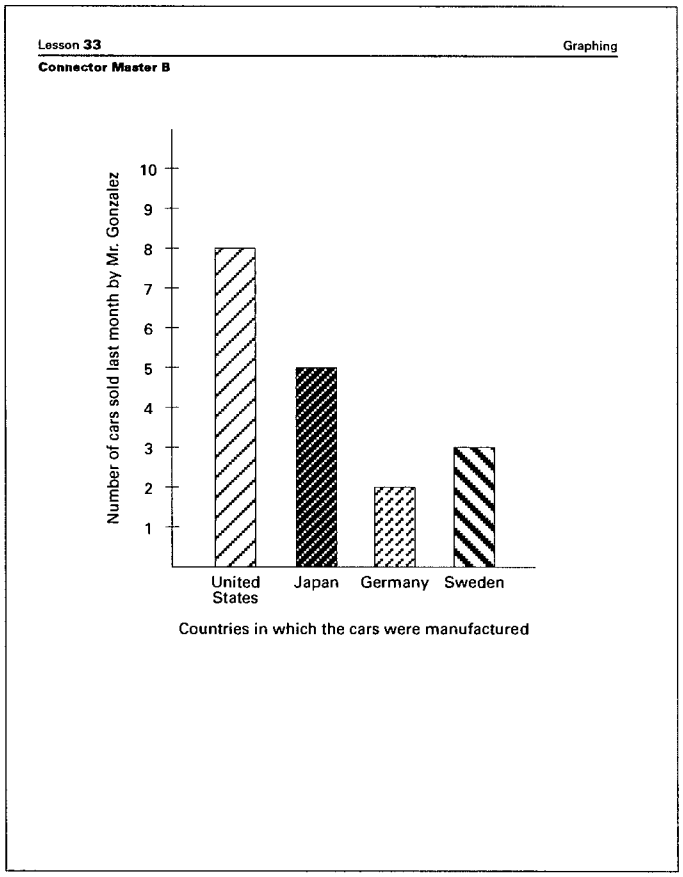
ACTIONS

2 Write the following thought starter on the overhead and ask the students to each complete it in writing:

What the word *graphing* means to me is...

and here are some examples...

3 Place a transparency of Connector Master B on the overhead. Ask the groups to record information they know as a result of looking at the bar graph. Discuss.



COMMENTS

2 If students are keeping journals, have them write their responses in them. This entry will provide useful information as you plan for the remainder of the lesson, and it will be useful for both you and the student to make comparisons after completing the lesson. At that time you may wish to have students reread this entry and add new thoughts they have regarding the meaning of graphing.

3 If groups have difficulty seeing the details of the graph, you may wish to give each group a copy of Connector Master B. There are many observations students could make. For example: Mr. Gonzalez sold a total of 18 cars (unless he also sold some from countries not listed on the graph); he sold $\frac{5}{8}$ as many Japanese cars as U.S. cars; the total number of German and Swedish cars he sold equals the number of Japanese cars he sold; $\frac{10}{18}$ of the cars he sold did not come from the United States; etc.

Focus Teacher Activity

OVERVIEW & PURPOSE

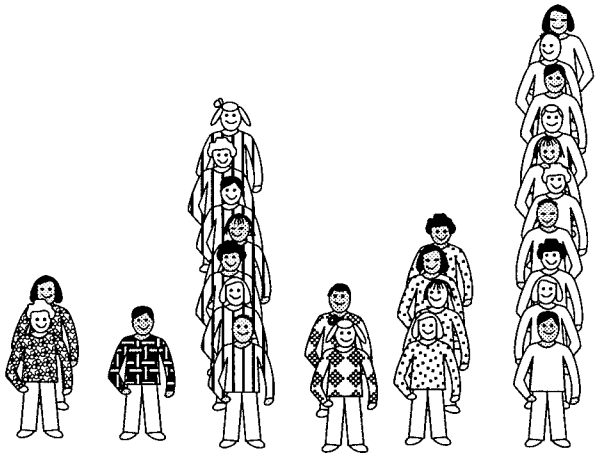
Students integrate their knowledge of angle measure, area, fractions, and probability to create graphs (bar, circle, rectangle, and Venn diagrams) that represent situations.

MATERIALS

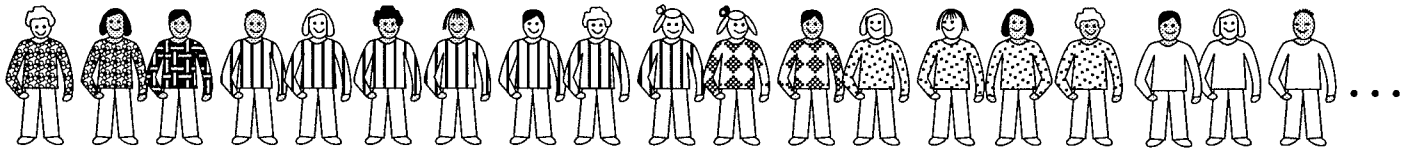
- ✔ Focus Student Activities 33.1 and 33.2, 1 copy of each per group and 1 transparency of each.
- ✔ Pattern blocks, 1 set per group.
- ✔ Orange segment strips, 4 strips per group of students.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Connector Master A, 1 copy of the completed survey per group.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Colored markers, several colors for each group.
- ✔ $^{10}/_{12}$-cm grid paper (optional), 2 sheets per group.

ACTIONS

1 Have the class line up in a "human bar graph" across the front of the classroom, grouping themselves according to the color of the shirts they are wearing, as illustrated below. Record the color groupings and the number of students in each grouping on the overhead.



Have the students rearrange themselves into a single line along one wall of the classroom, keeping themselves grouped according to the color of their shirts, as shown here:



COMMENTS

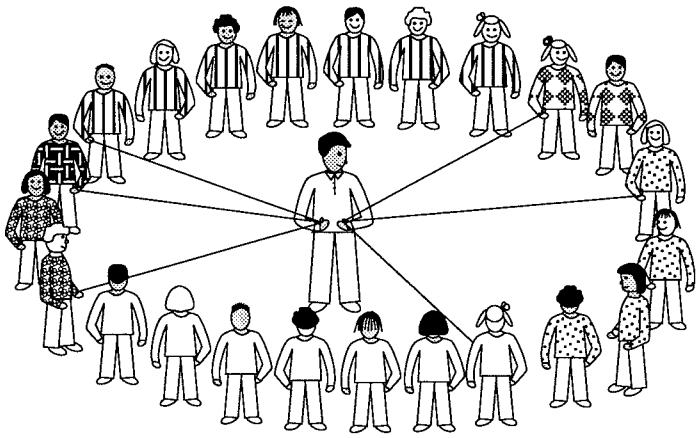
1 You may need to help with categorizing colors (e.g., plaid, mostly blue, etc.) to avoid having too many groups. The purpose of this action is to have the students physically experience the graphing process which they will later carry out using models and diagrams. If there isn't much variety in shirt color, you could group them according to birth months, shoe type, color of their home or apartment building, etc. If you change the topic, be sure to adjust the questions on Student Activity 33.1 (see Action 2) accordingly.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Have the line move to form a "human circle graph," without changing their color groupings. Stand in the center of the circle and use string to form central angles that subdivide the circle according to the color groupings. Next have the students sit on the floor (keeping the circle and string angles intact). Tape the string angles in place on the floor. While arranged in the circle, have the students make observations.



2 Have the students return to their small groups. Distribute one copy of Focus Student Activity 33.1 (2 pages) for each group to complete. Discuss their conclusions and reasoning, using a transparency of the activity.

COMMENTS

1 (continued.)
As the students sit down, have them space themselves as evenly as possible around the circle. Following are a few questions regarding the circle graph which may help to promote observations, and which can be answered without knowing the exact measure of the central angles:

Which color produces the largest central angle? the smallest? are any equal?

Without knowing the exact measures of the angles, can you tell if there any angle measures that are double another? triple? $\frac{1}{4}$? etc. How can you tell?

Which color group has a central angle that is closest to 180° (90°, 60°, etc.)? How many more shirts would be needed in that color group to form a 180° central angle (90°, 60°, etc.)? How can you tell?

Suppose a student is missing from the class, but is about to arrive late. What is the probability that student will be wearing a white shirt? a red one? a multicolored one? etc.

2 Unless the number of students in the class coincidentally divides 360, most fractions won't be "nice" and measuring angles is best done by approximation. The circle on Student Activity 33.1 is marked to show 12ths which can be used as a guide for approximations. For example, suppose there are 25 students in the class, and 7 students are wearing light blue shirts. Since the circle is marked to show 12ths, each 12th could be cut in half to show 24ths, and $\frac{7}{25}$ could be approximated by $\frac{7}{24}$. Each 24th would measure $360° \div 24 = 15°$, so $\frac{7}{25} \approx 7 \times 15° = 105°$.

The purpose here is not for students to get exact values, but rather to get reasonable approximations and to develop a "feel" for angle relationships on a circle. If students use calculators to compute angle measures, encourage them to discuss ways the relationships on the graph help them know what keys to press.

Focus Teacher Activity (cont.)

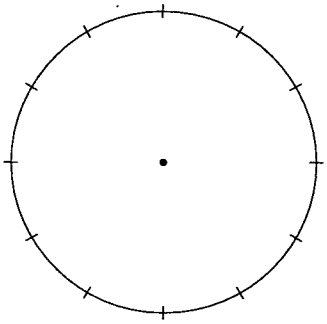
ACTIONS

Graphing Lesson 33

Focus Student Activity 33.1

NAME _____ DATE _____

1 Make a model that is a close *approximation* of the class "human circle graph" here:



Label your circular graph with the following:

a) the shirt color and number of students represented by each sector;

b) the fractional part of the circle covered by each sector;

c) a close *approximation* of the measure of each central angle.

2 Explain the methods or computations you used to approximate each angle measure for 1c) above.

(Continued on back.)

3 Place a transparency of Focus Master A on the overhead and give each group two orange segment strips. Ask the groups to devise a method of using a segment strip to model the Situation on Focus Master A. Discuss their methods of determining and modeling the number of votes each student received.

Graphing Lesson 33

Focus Master A

Situation

36 of the students in Ms. Manchester's class voted in the election for student council representative.

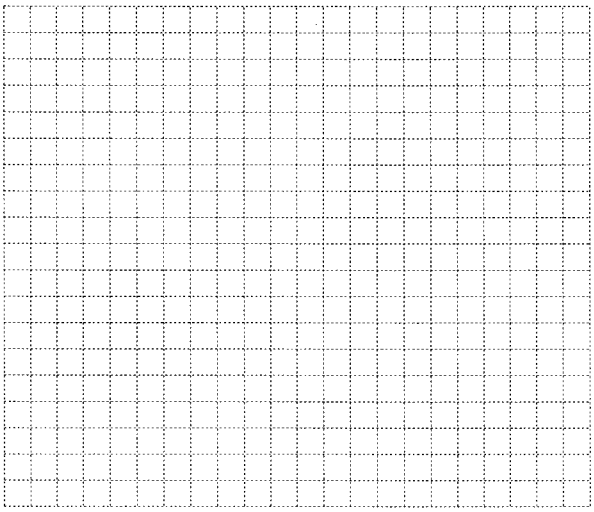
- Mary received 1/6 of the votes.
- Jill and Peter each received 1/18 of the votes.
- Sue received 4 times as many votes as Jill.
- Betsy received 1 vote more than 1/12 of the votes.
- Zach received the rest of the votes.

COMMENTS

Lesson 33 Graphing

Focus Student Activity 33.1 (cont.)

3 On this grid make a model of your "human bar graph." Label your bar graph so it is possible to tell what shirt color and number each bar represents.



3 Segment strips are available from The Math Learning Center. See Chapter 4 of *Starting Points* for ordering information. A transparent segment strip is useful for student demonstrations at the overhead.

Allow time for each group to decide the number of votes received by each student and to determine their own method of modeling this on the segment strip. The model at the bottom of this page shows the distribution of votes. Some students may compare such a linear representation to the process of lining up students in Action 1:

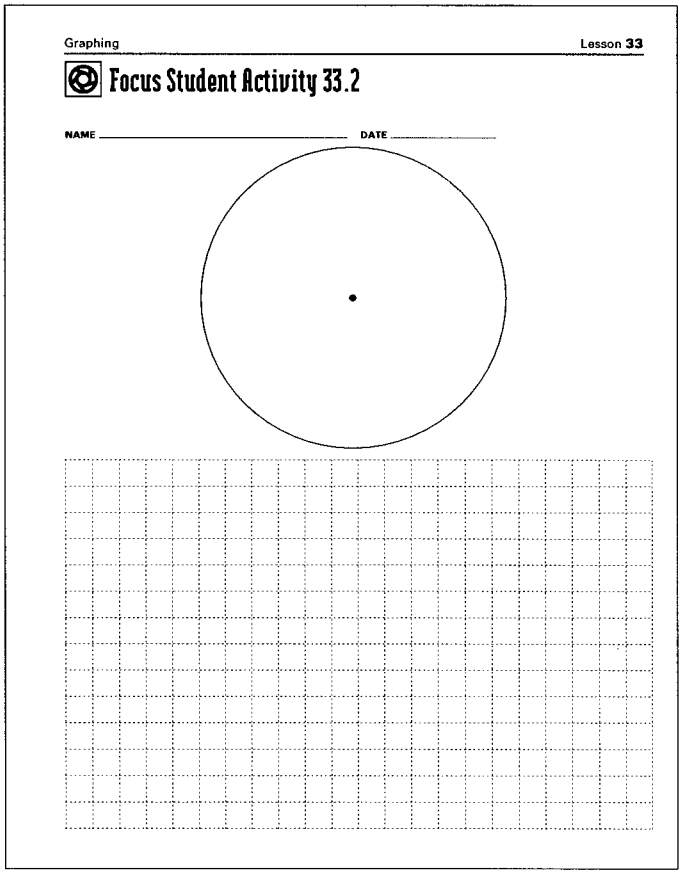
(Continued next page.)

M	M	M	M	M	M	J	J	P	P	S	S	S	S	S	S	S	B	B	B	B	Z	Z	Z	Z	Z	Z	Z	Z	Z	Z	Z	Z	Z

Focus Teacher Activity (cont.)

ACTIONS

4 Give each group a copy of Focus Student Activity 33.2 and ask them to form a circle graph and a bar graph that models the situation on Focus Master A, and to write as many mathematical conclusions as they can, based on relationships they can "see" in each graph. Discuss.

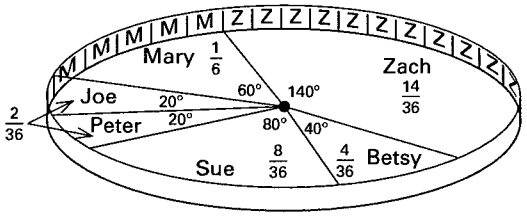

Graphing Lesson 33

Focus Student Activity 33.2

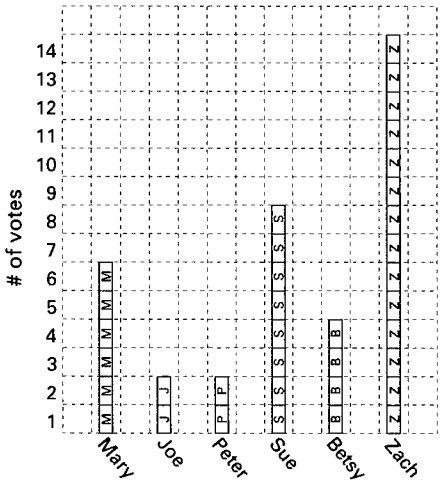
NAME _____ DATE _____

COMMENTS

4 Some groups may use pattern blocks or fraction relationships they observe to approximate the angles of their circle graphs. Notice also that the length of an orange segment strip with 36 segments exactly matches the circumference of the circle. Hence, angles could be marked by placing the strip along the circumference of the circle and marking off portions, as shown below. Since each 36th of the circle is 10°, then the measure of each angle can be found.



One way of forming a bar graph is to cut apart a segment strip model and tape each student's portion in a column of the grid paper, as shown here:



There are many observations groups could make by studying their bar and circle graphs. If needed, here are a few questions you might pose to groups as you circulate while they work: What part of the class voted for girls? for boys? How many more votes would Zach have needed to receive 2/3 of the votes? What was the average number of votes received by Betsy, Zach, and Mary? If Jana was absent on the day of the vote, who do you think she would have been most likely to vote for when she returned and why? How many times as many votes as Mary received did Sue receive? What would be the decimal value of each student's portion of the votes? etc.

Focus Teacher Activity (cont.)

ACTIONS

5 Give each group one copy of Focus Master B and two orange segment strips. Ask them to create a circle graph and a bar graph that represent the relationships (or approximations of them) in Situation 1. Next to their graphs have them write as many conclusions as they can based on mathematical relationships they "see" in the graphs.

Lesson 33 Graphing
Focus Master B

Situations

1. Before selecting refreshments for their class party, Mr. Miller's class conducted a survey of their favorite flavors of ice cream. Here were the results:

   13 like chocolate best
   7 prefer vanilla
   3 like strawberry best
   2 said they don't like ice cream

2. An elementary school has the following numbers of students in grades K-5:

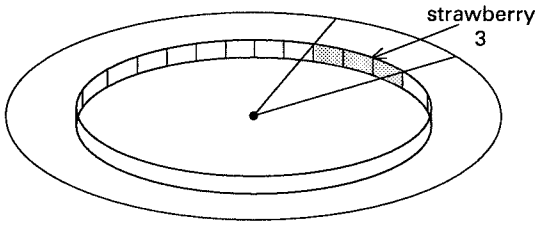
   grade K - 50
   grade 1 - 55
   grade 2 - 80
   grade 3 - 90
   grade 4 - 75
   grade 5 - 100

6 (Optional) Repeat Action 5 for Situation 2 on Focus Master B.

COMMENTS

5 Encourage students to make observations involving as many math concepts as possible; particularly those explored so far in this course.

Rather than providing grid paper (note that the length of one orange segment equals the length of a side of a square on $^{10}/_{12}$-cm grid paper in *Blackline Masters*) or circles in advance, allow the students time to decide what they need or to invent their own methods of forming circles and bars. For example, groups may cut apart a segment strip to form a bar graph. They may approximate a circle from a segment strip with 25 parts and use methods similar to those in Action 4 with that circle. Others may use a 25-segment strip to approximate a circle, place that inside another circle (e.g., a tracing of the circle on Student Activity 33.2) and use the segments to mark the sections of the circle graph:



Or, students may say 36 is about $1\frac{1}{2}$ of 25 so every $1\frac{1}{2}$ spaces of a 36-part segment strip could represent 1 student. Another method is to choose a strip of 100 parts and let every 4 parts represent 1 student. Thus, decimal values would be easy to calculate.

6 Groups might tape together a strip of 45 segments (since there are 450 students in the school) and let each segment represent 10 students.

This situation could also be given as homework or for completion in their journals or as the subject of an Eyewitness Book (see *Starting Points*).

Focus Teacher Activity (cont.)

ACTIONS

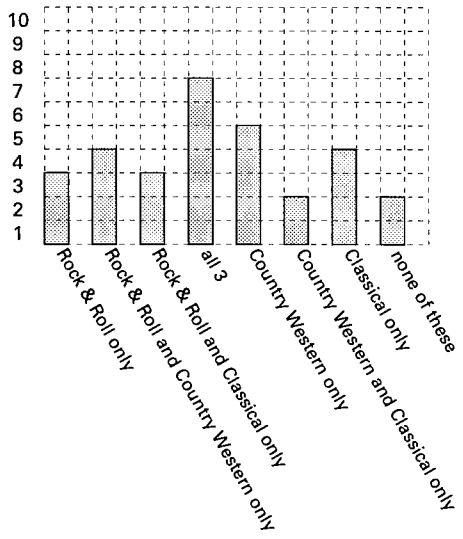
7 Give each group one copy of the survey results that you recorded on Connector Master A, colored markers, and a sheet of butcher paper. Ask them to do the following:

a) summarize the results of the survey on a bar graph, circle graph, Venn diagram, and a rectangular area graph;

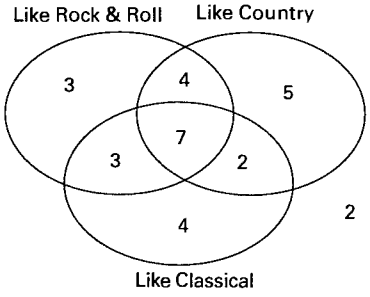
b) write several conclusions about the data based on mathematical relationships they see in their graphs.

Post and discuss their results. Ask students which representations they prefer and why.

Bar Graph



Venn Diagram



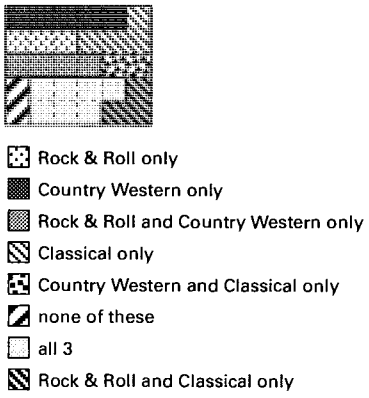
COMMENTS

7 Have orange segment strips and $^{10}/_{12}$-cm grid paper (see *Blackline Masters*) available for groups to use as they wish. If you plan to use the feedback process described at the end of this Comment, you may wish to ask groups to consider the criteria on the Assessment Guide (see *Starting Points*, or use another rubric developed in class) while preparing their posters. Note that rectangular area graphs were introduced in Lesson 12; and circle graphs, bar graphs, and Venn diagrams were explored in Lesson 32.

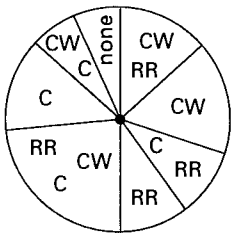
Shown below are examples of graphs of data collected by one class about their opinions of 3 music types. (See *Opening Eyes to Mathematics* for a similar activity regarding students' opinions about 2 topics.)

Note that there are 8 different possible "likes" for a set of 3 opinions like those listed in the Connector, and it is possible that some groups may double or triple count data by not considering all the possibilities. Rather than pointing this out in advance, allow groups time to wrestle with ideas about representing their data, to notice contradictions with other groups' representations, or to experience disequilibrium brought about by questions or observations made by other students or by you.

Rectangular Area Graph



Circular Area Graph



Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

Following is one way to have groups give feedback to each other about their completed posters:

a) Each group spends 5-7 minutes reviewing another group's poster and, using the Follow-up Assessment Guide (see *Starting Points*), completes the statements shown below about that poster (to save writing time you could preprint these thought starters):

We think your poster shows evidence of Exceptional Work in the following two (or more) ways...

Two ways you might enhance the quality of your poster are...

b) Each group tapes the statements they have written to the back of the poster and passes the poster to another group who repeats this process.

c) After each poster has been reviewed by 2-3 groups, return the posters to their creators.

d) Allow 10-15 minutes for groups to read their feedback and write completions of the following statements to attach to their posters (again to save time these thought starters could be copied earlier for groups):

The 3 aspects of our poster project we feel most proud of are...

The 3 most important ways we could improve our poster are...

The 5 most important things we learned during this poster project are...

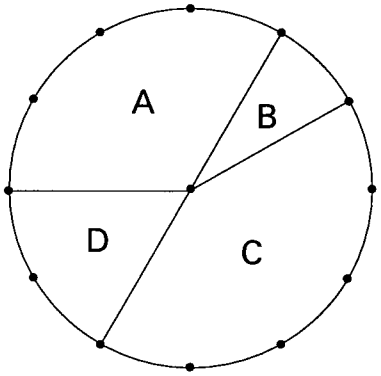
e) Have groups display their posters and share their responses to the last three statements.

TEACHER NOTES:

Follow-up Student Activity 33.3

NAME _____ **DATE** _____

1 a) Record what you think is the measure of each central angle on the following circle graph.



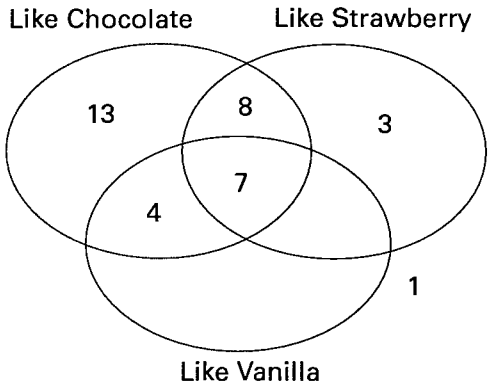
A: _____ B: _____

C: _____ D: _____

b) Tell how you decided the measure of angle A:

c) Make up a situation that could be represented by the above circle graph. Describe the situation here. Give enough details about the situation to show why the regions and angles would have to be the size they are.

2 Here is a Venn Diagram of a survey taken in another class about their opinions of chocolate, strawberry, and vanilla ice cream. In the space below, write ten or more interesting conclusions you can make by studying the mathematical relationships in the graph.

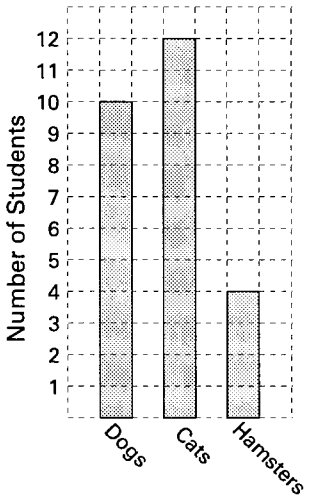


(Continued on back.)

Follow-up Student Activity (cont.)

3 On another sheet of paper make a circle graph, a bar graph, and a rectangular area graph of the data shown in the Venn Diagram in Problem 2. Label your graphs carefully, including approximations of the measure of each angle of your circle graph.

4 Ingrid drew the following graph to show the pets owned by students in her class. Shown at the right of the graph is some other information about the class.



- There are less than 30 students in the class.
- Seven students own both a dog and a cat.
- No one owns all 3 animals.
- No one owns both a hamster and a cat.
- Three students own only a hamster.
- Six students have no pets and no one has a pet other than a dog, cat, or hamster.

How many students are in the class? Explain how you decided this.

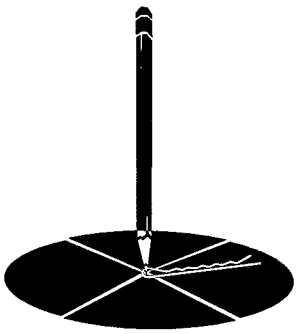
5 Conduct a survey of 20 people regarding their opinions about 3 things (other than ice cream flavors or other topics already surveyed in class). On another sheet, make a bar graph, circle graph, rectangular area graph, and Venn Diagram of your data. Also write 5 important conclusions revealed by the mathematical relationships in your graphs.

Graphing & Decision Making

Lesson 34

THE BIG IDEA

It is important that students have opportunities to gather data, and use their data as a basis for making predictions and decisions. The mathematical value of such experiences is enhanced by opportunities to explore the effects of the following on the students' levels of confidence about their predictions and decisions: sample size, sampling procedures, intuitions, methods of displaying and comparing data, theoretical relationships, and experimental outcomes.



CONNECTOR

OVERVIEW

Students discuss ways of collecting a data sample to use as a basis for making predictions and decisions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student posters from Lesson 33, Action 7.

FOCUS

OVERVIEW

Students use their understanding of fractions and angle measurement to create spinners. They use the spinners to explore experimental and theoretical probabilities and as a basis for making predictions and decisions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Pattern blocks, 1 set per pair of students.
- ✔ Focus Student Activities 34.1-34.3, 1 copy of each for each pair of students and 1 transparency of each.
- ✔ Bobby pins, 1 per pair of students.
- ✔ Colored markers for each pair of students.

FOLLOW-UP

OVERVIEW

Students create a spinner according to certain conditions and identify angle measurements and fractional parts covered by sectors of the spinner. They conduct an experiment on the spinner and compare experimental and theoretical results.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 34.4, 1 copy per student.
- ✔ Bobby pins, 1 per student.

LESSON IDEAS

PHILOSOPHY

Although it can be tempting to avoid dealing with "wrong answers," such as Spinner 3-A in Focus Action 4, there are many reasons to address them directly. When handled with respect and appreciation, exploring ideas that don't work can: provide insights and deepen understanding; raise issues that may not otherwise surface and, hence, uncover misconceptions and confusion; reinforce the notion that ideas that don't work are often more fruitful than ones that do work; encourage risk-taking; and build self-confidence.

LOOKING AHEAD

Sampling, confidence, and probability are also the focus of Lesson 45.

SELECTED ANSWERS

1. b) A: 135°
   B: 30°
   C: 60°
   D: 45°
   E: 90°

   d) A: $\frac{3}{8} = .375$
   B: $\frac{1}{12} = .08\overline{3}$
   C: $\frac{2}{12} = .1\overline{6}$
   D: $\frac{1}{8} = .125$
   E: $\frac{1}{4} = .25$

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students discuss ways of collecting a data sample to use as a basis for making predictions and decisions.

MATERIALS

✔ Student posters from Lesson 33, Action 7.

ACTIONS

1 Pose the following for groups to discuss (fill in the blank according to the topic of your survey from Lesson 33, Connector Master A).

Suppose you are in charge of deciding

_____.

What are reasonable methods of determining the whole student body's opinions without surveying every student in the school?

COMMENTS

1 If, for example, your survey in Lesson 33 was about music types you might ask students to "Suppose you are in charge of deciding the music to play at a school party," or if ice cream flavors were the survey topic you could ask them to "Suppose you are in charge of deciding the amount and flavors of ice cream to buy for the school picnic."

The intent here is for students to develop strategies for generating a sample that is *representative* of the entire student body and hence, more efficient to carry out. Discuss students' ideas about what makes them feel more or less confident in a sample (e.g., its size and randomness). Does the sample include a balance of boys and girls that is representative of the make-up of the student body? Does it represent a cross section of the grade levels? Is the sample biased in any way because of the time of day or location in which it is collected? etc. You could have students carry out a survey of the student body, using Extended Project B: Opinion Polls (see *Starting Points*) as the format for their work.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

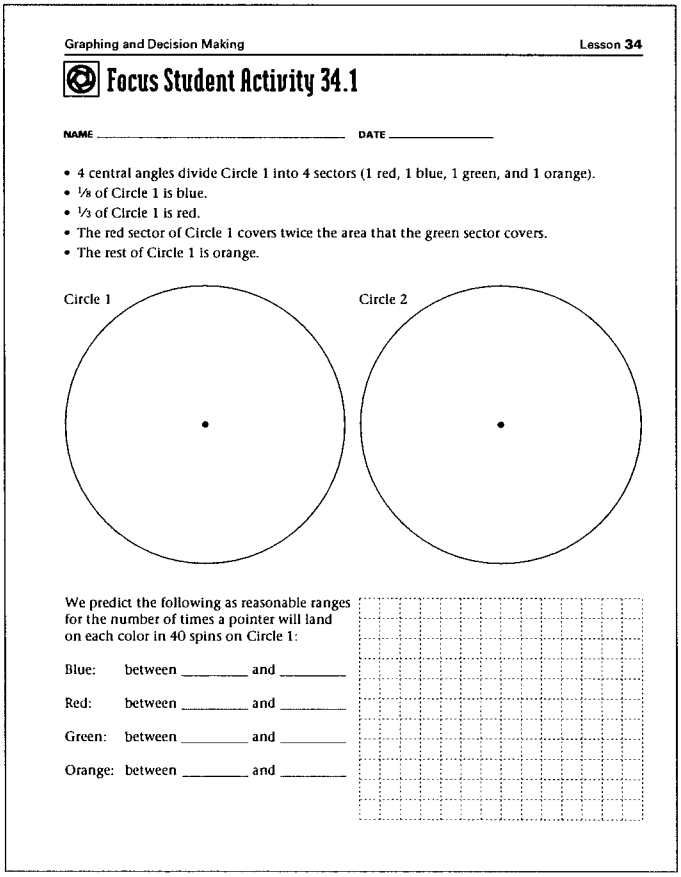
Students use their understanding of fractions and angle measurement to create spinners. They use the spinners to explore experimental and theoretical probabilities and as a basis for making predictions and decisions.

MATERIALS

- ✔ Pattern blocks, 1 set per pair of students.
- ✔ Focus Student Activities 34.1-34.3, 1 copy of each for each pair of students and 1 transparency of each.
- ✔ Bobby pins, 1 per pair of students.
- ✔ Colored markers for each pair of students.

ACTIONS

1 Divide the groups into pairs of students and give each pair a copy of Focus Student Activity 34.1, a set of pattern blocks, and colored markers. Ask the pairs to each subdivide (as accurately as possible) and color Circle 1 to show the relationships listed, and to use the relationships given to help them determine the measure of each central angle. Discuss their methods.



Graphing and Decision Making Lesson **34**

Focus Student Activity 34.1

NAME _____ DATE _____

- 4 central angles divide Circle 1 into 4 sectors (1 red, 1 blue, 1 green, and 1 orange).
- 1/8 of Circle 1 is blue.
- 1/3 of Circle 1 is red.
- The red sector of Circle 1 covers twice the area that the green sector covers.
- The rest of Circle 1 is orange.

Circle 1 Circle 2

We predict the following as reasonable ranges for the number of times a pointer will land on each color in 40 spins on Circle 1:

Blue: between _____ and _____

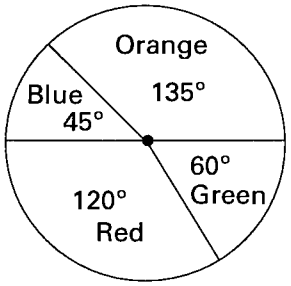
Red: between _____ and _____

Green: between _____ and _____

Orange: between _____ and _____

COMMENTS

1 Placing the students in pairs increases student involvement in Action 2. Pattern blocks are useful for forming angles on the circle (note that the term circle is used loosely here to include the region enclosed by the circle). Here is a diagram that shows the colors and angle measures for Circle 1:



Focus Teacher Activity (cont.)

ACTIONS

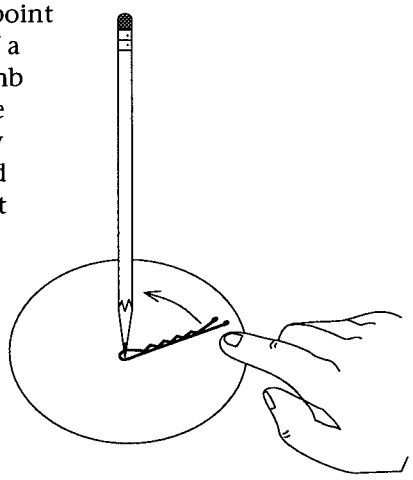
2 Demonstrate how to use a bobby pin as the pointer of a spinner on Circle 1. Ask the pairs to each predict a reasonable range for the number of times a pointer would land on each color if the pointer were spun 40 times, and to record their predictions on Student Activity 34.1. Discuss their predictions, reasoning, and level of confidence in their predictions.

3 Give each pair of students a bobby pin and have them carry out an experiment on Circle 1 by spinning the pointer 40 times and tallying their results. Ask them to make a circle graph on Circle 2 that closely approximates their experimental results and to label approximate angle measurements. Have them also use the grid to make a bar graph of their data.

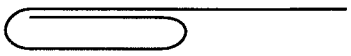
Post the groups' results where everyone can compare. Discuss students' ideas about differences between their predictions and outcomes; similarities and differences between their spinners and circle graphs; comparisons among the groups' results; and predictions about the overall class results.

COMMENTS

2 To use a bobby pin as a pointer for a spinner: place a pencil point through the looped end of the bobby pin and hold the pencil point at the center point of a circle; using the thumb and one finger of one hand, flick the bobby pin so it spins around the pencil point, as at the right.



A pointer can also be made by bending a small paper clip as shown here:



It is important not to pass judgment on students' predictions. Some groups may make random predictions, based on "gut feelings." Others may make guesses based on reasoning like "the red covers more area than the blue so it will land on red more." Still others will make predictions based on theoretical probability (e.g., "Since $\frac{1}{8}$ of the circle is blue and $\frac{1}{8}$ of 40 = 5, then a reasonable range for the number of times it could land on 40 blue could be 4-6."). If some predict very broad ranges, you could ask them to limit their ranges to, say, a difference of 4.

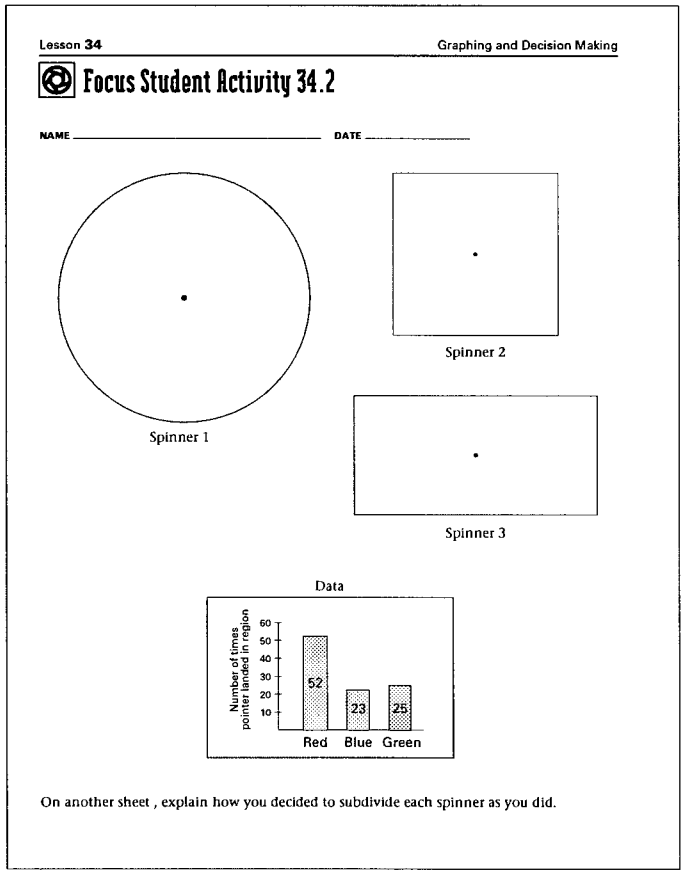
3 Students may wish to use pattern blocks to help with their angle approximations on Circle 2. Prior to posting the pairs' results, you could ask them to see what effect combining their data with another pair would have on their graph. Would it look more like the theoretical (i.e., Circle 1) data? Groups could also summarize the class results and investigate ways the overall class experimental results differ from the groups' results and from theoretical results.

If theoretical methods weren't used or discussed for predicting in Action 1, you might ask groups now to discuss ways they could have determined the theoretical outcomes. As discussed in Lesson 12, as more data is collected in an experiment, the results usually more closely resemble the theoretical outcomes. This idea is explored in more depth in Lesson 45.

Focus Teacher Activity (cont.)

ACTIONS

4 Give each group one copy of Focus Student Activity 34.2. Ask them to subdivide the three blank spinners so that in 100 random spins each spinner would be *highly likely* to produce data like that shown in the bar graph. Have volunteers show their ideas on a transparency of Student Activity 34.2. Discuss their reasoning.


Lesson 34 Graphing and Decision Making

Focus Student Activity 34.2

NAME _____ DATE _____

Spinner 1

Spinner 2

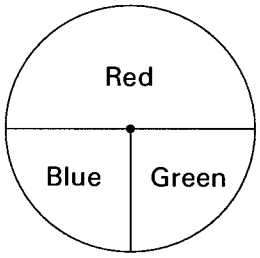
Spinner 3

Data

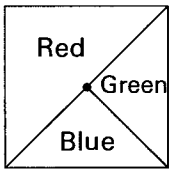
On another sheet, explain how you decided to subdivide each spinner as you did.

COMMENTS

4 Most students will probably subdivide Spinners 1 and 2 similar to the ones shown below. Some students may make small adjustments, decreasing the blue sector slightly, to more accurately reflect the numbers given in the bar graph. Keep in mind there are many spinners that would be highly likely to produce the data in the graph, as long as the central angles formed by the sectors are approximately 180°, 90°, and 90°.

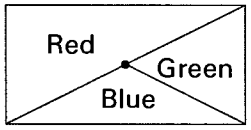


Spinner 1



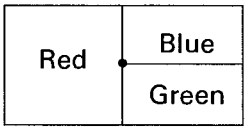
Spinner 2

Spinner 3 may generate more discussion. Spinner 3-A shown below is subdivided into areas that are representative of the distribution of colors; however, because of the obtuse angle in the blue region, the spinner should theoretically land in that region more than in the green region (in fact pattern blocks could be used to verify the green section has a central angle that is a little less than 60° while blue has a central angle that is a little more than 120°). Hence, Spinner 3-A would not be as likely to produce the data in the bar graph as Spinner 3-B and 3-C below.

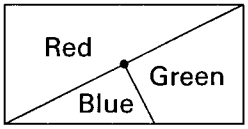


Spinner 3-A

Spinners 3-B or 3-C (see below) would be highly likely to produce results like those shown in the bar graph. Note that the green section in Spinner 3-C now covers more area but the pointer is equally likely to land on blue and green because both central angles are 90°.



Spinner 3-B



Spinner 3-C

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
	4 (continued.) If Spinner 3-A isn't suggested by a student you might pose it for exploration. If there is disagreement over which spinner is more likely to produce the data shown, groups could conduct experiments to test their ideas. For example, if students are unsure if Spinner 3-A would be Highly, Somewhat, or Not Likely to produce the data on 34.4, they might each conduct an experiment of 100 spins at home, and bring their data to class; determine a class average for each color; and then reach agreement on the likelihood of Spinner 3-A producing the given data. Keep in mind that experimental results may differ considerably from theoretical, even in a relatively large sample, and students may justifiably base their decisions about "likely" spinners on experimental results. Hence, even an experiment involving a large sample *could* suggest Spinner 3-A is highly likely to produce the data in the graph.
5 (Optional) Give each group another copy of Student Activity 34.2 and ask them to subdivide the blank spinners so that each spinner is *not likely* to produce the data shown in the bar graph. Discuss.	**5** There are many possibilities for highly unlikely spinners, as long as their color distributions are not close to ½ red, ¼ blue, and ¼ green. However, it is important to note that even "not likely" spinners *could* produce data close to that in the bar graph. You could have students repeat this action for spinners that are *somewhat likely* to produce the data shown in the bar graph.

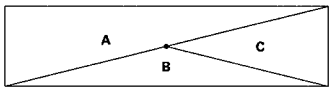
Focus Teacher Activity (cont.)

ACTIONS

6 (Optional) Give each group a copy of Focus Student Activity 34.3.

Focus Student Activity 34.3

NAME _____ DATE _____



Spinner Z

1 Complete the following statement: "If Spinner Z were spun 100 times, we think that, theoretically the pointer would land in sector A about _____ times, in sector B about _____ times, and in sector C about _____ times." Here is how we decided these numbers:

2 Each of the 23 students in Ms. Miller's class spun Spinner Z 100 times. Following is the average number of times the pointer landed on each part of the spinner in 100 spins: A, 51 times; B, 32 times; and C, 17 times.

Ms. Miller wrote the following 3 predictions on the board:

	A	B	C
Prediction 1	53	24	23
Prediction 2	51	41	8
Prediction 3	53	31	16

Before she spun Spinner Z another 100 times, Ms. Miller asked each student to vote for Prediction 1, 2, or 3. Students who voted for the prediction closest to her results would receive a prize. How would you vote and why?

COMMENTS

6 Some students may note that the areas of Sectors B and C on Spinner Z are equal and predict equal chances of the spinner landing in B and C. However, note that Sector A on Spinner Z is formed by a 180° angle, the central angle forming Sector B is a little less than 30°, and the central angle forming Sector C is a little more than 150°. Hence, theoretically for every one time the spinner lands in Sector C, it should land in B about 5 times, and in A about 6 times.

The situation described in Problem 2 is an example of a situation in which the experimental results, even after a relatively large sampling, are not what one would expect and students may find it difficult to decide whether to place their confidence in the theoretical or the experimental results.

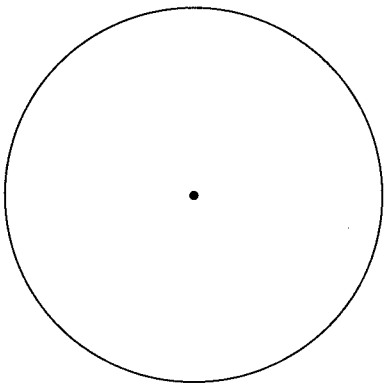
Prediction 1 represents an incorrect theoretical distribution based on the areas of the sectors of Spinner Z. Prediction 2 reflects the theoretical relationships in the spinner; and Prediction 3 reflects Ms. Miller's class experimental data. Either Prediction 2 or 3 could be argued as reasonable based on experimental or theoretical probabilities. Some students may distrust Ms. Miller's data, raising questions about sampling procedures. If so, this provides an opportunity to explore the students' ideas about randomness.

TEACHER NOTES:

Follow-up Student Activity 34.4

NAME _____ DATE _____

1 a) On the circle at the right, design a spinner with 5 sectors (A, B, C, D, and E) so that:



- $\frac{3}{8}$ of the area of the spinner is covered by sector A
- a central angle of 30° forms sector B
- sector C covers twice the area of sector B
- sector D covers $\frac{1}{3}$ as much as sector A
- sector E covers what is left

b) Record what you think should be the measure of the central angle that forms each sector on your spinner:

A_____ ; B_____ ; C_____ ; D_____ ; E_____

c) Tell how you decided the measures of the angles that form sectors A and E:

d) Tell what part of the circle is covered by each sector. Express each part as a fraction and as a decimal:

A_____ = _____; B_____ = _____; C_____ = _____;

D_____ = _____; E_____ = _____;

e) Explain how you determined the decimal value for A and B in d) above.

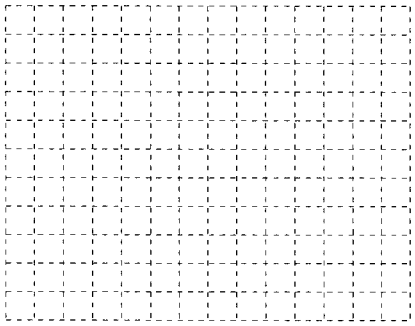
Continued on back.

Follow-up Student Activity (cont.)

2 Suppose the spinner you created in Problem 1 were spun 50 times. Draw and label 3 bar graphs so that:

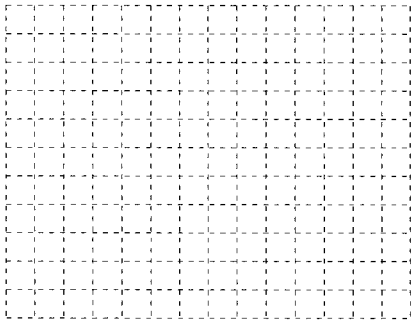
a) Bar Graph I shows data that *would not be likely* to come from 50 spins of your spinner;

Bar Graph I



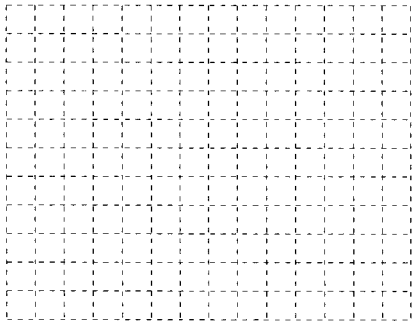
b) Bar Graph II shows data that would be *somewhat likely* to come from 50 spins of your spinner;

Bar Graph II



c) Bar Graph III shows data that would be *highly likely* to come from 50 spins of your spinner.

Bar Graph III



d) Explain how you decided on the data you used for Bar Graph III:

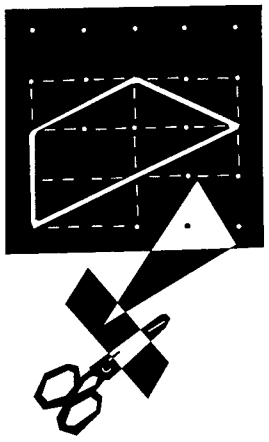
3 Conduct an experiment by spinning your spinner from Problem 1 50 times. On another sheet, make a bar graph, circle graph, and rectangular area model of your data. Explain how your data is like your predictions for Bar Graph III in 2c) above. How is it different and why do you think this is so?

Geoboard Area II

Lesson 35

THE BIG IDEA

Important insights and intuitions about the meaning of area are promoted by explorations on the geoboard. Subdividing and rearranging regions reinforces the idea of conservation of area. Enclosing a region in a rectangle and observing relationships between the area of the rectangle and the area of the region develop readiness for understanding and inventing formulas. Discussing a variety of methods for finding the area of geoboard regions broadens students' views and reinforces the idea that a problem can be solved in many ways.



CONNECTOR

OVERVIEW

Students recall methods developed in Lesson 19 for finding the area of a geoboard figure. They explore the effects changing the unit has on area measurements.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Scissors, 1 pair per student.
- ✔ Connector Master A, 1 copy of the top or bottom half per group of students and 1 transparency of the whole sheet for the teacher.
- ✔ Connector Master B, 1 copy per group and 1 transparency.
- ✔ 2-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.

FOCUS

OVERVIEW

Students discover and use formula-free methods for finding the area of geoboard regions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Geoboard, 1 per student.
- ✔ Geoboard or transparency of Geoboard Recording Paper (see *Blackline Masters*) for the overhead.
- ✔ Focus Masters A and B, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master C, 1 transparency.

FOLLOW-UP

OVERVIEW

Students find the areas of given geoboard regions and of regions they create.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 35.1, 1 copy per student.
- ✔ Geoboard Recording Paper, (see *Blackline Masters*), 1 sheet per student.

LESSON IDEAS

ASSESSMENT
Have students record their responses to Follow-up Problem 1 in their journals. Then have them review previous entries related to area concepts and discuss (orally or in writing) ways their views have developed.

TEACHER NOTES:

SELECTED ANSWERS

2. a) 10 square units
   b) 6 square units
   c) 6 square units

3. 7 square units

5. Unit A: area = $^{8.5}/_{16}$ or $^{17}/_{32}$
   Unit B: area = $^{34}/_{32}$ or $^{17}/_{16}$
   Unit C: area = $^{17}/_{3}$ or $5^{2}/_{3}$

Connector Teacher Activity

OVERVIEW & PURPOSE

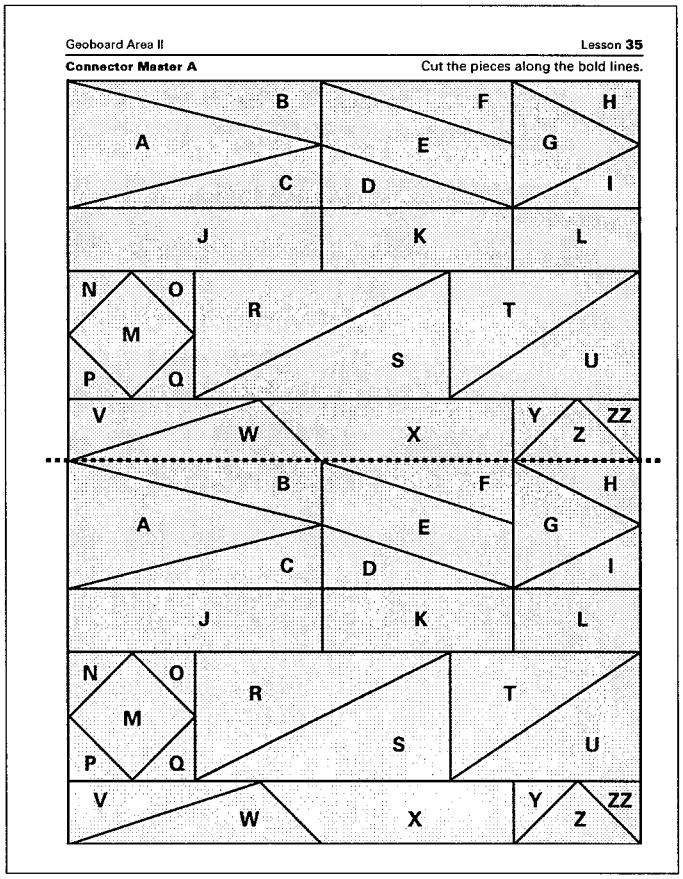
Students recall methods developed in Lesson 19 for finding the area of a geoboard figure. These preformula methods reinforce the concept of area.

MATERIALS

- ✔ Scissors, 1 pair per student.
- ✔ Connector Master A, 1 copy of the top or bottom half per group of students and 1 transparency of the whole sheet for the teacher.
- ✔ Connector Master B, 1 copy per group and 1 transparency.
- ✔ 2-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.

ACTIONS

1 Place the students in pairs or small groups and give each pair a copy of either the top or bottom half of Connector Master A, 1 sheet of 2-cm grid paper, and a pair of scissors. Have the pairs cut apart the pieces as indicated and sort the pieces according to their areas. Discuss.

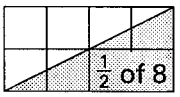


COMMENTS

1 Note that the top and bottom halves of Connector Master A are identical. Each group needs only one of these halves. To save time, cut these out and place them in envelopes prior to class. For reference at the overhead cut one transparent set of pieces apart and keep the other intact.

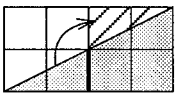
Students may sort pieces in a variety of ways, and some may pick different units of area than others. If students limit the ordering process to "eyeballing" pieces, you could encourage more conceptual discussion by asking them to "assign a number to each piece to indicate its area." It is not expected that students use formulas to determine areas. One method students may use is to place the pieces on the grid paper and apply reasoning that is based on the meaning of area, as explored in Lesson 19. For example, if 1 small square in Figure I below is 1 area unit, the shaded region shown in Figure I below is ½ of a rectangle with area 8, the shaded region has area 4. Alternatively, the area could be found by cutting the shaded region and rearranging the parts to form a square with 4 units of area, as illustrated in Figure II.

Figure I



or

Figure II



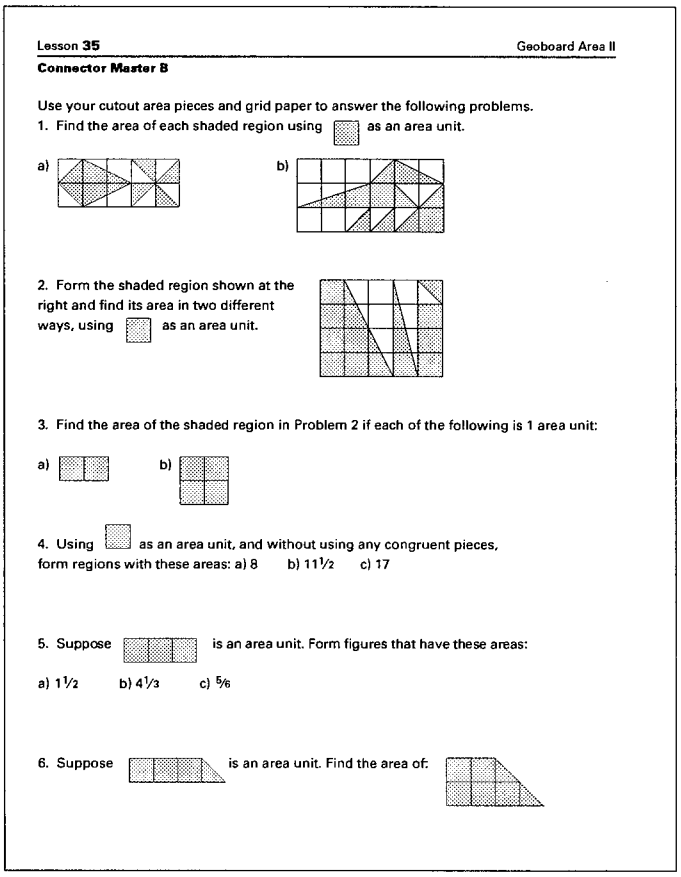
Using region N as ½ area unit (students may use other units), the areas of the other shaded regions on Master A are: A is 4; B, C, G, L, M, and W are 2; D, F, and V are 1½; E, K, T, and U are 3; H, I, and Z are 1; J, R, and S are 4; O, P Q, Y, and ZZ are ½; and X is 3½.

The method of enclosing a region in a rectangle to determine its area, and the idea that a region can be cut apart and rearranged while preserving area, are used extensively in the Focus activity.

Connector Teacher Activity (cont.)

ACTIONS

2 Give each group a copy of Connector Master B. Ask the groups to solve Problem 1. Have volunteers describe their reasoning at the overhead. Then have the groups complete the remaining problems on Master B. Discuss.



Lesson **35** Geoboard Area II

Connector Master B

Use your cutout area pieces and grid paper to answer the following problems.

1. Find the area of each shaded region using ▨ as an area unit.

a) b)

2. Form the shaded region shown at the right and find its area in two different ways, using ▨ as an area unit.

3. Find the area of the shaded region in Problem 2 if each of the following is 1 area unit:

a) b)

4. Using ▭ as an area unit, and without using any congruent pieces, form regions with these areas: a) 8 b) 11½ c) 17

5. Suppose ▭▭▭ is an area unit. Form figures that have these areas:

a) 1½ b) 4⅓ c) ⅚

6. Suppose ▭▭◺ is an area unit. Find the area of:

3 (Optional) Have each pair of students combine area pieces to form a region of their choice, choose a unit of area, and determine the number of their area units in the region. Then have them exchange regions with another pair of students and determine the area of each other's regions.

COMMENTS

2 Here are selected answers:

1a) 5 area units

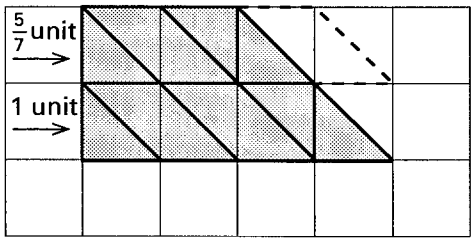
1b) $7\frac{1}{2}$ area units

2. $13\frac{1}{2}$ area units. Two common methods of determining this are: a) total the shaded parts and, b) subtract the area of the unshaded part from 20 (the area of the enclosing rectangle).

3a) $6\frac{3}{4}$ area units

3b) $3\frac{3}{8}$ area units

6. $1\frac{5}{7}$ area units, which can be determined by subdividing the unit and comparing the pieces, as shown below:



3 Circulate while pairs work. If time allows, invite pairs to pose regions whose areas can be determined by the class.

Focus Teacher Activity

OVERVIEW & PURPOSE

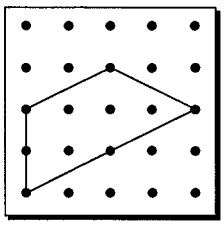
Students discover and use a variety of formula-free methods for finding the area of geoboard regions. These methods strengthen students' understanding of the concept of area and encourage the development of intuitions that will be useful later in understanding and inventing formulas and algorithms.

MATERIALS

- ✔ Geoboard, 1 per student.
- ✔ Geoboard or transparency of Geoboard Recording Paper (see *Blackline Masters*) for the overhead.
- ✔ Focus Masters A and B, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master C, 1 transparency.

ACTIONS

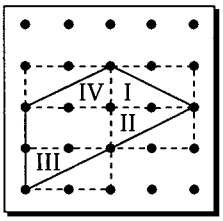
1 Place the students in groups and give each student a geoboard. Ask the students to each form the following geoboard region and to find at least two different ways to determine its area, assuming one small square on the geoboard is the area unit. Ask for volunteers to illustrate their methods at the overhead.



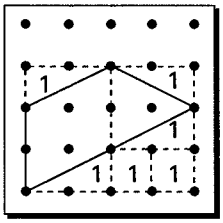
COMMENTS

1 Following are three methods that students may suggest. Discuss these and others that students devise.

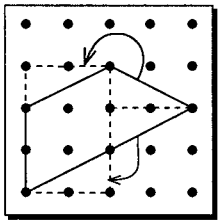
One method is to see that triangles I, II, III, and IV are each half of a 1×2 rectangle, so their areas are each 1 and the area of the trapezoid is 6 square units:



Another method is to enclose the whole trapezoid in a rectangle of area 12 and notice that the outside area is 6, so the inside area is $12 - 6$:



Some students may notice that the right side of the figure can be divided into two right triangles which can be moved to form a 2×3 rectangle:

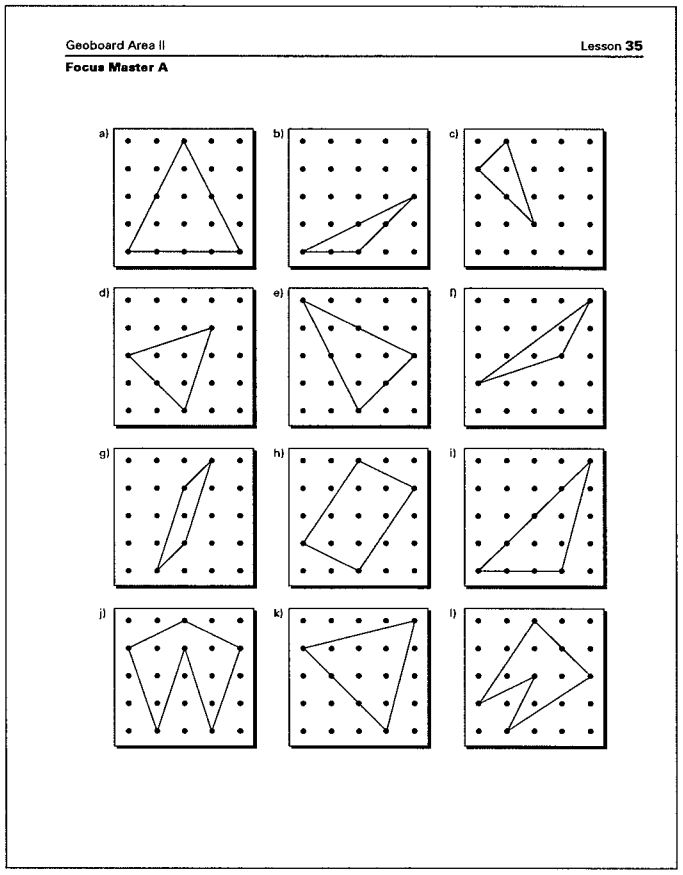


Still other students may begin by estimating (e.g., "I know its area is less than the area of a 3×4 rectangle.") and then continue to close in on the area. Acknowledge such approximations since they may give students having difficulty entry to the problem and a feeling of accomplishment. Listening to and trying out other students' strategies can then enable these students to develop more exact methods.

Focus Teacher Activity (cont.)

ACTIONS

2 Place a transparency of Focus Master A on the overhead and have students form and determine the areas of three or more selected regions from the transparency, using one small geoboard square as the area unit. Discuss.

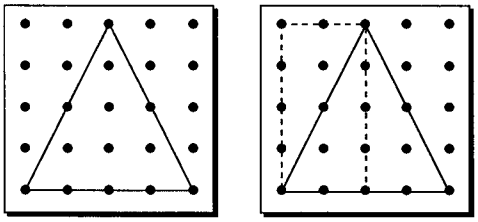


COMMENTS

2 To encourage the development of new strategies, it is helpful to have students share two or three methods for a couple of regions. If you feel students are stuck because of an unwillingness to try other methods you may wish to make suggestions such as, "determine whether the enclosing method (or another) works to find the area of this region." After discussing a few examples, turn groups loose to work on regions you assign, or allow them to choose two or three that look interesting. If some students finish before others, select additional regions from this sheet for them to investigate. There is more than one way to find the area of each of these regions.

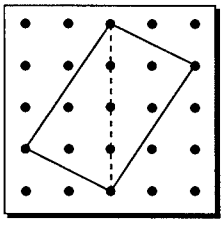
The areas of the regions on Master A in square units are: a) 8; b) 2; c) 2; d) 4; e) 6; f) $2\frac{1}{2}$; g) 2; h) 8; i) 6; j) 8; k) $7\frac{1}{2}$; and l) 6. Some of these regions could be assigned as homework, warm-up, or assessment problems for exploration on other days.

a) One way to find the area of region a) is to divide it into 2 congruent parts and find the area of 1 part, which is $\frac{1}{2}$ of a 2 by 4 rectangle. Thus, each part has area 4 and the original region has area $2 \times 4 = 8$, as shown below. A similar method of using lines of symmetry to subdivide a region and hence, simplify the problem would work for d), e), j), and l).

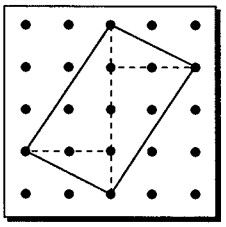


Before After

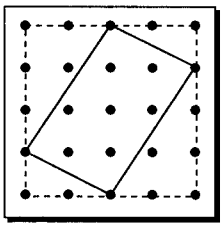
h) Here are three possible ways the area of region h) may be visualized:



as the sum of the areas of 2 congruent triangles (4 + 4 = 8);



as the sum of the areas of 4 triangles (1 + 3 + 1 + 3 = 8);



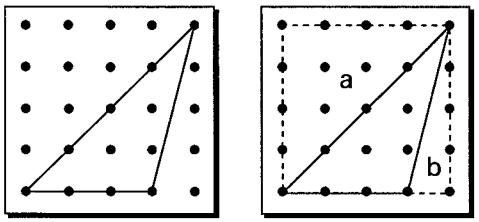
as the remaining area after the areas of 4 triangles are subtracted from the area of the whole geoboard (16 – 1 – 3 – 1 – 3 = 8).

Focus Teacher Activity (cont.)

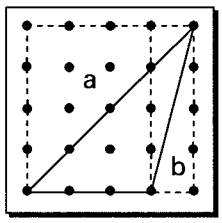
ACTIONS

COMMENTS

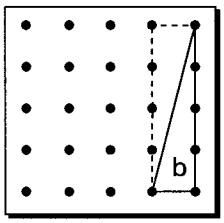
i) One method of finding the area of region i) is to enclose it in a rectangle (see diagram below) and subtract the unwanted area.



When the triangle is enclosed in a rectangle of area 16, one must subtract 8 (for unwanted region **a**) and subtract 2 (for unwanted region **b**), which leaves area 6 = (16 – 8 – 2) for the triangle. Note that the area of region **a** is $1/2$ the area of the enclosing rectangle and the area of region **b** is $1/2$ the area of a 1×4 rectangle. Some students may be confused by the overlapping regions created when they enclose region **b** in a 1×4 rectangle, as illustrated below:



One way to avoid such confusion is to form region **b** on a separate geoboard and then determine its area. This can clarify why the overlap of subregion **a** is not relevant to finding the area of **b**.

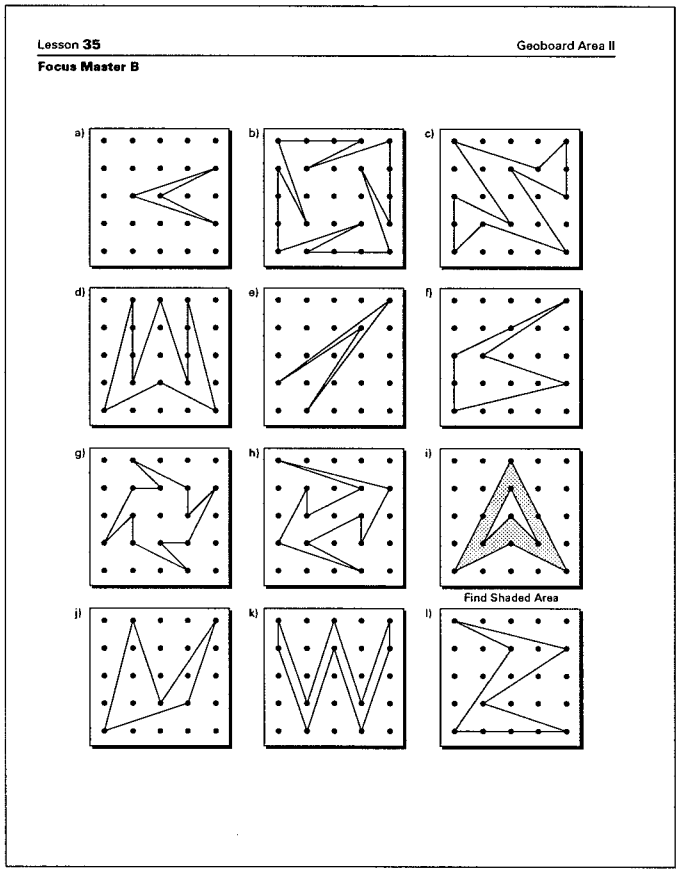


l) Note that finding the area of this region seems to be a "key" for many students to finding areas of regions on Master B. Hence, allow plenty of exploration time if you assign this problem.

Focus Teacher Activity (cont.)

ACTIONS

3 Place a transparency of Focus Master B on the overhead. Select three or more regions on the transparency for students to form and determine their areas, using the small geoboard square as the area unit. Discuss.



COMMENTS

3 If students are struggling, you may wish to assign more regions from Master A before assigning any of these. Master B has more challenging problems. However, the area of each region can be obtained by the methods that have been discussed. Copies of this sheet could also be given to students who want "challenge" problems to take home.

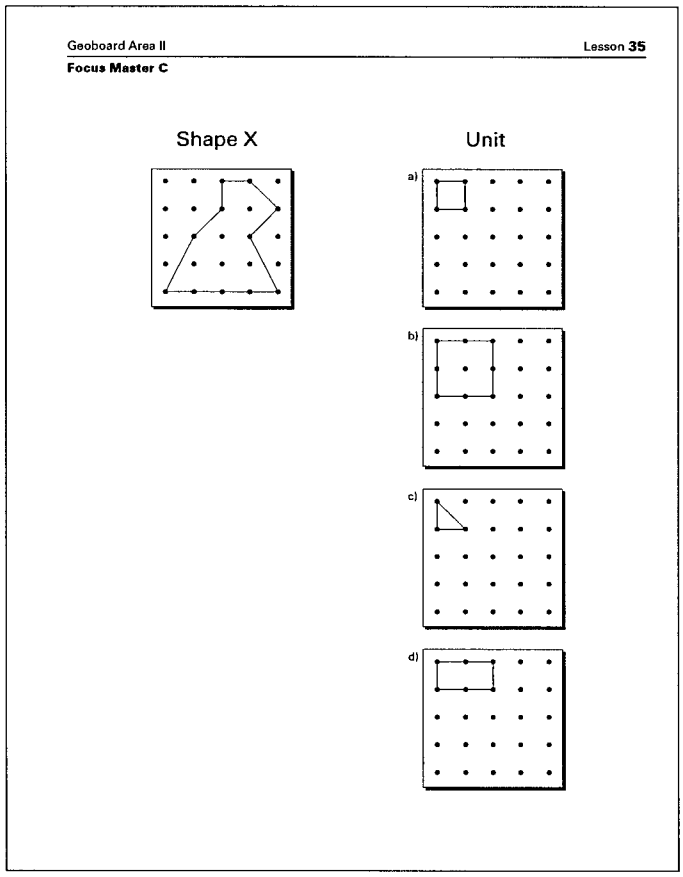
Encourage students to share their techniques. Watching others' methods gives new insights for those who have difficulty.

The areas in square units of the regions on this transparency are: a) 1; b) 14; c) 8; d) 7; e) 1; f) $5\frac{1}{2}$; g) 6; h) 5; i) 5; j) $5\frac{1}{2}$; k) 4; and l) $5\frac{1}{2}$.

Focus Teacher Activity (cont.)

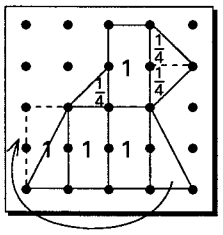
ACTIONS

4 Place a transparency of Focus Master C on the overhead and draw the figures shown below on the blank geoboards on Master C. Ask the students to form Shape X on their geoboards and determine its area using each of the units a)-d) as the unit of area. Discuss.



COMMENTS

4 Students will devise a variety of methods of determining the areas, which are: a) 9½, b) 2⅜, c) 19, and d) 4¾ area units. It is intended that fractional amounts be determined by conceptual methods (rather than by applying algorithms). For example, some students may find the area of Shape X using unit d) by subdividing the shape as shown here:

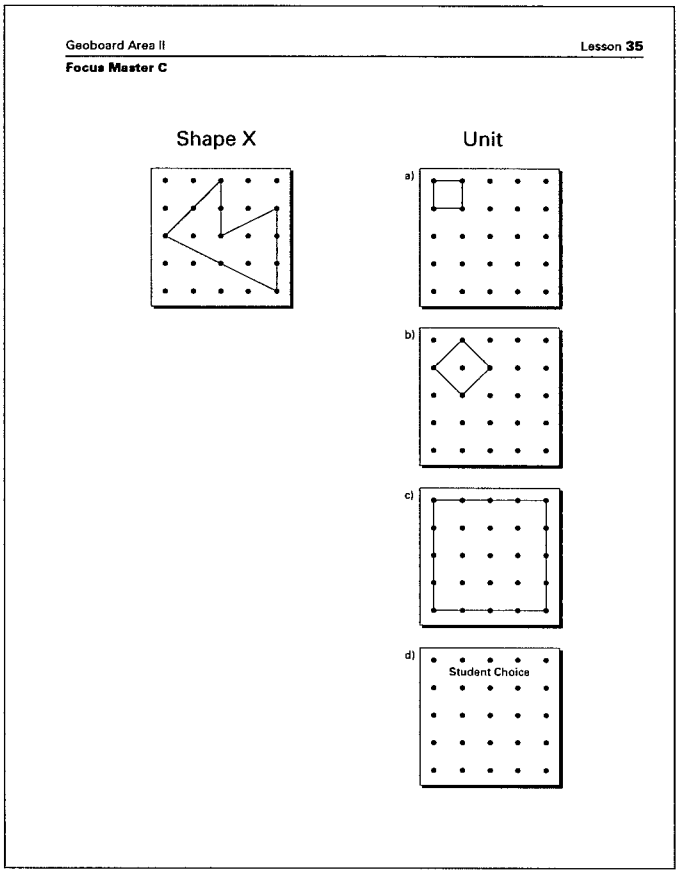


Others may reason that since unit d) is twice as large as unit a), it will take ½ as many units to cover Shape X using unit d) as it would take using unit a). Therefore, the area of Shape X is ½ of 9½ = 4¾ area units.

Focus Teacher Activity (cont.)

ACTIONS

5 Repeat Action 4 using the shape and units shown below:



6 (Optional) Have the students form and exchange "challenge" regions and determine their areas using a variety of area units.

COMMENTS

5 You may wish to invite volunteers to tell the area of Shape X based on the area unit they chose for d), and keeping their unit a secret, have the class determine units that produce that area.

The area of Shape X for the given units are a) 7; b) $3\frac{1}{2}$; c) $\frac{7}{16}$.

Regions on Focus Master A or Master B could also be posed for exploration (i.e., as Shape X) using a variety of units.

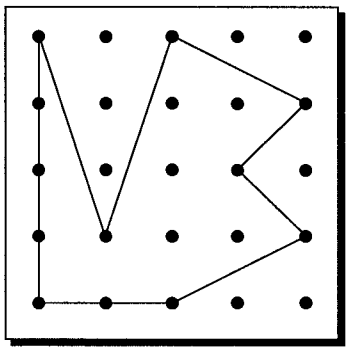
6 This could be carried out by having each student exchange their region with another student, by having groups exchange regions, or by having individuals pose regions to the whole class.

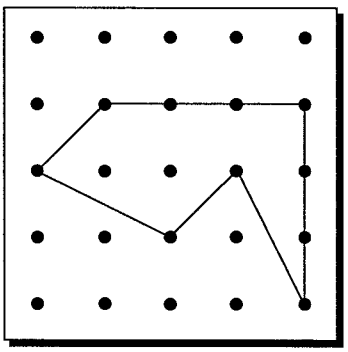
Follow-up Student Activity 35.1

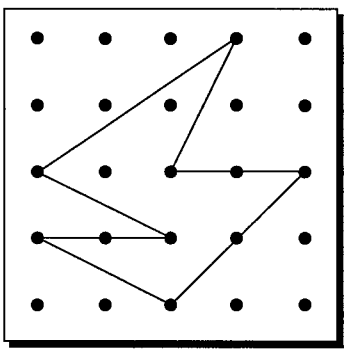
NAME _____ **DATE** _____

1 In your own words, explain the meaning of "area."

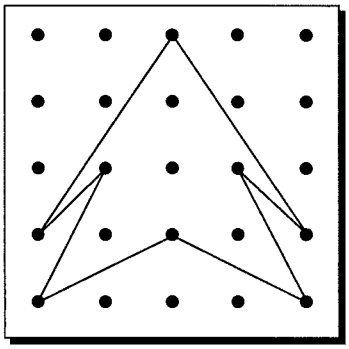
2 Determine the area of each geoboard region shown below. If needed, use the attached blank geoboard paper to explore your ideas. Identify the area unit(s) you use for each region.

a) 

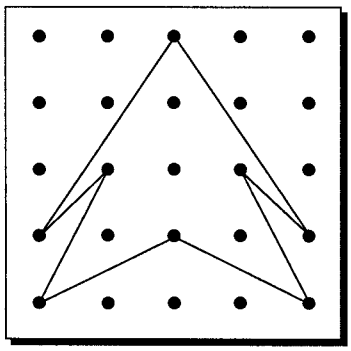
b) 

c) 

3 Find the area of this region using at least 2 different methods. Explain each method.



Method 1:

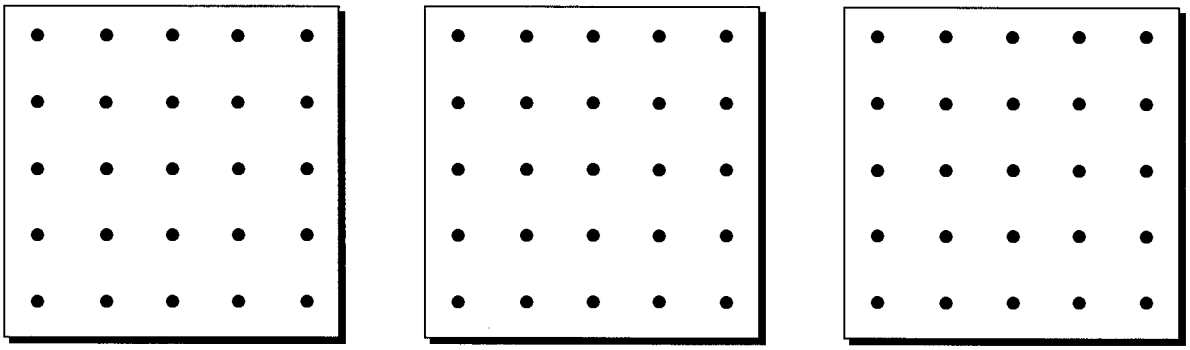


Method 2:

(Continued on back.)

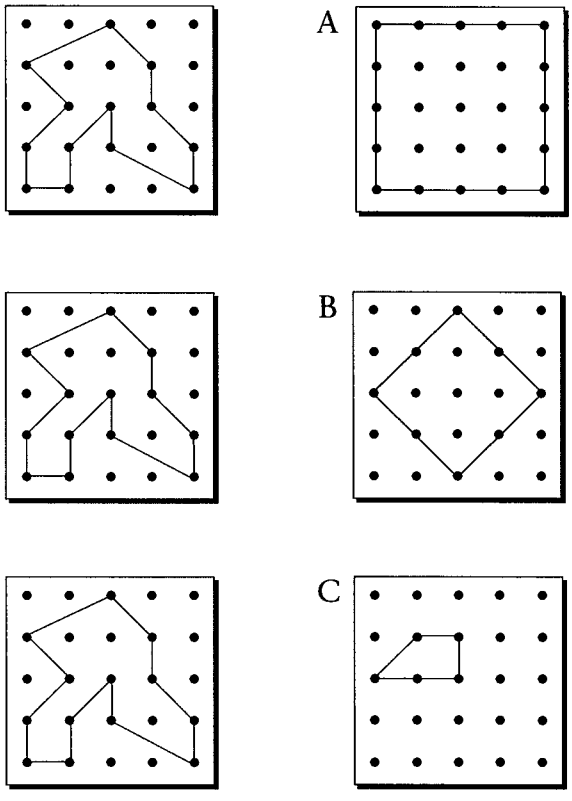
Follow-up Student Activity (cont.)

4 On the blank geoboards below sketch 3 different regions whose areas you think would be challenging to find. Find and record the area of each of your figures. (Be sure to indicate what you use as the area unit.)



5 Determine the area of Shape X using each of area units A, B, and C.

Shape X Unit

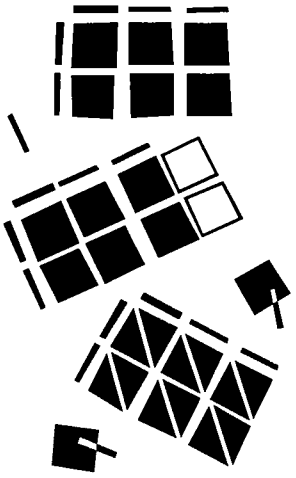


Dimension & Area II

Lesson 36

THE BIG IDEA

Understanding the distinction between linear and area measure is fundamental to understanding the area models for multiplication and division. Hands-on experiences with linear and area units help students clarify the distinctions and relationships between the two types of measures and prepare students for understanding multiplication and division.



CONNECTOR

OVERVIEW

Students explore the area and perimeter of a given rectangle using several square and nonsquare area units, and they discuss the advantages of using squares as area units and the lengths of the sides of squares as linear units.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Tile, 30 per student.
- ✔ Linear pieces (equal in length to the side of a tile, see Lesson 2), 30 per student.
- ✔ 1-inch grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ Connector Master A (2 pages), 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

Base ten area and linear pieces are used to find the area and dimensions of rectangles and to reinforce the distinction between linear measure and area measure.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Master C (optional), cut apart copies, 3 cards per group.
- ✔ Base ten grid paper, Version A, (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Base ten measuring tape (optional, see *Blackline Masters*), 1 per pair of students.
- ✔ Scissors and tape (optional), for each pair of students.
- ✔ Base ten area and linear pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students use base ten area and linear pieces to determine the area and dimensions of rectangles.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 36.1, 1 copy per student.
- ✔ Base ten grid paper, Version A, 2 sheets per student.
- ✔ Base ten area and linear pieces, paper copies for student use at home.

LESSON IDEAS

TIMING

This Connector may take a little more time than usual, depending on students' prior understanding of relationships and differences between area and length. The entire lesson (including Action 7) generally takes 3-4 hours of class time.

QUOTE

Area models are especially helpful in visualizing numerical ideas from a geometric point of view. ...Later, students can extend area models to the study of algebra, probability, dimension analysis in measurement situations, and other more advanced subjects.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

2. a) area, 324 square units
   b) dimensions, 24 × 24 linear units
   c) dimensions, 17 × 23 linear units
   d) dimensions, 21 × 33, linear units
   e) dimensions, 1.4 × 1.4 linear units
   f) dimensions, .3 × 3.8 linear units
   g) area, 8.28 square units

3. a) Dimensions: 5 × 7.5
      Area: 37.5
      Perimeter: 25

   b) Dimensions: $5 \times 8\frac{1}{3}$
      Area: $41\frac{2}{3}$
      Perimeter: $26\frac{2}{3}$

Connector Teacher Activity

OVERVIEW & PURPOSE

Students explore the area and perimeter of a given rectangle using several square and nonsquare area units, and they discuss the advantages of using squares as area units and the lengths of the sides of squares as linear units.

MATERIALS

- ✔ Tile, 30 per student.
- ✔ Linear pieces (equal in length to the side of a tile, see Lesson 2), 30 per student.
- ✔ 1-inch grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ Connector Master A (2 pages), 1 copy per group and 1 transparency.

ACTIONS

1 Arrange the students in groups and give each student a set of square tile and linear pieces. Write 3×6 on the overhead and ask the students to use the tile and linear pieces to show how they view the meanings of that expression. Discuss their ideas about other computations their models could represent.

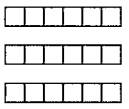
2 Use tile and linear pieces to form the rectangle shown here on the overhead and have the groups do the same.

Give each group a sheet of 1-inch grid paper and give each student a pair of scissors. On a transparency of 1-inch grid paper, sketch a)-d) below at the overhead. Ask the groups to determine, if possible, the area, dimensions, and perimeter of the above rectangle for each of these area units. Follow with a discussion of their ideas about relationships between area and linear units.

a) b) c) d) e)

COMMENTS

1 Be sure that the repeated addition concept (with area and/or linear pieces) and the area concept of multiplication are discussed. (The area concept provides the basis for the Focus activity.) For the repeated addition concept, students may view 3×6 as 3 groups of 6 (linear units or area units) or 6 groups of 3:



3 groups of 6

6 groups of 3

According to the area concept, 3×6 is viewed as a rectangle with dimensions 3 linear units by 6 linear units and with area 3×6 area units, as shown below.

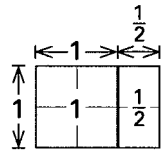
2 The intent of this action is for students to notice and wrestle with the issues involved in determining perimeter when the area unit is nonsquare and to establish the convention of using square area units. Some students may prefer to work with the tile and linear pieces; others may draw on the grid paper; and still others may cut out units to form the rectangle. For this reason, you may need to have extra sheets of grid paper available.

As introduced in Lesson 2, when the unit of area is a square, it is standard practice to call the length of the side of the square unit 1 linear unit. Based on this convention, the areas and perimeters of the rectangle using area units a), b), and c) are shown on the next page.

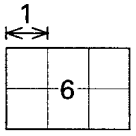
(Continued next page.)

Connector Teacher Activity (cont.)

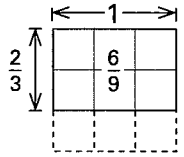
ACTIONS

a) 

Area = $1\frac{1}{2}$ square units
Perimeter = 5 linear units

b) 

Area = 6 square units
Perimeter = 10 linear units

c) 

Area = $\frac{6}{9}$ (or $\frac{2}{3}$) square units
Perimeter = $3\frac{1}{3}$ linear units

3 Ask the students to assume that for now a square is the unit of area and the length of its edge is the corresponding linear unit. Give each group a copy of Connector Master A (2 pages) and ask them to find the area and perimeter of each rectangle, based on the given lengths. Invite volunteers to share their methods for selected problems at the overhead.

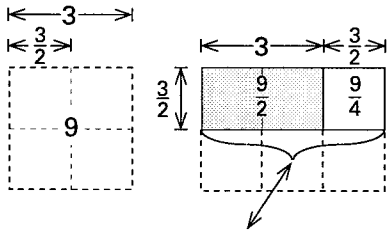
COMMENTS

2 (continued.)

Note, however, that some students may arbitrarily assign another length as the linear unit for a), b), and c). If so, notice that the area formula, area of a rectangle = base × height, does not hold. Although there is no mathematical "law" stating that the linear unit must be the length of the side of a square area unit, the area formula for rectangles holds only when this is so.

For example, when d) is 1 area unit, it is impossible to assign the value 1 linear unit to either of its dimensions and have their product equal 1. While most students will comfortably use unit d) to establish the area of the rectangle as 3 area units, they may experience "disequilibrium" over determining the perimeter of the rectangle. Similar issues arise with unit e). Hence, in most contexts area units are assumed to be squares whose sides have length 1 linear unit.

3 It may be helpful to encourage groups to use tile and linear pieces to explore the problems. On b)-f), some students may subdivide lengths and areas to form linear and area units, and then use those units to determine area and perimeter. Others may determine the length and area of one of the squares on the grid in order to calculate the answers. Still others may use the division model, as illustrated below for e).



Area: $9/2 + 9/4 = (9/4 + 9/4) + 9/4 = 27/4$

For discussion, you may wish to let the groups select the problems they found most interesting or most challenging, or the ones they are stuck on, etc.

Connector Teacher Activity (cont.)

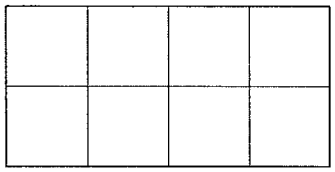
ACTIONS

Dimension and Area II Lesson **36**

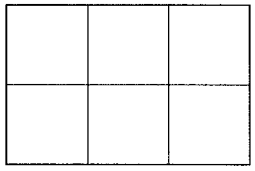
Connector Master A

Given: Find the area and perimeter:

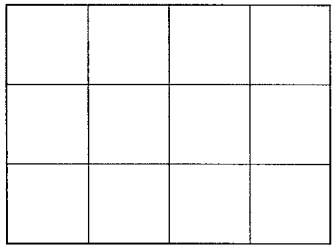
a) 1



b) 4



c) 9

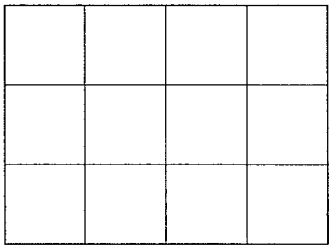


COMMENTS

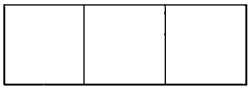
Lesson **36** Dimension and Area II

Connector Master A (cont.)

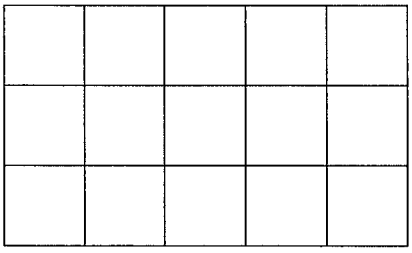
d) 10



e) 3



f) 3



TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Base ten area and linear pieces are used to find the area and dimensions of rectangles and to reinforce the distinction between linear measure and area measure. This provides a foundation for multiplying and dividing, which are explored in Lessons 37 and 38.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Master C (optional), cut apart copies, 3 cards per group.
- ✔ Base ten grid paper, Version A, (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Base ten measuring tape (optional, see *Blackline Masters*), 1 per pair of students.
- ✔ Scissors and tape (optional), for each pair of students.
- ✔ Base ten area and linear pieces for the overhead.

ACTIONS

1 Arrange the students in groups and give each student a set of base ten area and linear pieces. Ask the groups to determine the area and dimensions of each different area piece when:

a) the area of the large area piece square is 100 square units,

b) the length of the side of the small area piece square is .1 linear unit,

c) the large area piece square has area .01,

d) the length of the side of the large area piece square is .01.

COMMENTS

1 To stretch students' thinking, ask them to also predict the area and dimensions of a strip-mat, a mat-mat and a strip-let (see Lessons 13 and 25).

a) If the large square has area 100 square units, then its dimensions must be 10 linear units by 10 linear units. Hence, a small square has area 1 and dimensions 1×1. The other area pieces have area and dimensions as follows:

strip-let, $1 \times .1 = .1$;
strip of 10 small squares, $1 \times 10 = 10$;
strip-mat, $10 \times 100 = 1000$;
mat-mat, $100 \times 100 = 10{,}000$.

b) strip-let, $.01 \times .1 = .001$;
small square, $.1 \times .1 = .01$;
strip of 10 small squares, $.1 \times 1 = .1$;
large square, $1 \times 1 = 1$;
strip-mat, $1 \times 10 = 10$;
mat-mat, $10 \times 10 = 100$.

c) strip-let, $.01 \times .001 = .00001$;
small square, $.01 \times .01 = .0001$;
strip of 10 small squares, $.1 \times .01 = .001$;
large square, $.1 \times .1 = .01$;
strip-mat, $.1 \times 1 = .1$;
mat-mat, $1 \times 1 = 1$.

d) strip-let, $.0001 \times .001 = .0000001$;
small square, $.001 \times .001 = .000001$;

(Continued next page.)

Focus Teacher Activity (cont.)

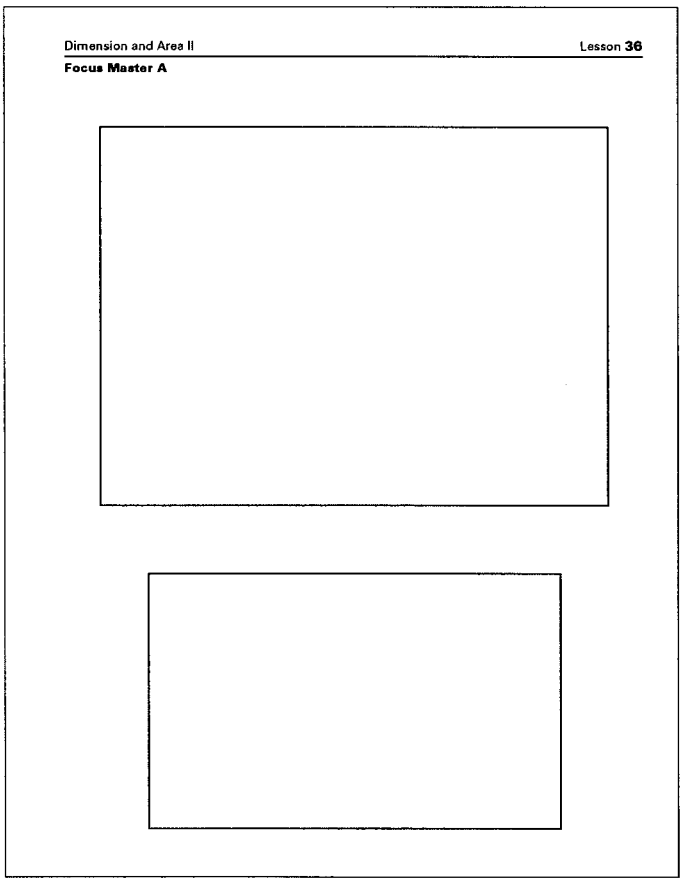
ACTIONS

COMMENTS

1 (continued.)
strip of 10 small squares, $.01 \times .001 = .00001$;
large square, $.01 \times .01 = .0001$;
strip-mat, $.01 \times .1 = .001$;
mat-mat, $.1 \times .1 = .01$;
strip-mat-mat, $.1 \times 1 = .1$;
mat-mat-mat, $1 \times 1 = 1$.

Some students may puzzle over the fact, for example, that a square with sides of length one tenth has area one hundredth. Some may make conjectures and observations about decimal placement. Encourage discussion of their ideas, but it isn't necessary to establish rules at this time. Rather, activities in this lesson are intended to help students develop intuitions and insights that are based on conceptual relationships.

2 Give each student a copy of Focus Master A. Ask them to devise a method of using their pieces to find the area and dimensions of the rectangle at the top. Have students share their conclusions and methods of counting the pieces on a transparency of Master A.



2 The intent here is to begin the development of a variety of strategies for mentally counting the area pieces that cover a rectangle.

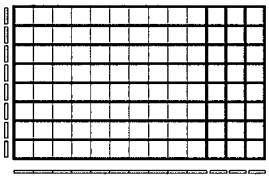
If students ask you to identify the units of area and length, suggest that they choose their own units. Otherwise, wait to see what comes up when students share at the overhead. Using the large area piece square as an area unit, the top rectangle on Master A can be covered with 1 unit, 8 tenths, and 12 hundredths. These pieces can be traded to get a minimal collection of 1 unit, 9 tenths, and 2 hundredths. So, the area is 1.92 square units. The rectangle can also be covered with other collections of decimal pieces, but the minimal collection will always be the same.

Using the length of the edge of the large area piece square as the linear unit, the dimensions of the rectangle are 1.2 linear units by 1.6 linear units, or 1.2×1.6. If the small area piece square and the length of its edge are the area and linear units, then the rectangle has dimensions 12 linear units by 16 linear units, and area 192 square units.

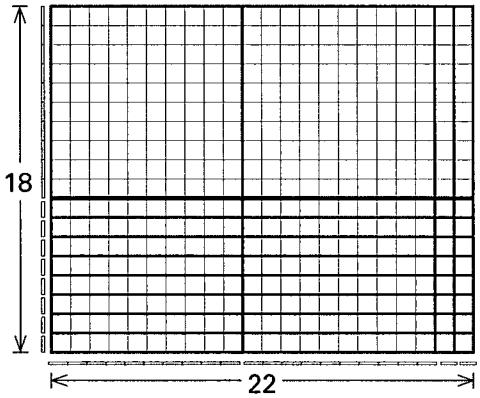
Focus Teacher Activity (cont.)

ACTIONS

3 Repeat Action 1 for the rectangle at the bottom of Focus Master A. Have volunteers show their methods at the overhead using both the large and small squares and their edges as units of area and length. Then ask the groups to record multiplication and division equations that represent each rectangle on Master A.



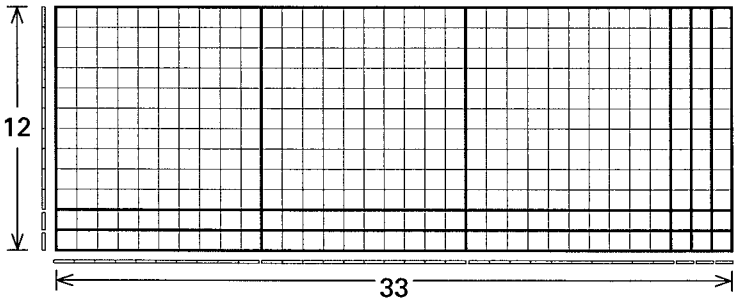
4 Ask each student to form a collection of area pieces that contains 3 hundreds, 9 tens, and 6 units. Have them form a rectangle using these pieces and then find its area and dimensions. After they have found one rectangle, ask them to find other rectangles that have the same area and can be formed with area pieces. Have volunteers model their ideas at the overhead and record multiplication and division equations that could represent each rectangle.

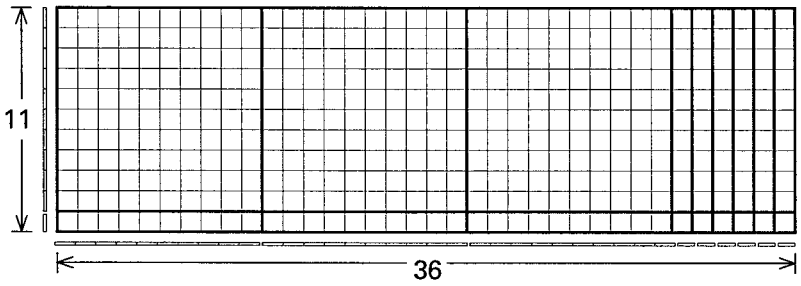


COMMENTS

3 There are many possibilities, depending on the area unit chosen. Using the large area piece square as the area unit and the length of its edge as the linear unit, the dimensions of this rectangle are .8 and 1.3 linear units. The area is 1.04 square units. Based on the area concept of multiplication and division (see Lesson 2), this rectangle can be represented by the equations $.8 \times 1.3 = 1.04$, $1.04 \div .8 = 1.3$, or $1.04 \div 1.3 = .8$. Alternatively, if the small area piece square is the unit, the rectangle has area 104 square units and dimensions 8 linear units by 13 linear units and is represented by the equations $8 \times 13 = 104$, $104 \div 13 = 8$, $104 \div 8 = 13$. Note that multiplication and division are the focus of Lessons 37 and 38.

4 The small square and its edge are the most convenient units for representing this collection. The area pieces can be arranged to form either a 12×33 rectangle or an 11×36 rectangle, showing that $12 \times 33 = 396$, $11 \times 36 = 396$, $396 \div 11 = 36$, $396 \div 36 = 11$, etc.





There are other area piece rectangles which have the same area. For example, if an equivalent collection is formed by exchanging a hundred for 10 tens, and a ten for 10 units, the resulting collection can be formed into an 18×22 rectangle.

Note that each pair of numbers that are dimensions of a rectangle with area 396 are factors of 396 (see Lesson 7).

Focus Teacher Activity (cont.)

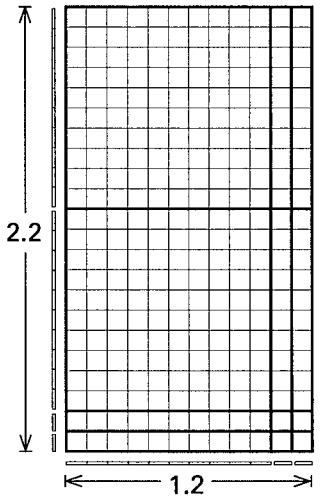
ACTIONS

5 Ask the groups to form a rectangle with the same areas as each of the following collections. Have them determine the area, dimensions, and perimeter of each rectangle. Discuss.

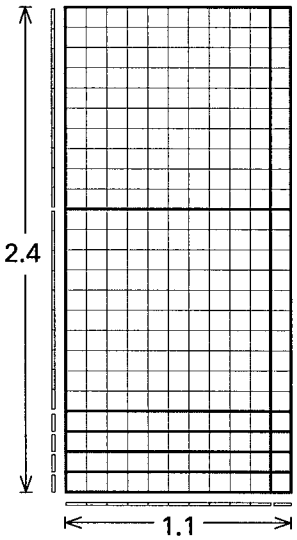
a) 2 units, 6 tenths, 4 hundredths;

b) 4 hundreds, 8 tens, 3 units;

c) 3 tenths, 8 hundredths.



Area: 2.64 square units
Perimeter: 6.8 linear units



Area: 2.64 square units
Perimeter: 7 linear units

6 Give each student a sheet of base ten grid paper, Version A. Place a transparency of Focus Master B on the overhead. Have the groups use their area and linear pieces (or draw sketches of them on the grid paper) to represent each situation. Ask them to write as many mathematical observations, particularly ones involving multiplication and division, as they can about each situation. Discuss.

COMMENTS

5 Encourage students to find possibilities by exchanging and rearranging the area pieces, rather than by using computation. It is intended that this activity emphasize conceptual relationships between the area and dimensions of a rectangle.

Below are some examples for the given collections. In each case other rectangles are possible, although exchanges of pieces may be required. Note that to represent both large and small numbers, it is necessary to vary the base ten piece that is used as the unit.

a) With the given area pieces, or their equivalent, the area of any rectangle constructed will be 2.64 square units. The rectangles shown at the left can be constructed with these pieces without exchanges.

By trading the 2 units for tenths and the 6 tenths for hundredths, several additional rectangles can be constructed.

b) Area: 483 square units. One possibility without trading is a 21 by 23 rectangle with perimeter 88 linear units.

c) Area: .38 square units. The pieces must be traded to form a rectangle. One possibility has dimensions .2 linear units by .19 linear units and perimeter .78 linear units.

If you feel students need additional explorations of this type, here are some possible collections: 3 units, 3 tenths, 6 hundredths; 3 hundreds, 8 units; 5 hundreds, 5 tens, 2 units; 3 units, 1 tenth, 2 hundredths.

6 Here are some possible observations:

a) A collection of 1 unit, 6 tenths, and 9 hundredths can be arranged into a square with sides of 1.3 linear units. Note that 1.3 is called the *square root* of 1.69 because it is the length of the side of a square with area 1.69 (see Lesson 5). The area of this square could be represented by $1.3 \times 1.3 = 1.69$ square units and each dimension has length $1.69 \div 1.3 = 1.3$ linear units.

b) The area is $1.7 \times 2.3 = 2.3 \times 1.7 = 3.91$ square units and the dimensions are $3.91 \div 1.7 = 2.3$ linear units and $3.91 \div 2.3 = 1.7$ linear units.

Focus Teacher Activity (cont.)

ACTIONS

Lesson **36** Dimension and Area II
Focus Master B

Situations

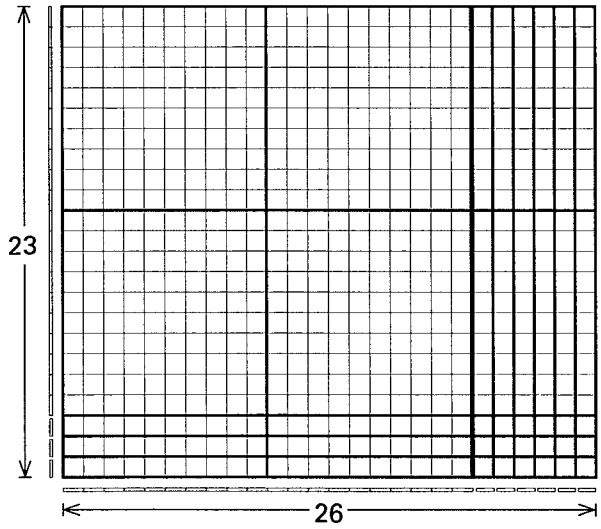
a) A square with area 1.69 square units.

b) A rectangle with dimensions 1.7 linear units by 2.3 linear units.

c) A square whose area is 529 square units.

d) A rectangle with area 598 and one dimension 23 linear units.

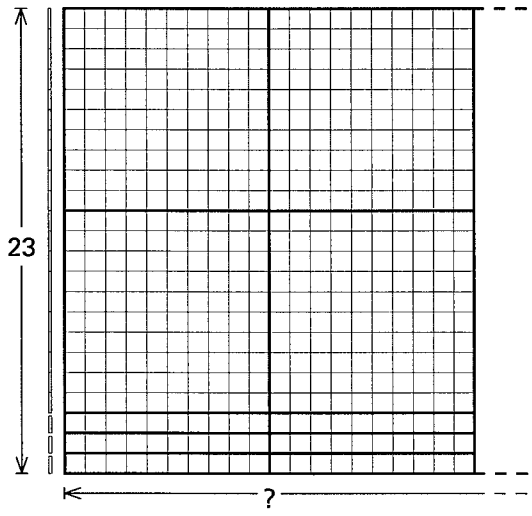
e) A rectangle with area 4.37 square units and one dimension 2.3 linear units.



COMMENTS

c) A collection of 5 hundreds, 2 tens, and 9 units cannot be formed into a square. However, by exchanging pieces a 23×23 square with area is 529 square units can be formed. Each side has length $529 \div 23 = 23$ linear units.

d) One way to form the desired rectangle is to start with an area piece collection of 5 hundreds, 9 tens, and 8 units; lay out linear pieces to represent a dimension of 23 linear units; and then arrange the area pieces to form a rectangle with one dimension 23 linear units, exchanging pieces as needed to complete. After arranging 4 hundreds and 6 tens, as shown here,



the remaining hundred is exchanged for 10 tens. Then additional columns of 2 tens and 3 units each are added to the rectangle until—with the exchange of 1 ten for 10 units—the supply of area pieces is exhausted.
The result is a rectangle whose missing dimension is $598 \div 23 = 26$ as shown at the left. The other dimension has length $598 \div 26 = 23$ linear units, and the area is $26 \times 23 = 23 \times 26 = 598$ square units.

e) A rectangle can be formed by trading 2 units for 20 tenths and 2 tenths for 20 hundredths. The missing dimension is $4.37 \div 2.3 = 1.9$ linear units. The other dimension is $4.37 \div 1.9 = 2.3$ linear units, and the area is $2.3 \times 1.9 = 4.37$ area units.

Focus Teacher Activity (cont.)

ACTIONS

7 (Optional) Place the students in pairs and give each pair a base ten measuring tape and 3 cards cut from Focus Master C. Have them find the indicated measurements. Discuss.

Dimension and Area II Lesson **36**

Focus Master C (Cut along solid lines.)

Using the *small* area piece square as a unit of area, and the length of its side as a linear unit, find the area and perimeter (to the nearest unit) of a sheet of notebook paper.	Select a base 10 unit and use it to find the area (to the nearest hundredth) of _____.
Using the *large* area piece square as a unit of area, and the length of its side as a linear unit, find the area (to the nearest hundredth) and perimeter (to the nearest tenth) of a sheet of notebook paper.	Select a base 10 unit and use it to find the length (to the nearest unit) of the _____ wall of the classroom.
Using the *small* area piece square as a unit of area, and the length of its side as a linear unit find the area and dimensions (to the nearest unit) of _____.	Select a base 10 unit and use it to find the dimensions (to the nearest hundredth) of _____.
Using the *large* area piece square as a unit of area, and the length of its side as a linear unit, find the area and dimensions (to the nearest unit) of _____.	Select a base 10 unit and use it to find the perimeter (to the nearest tenth) of _____.
Select the base 10 linear and area units of your choice and then use the base 10 pieces to find the area and perimeter (to the nearest unit) of _____.	

COMMENTS

7 A master for the base ten measuring tape is in *Blackline Masters*. If you wish to have students construct these, give each pair of students a pair of scissors and some transparent tape. To save time, prepare the measuring tapes prior to class.

Some examples of objects to measure include: the back (or front or side) wall of the classroom; the chalkboard tray; the door; a desktop; a file cabinet; a book jacket, etc. Area measures should be limited to objects which can be covered by at most 2 sets of base ten area pieces. Add your own instructions on the blank card.

The measuring tape is more convenient than linear pieces for measuring longer segments. Notice that, to allow for varying the linear unit there are no numbers written on the tape. If the length of the edge of a large area piece square is chosen as one linear unit (this also happens to equal 1 decimeter), students can measure lengths of objects from .1 linear units to 12.5 linear units, to the nearest tenth of a linear unit. If the length of the edge of the small area piece square is the unit (also 1 centimeter) then students can measure objects from 1 linear unit to 125 linear units.

Although the small units on the tape are centimeters, it is intended that emphasis be on linear and area units and their conceptual relationships. The metric and English measurement systems are explored in more detail in Lesson 43.

Follow-up Student Activity 36.1

NAME _____ **DATE** _____

1 On the attached base ten grid paper, draw a diagram of each of these situations. Next to each diagram, show the area and linear units you used.

a) A rectangle with dimensions 12 by 27.

b) A square whose area is 576.

c) A rectangle whose area is 391 and one dimension is 17.

d) A rectangle with area 693 and one dimension 21.

e) A square with area 1.96 square units.

f) A rectangle with area 1.14 square units and one dimension .3 linear units.

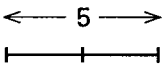
g) A rectangle with dimensions 2.3 linear units by 3.6 linear units.

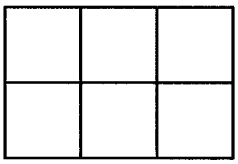
2 Determine the *dimensions*, *area*, and *perimeter* of each rectangle in Problem 1 by mentally counting the area and linear pieces you sketched. Record these measures on your drawings.

(Continued on back.)

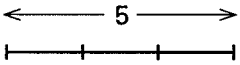
Follow-up Student Activity (cont.)

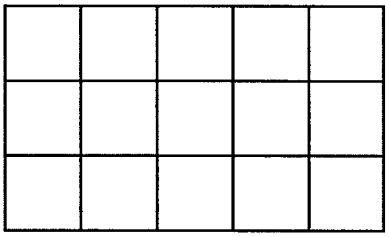
3 Use the given lengths to help you determine the dimensions, area, and perimeter of each of the rectangles below:

a) 



Dimensions: _____
Area: _____
Perimeter: _____

b) 



Dimensions: _____
Area: _____
Perimeter: _____

4 Explain the details of your methods of determining the area and perimeter of 3b).

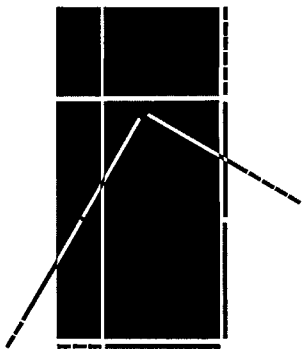
5 Explain ways that you think dimensions, area, and perimeter are related and ways they are different.

Whole Number & Decimal Multiplication Lesson 37

THE BIG IDEA

Working with the area model for multiplication enables students to see relationships between whole number and decimal products, strengthens their understanding of area and dimension, and lays important groundwork for future work with fractions, percent, probability, and algebra.

Experiences that involve building, sketching, and mentally visualizing rectangles can deepen students' understanding of multiplication, and develop their ability to calculate products.



CONNECTOR

OVERVIEW

Students explore the effects of multiplying and dividing one or both dimensions of an area unit by powers of ten. Students also investigate ways to relate their understanding of factors of whole numbers (see Lesson 7) to multiplication of decimals.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Decimal grids version A (see *Blackline Masters*), 2-3 copies per student and 1 transparency.
- ✔ Decimal grids version B (see *Blackline Masters*), 2 sheets per group.
- ✔ Connector Master A, 1 copy per group.
- ✔ Connector Master B (optional, see Comment 6), several copies per class.
- ✔ Connector Student Activity 37.1 (optional, 2 pages), 1 copy per student.
- ✔ Blank Product Chart (see Action 2), 1 per class.
- ✔ Mystery Rectangle (optional, see Comment 6), 1 enlarged copy for display.
- ✔ Scissors and tape, 1 of each per group.

FOCUS

OVERVIEW

Students use the area model for multiplication as a basis for inventing strategies for computing decimal products.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Student Activity 37.2, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 37.3, 1 copy per group and 1 transparency.
- ✔ Scissors, 1 per group.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Blank transparencies and overhead pens, 1 of each per group.

FOLLOW-UP

OVERVIEW

Students compute products by sketching rectangles. They write and solve word problems involving multiplication.

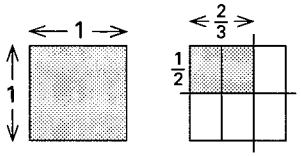
MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 37.4, 1 copy per student.

LESSON IDEAS

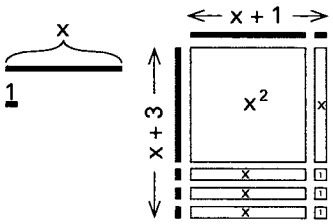
LOOKING AHEAD

The area model for multiplication, which was introduced in Lesson 2, is useful for modeling products involving whole numbers, decimals, fractions, and even algebra. In fact, any given rectangle can model several products, depending on the units of area and length chosen. In this lesson, students explore decimal and whole number products. In later lessons and other *Visual Mathematics* courses, students will explore fraction multiplication. This example shows an area representation of $1/2 \times 2/3$:



Area: $1 \times 1 = 1$ Area of shaded part: $1/2 \times 2/3 = 2/6$

The following example illustrates how the area model can give meaning to the product $(x + 3)(x + 1)$:



$(x + 3)(x + 1) = x^2 + 4x + 3$

Emphasis in this lesson is on developing understanding of the concept of multiplication and on developing background for mental computation.

FOCUS

Note that the sequence of actions in the Focus activity is intended to promote the development of meaningful models and strategies for multiplication, moving students from hands-on experiences with area pieces to sketching and imagining models to inventing symbolic representations of their models. Most students will be ready for this sequence; however, some students may need to continue using the area pieces in situations that call for sketching or imagining pieces. It is important to honor this need.

The visual representations explored in this lesson are used as a basis for exploring mental computation and estimation in *Visual Mathematics, Course II.*

To promote the development of mental strategies, regularly pose problems by sketching dimensions and asking students to imagine (and/or sketch) the completed rectangle, its area, dimensions, and perimeter.

QUOTE

Even though students can explore paper-and-pencil computations with numbers of any size and with various systems, they should not be expected to become proficient with paper-and-pencil computations with several digits. Do not include paper-and-pencil practice for proficiency with tedious computations, such as those with three-digit multipliers or divisors.

NCTM Standards

TIMING

Although the Connector of this lesson may be somewhat longer than in many lessons, it is important to limit the time spent on the entire lesson to about 5 hours.

If students struggle with ideas in this lesson, you might encourage more use of base ten pieces or grid paper for modeling products. Or, you might only assign 1 or 2 problems in each action and come back to other problems at a later time.

JOURNALS

After this lesson is completed, have the students sketch a 4.2×4.2 square and a 3.9×4.5 rectangle and, in their journals, make all the observations they can about mathematical relationships between the two figures. Suggest that students make observations with the intent to reveal as much as possible about their current understanding of decimals, decimal operations, and relationships between area and length.

SELECTED ANSWERS

2. a) $12 \times 50 = 600$
   b) $1.5 \times 6 = 9$
   c) $270 \times 320 = 86{,}400$
   d) $.8 \times 4.9 = 3.92$

3. a) 124
   b) 15
   c) 1180
   d) 11.4

Connector Teacher Activity

OVERVIEW & PURPOSE

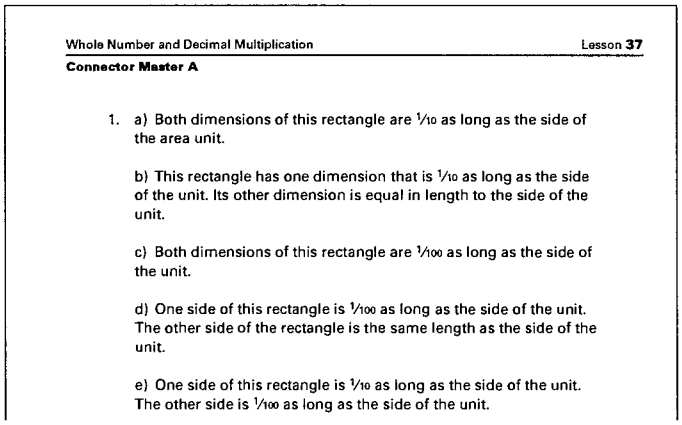
Students explore the effects of multiplying and dividing one or both dimensions of an area unit by powers of ten, thus reinforcing their understanding of relationships between linear and area measures and building intuitions about multiplication by powers of ten. Students also investigate ways to relate their understanding of factors of whole numbers (see Lesson 7) to multiplication of decimals.

MATERIALS

- ✔ Decimal grids version A (see *Blackline Masters*), 2-3 copies per student and 1 transparency.
- ✔ Decimal grids version B (see *Blackline Masters*), 2 sheets per group.
- ✔ Connector Master A, 1 copy per group.
- ✔ Connector Master B (optional, see Comment 6), several copies per class.
- ✔ Connector Student Activity 37.1 (optional, 2 pages), 1 copy per student.
- ✔ Blank Product Chart (see Action 2), 1 per class.
- ✔ Mystery Rectangle (optional, see Comment 6), 1 enlarged copy for display.
- ✔ Scissors and tape, 1 of each per group.

ACTIONS

1 Arrange the students in groups and give each student one decimal grid, version A. Tell the students that for now the large square on the grid is one area unit. Give each group a copy of Connector Master A. Ask the students to each imagine in their mind's eye (no pencils) what the rectangle described in Part 1 a) looks like, including its area and dimensions. Invite volunteers to show their mental images on a transparent decimal grid at the overhead. Repeat for b)-e) on Part 1.



Whole Number and Decimal Multiplication Lesson 37

Connector Master A

1. a) Both dimensions of this rectangle are 1/10 as long as the side of the area unit.

   b) This rectangle has one dimension that is 1/10 as long as the side of the unit. Its other dimension is equal in length to the side of the unit.

   c) Both dimensions of this rectangle are 1/100 as long as the side of the unit.

   d) One side of this rectangle is 1/100 as long as the side of the unit. The other side of the rectangle is the same length as the side of the unit.

   e) One side of this rectangle is 1/10 as long as the side of the unit. The other side is 1/100 as long as the side of the unit.

2 Place a blank "Product Chart" on the classroom wall and invite volunteers to record equations involving multiplication that are represented by the rectangles they discussed in Action 1.

COMMENTS

1 If students have difficulty picturing a rectangle, you might suggest they use a finger to trace (on their decimal grids) an imaginary outline of the rectangle. Encourage students to share methods of counting areas that they feel are especially easy to "see" in their mind's eye.

Following is a description of each rectangle:

a) a square with dimensions $1/10 \times 1/10$ and area $1/100$;

b) a strip made of 10 of the hundredths squares and having dimensions $1/10$ by 1 and area $1/10$;

c) a square with dimensions $1/100 \times 1/100$ and area $1/10{,}000$;

d) a 1 by $1/100$ rectangle made up of 10 of the thousandths strips;

e) a $1/10 \times 1/100$ rectangle with area $1/1000$.

2 A "Product Chart" can be made from a large sheet of butcher paper. Keeping a record of products that students compute throughout Lesson 37 (Connector and Focus) is intended to prompt student observations and

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Product Chart

$.1 \times .1 = .01$
$4 \times .1 = .4$
$.1 \times 1 = .1$
$(2 \times .1) + (2 \times 1) = .2 + 2 = 2.2$
$.01 \times .01 = .0001$
$4 \times .01 = .04$
$1 \times .01 = .01$
$.1 \times .01 = .001$

3 Repeat Action 1 for Part 2 of Connector Master A, but now using the smallest square on the decimal grid as the unit. Have students record products on the Product Chart from Action 2. When Part 2 is completed, repeat for Part 3, using the medium sized square as the area unit.

1. a) Both dimensions of this rectangle are $\frac{1}{10}$ as long as the side of the area unit.

   b) This rectangle has one dimension that is $\frac{1}{10}$ as long as the side of the unit. Its other dimension is equal in length to the side of the unit.

   c) Both dimensions of this rectangle are $\frac{1}{100}$ as long as the side of the unit.

   d) One side of this rectangle is $\frac{1}{100}$ as long as the side of the unit. The other side of the rectangle is the same length as the side of the unit.

   e) One side of this rectangle is $\frac{1}{10}$ as long as the side of the unit. The other side is $\frac{1}{100}$ as long as the side of the unit.

2. a) One side of this rectangle is 100 times as long as the side of the unit. The other side is equal in length to the side of the unit.

   b) Both sides of this rectangle are 10 times as long as the side of the unit.

   c) One side of this rectangle is 10 times as long as the unit. The other side of this rectangle is 100 times as long as the side of the unit.

3. a) One side of this rectangle is $\frac{1}{10}$ as long as the side of the unit and the other side is 10 times as long as the side of the unit.

   b) One side of this rectangle is $\frac{3}{10}$ as long as a side of the unit, and the other side of this rectangle is 10 times as long as a side of the unit.

   c) One side of this rectangle is the $\frac{2}{10}$ as long as the side of the unit. The other side of this rectangle is $\frac{5}{10}$ as long as the side of the unit.

COMMENTS

2 (continued.)
conjectures later. Shown at the left are examples of equations students may record from Action 1.

Note that students may also suggest using the commutative property (see Lesson 3) to produce additional products. Encourage this. You may wish to keep a marking pen attached to the chart and invite students to freely record products they find throughout the lesson. Hence, it is possible this chart will become quite long by the end of this lesson.

3 As students gain comfort with imagining these rectangles, less large group discussion may be necessary. Here are some equations that students may record for each:

2. a) $100 \times 1 = 100$;

   b) $10 \times 10 = 100$;

   c) $10 \times 100 = 1000$.

3. a) $.1 \times 10 = 1$;

   b) $.3 \times 10 = 3$;

   c) $.2 \times .5 = .1$.

Students may make conjectures about relationships they notice. Keep such discussion informal and based on visual relationships. If comments about adding zeroes or moving decimals come up, acknowledge them with comments such as, "I wonder why that works," or "I wonder how we could be sure that always works." It isn't necessary to establish rules. In fact, it is preferable to keep emphasis on developing intuitions. Once students know rules they are often reluctant to explore conceptual relationships that lead to understanding and the development of mental computation strategies.

Connector Teacher Activity (cont.)

ACTIONS

4 Give each group 2 sheets of decimal grids, version B, a pair of scissors, and a roll of tape. Ask the groups to form several noncongruent rectangles with area 1 square unit, assuming the large square on the decimal grids, version B, is the area unit and that rectangles are formed by cutting or drawing on grid lines. Have groups record the dimensions, perimeter, and area of each rectangle they form, and one or more multiplication equations associated with each rectangle. Discuss their observations and write equations on the Product Chart from Action 2.

5 Repeat Action 4 by having groups sketch noncongruent rectangles with area .45 square unit. Discuss.

COMMENTS

4 After groups have found several rectangles, you might encourage observations by asking groups to write 2 or 3 "we notice" statements regarding mathematical relationships they notice. Some groups may be surprised to note that perimeter can change when area remains constant. Others may note strategies like "we could always form new rectangles with area 1 by cutting one dimension in half and doubling the other dimension." Still others may simply note, "the product of the dimensions is always the area."

Here are the rectangles with area 1 unit that can be formed by cutting on the grid lines of the decimal grid, and one possible multiplication equation associated with each rectangle.

Dimensions	Perimeter	Area	Multiplication Equations
1 linear unit × 1 linear unit	4 linear units	1 square unit	$1 \times 1 = 1$
$.5 \times 2$	5	1	$.5 \times 2 = 1$
$.2 \times 5$	10.4	1	$.2 \times 5 = 1$
$.25 \times 4$	8.5	1	$.25 \times 4 = 1$
$.04 \times 25$	50.08	1	$.04 \times 25 = 1$
$.4 \times 2.5$	5.8	1	$.4 \times 2.5 = 1$
$.1 \times 10$	20.2	1	$.1 \times 10 = 1$
$.05 \times 20$	40.1	1	$.05 \times 20 = 1$
$.02 \times 50$	100.04	1	$.02 \times 50 = 1$
$.01 \times 100$	200.02	1	$.01 \times 100 = 1$

5 Some students may begin by identifying all the rectangles that contain 45 of the hundredths squares (using no partial squares). This is equivalent to finding all the factors of the whole number 45 (1, 45, 3, 15, 5, and 9) and then "changing the unit" to identify pairs of numbers whose products are .45. (This method suggests the common practice of multiplying two decimal numbers by disregarding the decimal points, multiplying them as though they were whole numbers, and then placing the decimal point.) This process produces the following rectangles: $.3 \times 1.5$; $.9 \times .5$; and $.1 \times 4.5$, which can then be subdivided and rearranged to form others, such as: $.15 \times 3$; $.25 \times 1.8$; $1 \times .45$; etc.

Connector Teacher Activity (cont.)

ACTIONS

6 (Optional) Distribute 1 copy of Connector Student Activity 37.1 (2 pages) to each student and display an enlarged copy of the Mystery Rectangle from page 2 on a classroom wall. Describe the activity, making sure that students understand how to use ordered pairs to identify the small squares. Ask the groups to discuss and determine what possibilities are suggested by the clue, "Tile B has area .56 square units."

Connector Student Activity 37.1

NAME _____ DATE _____

1 Your teacher has a Mystery Rectangle with the same area and dimensions as the 1.0 x 5.0 rectangle shown on the next page. However, your teacher's rectangle is covered (with no gaps or overlaps) by 21 noncongruent rectangular tiles. Each tile is bordered by grid lines and each tile is covered by a rectangular array of letters. Notice that Tile A has dimensions .4 linear unit by .2 linear unit, area .08 square unit, and it is covered by $4 \times 2 = 8$ A's. Below are a few clues about the Mystery Rectangle.

- Square (34, 5) contains an R, and Square (8, 10) contains a B.
- Tile B has area .56 square unit. It touches Tiles O, D, I, and J.
- The area of Tile C is double the area of Tile L.
- The area of Tile D is .24 square unit and its perimeter is 2.8 linear units.
- The area of Tile E is .24 square unit and one dimension is .8 linear unit.
- Tile F is not a square, but the number of F's that cover it is a square number.
- Tile G has area .45 square unit.
- The number of H's that cover Tile H is both prime and even.
- The area of Tile I is .22 square unit.
- Tile J has area .30 square unit.
- The area of Tile K is .63 square unit.
- The area of Tile L is .18 square unit.
- Tile M is a square.
- Tile N has area .15 square unit and perimeter 3.2 linear units.
- Tile O has area .36 square unit.

(Continued on back.)

COMMENTS

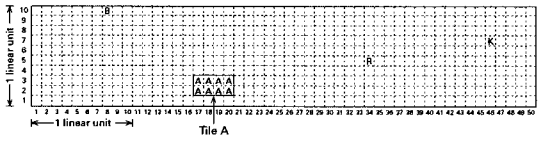
6 This activity reinforces the area concept of multiplication, provides practice with basic multiplication facts, and builds understanding of the role of basic facts in computing decimal products.

An enlarged Mystery Rectangle can be made by taping together pieces of 1-inch grid paper and sketching the rectangle shown on page 2 of Student Activity 37.1. The numbers along the base and side of the rectangle identify the columns and rows (not the grid lines). Hence, to locate square (8,10), beginning at the bottom left corner of the rectangle, count 8 spaces along the horizontal and then count up 10 spaces.

To generate enthusiasm, you might offer a prize to the group that first submits the correct tiling of the Mystery Rectangle, together with a correct listing of the dimensions, area, and perimeter of each rectangular tile. (You may wish to discourage random guessing by suggesting that two or three incorrect predictions of the tiling disqualify a group from winning).

- The area of Tile P is .12 square unit.
- A prime number of Q's cover Tile Q.
- The number of R's that cover Tile R is neither a prime nor a composite number.
- The perimeter of Tile S is 1 linear unit and its area is .04 square unit.
- The perimeter of Tile T is 1.6 linear units and one dimension is .3 linear unit.
- Tile U is a square. Its area is .11 square unit more than the area of Tile M.

Mystery Rectangle



The class task (which may take several days) is to determine, using as few additional clues as possible, what each rectangular tile looks like and where it is placed on the Mystery Rectangle. On another sheet of paper (or in your journal), write other information and possibilities that you think are revealed by the above clues.

2 Tomorrow your group will complete a Clue Request Card with 3 questions like the ones shown below. What do you think your group should ask and why?

a) Is square _____ part of Rectangle ____? I think it could be because:

b) Is square _____ part of Rectangle ____? I think it could be because:

c) Is square _____ part of Rectangle ____? I think it could be because:

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

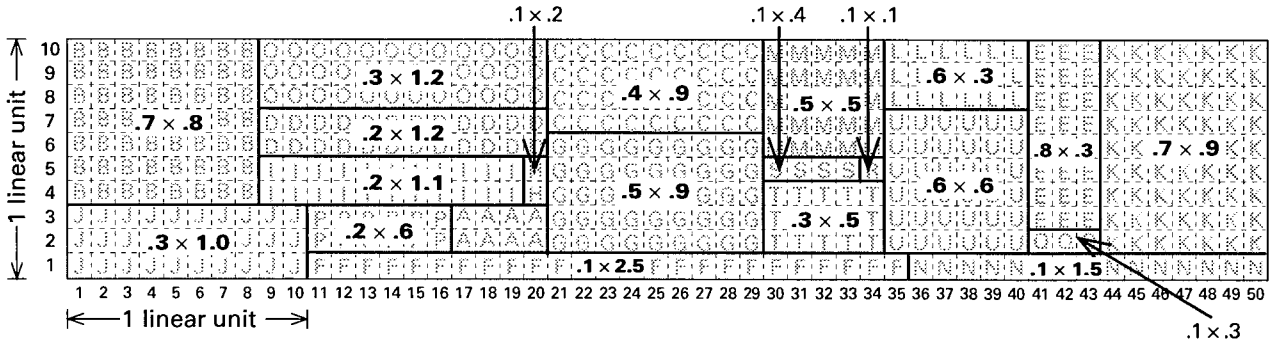
If you teach more than one math class, a butcher paper poster for each class can be posted and taken down at the beginning and end of each class period. Or, the activity will take less time if each class is allowed to use other classes' clues.

Rather than spending extended class time working on this activity, introduce the activity at the end of a class period and suggest that it will extend over several days as follows:

a) Each evening individual students determine possibilities suggested by each clue on Student Activity 37.1 and by other clues revealed in class. Each student brings to class their ideas about what clues they want their group to request and why.

b) At the beginning of each class period, give each group one Clue Request Card (cut from Connector Master B) and have them spend 5 minutes deciding what clues to request. Collect these. (Note that each group receives 1 Clue Request Card per day, so the number of cards needed depends on the class size and the time it takes for students to solve the problem.) Read aloud the clue requests and respond with a YES or NO. Encourage students to record these clues.

c) Later, post the Clue Request Cards (with a YES or NO written next to each clue request) on the wall and, for each YES, record the corresponding letter on the Mystery Rectangle. Groups may study and copy this information on their own time, before or after class each day or during the 5 minutes at the start of class. Shown below is the completed Mystery Rectangle to use as a basis for responding to the requests for clues.



TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use the area model for multiplication as a basis for inventing strategies for computing decimal products. Emphasis is placed on the development of meaningful strategies that later can be extended for use with fractions and algebra.

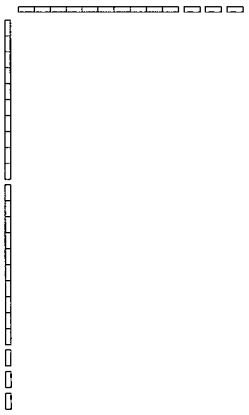
MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Student Activity 37.2, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 37.3, 1 copy per group and 1 transparency.
- ✔ Scissors, 1 per group.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Blank transparencies and overhead pens, 1 of each per group.

ACTIONS

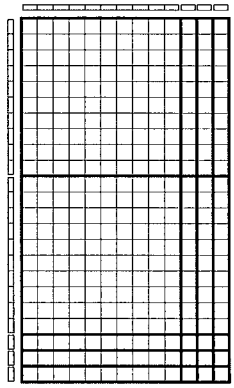
1 Arrange the students in groups and distribute base ten area and linear pieces to each student. Project a copy of the following arrangement of linear pieces on the overhead screen. Ask the students to form the same arrangement with their linear pieces.

Tell the students that these linear pieces represent the dimensions of a rectangle. Have the students privately imagine the area pieces that would "fill" the rectangle. Then ask them to verify their predictions by using their area pieces to fill in the rectangle, and to use the pieces to determine the area of the rectangle. Discuss their methods of counting the area pieces and their ideas about equations involving multiplication that could represent this rectangle. Record the equations on the Product Chart started in the Connector.



COMMENTS

1 Here is the completed rectangle:



If students ask you to identify the unit of area, suggest they select a unit or consider more than one possibility. For example, if the large area piece square is 1 area unit, then the rectangle has area 2.99 and represents 2.3×1.3 or 1.3×2.3. If the small area piece square is 1 area unit, then the rectangle has area 299 and represents 23×13 or 13×23.

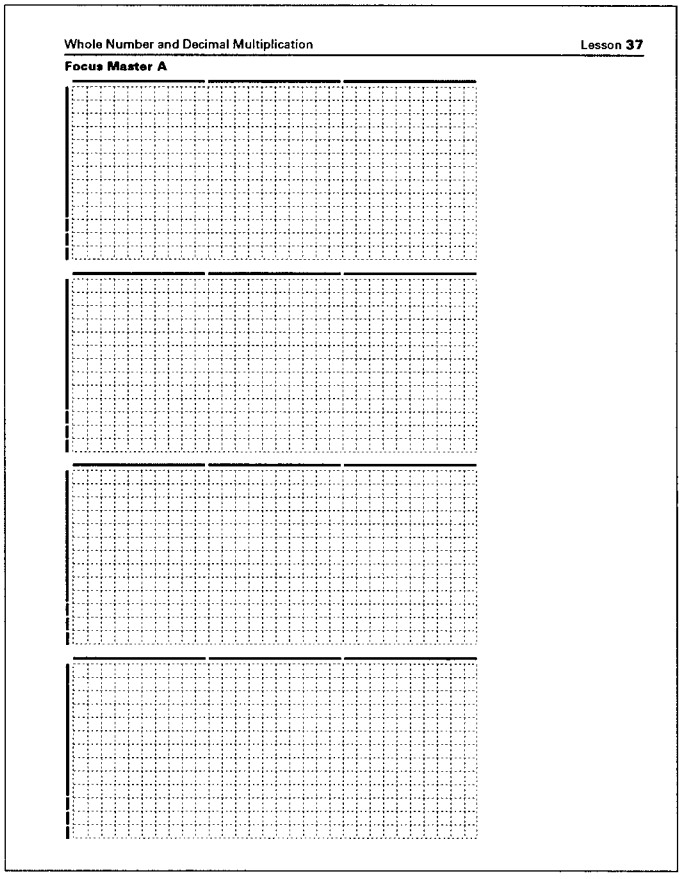
Some students may suggest products such as $23 \times (10 + 3)$, $(20 + 3) \times 13$, $2.3 \times (1 + .3)$, etc. Such products involve the distributive property (see Lesson 3) and are important to discuss (include these on the Product Chart).

Focus Teacher Activity (cont.)

ACTIONS

2 Give each group one copy of Focus Master A and tell them the length of the longer linear piece is 1 linear unit. Ask the groups to determine four different ways of viewing and counting the total area of the 1.3×3 rectangle, other than counting the pieces one by one. Have the students draw on the rectangles to illustrate their different methods of counting and ask them to write number statements that reflect each method.

Ask for volunteers to show their methods at the overhead on a transparency of Focus Master A. Have them point out methods they feel are easier to "see" mentally. Discuss students' ideas about other products represented by the rectangle on Master A (i.e., by changing the unit). Record all products discussed on the Product Chart.

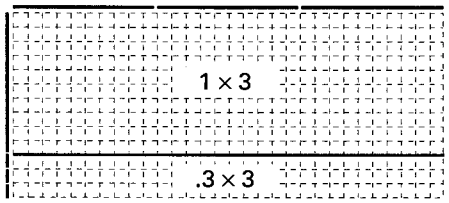

Whole Number and Decimal Multiplication Lesson 37

Focus Master A

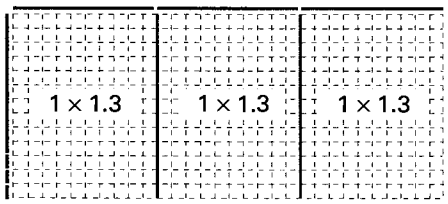
COMMENTS

2 Throughout this activity, it is assumed that the area unit is a square whose side has the length of the linear unit.

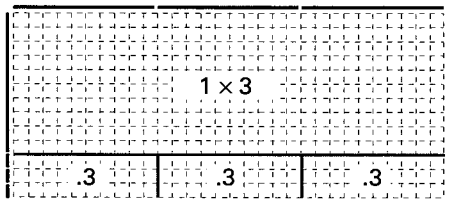
There are many ways that students may view the area of this rectangle (i.e., the product 3×1.3). Here are a few examples:



$1.3 \times 3 = (1 \times 3) + (.3 \times 3) = 3.9$



$1.3 \times 3 = (1 \times 1.3) + (1 \times 1.3) + (1 \times 1.3) = 3.9$



$1.3 \times 3 = 3 + (3 \times .3)$
$= 3.9$

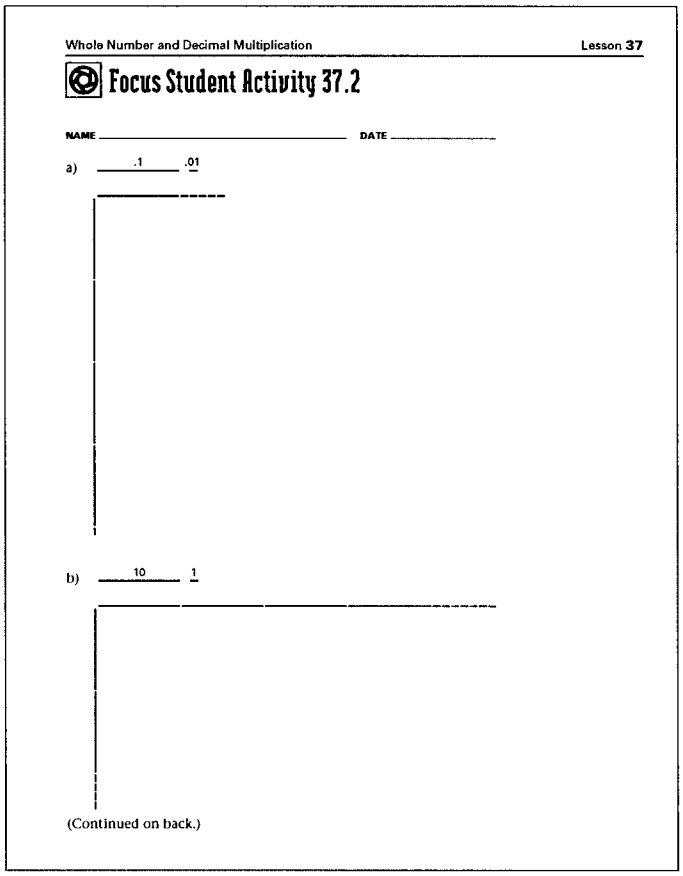
Note that in this lesson emphasis is placed on use of the area model for multiplication. This model is especially worthwhile to explore because it has extensions to fraction and integer multiplication, algebra, and beyond. In addition the area model provides a method of mentally "picturing" products that simplifies computing. For a situation which involves viewing a product as a repeated sum, a rectangular area model of the product also shows that sum. For example, in the second example above 1.3×3 is viewed as 3 groups of 1.3.

Although this activity provides an opportunity to reinforce students' understanding of the use of numerical statements to represent one's thoughts or actions (see Lesson 3), the main purpose here is to develop a variety of visual strategies for computing products.

Focus Teacher Activity (cont.)

ACTIONS

3 Give each student a copy of Focus Student Activity 37.2 and tell them the linear pieces shown represent the dimensions of rectangles. Ask the students to compute the areas of rectangles a)-d) by making a rough sketch of each rectangle, subdividing each rectangle to represent the way they "see" the area pieces that would cover it, and then adding the areas of the subregions. Have students also write number statements that reflect their thought processes. Ask for volunteers to show their ideas at the overhead. As number statements are given, record them on the Product Chart.


Whole Number and Decimal Multiplication Lesson 37

Focus Student Activity 37.2

NAME _____ DATE _____

a) .1 .01

b) 10 1

(Continued on back.)

d)

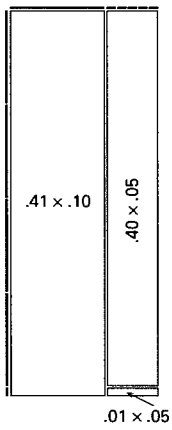
10	1	.3
4	.4	.12

$1.4 \times 11.3 = (1 \times 10) + (1 \times 1) + (1 \times .3) + (.4 \times 10) + (.4 \times 1) + (.4 \times .3)$
$= 15.82$

COMMENTS

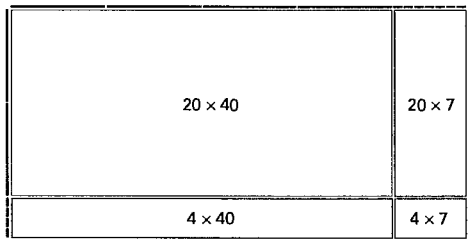
3 Keep emphasis on imagining subregions whose areas are easier for students to compute, but allow students who need more concrete experiences to use base ten pieces or grid paper. (Note that students' number statements will not necessarily be as detailed as the ones shown below.)

a)



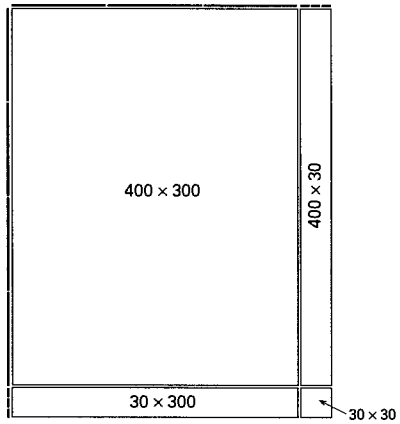
$.41 \times .15 = (.41 \times .10) + (.40 \times .05) + (.01 \times .05)$
$= .041 + .020 + .0005$
$= .0615$

b)



$24 \times 47 = (20 \times 40) + (20 \times 7) + (4 \times 40) + (4 \times 7)$
$= 800 + 140 + 160 + 28$
$= 1128$

c)



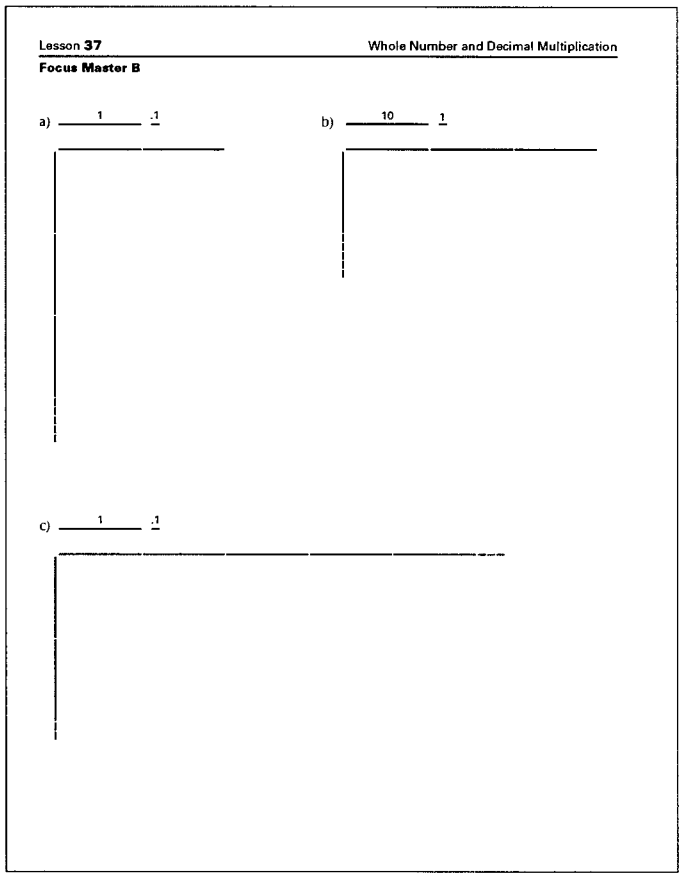
$430 \times 330 = (400 + 30) \times (300 + 30)$
$= (400 \times 300) + (400 \times 30) + (30 \times 300) + (30 \times 30)$
$= 120{,}000 + 12{,}000 + 9{,}000 + 900$
$= 141{,}900$

Focus Teacher Activity (cont.)

ACTIONS

4 Place a transparency of Focus Master B on the overhead, revealing a) only. Ask the students to imagine in their mind's eye the area pieces that cover the rectangle whose dimensions are shown in a) and to mentally count those area pieces to determine the area of the rectangle. Have volunteers illustrate their mental pictures by sketching each subdivision they "saw" in the rectangle. Discuss number statements that could represent each way of viewing 3.5×2.

Repeat for b) and/or c). Invite volunteers to record equations that illustrate their methods on the Product Chart.

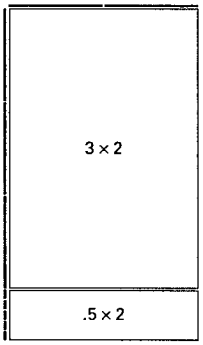


COMMENTS

4 If some students have difficulty imagining the pieces, encourage them to use their area pieces to form the rectangles, or base ten grid paper (see *Blackline Masters*) to draw the rectangles, or give them a copy of Focus Master B to sketch on. The intent here is to move those students who are ready to the use of "mental sketches."

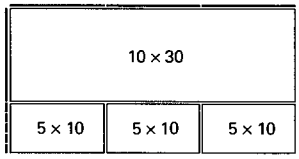
Note that the unit changes on these rectangles, and you may wish to draw students' attention to this. There are many ways of viewing each area. Students may imagine individual pieces or they may "see" regions formed by groups of pieces. Following is one possibility for viewing each of the products represented on Focus Master B. Encourage students to discuss mental strategies they use to add the areas of the subregions (see Lesson 31).

a)



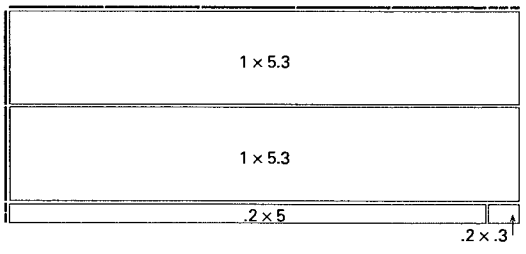
$3.5 \times 2 = (3 \times 2) + (.5 \times 2) = 7$

b)



$15 \times 30 = (10 \times 30) + ((5 \times 10) \times 3)$
$= 300 + (50 \times 3)$
$= 450$

c)



$2.2 \times 5.3 = 5.3 + 5.3 + (.2 \times 5) + (.2 \times .3)$
$= 11.66$

Focus Teacher Activity (cont.)

ACTIONS

5 Select one or more of the following multiplications for the students to solve by sketching (or imagining) a rectangle and counting its area. Have students record the corresponding equations on the Product Chart and encourage students to make observations and conjectures about relationships they notice on the chart.

a) 45×27

b) 15.2×3

c) 310×460

d) $.1 \times .11$

e) $.02 \times .03$

f) 1.4×2.5

g) 1234×243

h) $.46 \times .95$

i) 105×202

j) 2.9×5

k) 10×5.4

l) 500×60

m) 6.7×1000

n) $.01 \times 72$

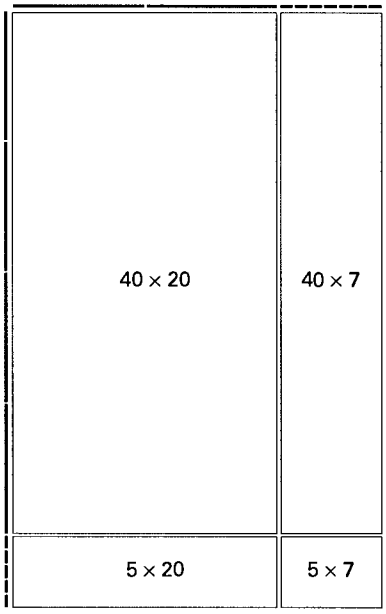
COMMENTS

5 As mentioned earlier, provide base ten pieces or grid paper for students who need them. Groups could explore and sketch their ideas for one or more of these problems on a sheet of poster paper. Select problems according to the needs and interest level of your students. For example, if you think students are ready for additional discussion about the effect of multiplying by powers of 10, explore problems d), k), l), m), and n).

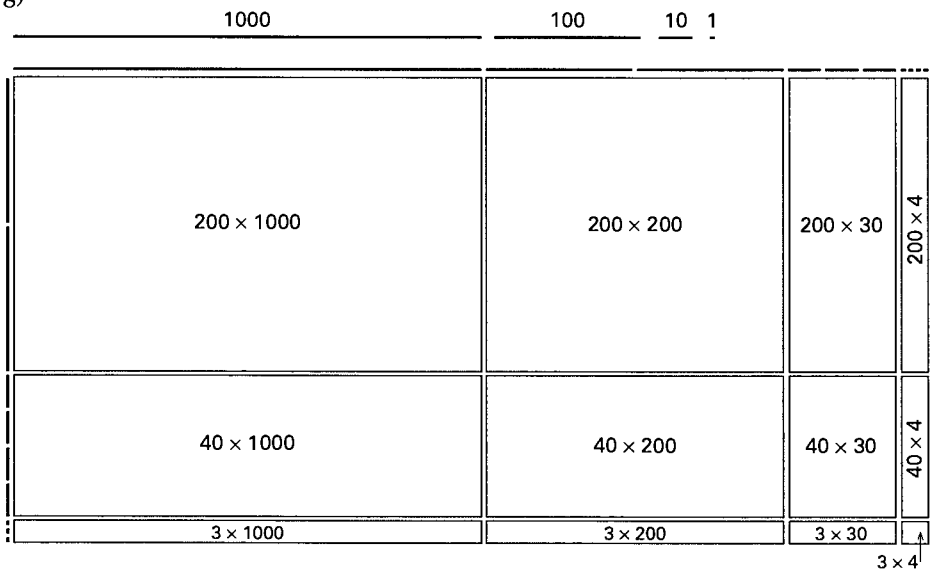
Note that the products involving numbers with 3 and 4 digits are posed here so that students can explore ways the area model extends to products involving any sized numbers. However, it is important to stress the fact that tedious computations are more appropriately carried out on the calculator.

Below are examples that illustrate how a sketch could be used to compute products a), g), i), and j) (drawings are not to scale).

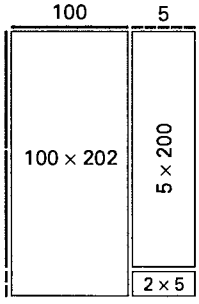
a)



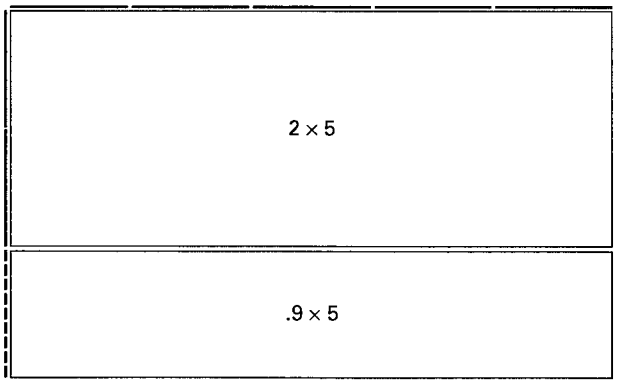
g)



i)



j)



Focus Teacher Activity (cont.)

ACTIONS

6 Write the problem 2.6×3.4 on the overhead and ask each student to each compute the product by sketching a diagram. Then have each group do the following:

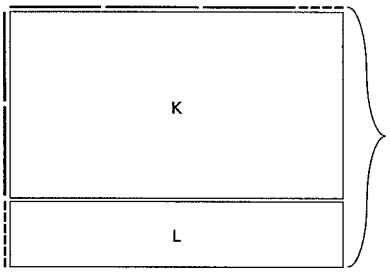
a) Each student shares their methods with their groupmates.

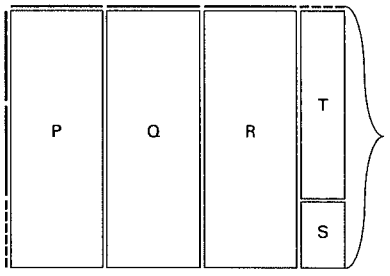
b) The groups each select *one* of the visual methods used in their group and devise a way of using numbers and arithmetic symbols only (no words or diagrams) to communicate that method so someone reading their symbolic statements could reproduce the students' visual procedures.

c) Using a blank transparency and an overhead pen, each group records only their symbolic statements.

d) Volunteers place their group's statements on the overhead; before the volunteers explain the thinking associated with their statements, other groups speculate about the visual methods the statements represent. Examine and discuss several recordings.

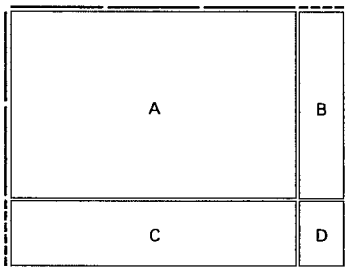
Discuss the students' ideas about whether each visual method and recording system generalizes (i.e., do they work on all multiplication problems).





COMMENTS

6 Circulate as individuals share their methods within their groups. If needed to promote variety, prompt groups to record specific methods you hear mentioned. Here are 3 possibilities for determining the area of a 2.6×3.4 rectangle:



In this example, the rectangle is divided into 4 rectangular regions. The area of the original rectangle is obtained by adding the areas of rectangles A, B, C, and D (sometimes referred to as *partial products*):

$$\begin{array}{lrcr} A & 2.0 \times 3.0 & = & 6.00 \\ B & 2.0 \times .4 & = & .80 \\ C & .6 \times 3.0 & = & 1.80 \\ D & .6 \times .4 & = & .24 \\ & & & \overline{8.84} \end{array}$$

Alternatively, the rectangle could be divided into 2 regions K and L, as shown at the left, and its area obtained by adding these partial products:

$$\begin{array}{lrcrcr} K & 2.0 \times 3.4 & = & (2.0 \times 3.0) + (2.0 \times .4) & = & 6.80 \\ L & .6 \times 3.4 & = & (.6 \times 3.0) + (.6 \times .4) & = & \underline{2.04} \\ & & & & & 8.84 \end{array}$$

A third method (shown at the left) is to subdivide the rectangle into three 1.0×2.6 rectangles, a $.4 \times 2.0$ rectangle, and a $.4 \times .6$ rectangle to get the following partial products:

$$\begin{array}{lrcr} P & 1.0 \times 2.6 & = & 2.60 \\ Q & 1.0 \times 2.6 & = & 2.60 \\ R & 1.0 \times 2.6 & = & 2.60 \\ S & .4 \times .6 & = & .24 \\ T & 2.0 \times .4 & = & .80 \\ & & & \overline{8.84} \end{array}$$

It is important to emphasize that the visual procedures students have developed are valid algorithms for computing products. The usual procedure presented in textbooks represents one way of viewing the area of a rectangle, but it is generally memorized without regard to its visual representation. Hence, unless students are

Focus Teacher Activity (cont.)

ACTIONS

7 Give each group one copy of Focus Student Activity 37.3 to complete. Discuss.

Whole Number and Decimal Multiplication Lesson **37**

Focus Student Activity 37.3

NAME _____ DATE _____

1 For each of a)-d) below, write an interesting word problem so that the solution to each problem involves multiplying 2 of the numbers listed below. You may only use a number once.

1.3, 210, .3999, 5.3, 16.7, 10.2, 25, 10

a) In your opinion, mental multiplication would be most appropriate for solving this word problem:

b) You think the calculator would be the most appropriate method for multiplying the numbers in this word problem:

c) You feel estimation would be the most appropriate method for finding the answer to this word problem:

d) A paper-and-pencil method (such as a diagram or a numerical representation of a diagram) would be your preferred method for computing the product in this problem:

(Continued on back.)

COMMENTS

curious, it isn't necessary to "show" the standard algorithm. (Note the standard textbook algorithm can be related to the method used in the first example above.) It is important that students calculate products by using the methods that make sense to them or that best fit a particular problem.

7 It is important that students view mental, paper and pencil, calculator, and estimation strategies as *options* for computing. Certain options may be viewed as more practical in some situations than others and some students may view a computation as reasonable for mental calculation while others may feel the same computation is more reasonably computed by paper and pencil or calculator. The purpose of this action is to encourage students to talk about the reasoning behind their choice of options. Note that in *Visual Mathematics, Course II* emphasis is placed on developing strategies for mentally computing and estimating products.

Lesson **37** Whole Number and Decimal Multiplication

Focus Student Activity 37.3 (cont.)

2 Write 3 computations for each of the following (using any numbers you wish, as long as each number has *at least* 2 digits):

a) You prefer to solve these 3 multiplications using a calculator:

b) You think that using a diagram or numerical representation of a diagram would be reasonable for these 3 multiplications:

c) You think these 3 multiplications are especially appropriate for solving mentally:

3 List 3 reasons you think it is important to know how to estimate products:

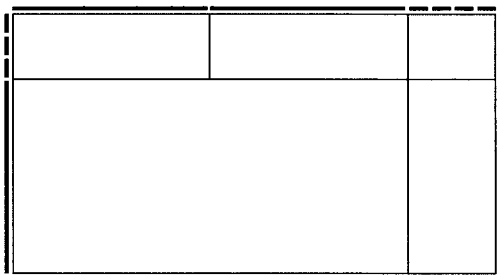
4 Explain how you would determine a reasonable estimate of the product .3792 x 73.0099 (without computing the product first).

TEACHER NOTES:

Follow-up Student Activity 37.4

NAME _____ **DATE** _____

1 Simon said that he used the following diagram to compute 13×24. Write an equation that describes what you think might have been his thought processes.



2 Each arrangement of linear pieces below represents the dimensions of a rectangle. Imagine covering each rectangle with area pieces and mentally adding the areas of those pieces to find the area of the rectangle. Then sketch the rectangle to show how you counted the area and write equations to describe your thought processes.

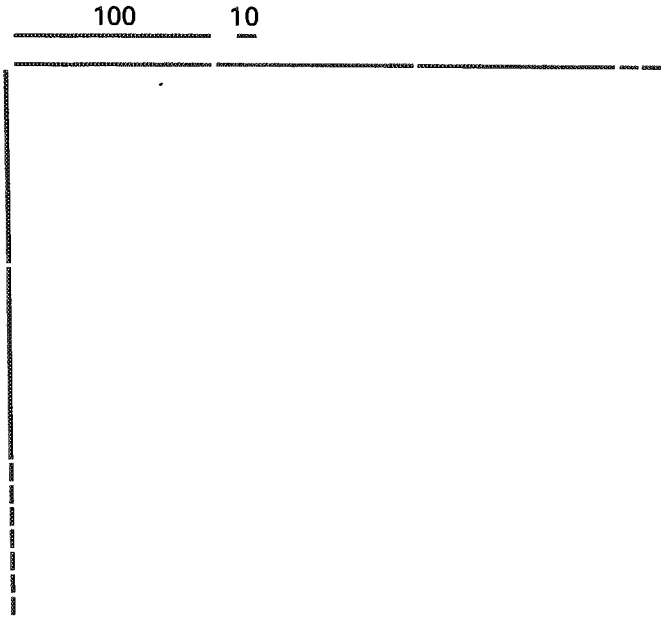
a) 10 1

b) 1 .1

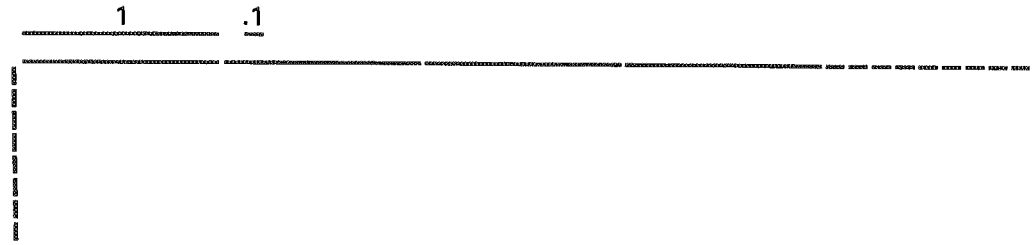
(Continued on back.)

Follow-up Student Activity (cont.)

c)



d)



3 Determine and record the perimeter of rectangles a)-d) in Problem 2. Explain the method you used to find the perimeter of d).

4 Reggie plans to cover the floor of her room with carpet. The dimensions of the room are 3.6 meters by 4.2 meters. On another sheet of paper, write Reggie a letter that tells her how much carpet to buy. Provide a convincing explanation of why that is the right amount. Include diagrams to support your explanation.

5 Each of the following is the answer to a multiplication problem. For each answer in a)-d), on the attached grid paper write an interesting word problem that involves multiplying to find that answer. Show diagrams that illustrate how to find the answer to each of your problems.

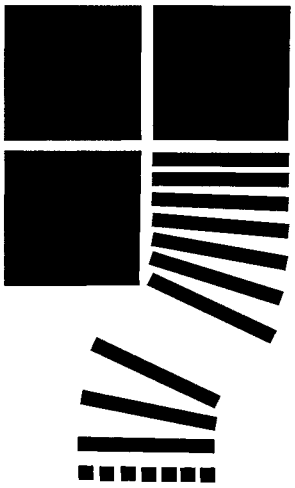
a) .48 b) 38000 c) 5.4 d) 10.8

Whole Number & Decimal Division

Lesson 38

THE BIG IDEA

Using base ten pieces to explore the concept of division can enhance students' understanding of place value and number sense; promote insights regarding relationships between area and length; and enable students to model situations involving division and solve problems about those situations.



CONNECTOR

OVERVIEW

Students use base ten pieces to explore different interpretations of the meaning of 782 ÷ 34. They write word problems based on these interpretations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces for each student.
- ✔ Blank transparencies, one 8½" × 3" strip of transparency per group.
- ✔ Overhead pens, 1 per group.
- ✔ Base ten area and linear pieces for the overhead.

FOCUS

OVERVIEW

Students use base ten area and linear pieces to explore the meanings and methods of division. These experiences enable students to model and draw conclusions about situations involving division.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area and linear pieces, including toothpick pieces, for each student.
- ✔ Base ten grid paper, version A, 1-3 sheets per student and 1 transparency.
- ✔ Butcher paper (optional), 1 sheet per group.
- ✔ Marking pens (optional), 1 or more per group.
- ✔ Focus Master A, 1 transparency.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Blank Quotient Chart (see Action 3), 1 per class.

FOLLOW-UP

OVERVIEW

Students use base ten area and linear pieces to determine quotients. They demonstrate their understandings of decimal and whole number multiplication and division to an adult.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 38.1, 1 copy per student.
- ✔ Base ten area and linear pieces, 1 set per student.
- ✔ Base ten grid paper, version A, 2 sheets per student.

LESSON IDEAS

ASSESSMENT

Have students review their Work-folios and journals (see *Starting Points*) and look for evidence of growth in their understanding of decimals and decimal operations. One way to do this is to have them select two examples (a "before" and an "after") to illustrate their growth. Have them write an explanation of their rationale for selecting the papers.

LOOKING AHEAD

Throughout this lesson emphasis is placed on developing students' understanding of the meaning of division. In *Visual Mathematics, Course II* emphasis shifts to using these conceptual models as a basis for estimation and mental strategies of division.

FOLLOW-UP

It may be helpful to remind students that you want them to use the base ten pieces to find the answers, not to compute (e.g., with a calculator) the answers and then draw a diagram of the result.

You may wish to allow some class time for brainstorming and organizing ideas for students' presentations (see Problem 4).

QUOTE

It is no longer necessary or useful to devote large portions of instructional time to performing routine computations by hand... Students need more experiences in developing procedures and evaluating their work and in interpreting the results of computations done by machines.

NCTM Standards

SELECTED ANSWERS

1. a) There are many possibilities. For example, students may describe how they formed a rectangle with area 4.8 square units and one dimension 1.2 and then determined the other dimension, $4.8 \div 1.2 = 4$. Or, some students may explain their methods of finding the number of groups of 1 unit and 2 tenths in 4 units and 8 tenths.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students use base ten pieces to explore different interpretations of the meaning of 782 ÷ 34. They write word problems based on these interpretations.

MATERIALS

- ✔ Base ten area and linear pieces for each student.
- ✔ Blank transparencies, one 8½" × 3" strip of transparency per group.
- ✔ Overhead pens, 1 per group.
- ✔ Base ten area and linear pieces for the overhead.

ACTIONS

1 Arrange the students in groups and give each student a set of base ten area and linear pieces. Write 782 ÷ 34 on the overhead and ask the groups to devise methods of using their area and/or linear pieces, combined with their understanding of the meaning of division, to determine this quotient.

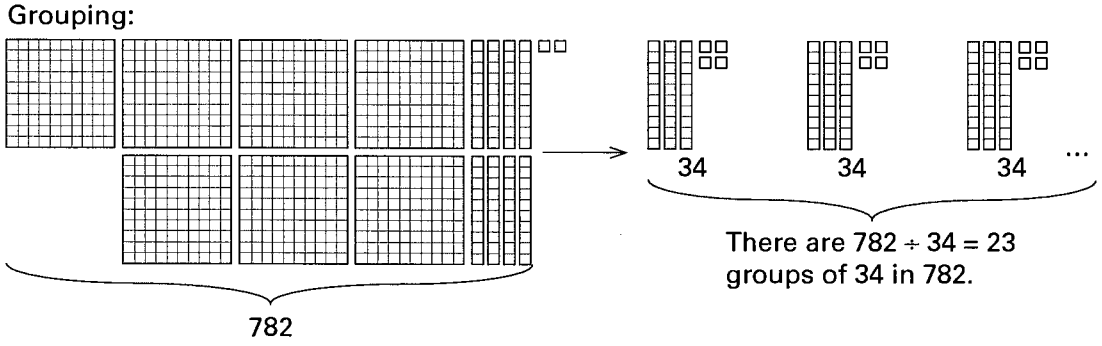
Have volunteers illustrate their methods and reasoning at the overhead. Ask them to show how they see the numbers 782 and 34 in their models, and to relate the quotient 782 ÷ 34 to the models.

COMMENTS

1 It is important to emphasize *using the number pieces, or sketches of pieces, to determine the quotient,* rather than finding the quotient (e.g., using the calculator) and then representing the results with the pieces. The intent here is to develop insights and intuitions about the concept of division rather than to focus on the answers.

Students will probably suggest a variety of methods involving the grouping, sharing, and/or area concepts of division (see Lesson 2). Following are some possible lines of reasoning about 782 ÷ 34 (if these don't all come up, note in Action 2 there are opportunities to bring them up).

Based on the grouping (subtractive) method of division, 782 ÷ 34 is viewed as: arranging a collection of number pieces with the value 782 in as many groups of value 34 as possible; there are 782 ÷ 34 groups. Students may begin with a collection of 7 hundreds, 8 tens, and 2 units and then form as many groups of 3 tens and 4 units as possible, exchanging pieces as needed (it is also possible to begin by exchanging all pieces for units and then form groups of 34 units) until all the pieces are used.



(Continued next page.)

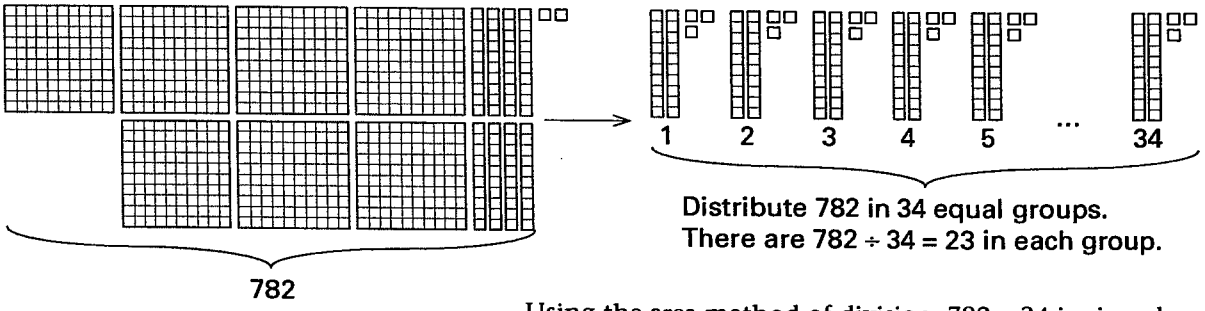
Connector Teacher Activity (cont.)

ACTIONS

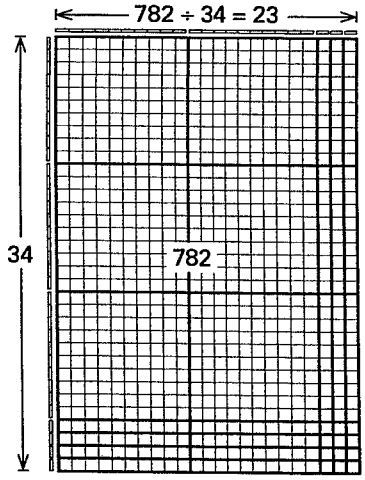
COMMENTS

1 (continued.)
Using the sharing (partitive) method of division, 782 ÷ 34 is viewed as follows: arrange a collection whose value is 782 in 34 equal groups; the value of each group is 782 ÷ 34.

Sharing:



Using the area method of division, 782 ÷ 34 is viewed as follows: if the area of a rectangle is 782 square units and one dimension is 34 linear units, then the other dimension is 782 ÷ 34 linear units. An area consisting of 7 hundreds, 8 tens, and 2 units can be arranged to form a rectangle with one dimension 34 linear units if 1 hundred is exchanged for 10 tens and 1 ten is exchanged for 10 units, as illustrated below.



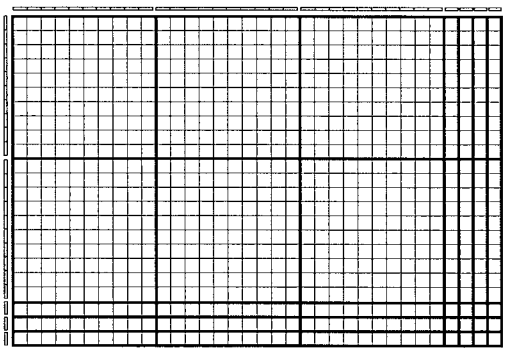
Note that the rectangle shown above also reveals there are 782 ÷ 34 = 23 groups of 34 units in 782 (each column is one group of 34). Similarly, one can see there are 34 groups of 782 ÷ 34 = 23 units in 782 (each row is one of the 34 groups).

Connector Teacher Activity (cont.)

ACTIONS

2 If the grouping, sharing, and area methods weren't each suggested by students in Action 1, ask the students to try them now. Discuss.

3 Use area and linear pieces to form the rectangle shown below on the overhead and ask the groups to each create a word problem involving division that could be modeled by this rectangle. Give one $8\frac{1}{2}" \times 3"$ strip of transparency and an overhead pen to each group and ask them to record their problems. Place these on the overhead and ask the students to classify the problems according to: grouping method, sharing method, or area method. If needed, ask the groups to write other problems to specifically fit one or more of these classifications.



COMMENTS

2 See Comment 1 for an illustration of $782 \div 34$ using these methods.

The area method of division is particularly useful since it has applications that extend to algebra and beyond. Further, both the grouping and sharing interpretation of division are "visible" in an area model, and one rectangle can represent several quotients, depending on the choice of unit. For example, the rectangle shown in Comment 1 could also represent $7.82 \div 3.4$, if the 10×10 base ten piece is the unit of area and its side is the unit of length.

3 Students may have difficulty remembering the methods by name (it is easy to mix up grouping and sharing). Hence, you might write a description of each method on the board or a sheet of butcher paper for reference by students.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use base ten area and linear pieces to explore the meanings and methods of division. These experiences enable students to model and draw conclusions about situations involving division.

MATERIALS

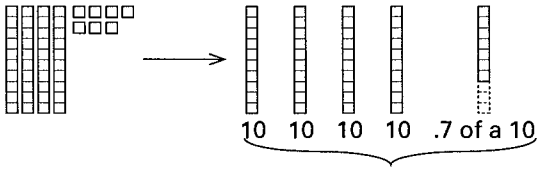
- ✔ Base ten area and linear pieces, including toothpick pieces, for each student.
- ✔ Base ten grid paper, version A, 1-3 sheets per student and 1 transparency.
- ✔ Butcher paper (optional), 1 sheet per group.
- ✔ Marking pens (optional), 1 or more per group.
- ✔ Focus Master A, 1 transparency.
- ✔ Base ten area and linear pieces for the overhead.
- ✔ Blank Quotient Chart (see Action 3), 1 per class.

ACTIONS

1 Arrange the students in groups and give base ten area and linear pieces to each student. Write the problem 47 ÷ 10 on the overhead and ask the students to devise ways to use their base ten pieces to compute the quotient. As students share their methods, have them point out how they see the numbers 47, 10, and 47 ÷ 10 in their model.

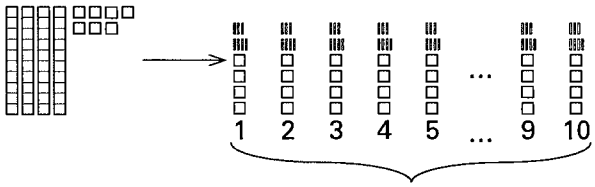
COMMENTS

1 One method (grouping) of using the pieces to compute 47 ÷ 10 is to arrange a collection of 4 tens and 7 units in groups of 10, as shown below:



There are 4 groups of 10 and .7 of a 10 in 47 (i.e., there are 47 ÷ 10 = 4.7 tens in 47).

Another method (sharing) is to arrange a collection of 47 units in 10 equal groups, first by distributing 4 units to each group, then by exchanging the leftover 7 units for 70 tenths and distributing 7 tenths to each group, as shown below:



There are 10 equal groups of 47 ÷ 10 = 4.7

One way of using the area method to arrange a collection of 47 units in a rectangle with one dimension 10 is shown on the next page.

(Continued next page.)

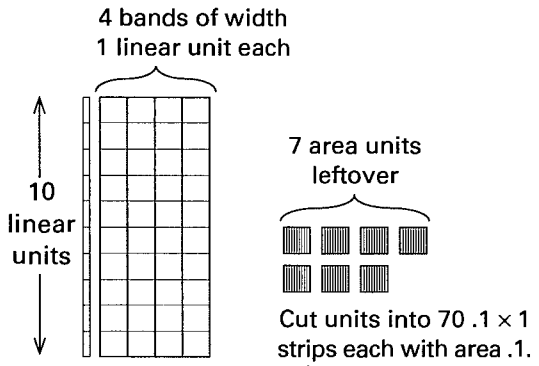
Focus Teacher Activity (cont.)

ACTIONS

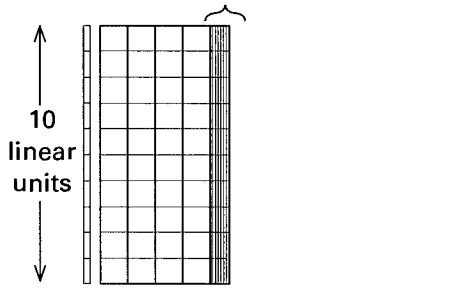
COMMENTS

1 (continued.)

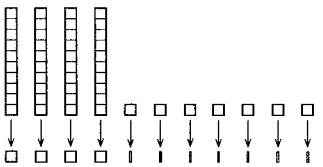
Step 1: Start with an area piece collection of 4 tens and 7 units. Lay out a dimension of 10 linear units. Form 4 bands of width 1 linear unit each, with 7 area units left over:



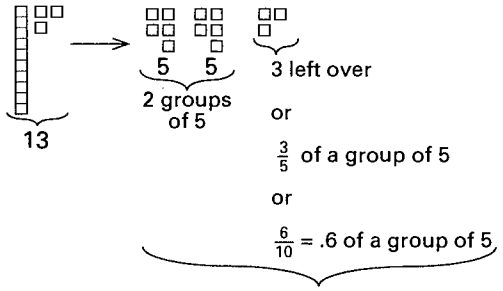
Step 2: Exchange each of the 7 left over area units for 10 tenths (e.g., toothpick pieces) and distribute the 70 tenths in 7 bands of width .1 linear unit to show 47 ÷ 10 = 4.7.



Yet another method that may come up is to view 47 ÷ 10 as 1/10 of 47, by replacing each area piece with a new piece that is one tenth as large, as shown here:



2 Repeat Action 1 for 13 ÷ 5.



There are 13 ÷ 5 = 13/5 = 2 3/5 = 2.6 groups of 5 in 13.

2 Once again, students may use a variety of methods. In the example shown at the left, the grouping method is used to arrange 13 units in groups of 5. Notice there are 2 groups of 5 with 3 units left over. Hence, one could say there are 13 ÷ 5 = 2 groups of 5 in 13, with a remainder of 3. Or, one could note that the remaining 3 units are 3/5 of a group of 5 units.

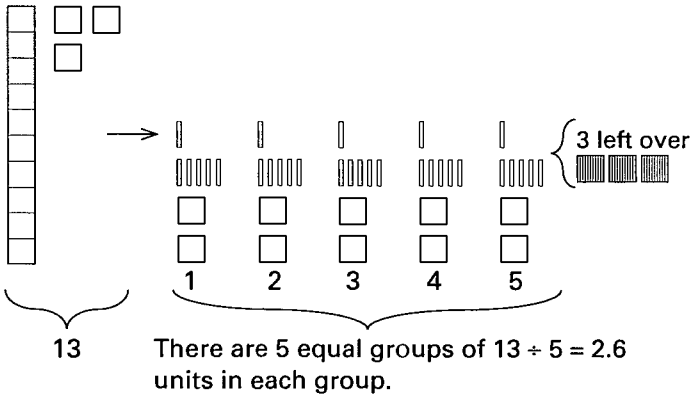
In the following example of the sharing method, 13 units are arranged in 5 equal groups of 13 ÷ 5 units. Notice each group receives 2 units, with 3 units left over. These 3 leftover units could be divided into 5 equal parts of 3 ÷ 5 = 3/5 (see Lesson 28, Division Concept of a Fraction), so each group receives 13 ÷ 5 = 2 3/5

Focus Teacher Activity (cont.)

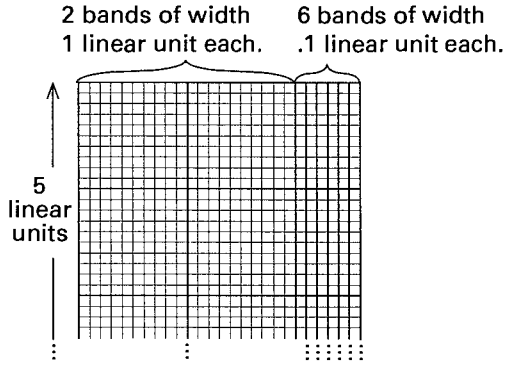
ACTIONS

COMMENTS

units, or, if the 3 left over were divided into 10 equal parts, one could see there are $2\frac{6}{10} = 2.6$ units in each group.



One way that a collection of 13 units can be arranged to form a rectangle with one dimension 5 linear units is shown here:



3 Repeat Action 1 for one or more of the following computations, recording equations involving division on a Quotient Chart that is similar to the Product Chart formed in Lesson 37. Have students point out how each number in their equations relates to their model. Ask the students to describe situations in which it would be necessary to find each quotient they explore.

a) 8 ÷ 10

b) 50 ÷ 20

c) 3.08 ÷ 1.4

d) 2.3 ÷ 100

e) 290 ÷ 13

f) 2.7 ÷ .6

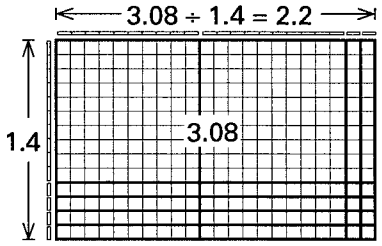
3 To develop students' comfort with several methods, you may wish to ask them to try specific methods on some problems, and to try more than one method on other problems. Students may feel some methods are more convenient or logical for certain computations and/or contexts than for others. For example, one is more likely to encounter a situation involving an area of 8 and one dimension 10, or a quantity of 8 to be divided into 10 equal parts than a situation that requires placing 8 objects in groups of 10.

The point here is to continue exploring the meanings of division and to find quotients using methods that are based on those meanings. If students bring up algorithms or rules about moving the decimal, you might suggest that a purpose of this lesson will be to see how and why rules work, and perhaps for the students to

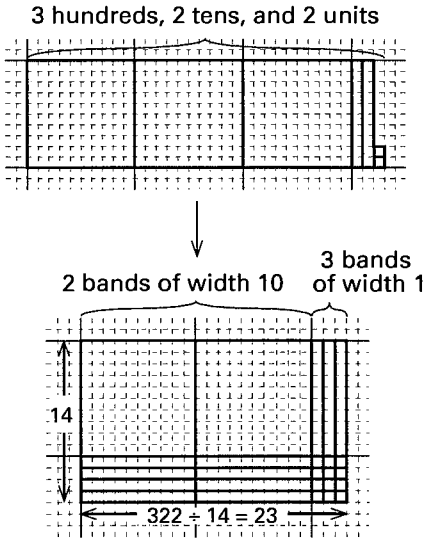
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



4 Distribute 1 sheet of base ten grid paper, version A to each pair of students and ask them to draw a rectangle that will enable them to find the quotient 322 ÷ 14. Discuss their methods.



COMMENTS

3 (continued.)
invent rules and algorithms of their own. Keeping a list throughout the lesson of all division equations that students explore may help students to formulate conjectures later. Here are examples of equations that students may suggest:

a) $8 \div 10 = .8$

b) $50 \div 20 = 2.5$

c) $3.08 \div 1.4 = 2.2$, as shown in the rectangle at the left.

d) $2.3 \div 100 = .023$

e) $290 \div 13 = 22$, with remainder 4, or $22\frac{4}{13}$. Note that using the pieces to compute the decimal equivalent of $\frac{4}{13}$ ($.\overline{307692}$) is tedious, and therefore, this would be an appropriate computation to also examine on the calculator.

f) $2.7 \div .6 = 4.5$

4 Once again the emphasis is on developing conceptual understanding by using a model to determine the quotient rather than computing and then sketching.

This quotient may be found by sketching a rectangle which has an area of 322 and a dimension of 14. The other dimension is the quotient.

If students have difficulty with sketching, encourage them to use the pieces, and then repeat their procedures on grid paper. It may also help to suggest that they enclose a region whose area is 3 hundreds, 2 tens, and 2 units (as shown at the left) and then draw a rectangle which encloses the same area and has a side length of 14 linear units. One way to form this rectangle is to first determine how many bands of width 10 linear units and length 14 linear units can be incorporated into the rectangle. Each of these bands has an area of 1 hundred and 4 tens, or 140 square units. Two bands of width 10 provide an area of 2 hundreds and 8 tens, or 280 square units. Adding another band of width 10 creates too large an area, so bands of width 1 and edge 14 are added until an area equivalent to 322 square units in obtained. Since it takes 2 bands of width 10 and 3 bands each of width 1 to form and area of 322, $322 \div 14 = 23$, as shown at the left.

Focus Teacher Activity (cont.)

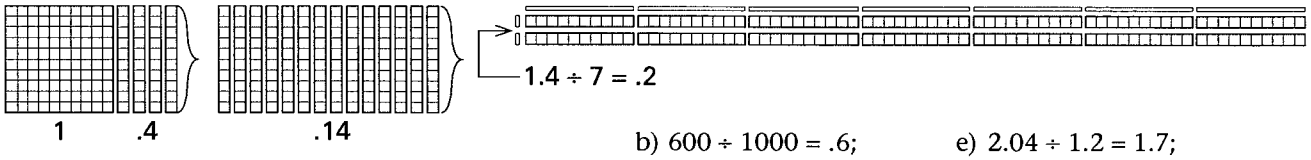
ACTIONS

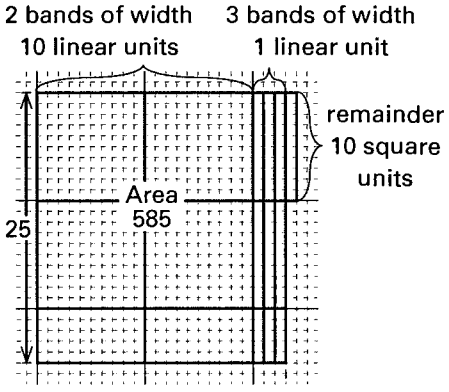
5 Ask the students to identify other quotients that could be represented by the 23 × 14 rectangle they formed in Action 4. Record the corresponding equations on the Quotient Chart started in Action 3.

6 Ask the pairs of students to devise methods of using their area and linear pieces or sketches of pieces (either rough sketches or on base ten grid paper) to determine one or more of the following quotients. Discuss and record the corresponding equations. Ask students to use the Quotient Chart and their experiences so far in this lesson to make observations and generalizations about division.

a) 1.4 ÷ 7 c) 1000 ÷ 600 e) 2.04 ÷ 1.2

b) 600 ÷ 1000 d) 2.99 ÷ 2.3 f) 585 ÷ 25





COMMENTS

5 There are many possible quotients represented by this rectangle. For example, this rectangle also illustrates 322 ÷ 23 = 14. By changing the area unit to the large square, the rectangle could also represent 3.22 ÷ 1.4 = 2.3 and 3.22 ÷ 2.3 = 1.4. If the large square is 1/100, .0322 ÷ .14 = .23 and .0322 ÷ .23 = .14. Or, if the large area piece square has dimensions 100 × 100 and area 10,000, then the rectangle represents 32,200 ÷ 140 = 230 and 32,200 ÷ 230 = 140.

6 It is not intended that all of these problems necessarily be assigned. Rather select according to the types of problems students find challenging. Keep emphasis on developing conceptually based strategies.

a) One method of dividing 1.4 by 7 is to exchange the unit and 4 tenths for 14 tenths, and then arrange these pieces in a rectangle with one dimension 7, as shown here:

b) 600 ÷ 1000 = .6; e) 2.04 ÷ 1.2 = 1.7;

c) 1000 ÷ 600 = $1.\overline{6}$; f) 585 ÷ 25 = 23.4.

d) 2.99 ÷ 2.3 = 1.3;

If students raise questions about the standard long division algorithm, you could invite them to create algorithms to represent their methods of computing 585 ÷ 25 (or another quotient) and then discuss their ideas about relationships between the standard algorithm and the method of computing quotients by forming rectangles (see next page). We don't believe, however, that it is necessary for students to memorize or use the standard textbook procedure. If it is taught, it is important that students view this procedure as one of several options for calculating quotients, and that emphasis be placed on understanding relationships between the procedure and the concept of division.

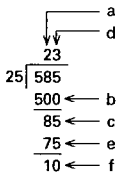
The diagram at the left illustrates the area method of computing 585 ÷ 25 by finding the missing dimension of a rectangle with area 585 and one dimension 25.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Remainder expressed as a whole number:



a) record the maximum number of bands of width 10,

b) record the number of units in the bands of width 10,

c) determine the number of units left to be distributed,

d) record the maximum number of bands of width 1,

e) record the number of units in the bands of width 1,

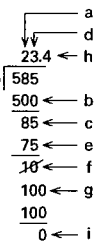
f) determine the remainder.

COMMENTS

6 (continued.)
To express the remainder as a decimal, exchange the remainder for 100 tenths. Four bands of width .1 each and length 25 (i.e., each band contains 25 tenths) will equal 100 tenths, and hence, the quotient is 23.4.

A recording of these steps might look like the following:

Remainder expressed as a decimal:



a)-e) See explanation at the left.

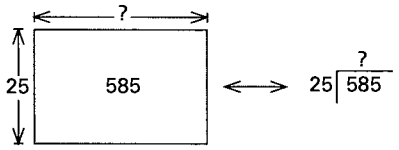
f) Determine the number of units left to be distributed,

g) exchange the units left for tenths,

h) place a decimal and record the maximum number of bands of width .1,

i) determine the number of tenths left to be distributed.

The above procedures can be extended to finding quotients of larger numbers. However, such quotients are more efficiently found with the use of a calculator. On most calculators, a remainder will appear as a decimal. Note the similarity between a sketch of a rectangle with a missing dimension and the notation for a long division with a missing quotient, as illustrated below.



7 Place a transparency of Focus Master A on the overhead, revealing only Situation a) and ask the groups to make a model (or sketch of a model) that illustrates the relationships described in the situation. Then ask them to make as many mathematical conclusions as they can about the situation by observing their model. Have them use calculators if needed to carry out computations suggested by their diagrams. Repeat for one or more of Situations b)-e).

7 The intent here is for students to use their diagrams and models as a basis for determining what calculations to carry out (e.g., on calculators) in order to make mathematical conclusions about the situation.

Note that it isn't possible to tell what conclusions students will make about each situation since there are no questions posed. If groups have difficulty with a particular situation, you could pose questions about their models.

Focus Teacher Activity (cont.)

ACTIONS

Whole Number and Decimal Division Lesson **38**
Focus Master A

a) Jorge lives 2.7 miles from the school. Briann lives 1.3 times as far from school as Jorge. Julie lives ½ as far from the school as Briann.

b) Lindsay's dad paid $17 for 13.2 gallons of gasoline. Jennie's dad paid $18 for 15 gallons.

c) For the picnic Erica bought 3.4 pounds of hamburger which cost $1.67 per pound. Iain spent $5.90 on 3.6 pounds of hamburger.

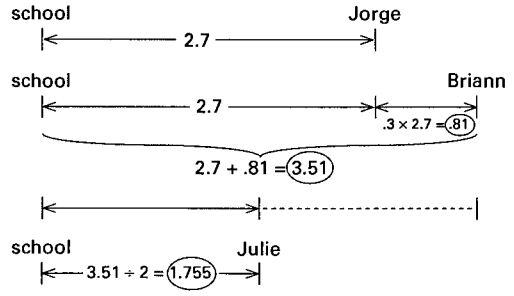
d) Dustin's favorite ice cream is on sale for $3.89 per gallon. He has $20 and wants to buy as much ice cream as possible.

e) Dylan has 3 pieces of rope that are each 15.93 meters long. He needs 16 pieces of rope that are each 2.7 meters long.

f) Kevin ran a total of 657 miles during the past 36 days. His goal is to run 1000 miles in 60 days.

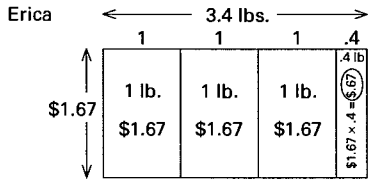
COMMENTS

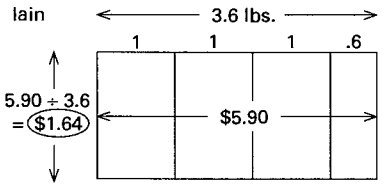
a) Some students may use linear pieces to represent Situation a), while others may make rough sketches, such as the following (the circled numbers were computed using a calculator):



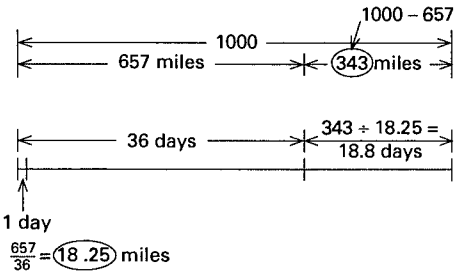
Questions you might ask about Situation a) (for calculation by calculator or other method) include: How far does each student live from school? How much farther from the school does Briann live than Julie? How many times farther from the school does Jorge live than Briann? etc.

c) Below is an example of a student's area representation of the relationship in this situation:





f) A diagram of situation f) might look like the following:



Focus Teacher Activity (cont.)

ACTIONS

8 (Optional) Give each group a sheet of butcher paper and a marking pen. Ask the groups to write a list of 8-10 important ideas that will help the reader *understand* division. Suggest they include diagrams to support their explanations. Post and discuss.

COMMENTS

8 Students may base their ideas on illustrations that have come up in class discussions, observations about the Quotient Chart, personal AHA's that occurred while using the pieces, ideas that were especially challenging early in the lesson, rules or algorithms students have invented (or understand for the first time), etc.

To help students focus you might suggest an audience for the posters, such as: parents coming for a parent meeting; younger students or another class that hasn't yet studied division; a parent newsletter; a classmate who was absent for the division lesson; the authors of *Visual Mathematics* (please send copies to us!) for consideration as part of a new lesson, etc.

Follow-up Student Activity 38.1

NAME _____ **DATE** _____

1 Use area and linear pieces to find the quotients in these problems. Then sketch a diagram of your methods and solutions on the attached base ten recording paper. Next to each diagram, write an explanation of your reasoning and methods.

a) $4.8 \div 1.2$ b) $344 \div 16$ c) $2.73 \div 1.2$

2 Use a calculator to find $57904 \div 112 =$ _____.

a) Tell which keys you pressed, in the order that you pressed them:

b) Do you think your calculator quotient is reasonable? Describe the strategies you used to decide about its reasonableness.

c) Write an interesting word problem for which you need to compute $57904 \div 112$ to solve the problem.

(Continued on back.)

Follow-up Student Activity (cont.)

Dear _____,

Part of my math assignment today is to spend 20-30 minutes presenting to an adult my understanding of multiplication and division of decimals and whole numbers. Would you please take the time to listen to my presentation? The purpose of explaining to you is so I can clarify my thinking about these concepts and identify places I still have questions, so we can explore those questions more in class.

I will use the base ten pieces we use in class to show you how I view the meanings of multiplication and division and to show you ideas and methods that I think are especially interesting, easy, and/or challenging. If I get stuck or have difficulty explaining an idea, would you remind me to write it down so that I can take my questions back to class and investigate them some more?

When I am finished, please write a note to my teacher, listing three positive comments about my presentation. Thanks for taking the time to listen to my thinking!

Student Signature _____

_____ spent _____ minutes helping me understand her/his thinking about multiplication and division of decimals. Following are three positive comments about that presentation.

1.

2.

3.

Other positive thoughts:

Adult Signature _____ Date _____

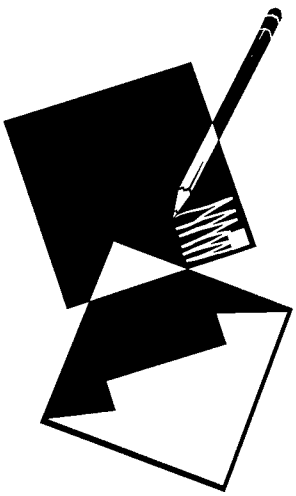
Modeling Percent

Lesson 39

THE BIG IDEA

Base ten area pieces provide a hands-on model for exploring the meaning of percent.

Using base ten pieces to represent various situations involving percent enables students to observe mathematical relationships. Such experiences foster the development of insights and strategies regarding solving several types of percent problems.



CONNECTOR

OVERVIEW

Students describe their current views of the meaning of percent, situations in which they have encountered percent, and questions they have about this concept.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).

FOCUS

OVERVIEW

The meaning of percent is introduced using base ten area pieces. By assigning different values to the 10×10 square, various percent problems can be modeled.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Base ten area pieces, 1 set per student.
- ✔ Focus Student Activity 39.1, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 39.2, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.
- ✔ Percent grids (optional, see *Blackline Masters*), 2-3 sheets per group and 1 transparency.
- ✔ Base ten pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students use percent grids to model and solve percent problems.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 39.3, 1 copy per student.
- ✔ Percent grids (see *Blackline Masters*), 2 sheets per student.

LESSON IDEAS

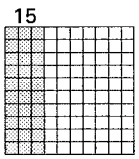
JOURNALS
Students could record their responses for both Action 1 of the Connector and Problem 1 of the Follow-up in their journals. You could also ask them to compare their responses and discuss ways their understanding has changed during this lesson.

FOLLOW-UP
To help the students prepare for presenting Problem 3 on the Follow-up, you might have the students brainstorm ideas and practice their presentations with groupmates.

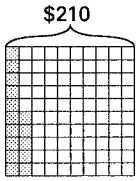
TEACHER NOTES:

SELECTED ANSWERS

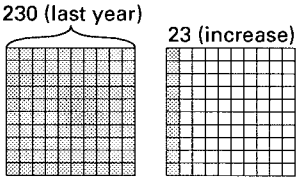
2. a) Observations: 70% did not have knee injuries. The total number of football players was 50. Thirty-five players did not have knee injuries.



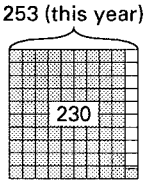
b) The amount of discount is $31.50. The sale price is $178.50. The customer will pay 85% of the regular price.



c) Comparing the increase to last year's enrollment; the 23 additional students is a 10% increase.



Comparing the number of students from last year to the number from this year; the 230 students from last year is $^{230}/_{253}$ or approximately 90.9% of the 253 students from this year.



Connector Teacher Activity

OVERVIEW & PURPOSE

Students describe their current views of the meaning of percent, situations in which they have encountered percent, and questions they have about this concept. Their ideas can be useful in planning for the Focus activity and in measuring growth in their understanding of percent after completing the Focus activity.

MATERIALS

✔ Student journals (optional).

ACTIONS

1 Have the students spend about 5 minutes writing a response to the following thought starters:

I think the meaning of percent is...

Some situations I have experienced or noticed that involve percent are...

One question I have about percent is...

2 Discuss the students' ideas and, if appropriate, use their ideas as a basis for beginning discussion of the Focus activity (see Focus Comment 1).

COMMENTS

1 You may wish to let students know that you are only interested in how they think about the meaning of percent at this point in time and that you have no expectations about what they "should" know. Some students may have explored the concept in other classes while others may only have intuitive notions based on observations they have made about advertisements and conversations they have heard.

This information can be useful for planning for implementation of the Focus activity and for monitoring students' growth during and after the activity. After completing this lesson, students can make comparisons to see how their understanding of percent is developing.

2 Some students may have misconceptions about the meaning of percent, while others may make suggestions that reflect a better understanding. Rather than trying to change students' views at this point, suggest that you would like them to observe ways their ideas develop during the Focus activity. To monitor this development, perhaps they could make self-initiated journal entries each time an "AHA!" about percent occurs or when they are confused.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

The meaning of percent is introduced using base ten area pieces. By assigning different values to the 10 × 10 square, various percent problems can be modeled.

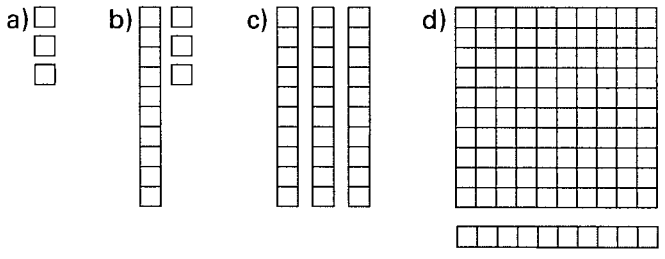
MATERIALS

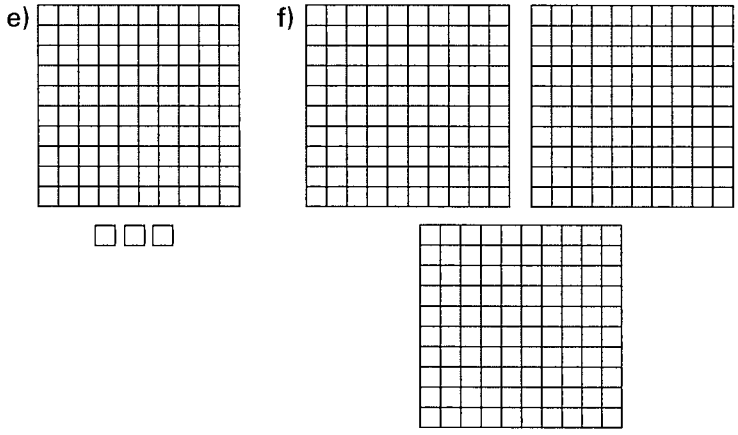
- ✔ Base ten area pieces, 1 set per student.
- ✔ Focus Student Activity 39.1, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 39.2, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 transparency.
- ✔ Percent grids (optional, see *Blackline Masters*), 2-3 sheets per group and 1 transparency.
- ✔ Base ten pieces for the overhead.

ACTIONS

1 Distribute base ten area pieces to each student. Tell the students that when something is divided into 100 equal parts, one part is called 1%. Therefore, the small base ten square is 1% of the large base ten square, and the large square is 100%.

Form the collection in a), shown below, on the overhead. Ask the students to form this collection and to determine what percent that collection represents if the large 10 × 10 square represents 100%. Discuss. Repeat for the collections b)-f).





COMMENTS

1 This may be a good time to show the students percent notation. You might tell the students how percent notation developed over time. For example, in 15th century Italy 20% was written variously as *XX p cento*, *XX p c^0*, and *20 p 100*. By the 17th century it had developed to the form per $\frac{0}{0}$, then to simply $\frac{0}{0}$, and finally to the modern form %.

The collections shown represent the following percentages:

a) 3%

b) 13%

c) 30%

d) 110%

e) 103%

f) 300%

Focus Teacher Activity (cont.)

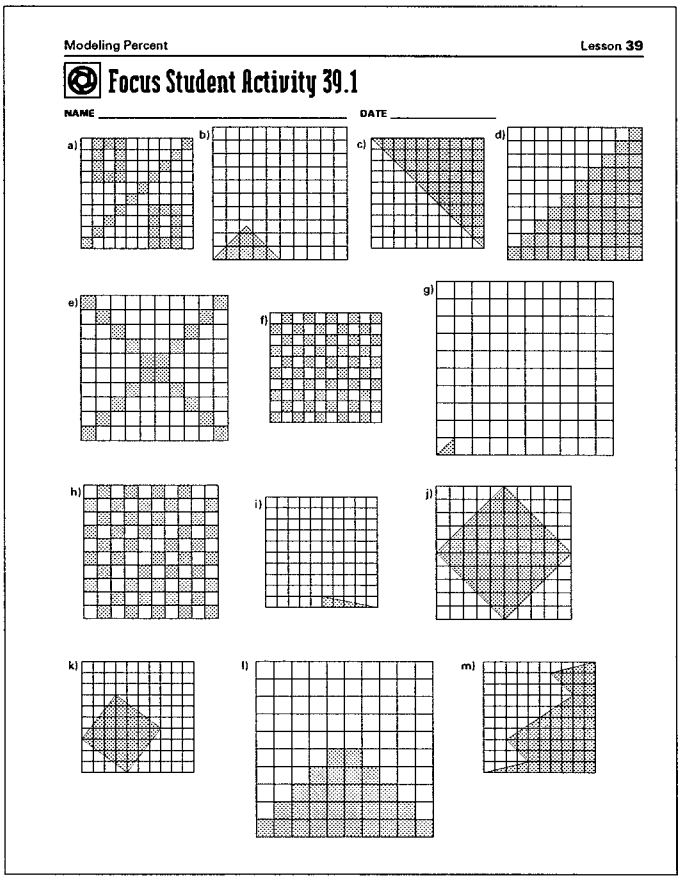
ACTIONS

2 Using the 10 × 10 square as 100%, ask the students to form a collection of pieces which represents the percentage in a) below. Repeat for percentages b)-f). Discuss.

a) 13% c) 125% e) 303%

b) 99% d) 222% f) ½%

3 Distribute 1 copy of Focus Student Activity 39.1 to each student and ask them to determine what percent of each 10 × 10 grid is shaded. Discuss the different ways students determined the percents.


Modeling Percent Lesson 39

Focus Student Activity 39.1

NAME _____ DATE _____

a) b) c) d)

e) f) g)

h) i) j)

k) l) m)

COMMENTS

2 The last percentage, ½%, usually creates some discussion. Some students may incorrectly represent it as ½ of the 10 × 10 square instead of ½ of a small base ten square or 1% piece.

3 The grids are various sizes but each is divided into 100 equal parts. You may wish to make a transparency of the activity sheet, show one of the grids on the overhead, and let the whole class discuss different ways to determine the percent shaded. For example, in figure d) some may see the shaded part as the sum 1+2+3+4+5+6+7+8+9+10 squares or 55%. Others may draw a diagonal starting at the lower left corner and see the shaded part as ½ of the 10 × 10 plus the 10 halves that are cut off by the diagonal. Some students may cover the last column and see the remaining shaded part as ½ of a 9 × 10 array then add the 10 squares that were covered. And there are other ways!

In Figure m), some students may dissect the shaded sections into smaller parts and compute the area as the sum of the smaller areas. Others may see that the shaded and unshaded portions are congruent. That is, if the grid is rotated 180° about its center, the shaded and unshaded parts coincide.

Encourage students to devise their own counting strategies for determining the percentages shaded. Each of the figures can be approached in different ways. The amounts shaded are:

a) 30% f) 50% k) 25%

b) 6¼% g) ½% l) 30%

c) 50% h) 45% m) 50%

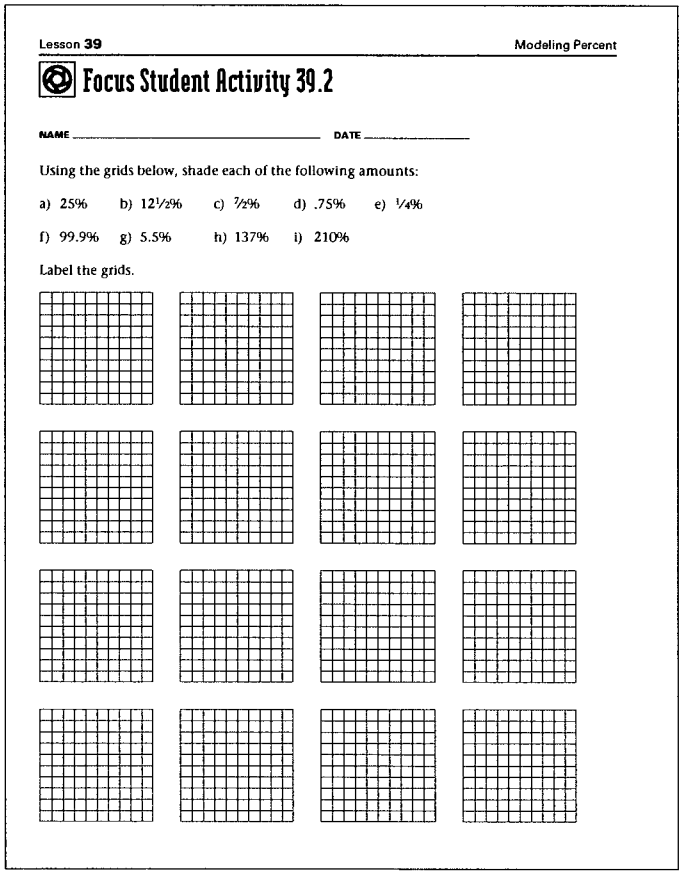
d) 55% i) 2½%

e) 20% j) 50%

Focus Teacher Activity (cont.)

ACTIONS

4 Distribute one copy of Focus Student Activity 39.2 to each student and ask the students to carry out the instructions on the sheet. Ask volunteers to share different ways of representing the given percents. Invite the students to shade one or more of the remaining blank grids to show percents of their choice.



Lesson 39 Modeling Percent

Focus Student Activity 39.2

NAME _____ DATE _____

Using the grids below, shade each of the following amounts:

a) 25% b) 12½% c) ½% d) .75% e) ¼%

f) 99.9% g) 5.5% h) 137% i) 210%

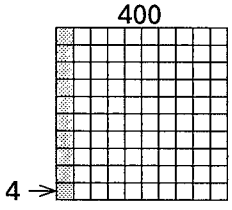
Label the grids.

5 Place a transparent 10 × 10 square on the overhead and ask the students to imagine that the square has a value of 400 and that this value is spread evenly among all the small squares. Ask them to determine the value of 10 small squares. Discuss the different ways in which students arrive at their answers.

COMMENTS

4 Here the students are asked to shade a given percent of a 10 × 10 square. You may wish to make a transparency and have students come to the overhead to share their sketches with the class.

5 It may be helpful for the students if you write the value 400 above the 10 × 10 square. Since the value 400 is divided evenly among the small squares, each square has a value 400 ÷ 100 = 4, so 10 squares have value 40. Or, 10 squares comprise 1/10 of the grid and 1/10 of 400 is 40.



Focus Teacher Activity (cont.)

ACTIONS

6 Repeat Action 5 using the value 400 but ask them to determine the value of 5 small squares, 16 small squares, 96 small squares, 125 small squares. Discuss.

7 Have the students use their base ten pieces to represent 25% of the 10 × 10 square. Tell them to imagine that the 10 × 10 square has a value of 300 and that this value is spread evenly among all the little squares. Ask them to determine the value of the collection representing 25%. Discuss.

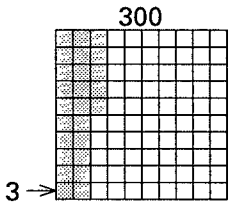
8 Repeat Action 7 using the same value, 300, for the 10 × 10 square, but changing the percentages to 10%, 13%, 75%, 100%, 1%, 130%, and 210%, respectively. Discuss.

9 Once again assigning a value of 300 to the 10 × 10 square, ask the students to determine a collection of pieces that has a value of 90. Then ask them to determine what percent of the 10 × 10 square the value 90 represents.

COMMENTS

6 In each case one could use the value of one small square, 4, and multiply that by the number of squares. But, there are other ways and here is one alternative method for each: 5 small squares are 1/20 of the whole grid and have a value of 20; 16 small squares are 4 squares short of 20 squares or have value 16 less than 80; since 96 squares are 4 squares less than 100 squares they have a value 400 – 16 = 384; 125 squares have the value of a whole 10 × 10 square (400) plus 1/4 of a 10 × 10 square (100).

7 25% of the 10 × 10 square can be represented by 2 strips and 5 units. Each strip has value 30, 5 units have value 15, so the total value is 30 + 30 + 15 = 75. Another way is to determine that each square has value 3, so 25 squares have value 75. These actions can be summarized by saying: 25% of the value 300 is 75, or 25% of 300 is 75, as illustrated in the following diagram.



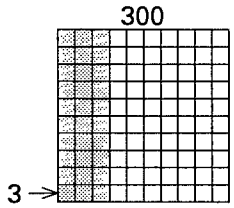
8 As in Action 7 there are several ways to determine the value of each given percent.

percent	10%	13%	75%	100%
values	30	39	225	300

percent	1%	130%	210%
values	3	390	630

Summary statements: 13% of the value 300 is 39; 210% of the value 300 is 630; etc.

9 Since each small square has value 3 it would take 30 small squares or 30% of the 10 × 10 square to have value 90. This can be summarized as: 90 is 30% of 300; or 30% of 300 is 90.



Focus Teacher Activity (cont.)

ACTIONS

10 Repeat Action 9 using the same value, 300, but changing the value of the collection of pieces to 75, 99, 240, and 324, respectively.

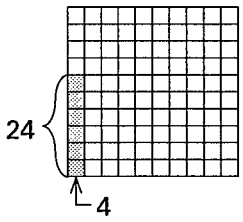
11 Ask the students to imagine that 6% of the 10×10 square has a value of 24 and to determine the value of the 10×10 square. Ask them to write a percent statement using the values they have determined. Discuss.

12 Repeat Action 11 when 12% has a value of 18; when 55% has a value of 110; and when 150% has a value of 180.

COMMENTS

10 A value of 300 for the whole 10×10 square means that 10 small squares have value 30, 20 small squares have value 60 and 25 small squares have value 75. So the value 75 represents 25% of the total value of the 10×10 square (300). The values 99, 240, and 324 are represented by collections of 33 small squares, 80 small squares and 108 small squares, respectively. (Or, they represent 33%, 80%, and 108%, respectively of the total value, 300).

11 One way to think of this is that if 6% (6 squares) has a value 24 then 1% (1 square) has value 4 and the 10 by 10 square has value 400. This situation really answers the question: If 6% of a value is 24, what is that value? Or, 6% of what value is 24? Here are two percent statements that can be written about this situation: 6% of 400 is 24; 24 is 6% of 400.



12 Here is one possible way to determine the value of 100% for each situation.

When 12% has a value of 18, 4% has a value of 6, so 100% has a value of 150 (25×6). (18 is 12% of 150.)

When 55% has a value of 110, 1% has value 2, so 100% has a value of 200. (110 is 55% of 200.)

When 150% has a value of 180, 50% has a value of 60, so 100% has a value of 120. (180 is 150% of 120.)

Focus Teacher Activity (cont.)

ACTIONS

13 Distribute a copy of Focus Master A to each group. Ask the students to think of the 10 × 10 square as 100% and to use their base ten pieces to represent Situation a) from Focus Master A. Have volunteers show their models at the overhead and explain mathematical observations they can make by looking at their model.

Modeling Percent Lesson 39
Focus Master A

Situations

a) 65% of the company's 160 employees were women.

b) 280 of the school's 800 students were absent.

c) I bought this CD player for $126 at a 30% off sale.

d) The school population this year in Alstown is 810, which is 135% of the school population for last year.

e) This year the charity benefit had a goal of $8000 and it raised 125% of its goal.

f) Last year the charity benefit raised $6900 which was 115% of its goal.

g) In the middle school, 129 students were absent at least one day during the school year. 129 students is 86% of the student body.

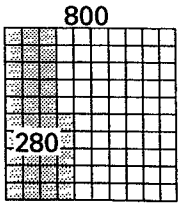
h) The value of one stamp in the collection has risen 232% and is now worth $83.

i) In the election for student body president, 33⅓% more students voted for Mariah than for Carey. 216 students voted for Carey.

j) 336 people tried out for the 126 parts in the musical.

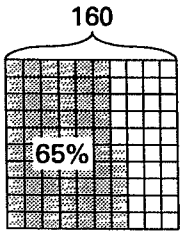
k) Based on his experiment, Nick says the probability of drawing a green tile out of his bag is 45%. There are 18 green tile in his bag.

14 Repeat Action 13 by choosing other situations from Focus Master A. Emphasize the use of base ten pieces to represent each situation.



COMMENTS

13 Six strips and 5 small squares represent 65% of the 10 × 10 square and the square has value 160, as indicated in the diagram. Even though the situation poses no questions, students may respond as though they were asked a question. For example, after explaining the model the student may say, "There are 104 women employees," or "35% of the employees are not women," or "56 employees are not women."

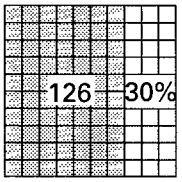


14 Students will benefit by using the pieces to model these situations. If groups work on several situations before discussion occurs, then provide them with 2-3 sheets of percent grids (see *Blackline Masters*) for sketching a record of their models. If some groups have difficulty getting started making observations, you might pose questions for them to explore. For example, regarding b) you might ask: What percent were absent? What percent were present? How many students are 1% of the student body? etc. Here are diagrams which represent each of the situations and at least one way to describe the diagrams.

b) If the 10 × 10 square has value 800, each strip has value 80 and 3½ strips or 35 small squares has value 280. Hence, 35% were absent and 65% (520 students) were present.

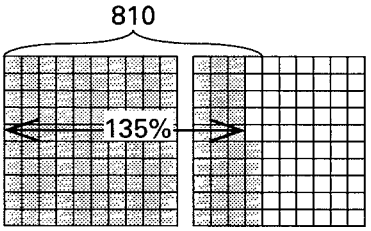
Focus Teacher Activity (cont.)

ACTIONS

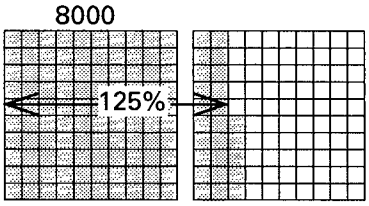


COMMENTS

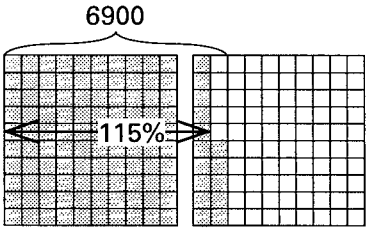
c) If the entire 10 × 10 square represents the full price, then after 30% has been deducted the part of the grid remaining represents a value of 126. Hence, 7 strips have value \$126 so each strip has value \$18 and the total value is \$180. The value of the discount is \$54. This is equivalent to saying 30% of \$180 is \$54 or 70% of \$180 is \$126.



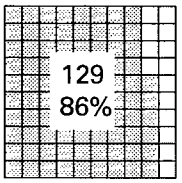
d) 135% has value 810. When the value 810 is distributed equally among 135 squares each square has value 6, so last year there were 600 students; there are 210 more students this year.



e) 100% has value \$8000. Each strip has value \$800 so the charity raised \$2000 more than the goal. Altogether they raised a value of \$10,000.



f) 115% amounts to \$6900. 23 half-strips have value \$6900 so each half-strip has value \$300. A strip then has value \$600 and the 10 × 10 square has value \$6000.



g) 86% has value 129. When the value 129 is distributed equally among 86 squares each has value 1.5, so 1% has value 1.5 and 100% has value 150. Therefore, 21 students had perfect attendance.

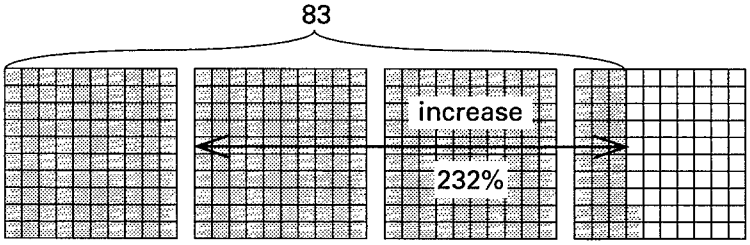
(Continued next page.)

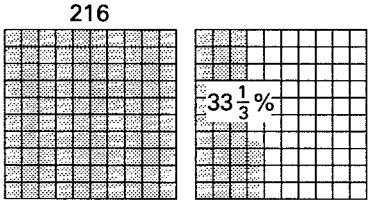
Focus Teacher Activity (cont.)

ACTIONS

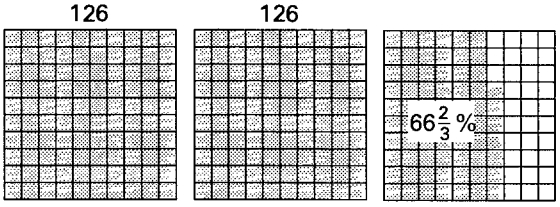
COMMENTS

14 (continued.)
h) When the original value is represented by a 10×10 square, that must be increased, or supplemented, by 232%. The total value, original plus increase, is 83. Hence, 332% has value 83 so 1% has value $83 \div 332 = .25$ and 100% has value 25. The original value has increased by a value of $232 \times .25 = 58$.

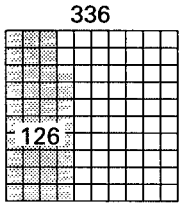




i) If a 10×10 square has value 216, that must be increased by $33\frac{1}{3}\%$. If the 10×10 square has value 216 then $33\frac{1}{3}$ squares have value $\frac{1}{3}(216) = 72$, and thus, $216 + 72 = 288$ students voted for Mariah. This is equivalent to saying $133\frac{1}{3}\%$ of 216 is 288 and 288 is $33\frac{1}{3}\%$ more than 216.



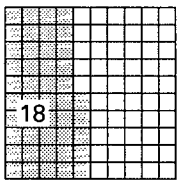
or



j) If a 10×10 square has value 126 and represents 100% then each small square has value 1.26 and $336 \div 1.26 = 266\frac{2}{3} = 266.\overline{6}$ squares have value 336.

Or, if a 10×10 square has value 336 and that represents 100% then each small square has value 3.36, and $126 \div 3.36 = 37.5$ small squares have value 126.

Some statements that describe the relationships in this situation are: 336 is $266\frac{2}{3}\%$ of 126; 126 is $37\frac{1}{2}\%$ of 336; 84 is $66\frac{2}{3}\%$ of 126; $37\frac{1}{2}\%$ of the 336 who tried out could have parts; $62\frac{1}{2}\%$ of those who tried out could not have parts.



k) If a 10×10 square is 100%, then 45% is 45 squares which have value 18.

Since there are 9 groups of 5 squares in 45, each group of 5 squares has value $18 \div 9 = 2$ and the whole square has value $2 \times 20 = 40$ (since there are 20 groups of 5 in 100). Hence, there must be 40 tiles in the bag (the amount represented by the entire square). Each small square has value $\frac{18}{45} = .4$, so the entire 10×10 square has value $.4 \times 100 = 40$.

Follow-up Student Activity 39.3

NAME _____ DATE _____

1 Write an explanation, in your own words, of the meaning of percent.

2 Use your base ten pieces to build a model of the relationships in each of the following situations. Then sketch your model on the attached sheet of percent grids. Next to each diagram, write at least two different mathematical observations that you can make by looking at your model. (Be sure to label each diagram.)

a) 30% of the football players injured a knee during the football season. 15 players had knee injuries this season.

b) A television that usually sells for $210 is on sale for 15% off the regular price.

c) Last year 230 students joined the choir. This year 253 students joined.

d) The drama students need to raise $2000 for a class trip. They have raised $880 so far.

e) Dylan has earned $115 mowing lawns. This is 46% of what he needs to buy the bike he wants.

f) In an experiment, Willis' spinner landed on purple 32% of the time. He spun the spinner a total 75 times.

g) This year tickets to the concert cost $15, which is 125% of the cost last year.

(Continued on back.)

Follow-up Student Activity (cont.)

3 Spend 20 or more minutes teaching an adult about the meaning of percent. Use your base ten pieces to give examples and illustrate your thinking. Then pick three situations from Problem 2 and explain to the adult:

a) your model for each situation,

b) your mathematical observations about each model.

Be sure they understand your thinking. In this space have the adult write 3 "I appreciate..." and 1 "I wish..." statement about your presentation.

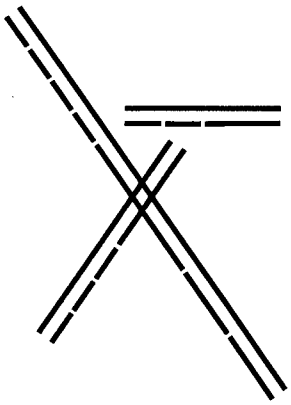
4 Find at least three different advertisements or articles in the newspaper or a magazine that involve percent. Attach them to this assignment.

Linear Model For Fractions

Lesson 40

THE BIG IDEA

Fraction situations that involve changing the unit challenge students to be flexible in their thinking about fractions and help build conceptual understanding. Explorations that involve comparing and combining portions of a unit can help students understand fraction equivalence and develop intuitions about operations with fractions.



CONNECTOR

OVERVIEW

Students describe their current views of the meanings of a fraction. They examine fractional relationships among several different lengths and explore the effects of changing the linear unit.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set containing 2 strips each of the colors blue, pink, orange, yellow, green, and white per student.
- ✔ Scissors, 1 pair per student.
- ✔ Segment strips (in the same colors as student sets) for the overhead.

FOCUS

OVERVIEW

Students investigate equivalent fractions, explore fractions viewed as portions of a whole and as quotients, and develop readiness for operations with fractions by combining and comparing segment strips. Emphasis is placed on the effects of changing the linear unit.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set (see Connector) per student plus extras.
- ✔ Scissors, 1 pair per student.
- ✔ Focus Masters A and B, 1 transparency of each.

FOLLOW-UP

OVERVIEW

Students use segment strips to construct and sketch lengths.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 40.1, 1 copy per student.
- ✔ Segment strips, 1 standard set per student.

LESSON IDEAS

ASSESSMENT
Follow-up Problem 5 could be placed in students' Growth-folios. As an alternative, the letter could be addressed and delivered by students to parents.

JOURNALS
The journal entry for the Connector could be compared to the entry for Lesson 8. After this lesson is completed students could reflect in their journals on ways their thinking about fractions has developed since Lesson 8. They could also write about questions they still have.

FOLLOW-UP
Rather than sketching diagrams, students could glue or tape segment strips on their Follow-up.

LOOKING AHEAD
Before starting this lesson with students, you may wish to read Lessons 40-44 in order to get the "big picture" of the approach and purposes of these fraction lessons. This may help with planning and pacing the lessons.

SELECTED ANSWERS

1. The length of 3 blue is $2\frac{1}{4}$ linear units.

2. One way of viewing this is based on the division concept of a fraction, which suggests $7 \div 4$ mean $\frac{1}{4}$ of 7 or $\frac{7}{4}$. Hence, 1 green segment has a length of $\frac{7}{4}$ units.

3. Four pink segments have a length of 5 units.

4. a) $\frac{11}{12}$ units.
   b) 1 linear unit = $1\frac{1}{3}$ blue = 4 orange = 1 yellow.
   c) The green segment is the mystery unit.
   d) The yellow segment is the mystery unit.
   e) The mystery unit is $2\frac{2}{3}$ green units.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students describe their current views of the meanings of a fraction. They examine fractional relationships among several different lengths and explore the effects of changing the linear unit.

MATERIALS

- ✔ Segment strips, 2 strips of each of the colors blue, pink, orange, yellow, green, and white per student.
- ✔ Scissors, 1 pair per student.
- ✔ Segment strips (in the same colors as student sets) for the overhead.

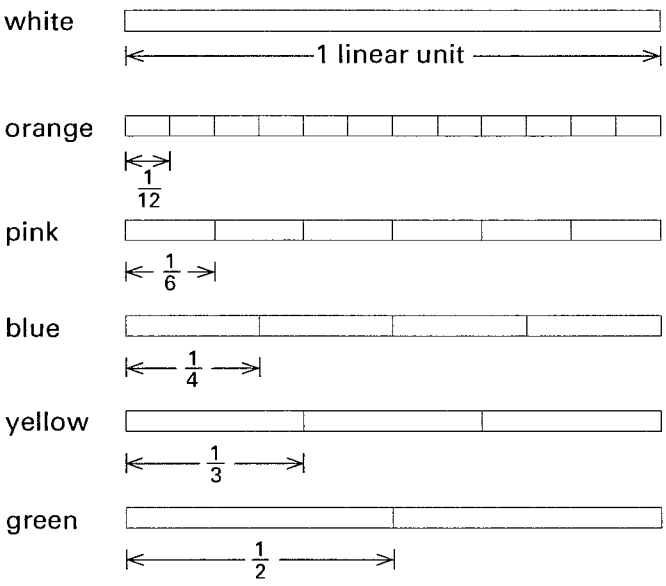
ACTIONS

1 Write the following thought starter on the overhead for the students to complete in their journals:

> **Here is how I think about the meaning of a fraction, along with a few examples to illustrate my understanding:**

2 Place the students in pairs and distribute segment strips and a pair of scissors to each student. Cut off one transparent white segment, place it on the overhead, and tell the students that for now this is 1 linear unit.

Ask the pairs to determine the length of 1 segment of each of the other colors. (Tell them it is okay to mark on or cut lengths.) Have volunteers use transparent strips at the overhead to illustrate their methods and reasoning.



COMMENTS

1 If students are not keeping journals, have them write on a separate sheet and turn it in to you. Their responses will provide you insights about their thinking at this time and help with planning for Lessons 40-44.

2 Distribute strips in blue, pink, orange, yellow, green, and white only. The purple and red strips will be used in Lessons 43 and 44. (Segment strips can be purchased from The Math Learning Center.) The reason for using such narrow strips is to reinforce the idea that these strips represent lengths and theoretically do not have area. A lid of a shoe box or other shallow container is useful for keeping pieces together on students' work spaces. Have a collection of extra strips available for students to use as needed.

If the length of 1 white segment is 1 linear unit, then the length of 1 orange = $1/12$, 1 pink = $1/6$, 1 blue = $1/4$, 1 yellow = $1/3$, and 1 green = $1/2$ linear unit. Encourage students to share a variety of methods of determining these.

Connector Teacher Activity (cont.)

ACTIONS

3 Repeat Action 2:

a) if the length of 1 yellow segment is 1 linear unit;

b) if the length of 1 orange segment is $\frac{1}{8}$ of a linear unit.

COMMENTS

3 a) If the length of 1 yellow segment is 1 linear unit, then the lengths of the other colored segments are: 1 orange = $\frac{1}{4}$, 1 pink = $\frac{2}{4}$, 1 blue = $\frac{3}{4}$, 1 green = $1\frac{1}{2}$, and 1 white = 3 linear units. There are a variety of ways of determining these. Some students may use the part-to-whole concept of a fraction and count the number of copies of each segment needed to match 1 yellow. Other students may use the division concept of a fraction. For example, since 3 yellow segments are equal in length to 2 green, then 1 green must have length $3 \div 2 = \frac{3}{2}$ linear units.

b) Acknowledge several methods. Some students may count the number of orange segments in each different colored segment and describe all lengths in terms of eighths. Hence, *improper fractions* such as $\frac{12}{8}$ may come up. (Note that any fraction whose numerator is greater than or equal to the denominator is called improper; otherwise, it is called a *proper fraction*. When an improper fraction is rewritten as a combination of a whole number and a proper fraction (e.g., $1\frac{2}{3}$), the new number is called a *mixed number*). It isn't necessary that students "reduce" fractions or express improper fractions as mixed numbers. Rather, encourage students to use fractions that reflect the relationships they "see" among the strips.

If some students have difficulty with this activity, experiences with Fraction Bars (see *Opening Eyes to Mathematics*) can provide background for working with the segment strips.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students investigate equivalent fractions, explore fractions viewed as portions of a whole and as quotients, and develop readiness for operations with fractions by combining and comparing segment strips. Emphasis is placed on the effects of changing the linear unit.

MATERIALS

- ✔ Segment strips, 2 of each color per student plus extras.
- ✔ Scissors, 1 pair per student.
- ✔ Focus Masters A and B, 1 transparency of each.

ACTIONS

1 Arrange the students in pairs and give each student a set of segment strips and a pair of scissors. Place a strip of 3 blue segments on the overhead and tell the students that for now this strip has length 1 linear unit. Have the pairs use this unit to determine the length of one or more of the following:

a) 5 orange segments

b) 5 blue segments

c) 2 green segments

d) student choice

Discuss their methods.

COMMENTS

1 Since the unit is changed throughout this activity, place and label a transparent copy of the unit on the overhead each time it is changed.

It is helpful to pose one of these problems at a time and have volunteers share their reasoning at the overhead, so that students can take advantage of methods they observe. Encourage students to cut off segments if it is helpful (be sure to have extra strips of each color available).

a) Since 9 orange segments are equal in length to 3 blue segments (1 linear unit), the length of each orange segment is $\frac{1}{9}$ linear unit and 5 orange segments have length $\frac{5}{9}$ linear unit. Some students may also report the length as $\frac{1}{3} + \frac{2}{9}$ (1 blue and 2 orange), or using a complex fraction, as $\frac{1\frac{2}{3}}{3}$.

b) Since 3 blue segments form 1 linear unit, each blue segment has length $\frac{1}{3}$ linear unit and 5 blue segments have length $1\frac{2}{3}$ linear units. This could also be described as $\frac{5}{3}$ ("five thirds") linear units.

c) The length of 2 green segments is $1\frac{1}{3} = \frac{4}{3}$ linear units. One line of reasoning is that 1 green segment is equal in length to 2 blue segments or $\frac{2}{3}$ linear unit, so 2 green segments have length $\frac{4}{3}$ linear units.

Focus Teacher Activity (cont.)

ACTIONS

2 Place a transparency of Focus Master A on the overhead, revealing Part 1 for groups to complete. Discuss their methods and reasoning.

Linear Model For Fractions Lesson **40**

Focus Master A

1. Find the length of 3 blue segments if 1 linear unit is the length of:

   a) 1 white segment
   b) 1/2 white segment
   c) 1/4 white segment
   d) 2 white segments
   e) 4 white segments
   f) your choice

3 Reveal Part 2a) of Focus Master A for groups of students to complete. Give each group a blank sheet of paper for recording. Discuss as needed. Then repeat for 2b) and/or 2c).

Linear Model For Fractions Lesson **40**

Focus Master A

1. Find the length of 3 blue segments if 1 linear unit is the length of:

   a) 1 white segment
   b) 1/2 white segment
   c) 1/4 white segment
   d) 2 white segments
   e) 4 white segments
   f) your choice

2. Record several different combinations of segments (using 1 or more colors) that have a total length of 1/2 linear unit using the following:

   a) 1 linear unit is the length of 8 pink segments.
   b) 1 linear unit is the length of 9 pink segments.
   c) 1 linear unit is the length of 1 white segment.

COMMENTS

2 The purpose here is to provide experiences with determining lengths while varying the linear unit. Some students may notice that although the *actual* length of a segment does not change, as the linear unit increases/decreases in length the measure of the length of the segment decreases/increases. Here is the length of 3 blue segments for each unit: a) 3/4 linear unit; b) 3/2 or $1\frac{1}{2}$ linear units; c) 3 linear units; d) 3/8 linear unit; e) 3/16 linear unit.

Some students may be interested in using patterns and relationships they observe to predict the length of 3 blue for linear units such as: 25 white segments, 100 white segments, 1/20 white segment, etc.

3 Rather than suggesting recording methods, invite groups to use whatever method they choose. While some groups may record combinations as sums and differences of fractions, others may use recording systems such as 1Y, 4O (for 1 yellow and 4 orange). Some may make organized charts and others may make random lists. Acknowledge all methods that work for students.

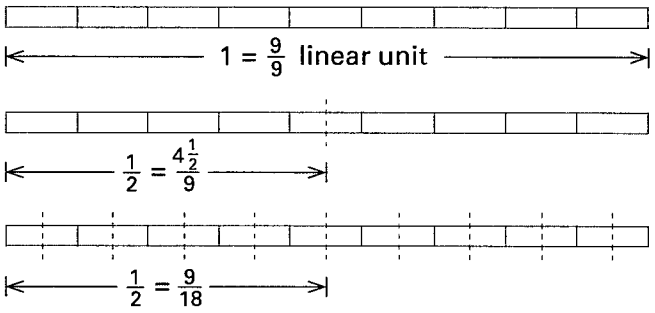
Note that a copy of this Focus Master could be given to each student and parts of it could be completed as homework. This would be appropriate if, for example, groups have worked on 2a) in class and you wish to have students work independently on 2b) and/or c).

a) If 8 pink segments have length 1 linear unit, 4 pink have length 1/2, as do 2 yellow and 8 orange.

After the groups' lists are completed, you might ask them to add a column to their lists, and in that column to restate each combination they recorded using fractions in place of words or letters. For example, the fraction 3/4 could be used in place of "1 white," 3/8 could be used in place of "1 green," or 4/8 (1/2) could be used in place of "4 pink." Some students may suggest that 1 yellow attached end-to-end with 4 orange shows that $1/2 = 1/4 + 4/16$, or the difference between 2 green and 2 pink is 4 pink showing $6/8 - 2/8 = 1/2$. It is important to discuss such combinations. However, keep discussion of the operations at an intuitive and conceptual level. That is, rather than discussing rules for adding and subtracting fractions, emphasize the use of symbols to record what students think and observe about the segments.

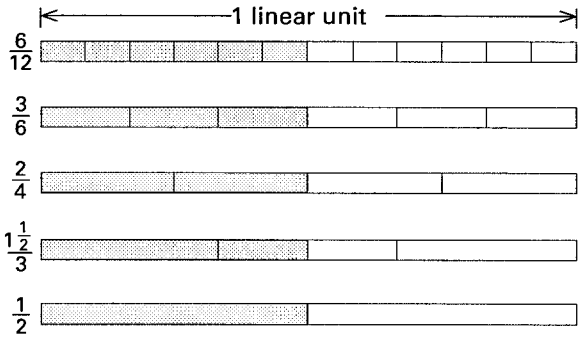
Focus Teacher Activity (cont.)

COMMENTS



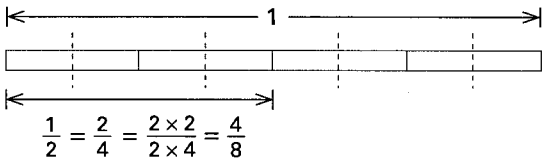
b) The use of complex fractions to represent $\frac{1}{2}$ may come up. For example, if 9 pink segments are 1 linear unit, then each segment is $\frac{1}{9}$ linear unit and $4\frac{1}{2}$ pink are $\frac{4\frac{1}{2}}{9} = \frac{1}{2}$ linear unit. This is correct, although a fraction is most commonly expressed with a whole number as the numerator and a nonzero whole number as the denominator. Viewing $\frac{1}{2} = \frac{4\frac{1}{2}}{9} = \frac{9}{18}$ can help students develop insights and motivate conjectures about methods of forming *equivalent fractions* (see Lesson 8).

Different colored segments could also be combined here to form $\frac{1}{2}$. For example, if 9 pink segments are 1 linear unit, then 1 green segment plus 1 blue segment are $\frac{1}{2}$ linear unit, i.e., $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ linear unit.

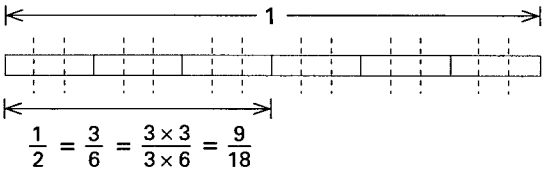


c) Using this linear unit, it is possible to show that $\frac{6}{12}$, $\frac{3}{6}$, $\frac{2}{4}$, $\frac{1\frac{1}{2}}{3}$ and $\frac{1}{2}$ are all *equivalent* fractions. Each of the strips at the left is shaded to show $\frac{1}{2}$ linear unit.

Note that fractions that represent equal portions of the unit and can be represented using all pieces of one color (e.g., $\frac{6}{12}$, $\frac{3}{6}$, $\frac{2}{4}$, $\frac{1\frac{1}{2}}{3}$, and $\frac{1}{2}$) are *equivalent* fractions, whereas expressions that represent combinations of two or more colors, such as $\frac{3}{12} + \frac{1}{4}$ or $\frac{1}{6} + \frac{4}{12}$ are *equal* to $\frac{1}{2}$.



You may wish to discuss ways to determine other fractions equivalent to $\frac{1}{2}$. One method is to divide each part of a linear unit into an equal number of parts. For example, if 4 blue segments are equal in length to 1 linear unit, then 2 blue segments have length $\frac{2}{4} = \frac{1}{2}$ linear unit. Dividing each part of the linear unit into 2 equal parts produces a length with $2 \times 4 = 8$ equal parts. Half of this length has $2 \times 2 = 4$ equal parts, as shown at left.



Similarly, dividing each part of a length equal to 1 linear unit into 3 equal parts triples the total number of parts in the unit, and triples the number of parts in $\frac{1}{2}$ linear unit. For example, if 6 pink segments are equal in length to 1 linear unit, then 3 pink segments have length $\frac{3}{6} = \frac{1}{2}$ linear unit. Dividing each part of a unit into 3 equal parts produces a length with $3 \times 6 = 18$ equal parts. Half of this length has $3 \times 3 = 9$ equal parts, as illustrated at left.

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS

4 Reveal Part 3 of Focus Master A for groups to complete. Discuss the students methods and reasoning.

Linear Model For Fractions Lesson **40**

Focus Master A

1. Find the length of 3 blue segments if 1 linear unit is the length of:

   a) 1 white segment
   b) 1/2 white segment
   c) 1/4 white segment
   d) 2 white segments
   e) 4 white segments
   f) your choice

2. Record several different combinations of segments (using 1 or more colors) that have a total length of 1/2 linear unit using the following:

   a) 1 linear unit is the length of 8 pink segments.
   b) 1 linear unit is the length of 9 pink segments.
   c) 1 linear unit is the length of 1 white segment.

3. Suppose that 1 linear unit is the length of 10 blue segments. Find several different combinations of segments (using 1 or more colors) that have total length 1/3 linear unit. Invent a way to use numbers and arithmetic symbols only to record these combinations.

4 Remind students that arithmetic symbols include: +, −, ×, ÷, =, and ( ). Some examples of fractions equivalent to $\frac{1}{3}$ are: 10 orange, or $\frac{10}{30}$; 5 pink, or $\frac{5}{15}$ (or $5 \times \frac{2}{30}$); $3\frac{1}{3}$ blue, or $3\frac{1/3}{10}$; $1\frac{2}{3}$ green, or $1\frac{2/3}{5}$; $\frac{20}{60}$ (formed by dividing each orange segment into 2 equal segments); $\frac{30}{90}$; etc.

Some students may suggest fractions such as $\frac{2\frac{1}{2}}{7\frac{1}{2}}$. In this case, $2\frac{1}{2}$ yellow have length $\frac{1}{3}$ linear unit and there are $7\frac{1}{2}$ yellow in 1 linear unit. The fraction $\frac{2\frac{1}{2}}{7\frac{1}{2}}$ is not a common format for fractions but it does represent a part-to-whole view of $\frac{1}{3}$ using yellow segments. Still others may suggest combinations such as: 1 yellow and 6 orange, or $\frac{1}{7\frac{1}{2}} + \frac{6}{30}$, or $\frac{2}{15} + \frac{6}{30}$ (depending on how they view 1 yellow); 1 white − 2 orange, or $\frac{4}{10} - \frac{2}{30}$, etc.

5 Place a transparency of Focus Master B on the overhead and reveal problem 1a) for groups to solve. Have volunteers discuss their methods at the overhead. Repeat for 1b) and/or 1c).

Lesson **40** Linear Model For Fractions

Focus Master B

1. Suppose that 4 blue segments have total length 5 linear units. Determine the length of:

   a) 1 blue segment
   b) 1 green segment
   c) your choice

5 You may wish to have groups explore and discuss one problem at a time so they can benefit from each other's ideas. Encourage students to look for more than one way to think about each problem. The intent of the problems on Master B is to bring up the division concept of a fraction (see Lessons 8, 9, and 28).

1. a) This could be determined using the division concept of a fraction: If 4 blue segments have length 5 linear units then 1 blue has length $5 \div 4 = \frac{5}{4}$ or $\frac{1}{4}$ of 5 linear units. Or, some may reason that 2 segments have length $\frac{1}{2}$ of $5 = 2\frac{1}{2}$, and so 1 segment has length $\frac{1}{2}$ of $2\frac{1}{2} = 1\frac{1}{4}$.

b) 1 green could be viewed as equal to 2 blue segments, $\frac{5}{4} + \frac{5}{4} = \frac{10}{4}$ linear units. Or, 1 green could be viewed as $\frac{1}{2}$ of 4 blue and hence, with length $5 \div 2 = \frac{5}{2}$ linear units.

Focus Teacher Activity (cont.)

ACTIONS

6 Have the groups investigate and discuss one or more of problems 2-5 on Focus Master B.

Lesson 40 Linear Model For Fractions
Focus Master B

1. Suppose that 4 blue segments have total length 5 linear units. Determine the length of:

   a) 1 blue segment

   b) 1 green segment

   c) your choice

2. Suppose 5 yellow segments have total length 7 linear units. What is the length of 12 pink?

3. If 3 green segments have total length 2 linear units, show how to draw a length of $3\frac{1}{3}$ linear units.

4. Cut off a pink strip containing 11 pink segments. Tell the length of 1 pink segment if the total length of 11 segments is:

   a) 22 linear units c) 8 linear units

   b) 27 linear units d) 1 linear unit

5. Let 1 orange segment have length $\frac{20}{7}$ linear units. Show these lengths:

   a) 20

   b) 60

   c) $\frac{60}{7}$

COMMENTS

6 2. Since 10 pink segments are equal in length to 5 yellow segments (i.e., 7 linear units), then 1 pink segment has length $7 \div 10 = \frac{7}{10}$ linear unit. Hence, 12 pink has length $7 + \frac{7}{10} + \frac{7}{10} = 7 + \frac{14}{10}$. Some students may be satisfied with $7\frac{14}{10}$, while others may suggest this is equivalent to $8\frac{4}{10}$. Still others may suggest $\frac{84}{10}$.

3. One possible line of reasoning follows. If 3 green have length 2 linear units, then 1 green has length $2 \div 3 = \frac{2}{3}$ linear unit and 2 green have length $\frac{2}{3} + \frac{2}{3} = \frac{4}{3} = 1\frac{1}{3}$ linear units. Hence, 5 green have length $3\frac{1}{3}$ linear units.

4. a) $\frac{22}{11}$ b) $\frac{27}{11}$ c) $\frac{8}{11}$ d) $\frac{1}{11}$

5. a) If one orange has length $\frac{20}{7}$ linear units then it represents one part of a length of 20 that has been divided into 7 equal parts. Hence, 7 orange must have length 20.

b) If 7 orange have length 20 linear units, then $3 \times 7 = 21$ orange have length $3 \times 20 = 60$ linear units.

c) 3 orange or 1 blue segment.

TEACHER NOTES:

Follow-up Student Activity 40.1

NAME _____ DATE _____

For this activity, you need to get a set of segment strips from your teacher.

1 Suppose 2 pink segments have total length 1 linear unit. Draw a diagram to show how to use this information to determine the length of 3 blue segments.

2 Suppose 4 green segments have a total length of 7 linear units. Draw a diagram and explain how to use this information to determine a length of $\frac{7}{4}$.

3 Suppose the length of 1 pink segment is $\frac{5}{4}$ linear units. Show and explain how to use this information to draw a length of 5 linear units.

(Continued on back.)

Follow-up Student Activity (cont.)

4 For each of the following conditions, draw and label the length described (make sure your sketch shows how your length satisfies the conditions).

a) Suppose the length of 1 white segment is 1 linear unit. Mystery Length I is formed by $3\frac{2}{3}$ blue segments placed together end-to-end. How long is Mystery Length I?

b) Suppose 3 blue segments have total length $2\frac{1}{4}$ linear units. Draw 1 linear unit.

c) Suppose the length of 1 white segment is 1 linear unit. $2\frac{1}{2}$ copies of Mystery Length II placed end-to-end measure a total of $1\frac{1}{4}$ linear units. How long is Mystery Length II?

d) Suppose 4 yellow segments have length 3 linear units and Mystery Length III is $3 \div 4$ linear units. Draw Mystery Length III.

e) Suppose the length of 1 green segment is 1 linear unit. The total length of 6 pink segments is $\frac{3}{4}$ as long as Mystery Length IV. How long is Mystery Length IV?

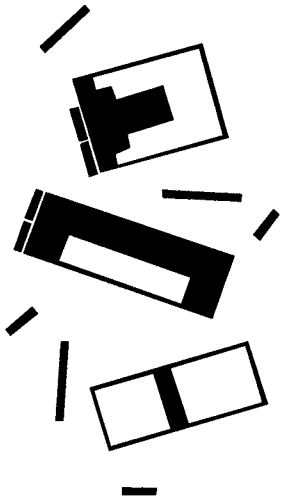
5 On another sheet, write a letter to a student who wasn't in class for the explorations with the segment strips. Send them a set of colored segment strips and explain how you think they can be used to understand the meaning of equivalent fractions and ways to determine them.

Area Model For Fractions

Lesson 41

THE BIG IDEA

Rectangles whose dimensions include portions of linear units often have areas which include portions of square units. Using segment strips and grids to explore such rectangles can lead to insights and generalizations about area and linear measure, develop understanding of fraction concepts, and lay groundwork for fraction operations.



CONNECTOR

OVERVIEW

Given various linear units students use their knowledge of relationships between linear and area measure to construct and determine the areas of squares.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Segment strips (see Lesson 40, 2 of each color except red and purple), per student.
- ✔ Segment strips for the overhead.

FOCUS

OVERVIEW

Students use segment strips and grids to determine the area and perimeter of rectangular regions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per student, 3 sheets per group of students, and 1 transparency.
- ✔ Segment strips (see Connector), 1 standard set per student.
- ✔ Colored markers, 4 colors per group.
- ✔ Focus Student Activity 41.1, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ Segment strips for the overhead.

FOLLOW-UP

OVERVIEW

Students use lengths and grids to model, compare, and combine fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 41.2, 1 copy per student.
- ✔ Segment strips, 1 standard set per student.
- ✔ 10/12-cm grid paper, 3 sheets per student.

LESSON IDEAS

TIMING
This lesson generally takes from 2 to 3 hours of class time. The intent is to use area and length as a context for reexploring the concept of a fraction.

TEACHER NOTES:

SELECTED ANSWERS

1. blue $\frac{1}{3}$ orange $\frac{1}{6}$
   red $\frac{1}{3}$ purple $\frac{1}{36}$
   green $\frac{1}{18}$ none $\frac{1}{12}$

   Note it isn't necessary that fractions be "reduced," rather they should reflect how students "see" the colored parts.

2. blue $\frac{12}{64}$ orange $\frac{6}{64}$
   red $\frac{12}{64}$ purple $\frac{1}{64}$
   green $\frac{2}{64}$ none $\frac{31}{64}$

3. Observations will vary and may refer to the 2 figures and/or the shaded parts of the figures.

4. a) $\frac{42}{144}$
   b) $3\frac{1}{4}$
   c) $\frac{1}{6}$

Connector Teacher Activity

OVERVIEW & PURPOSE

Given various linear units students use their knowledge of relationships between linear and area measure to construct and determine the areas of squares. They apply their knowledge of the meaning of fractions as parts of a whole to finding the areas of portions of squares.

MATERIALS

- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Segment strips, 1 standard set (see Lesson 40, Connector, 2 of each color except red and purple) per student.
- ✔ Segment strips for the overhead.

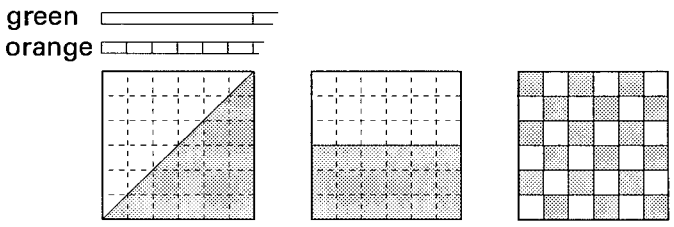
ACTIONS

1 Place the students in small groups and distribute one sheet of 10/12-cm grid paper and a set of segment strips to each student. Tell the students that for now the length of 1 green segment is 1 linear unit. Ask each student to draw a square with area 1 square unit on their grid paper, and to shade 1/2 of this square unit. Have volunteers show a few different shaded squares at the overhead and verify that each shading is 1/2 square unit.

2 Ask the students to suppose the linear unit is changed to the length of 1 white segment. Now what is the area of the square formed in Action 1? What is the area of the shaded region? What fractional part of the area of the square is shaded? Have the groups discuss their ideas and then share with the large group.

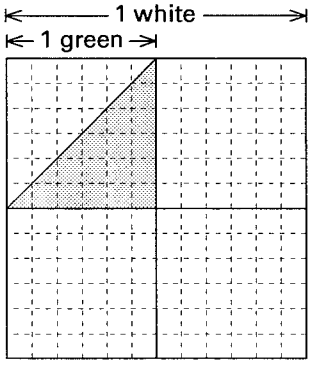
COMMENTS

1 Before asking students to shade 1/2 of their squares, you may need to have volunteers demonstrate how they formed a unit square on the grid paper. Throughout this and subsequent lessons, we use the convention that 1 linear unit is the length of the edge of 1 square unit. Notice that the length of the side of each small square on the grid paper is equal to the length of one orange segment. Hence, using the length of a green segment as the linear unit produces a square unit consisting of 36 small squares. Shading 1/2 of the square can be done in many different ways. Here are 3 examples:



(Notice that it is possible to shade 1/2 of a square in the checkerboard fashion shown above only when the number of small squares along each dimension is even.)

2 If the linear unit is changed to the length of 1 white segment, then the square drawn in Action 1 has area 1/4 square unit, as shown in the example below:



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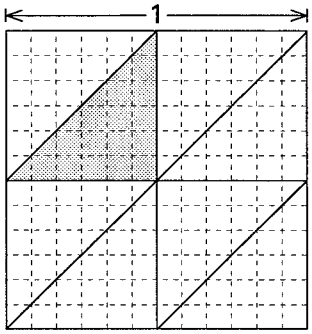
Connector Teacher Activity (cont.)

ACTIONS



3 Suggest that you have a "mystery linear unit" in mind and that, based on this mystery unit, the area of the shaded region of their square formed in Action 1 is 4½ square units. Ask the groups to determine your mystery linear unit. Discuss.

4 (Optional) Have each group select a different linear unit (from their segment strips) to use for this action. Then ask each group to draw 2 differently-sized squares, to shade ¾ of the area of each, and to determine the area of the shaded part of each square, based on their linear unit.

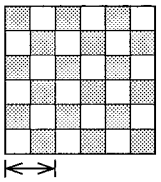
Invite a group to draw one of their shaded squares at the overhead and, without telling the linear unit, record the area of the shaded part. Invite the other groups to identify the "mystery linear unit" associated with the shaded square. Discuss their methods and reasoning. Repeat as time allows.

COMMENTS

2 (continued.)
Notice that although the shaded region covers ½ of the original square, the shaded region now has area ⅛ square unit. This can be shown by subdividing the unit square, as in the example shown at the left.

3 Allow time for groups to wrestle with this question. Some students may struggle with the idea that, for a given figure, the value of the area increases as the length of the linear unit decreases and vice versa. There are several opportunities to explore this idea in this lesson.

Here is an example of one student's diagram and line of reasoning for this problem:



1 linear unit
(1 pink segment)

"Since the 18 shaded squares have total area 4½ square units the 18 unshaded squares must also have area 4½, and the large square must have area 4½ + 4½ = 9 square units. So, the large square has sides of length 3 linear units, and 1 linear unit must be ⅓ the length of the side of the large square. This is the length of 1 pink segment (also ⅓ green or 2 orange segments)."

4 Although the *part* of the square that is shaded remains constant (¾), the *value* of the area of the shaded part changes as the units of length and area change.

Students may find it helpful to draw each square in question on grid paper in order to determine the unit.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use segment strips and grids to determine the area and perimeter of rectangular regions. Given regions that are partially shaded, students use their understanding of fractions together with their strategies for finding area (see Lesson 19) to determine the area of the shaded part of each region. This activity provides a context for strengthening understanding of area and linear relationships and for building intuitions about fraction operations which are explored further in Lessons 42 and 43.

MATERIALS

- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per student, 3 sheets per group of students, and 1 transparency.
- ✔ Segment strips, 1 standard set per student.
- ✔ Colored markers, 4 colors per group.
- ✔ Focus Student Activity 41.1, 1 copy per student and 1 transparency.
- ✔ Focus Master A, 1 copy per group and 1 transparency.
- ✔ Segment strips for the overhead.

ACTIONS

1 Place the students in small groups and give a sheet of 10/12-cm grid paper and a set of segment strips to each student. Give 4 colored markers to each group. Tell the students that for now the length of 3 blue segments is 1 linear unit.

Ask each student to: draw (on the grid paper) a square with area 1 square unit; shade 5/9 of the square in one color; shade 3/4 of what is left another color; and determine what part of the square is not colored. Discuss different methods of reasoning about the area of the uncolored region.

COMMENTS

1 It is expected that students use their intuitions and conceptual understandings of fractions, rather than paper-and-pencil algorithms throughout this activity. More discussions of computation occur in Lesson 42-44 and in *Visual Mathematics, Courses II-IV.*

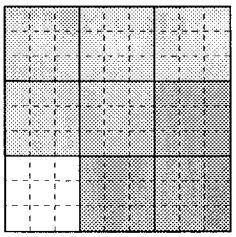
One method of determining the unshaded amount is shown below:

a) Subdivide the square into 9 equal regions.

b) Color 5 out of 9 regions, leaving 4 uncolored regions.

c) Color 3 out of the 4 uncolored regions, leaving 1 uncolored region.

d) The uncolored region represents 1 out of 9 total regions, or 1/9 of the unit square.



Focus Teacher Activity (cont.)

ACTIONS

2 Tell the students that the length of 1 pink segment is now the linear unit. Ask them to use this linear unit and determine the following about the square they formed in Action 1:

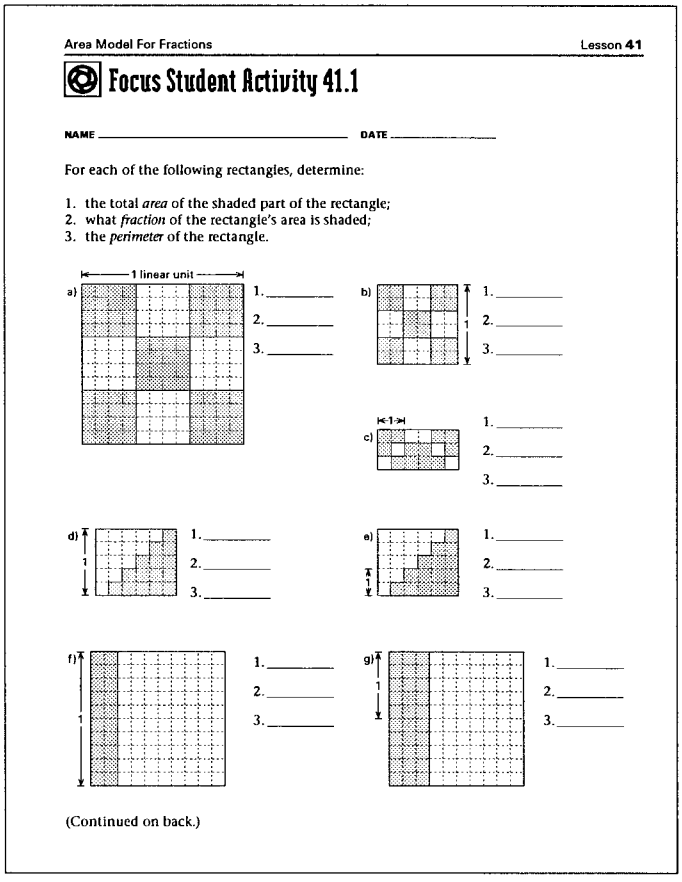
a) the total *area* of the colored parts of the square;

b) what *fraction* of the area of the square is colored;

c) the *perimeter* of the square.

Discuss their methods and reasoning.

3 Give each student a copy of Focus Student Activity 41.1. Have them find the requested information for figures a)-e). Discuss. Repeat for figures f)-j), and then k)-n).



Area Model For Fractions Lesson 41

Focus Student Activity 41.1

NAME _____ DATE _____

For each of the following rectangles, determine:

1. the total *area* of the shaded part of the rectangle;
2. what *fraction* of the rectangle's area is shaded;
3. the *perimeter* of the rectangle.

a) 1 linear unit 1. _____ 2. _____ 3. _____

b) 1. _____ 2. _____ 3. _____

c) 1. _____ 2. _____ 3. _____

d) 1. _____ 2. _____ 3. _____

e) 1. _____ 2. _____ 3. _____

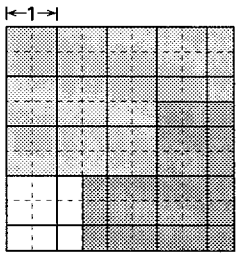
f) 1. _____ 2. _____ 3. _____

g) 1. _____ 2. _____ 3. _____

(Continued on back.)

COMMENTS

2 Using the length of 1 pink segment as the linear unit, the area of the colored region is 18 square units; $^{8}/_{9}$ or $^{16}/_{18}$ is the part of the square that is colored; and the perimeter is 18 linear units (each side of the square has length $4\frac{1}{2}$ linear units).



3 You may wish to assign only selected figures from a)-n), based on the needs, interest, and comfort of your students.

Encourage students to talk over their ideas and questions with their groupmates as they work. During class discussion emphasize strategies students use rather than answers, which are listed below (areas are in square units and perimeters are in linear units):

a) $^{5}/_{9}$, $^{5}/_{9}$, 4;

b) $^{5}/_{9}$, $^{5}/_{9}$, 4;

c) 3, $^{4}/_{6}$, 9;

d) $^{15}/_{25}$, $^{1}/_{2}$, $4\frac{2}{5}$;

e) $3\frac{3}{4}$, $^{1}/_{2}$, 11;

f) $^{2}/_{10}$, $^{2}/_{10}$, 4;

g) $1\frac{1}{5}$, $^{3}/_{10}$, 8;

h) $^{5}/_{8}$, $^{1}/_{4}$, 7;

i) $^{1}/_{4}$, $^{1}/_{2}$, 3;

j) $^{3}/_{8}$, $^{1}/_{6}$, 6;

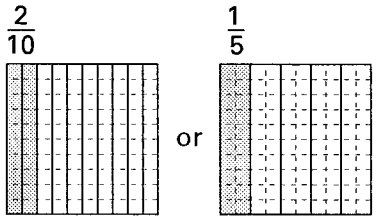
k) 30, $^{30}/_{60}$, 46;

l) $3\frac{1}{3}$, $^{1}/_{2}$, $15\frac{1}{3}$;

m) $^{25}/_{64}$, $^{25}/_{64}$, 4;

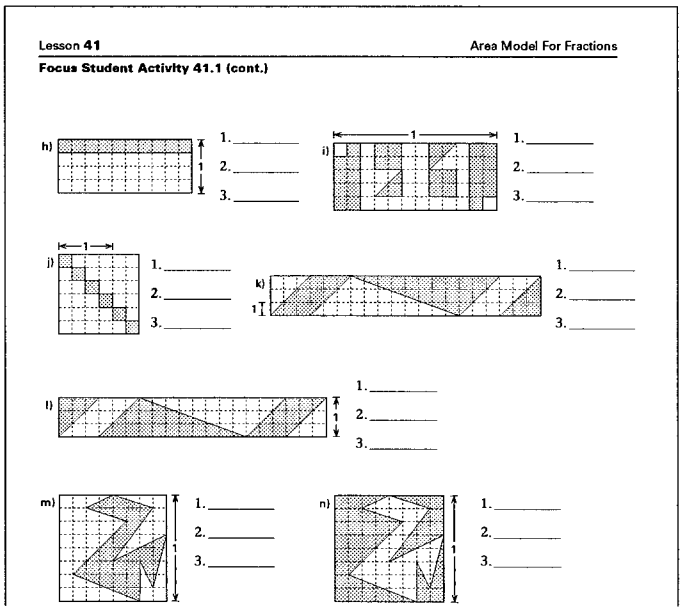
n) $^{39}/_{64}$, $^{39}/_{64}$, 4.

Students may suggest equivalent fractions to those listed above depending on how they view each region. For example, some students may view the area of the shaded part of rectangle f) as $^{2}/_{10}$ while others may see it as $^{1}/_{5}$, as illustrated below:

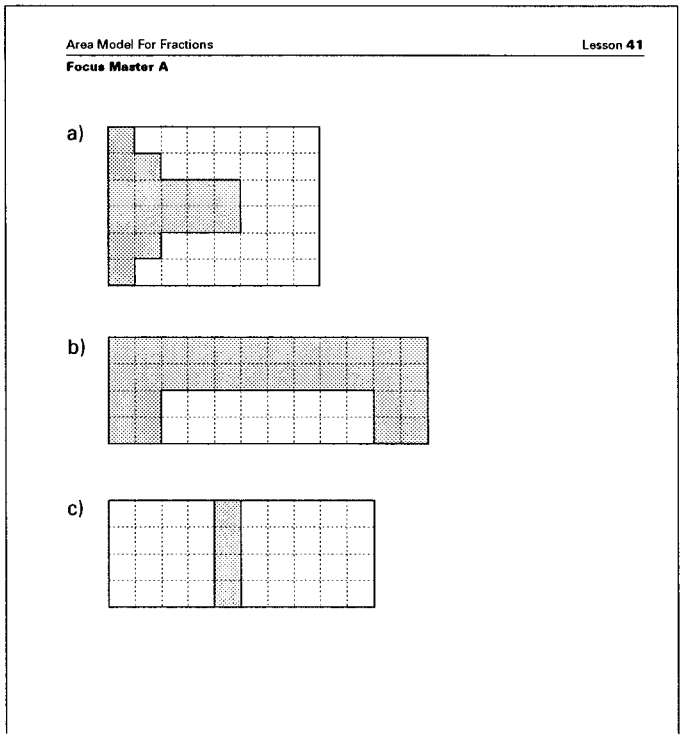


Focus Teacher Activity (cont.)

ACTIONS


Lesson 41 Area Model For Fractions
Focus Student Activity 41.1 (cont.)
h) 1. 2. 3.
i) 1. 2. 3.
j) 1. 2. 3.
k) 1. 2. 3.
l) 1. 2. 3.
m) 1. 2. 3.
n) 1. 2. 3.

4 Tell the students that for now the length of 2 yellow segments is 1 linear unit. Give each group a copy of Focus Master A and ask them to make as many mathematical observations as they can about the three rectangles, including ways they are alike and ways they are different. Allow the groups plenty of time to explore relationships and to record their observations. Discuss.


Area Model For Fractions Lesson 41
Focus Master A
a)
b)
c)

COMMENTS

It is not important that fractions be "reduced" (placed in *lowest terms*), although you may wish to point out that when two different subdivisions of a region are represented by equivalent fractions, one is said to be a "reduced" form of the other if it contains fewer (i.e., larger) parts. Hence, $\frac{1}{5}$ is a reduced form of $\frac{2}{10}$ in the preceding example.

4 It is impossible to predict what students may observe. For example, some groups may compare or compute the sums and differences of the areas and perimeters of the rectangles. They may notice that rectangles a) and b) have the same areas but different perimeters, while a) and c) have different areas and the same perimeters. Groups may also compare fractional parts of the rectangles that are shaded. Others may compute the decimal (or percent) that represents the fraction of each shape that is shaded. Still other groups may compute what fractional part the shaded portion of one rectangle is of the shaded portion of another rectangle. And so on.

Encourage students to explore their own "what-if" questions (e.g., what if we combined the shaded regions; which is greater, the total shaded or the total unshaded; etc.).

TEACHER NOTES:

Follow-up Student Activity 41.2

NAME _____ DATE _____

1 Let the length of 1 green segment be 1 linear unit. Draw a unit square on the attached grid paper. Then do the following:

Color $\frac{1}{3}$ of the square blue.
Color $\frac{1}{2}$ of the uncolored part red.
Color $\frac{1}{6}$ of what is now uncolored green.
Color $\frac{3}{5}$ of what is now uncolored orange.
Color $\frac{1}{4}$ of what is left purple.
Cut out your colored square and tape it at the right.

Now tell what part of the unit square is not colored: _____

Tell the area of the region covered by each color.

Blue: _____ Orange: _____

Red: _____ Purple: _____

Green: _____

2 Suppose the linear unit is the length of 2 yellow segments. Using this linear unit, what is the area of each differently-colored region on your drawing in Problem 1? Explain or draw a diagram to show how you decided these areas.

Blue: _____

Red: _____

Green: _____

Orange: _____

Purple: _____

(Continued on back.)

Follow-up Student Activity (cont.)

3 Suppose the length of 1 yellow segment is $\frac{2}{3}$ linear unit. Write at least 6 mathematical observations that involve fractions and are about relationships between Figure A and Figure B below.

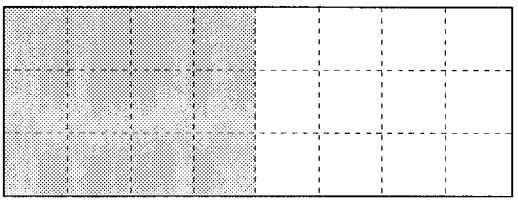


Figure A

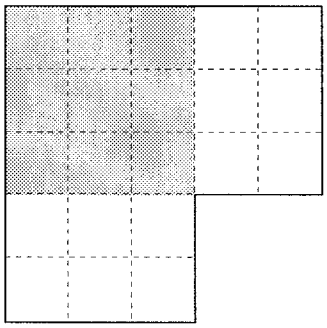


Figure B

4 Suppose the length of 3 yellow segments is 1 linear unit. On the attached grid paper draw a polygon with area 1 square unit. Subdivide your polygon into 4 colored regions (red, blue, green, and yellow) so that:

- There are no gaps or overlaps of colors.
- $\frac{1}{6}$ of the polygon is red.
- $\frac{3}{8}$ of the polygon is blue.
- $\frac{3}{11}$ of what is neither red nor blue is green.
- There is $2\frac{2}{3}$ as much yellow as there is green.

Next to your polygon, please do the following:

a) tell what part of the polygon is covered by the red and green regions combined;

b) tell how many red regions it would take to exactly cover a blue region and explain how you determined this;

c) determine the difference between the area of the yellow region and the area of the red region and explain your methods;

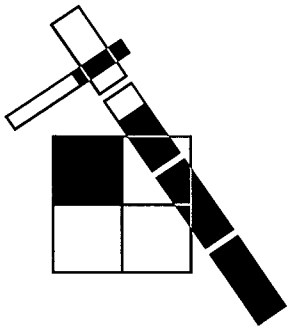
d) write three more interesting questions involving fractions about your colored polygon and then answer your questions.

Modeling Fraction Operations

Lesson 42

THE BIG IDEA

Fraction lengths and areas provide a concrete setting in which to explore situations involving fractions. Many insights about fraction operations can emerge from observing and discussing models of such situations.



CONNECTOR

OVERVIEW

Students record and discuss their current understandings of the meanings of the basic operations. Students also use their knowledge of the part-to-whole and division concepts of a fraction to relate fractions to areas and lengths.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Student journals (optional).
- ✔ Connector Student Activity 42.1, 1 copy per group of students and 1 transparency.

FOCUS

OVERVIEW

Students relate their actions with segment strips and grids to their understanding of the meaning of the four basic operations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set (2 of each color except red and purple) per student.
- ✔ $^{10}/_{12}$-cm grid paper (see *Blackline Masters*), 4 sheets per group and several transparencies.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group of students and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Segment strips for the overhead.
- ✔ Colored markers and tape or glue for each group.
- ✔ Blank paper, several sheets per group (see Comments 2 and 3).

FOLLOW-UP

OVERVIEW

Students use segment strips and grids to create and solve fraction computation problems.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 42.2, 1 copy per student.
- ✔ Segment strips, 1 standard set per student.
- ✔ $^{10}/_{12}$-cm grid paper, 4 sheets per student.

LESSON IDEAS

JOURNALS

After this lesson is completed have the students complete the following thought starters in their journals:

"Regarding computing with fractions, something I understand for the first time or feel more confident about is..."

"What has helped my understanding or confidence most is..."

"What I still wonder about is..."

LOOKING AHEAD

Keep emphasis on understanding relationships and working with models. These experiences will lead to generalizations and the invention of algorithms in Lesson 44 and in *Visual Mathematics, Courses II-IV.*

FOLLOW-UP

Because responses may vary widely on these problems, you may wish to involve students in providing feedback to each other before collecting their work. See "Using Follow-ups" in *Starting Points* for ideas.

SELECTED ANSWERS

Responses will vary.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students record and discuss their current understandings of the meanings of the basic operations. Students also use their knowledge of the part-to-whole and division concepts of a fraction to relate fractions to areas and lengths. In the Focus activity, they will apply these ideas to computations with fractions.

MATERIALS

- ✔ Student journals (optional).
- ✔ Connector Student Activity 42.1, 1 copy per group of students and 1 transparency.

ACTIONS

1 Write one or more of the following thought starters on the overhead and ask the students to spend 10-15 minutes writing their ideas. Discuss.

The way I think about the meaning (or meanings) of each operation (addition, subtraction, multiplication, and division) is...

Here are diagrams that illustrate each different way I have of viewing the meaning of each operation:

Some ways my understanding of the meanings of the operations has been clarified or developed this year include:

2 Place the students in pairs and give each pair a copy of Connector Student Activity 42.1 (see next page). Ask the pairs to complete 1a). Discuss and then have them finish the activity sheet. Have volunteers demonstrate their reasoning about selected problems at the overhead.

COMMENTS

1 Their ideas could be recorded in their journals, providing documentation of the development of their understanding. After students complete their responses to the first two thought starters, you could have them look back at prior journal entries to make comparisons.

It is important to discuss the difference and take-away methods of subtraction, the area and repeated addition methods of multiplication, and the grouping, sharing, and area methods of division. These were first introduced in Lesson 2, Basic Operations, and are drawn upon extensively in the Focus activity. You may wish to encourage students to add to their journal entries new ideas, and/or examples, that come up during discussion.

2 There are many possible ways of viewing the areas in 1a)-1f). A few possibilities are listed below. Students who have difficulty writing equations might write their observations in words first.

1a) Students may notice that each of the first two shaded rectangles has area $\frac{1}{2}$ square unit and the third rectangle has area $\frac{7}{8}$ of $\frac{1}{2}$. Some equations to represent the shaded part might be: $\frac{1}{2} + \frac{1}{2} + (\frac{7}{8} \times \frac{1}{2}) = 1\frac{7}{16}$; $2\frac{7}{8} \times \frac{1}{2} = 1\frac{7}{16}$; $\frac{1}{2} + \frac{1}{2} + \frac{7}{16} = 1\frac{7}{16}$; etc. Note that if students write $\frac{1}{2} + \frac{1}{2} + (\frac{7}{8}$ of $\frac{1}{2}) = 1\frac{7}{16}$, you may wish to point out that words are not generally used in equations; however, if using the word "of" helps students understand their recordings, you may wish to allow it for now.

1b) Some examples of equations that may come up include: $\frac{12}{64} + \frac{12}{64} + \frac{9}{64} = \frac{33}{64}$; $(2 \times \frac{12}{64}) + \frac{9}{64} = \frac{33}{64}$; $\frac{12}{64} + \frac{12}{64} + (\frac{3}{4} \times \frac{12}{64}) = \frac{33}{64}$; etc.

1c) $\frac{7}{16} \times 1 = \frac{7}{16}$; $\frac{1}{4} + \frac{3}{16} = \frac{7}{16}$; $\frac{3}{8} + (\frac{1}{2} \times \frac{1}{8}) = \frac{7}{16}$; etc.

1d) $\frac{7}{16} \times 2 = \frac{14}{16}$; $\frac{7}{16} + \frac{7}{16} = \frac{14}{16}$; etc.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Modeling Fraction Operations — Lesson 42

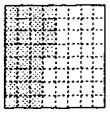
Connector Student Activity 42.1

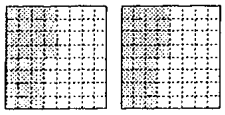
NAME _____ DATE _____

1 Suppose that this length ⟷ is 1 linear unit and is the length of the side of 1 square unit. For each of the following problems find the total area of the shaded parts. Write at least 2 different equations that could represent ways of "seeing" each total area.

a)

b)

c) 

d) 

e)

f)

(Continued on back.)

COMMENTS

2 (continued.)
1e) $7/16 \times 1/8 = 3^{1/2}/64$; $3^{1/2}/8 \times 1/8 = 7/128$; etc.

1f) "1/4 of 2 is 2/4 of 1" or $1/4 \times 2 = 2/4 \times 1$; $16/64 + 16/64 = 32/64$; etc.

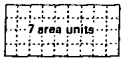
2a-b) One way to solve these is to use the division concept of a fraction, e. g., viewing 7/4 area units as $7 \div 4$ or 1/4 of 7 and 7/3 as $7 \div 3$ or 1/3 of 7.

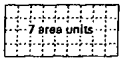
3d) This problem is a linear version of 2a), and hence, 7/4 linear units can be found by dividing a length of 7 into 4 equal lengths of $7 \div 4 = 1/4$ of $7 = 7/4$.

Lesson 42 — Modeling Fraction Operations

Connector Student Activity 42.1 (cont.)

2 Shade the indicated part of each of these rectangles.

a) 7/4 area units: 

b) 7/3 area units: 

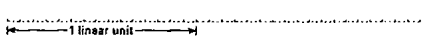
c) 5/6 area unit:

d) 19/12 area unit:

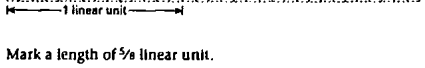
e) 6/5 area units:

3 For each problem below, mark the indicated lengths.

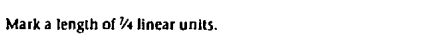
a) Mark a length of 1 3/5 linear units on the segment below.



b) Mark a length of 2 2/7 linear units below.



c) Mark a length of 5/8 linear unit.



d) Mark a length of 7/4 linear units.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students relate their actions with segment strips and areas to their understanding of the meanings of the four basic operations. This prepares students for Lesson 43, where they model and draw conclusions about everyday situations involving fractions, and for Lesson 44, where they invent algorithms for adding and subtracting fractions.

MATERIALS

- ✔ Segment strips, 1 standard set (2 of each color except red and purple) per student.
- ✔ $^{10}/_{12}$-cm grid paper (see *Blackline Masters*), 4 sheets per group and several transparencies.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group of students and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Segment strips for the overhead.
- ✔ Colored markers and tape or glue for each group.
- ✔ Blank paper, several sheets per group (see Comment 2).

ACTIONS

1 Arrange the students in groups of 2-4. Distribute a set of segment strips, 1 sheet of $^{10}/_{12}$-cm grid paper and scissors to each group. Place a transparency of Focus Master A on the overhead for the groups to complete. Have volunteers sketch their results on transparencies of $^{10}/_{12}$-cm grid paper and discuss their methods and reasoning at the overhead.

Modeling Fraction Operations Lesson 42

Focus Master A

A rectangle has dimensions $^{4}/_{5}$ linear unit by $^{2}/_{3}$ linear unit. Use segment strips and $^{10}/_{12}$-cm grid paper to determine the following about the rectangle:

a) its perimeter,

b) its area,

c) the difference between its length and width.

2 Give each group 2-3 additional sheets of $^{10}/_{12}$-cm grid paper, several blank sheets of paper, some tape or glue, colored markers, and a copy of Focus Master B. Ask the groups to create a model of Situation a) and to tape or glue the model to a blank sheet of paper. Then ask the groups to write (under their models) several observations about mathematical relationships they notice in their model. Have volunteers share their models and observations. For selected verbal observations that are given by students, ask the class to invent equations to represent those observations.

COMMENTS

1 Note that issues related to understanding relationships and differences between area and length resurface here, and groups may be challenged by deciding the size of the unit to use. Hence, allow groups plenty of time to explore this rectangle before large group discussion.

To save time, you may wish to distribute transparencies of $^{10}/_{12}$-cm grid paper and overhead pens to selected groups and have them record their rectangles prior to coming to the overhead.

The perimeter of this rectangle is $2^{14}/_{15}$ linear units and the area is $^{8}/_{15}$ square unit (accept fractions in whatever form represents how students "see" the area and perimeter). The difference between the length and width is $^{2}/_{15}$ linear unit.

If some groups finish before others are ready to discuss, pose other rectangles, such as $1^{1}/_{7} \times {}^{3}/_{4}$ and ${}^{5}/_{6} \times {}^{3}/_{5}$, and have groups find the area, perimeter, and difference between the length and width of each of these.

2 The purpose here is to get students to reveal their thinking as they explore and model an open-ended situation about which many mathematical observations are possible. These observations may differ among the groups, depending on their models and on their views of the meanings of the operations. To encourage collaboration and careful thought, have the groups record their observations using a marking pen so they can post the sheets (standard $8\frac{1}{2}" \times 11"$ or $8\frac{1}{2}" \times 14"$ sheets work fine) containing their models and observations for viewing and comparison by the other groups.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson 42 Modeling Fraction Operations
Focus Master B

Situations

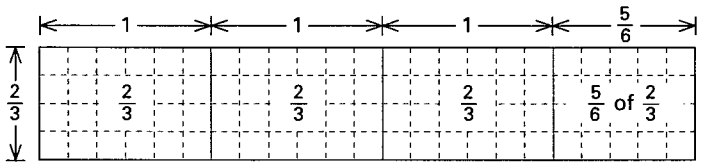
a) $3\frac{5}{6}$ copies of a rectangle with area $\frac{2}{3}$ are joined together to form a large rectangle.

3 Ask the groups to form models and make mathematical observations for three or more of Situations b)-l) from Focus Master B (you pick or have them pick). Ask them to sketch or tape each model on a separate sheet of blank paper and to record their observations next to their models. Encourage students to write, whenever possible, equations to represent their observations. Have volunteers post and explain their results.

COMMENTS

2 (continued.)
If groups have difficulty getting started with observations, you might ask a group to sketch their model at the overhead and then ask the class, "What can you say about this model?" Record several observations as they are suggested by students, and then ask the class to work together to invent equations or number statements that represent selected observations.

Notice that no unit is defined for this situation. Some students may, for example, choose the length of 6 orange as the linear unit and form the following model (note students may form other rectangles that satisfy the conditions in Situation a) but have different dimensions than those shown):



To initiate observations you might also ask questions such as: What is the difference between the length and width of the large rectangle shown? The length is how many times as long as the width? What is the area of the rectangle? etc.

Keep in mind that observations will vary. It is intended that all fraction operations be computed here by counting areas and lengths rather than by algorithms.

3 Rather than discussing each situation orally, you might post together all the groups' sheets of observations for each assigned situation and allow time for students to study and compare the different ideas. Then discuss their thoughts. Some of these situations could be posed as homework.

Note that it is necessary to use different units for several of these problems in order to conveniently represent the given fractions. You may occasionally wish to add an observation or pose a question about a relationship that you notice in a model. Here are one or two possible observations about each situation. It is not necessary that these come up; students may make others.

b) There are $3\frac{1}{3}$ regions of area $\frac{3}{5}$ contained in 2 square units, or based on the grouping method of division, $2 \div \frac{3}{5} = 3\frac{1}{3}$.

Focus Teacher Activity (cont.)

ACTIONS

Lesson 42 Modeling Fraction Operations
Focus Master B

Situations

a) $3\frac{5}{6}$ copies of a rectangle with area $\frac{2}{3}$ are joined together to form a large rectangle.

b) A rectangle with area 2 is subdivided into regions with area $\frac{3}{5}$ square unit.

c) A rectangle has area $1\frac{1}{3}$ square units. A certain portion of this rectangle has area $\frac{1}{2}$ square unit.

d) This segment is $1\frac{1}{2}$ times as long as $2\frac{1}{10}$ linear units.

e) This rectangle has area $\frac{1}{3}$ square unit and one dimension $\frac{2}{3}$ linear unit.

f) A segment with length $2\frac{3}{4}$ linear units is subdivided into lengths of $\frac{2}{6}$ linear units.

g) A rectangle has dimensions $1\frac{2}{5}$ linear units by $2\frac{1}{4}$ linear units.

h) One segment has length $\frac{1}{2}$ linear unit and another segment has length $2\frac{7}{8}$ linear units.

i) One rectangle has area $1\frac{2}{3}$ square units. Another rectangle has area $2\frac{4}{12}$ square units.

j) Each of these 5 rectangles has area 1 square unit and each has 1 dimension less than 1 linear unit.

k) $\frac{1}{2}$ of $\frac{1}{3}$ is equal to $\frac{1}{3}$ of $\frac{1}{2}$. Here are 5 additional pairs of fractions for which this relationship is also true.

l) You make up 3 situations to model.

4 Write the following on the overhead.

a) $\frac{1}{2} + \frac{1}{5}$ c) $\frac{1}{2} \times \frac{1}{5}$

b) $\frac{1}{2} - \frac{1}{5}$ d) $\frac{1}{2} \div \frac{1}{5}$

Ask the groups to invent ways to use segment strips and/or $\frac{10}{12}$-cm grids to carry out the above computations. Encourage groups to find more than one way to use lengths and/or areas to compute the answers. Have volunteers share their group's ideas at the overhead.

COMMENTS

c) The portion of the rectangle with area $\frac{1}{2}$ square unit is $\frac{3}{8}$ of the rectangle, so $\frac{3}{8}$ of $1\frac{1}{3}$ is $\frac{1}{2}$, or $\frac{3}{8} \times 1\frac{1}{3} = \frac{1}{2}$.

d) The length of the segment is $2\frac{1}{10} + \frac{1}{2}$ of $2\frac{1}{10}$, or $3\frac{3}{20}$ linear units; i.e., $2\frac{1}{10} + (\frac{1}{2} \times 2\frac{1}{10}) = 3\frac{3}{20}$, or $1\frac{1}{2} \times 2\frac{1}{10} = 3\frac{3}{20}$.

e) The other dimension is $\frac{1}{3} \div \frac{2}{3} = \frac{1}{2}$ linear unit.

f) There are $2\frac{3}{4} \div \frac{2}{6} = 8\frac{1}{4}$ lengths of $\frac{2}{6}$ linear units contained in $2\frac{3}{4}$ linear units.

g) the area of the rectangle is $1\frac{2}{5} \times 2\frac{1}{4} = 3\frac{3}{20}$ square units; the perimeter is $1\frac{2}{5} + 1\frac{2}{5} + 2\frac{1}{4} + 2\frac{1}{4} = 7\frac{3}{10}$ linear units.

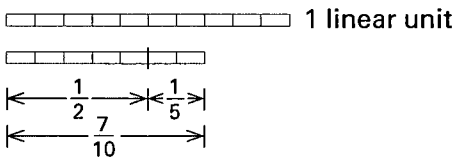
h) Their total length is $\frac{1}{2} + 2\frac{7}{8} = 3\frac{3}{8}$ linear units; the longer segment is $2\frac{7}{8} \div \frac{1}{2} = 5\frac{3}{4}$ times as long as the shorter one; $5\frac{3}{4}$ copies of a length of $\frac{1}{2}$ have total length $2\frac{7}{8}$ linear units, or $5\frac{3}{4} \times \frac{1}{2} = 2\frac{7}{8}$ linear units.

i) The difference in their areas is $2\frac{4}{12} - 1\frac{2}{3} = \frac{2}{3}$ square unit.

j) Here are the dimensions of 5 possible rectangles: $\frac{1}{2} \times 2$; $\frac{1}{3} \times 3$; $\frac{3}{2} \times \frac{2}{3}$; $\frac{5}{6} \times \frac{6}{5}$; and $\frac{3}{5} \times 1\frac{2}{3}$. Whenever one dimension is less than 1 linear unit and the area is 1 square unit, then the other dimension is greater than 1 linear unit. Notice that the dimensions of each of these rectangles must be reciprocals because their product (i.e., the area of each rectangle) is 1; this illustrates the fact that for any fraction $\frac{a}{b}$ (with $a \neq 0$ and $b \neq 0$) and its reciprocal $\frac{b}{a}$, $\frac{a}{b} \times \frac{b}{a} = 1$).

k) $\frac{1}{2} \times \frac{1}{5} = \frac{1}{5} \times \frac{1}{2}$, $\frac{2}{3} \times \frac{1}{4} = \frac{1}{4} \times \frac{2}{3}$, $\frac{3}{4} \times \frac{2}{3} = \frac{2}{3} \times \frac{3}{4}$, etc. This illustrates the commutative property (see Lesson 3) for multiplication of fractions: $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$.

4 a) To solve $\frac{1}{2} + \frac{1}{5}$ using linear representations of the fractions, it is necessary to use a linear unit that can be divided into 2 equal parts (to show halves) and into 5 equal parts (to show fifths). For example, if the length of 10 segments (of any color) is 1 linear unit, then it is possible to model $\frac{1}{2} + \frac{1}{5}$, as shown below:



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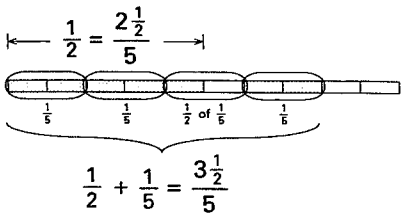
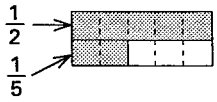
Focus Teacher Activity (cont.)

ACTIONS



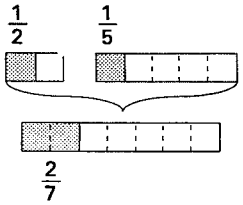
COMMENTS

4 (continued.)
Similarly, to carry out this computation using area representations of the fractions, it is necessary to use a grid that can be subdivided into 2 regions of equal area and 5 regions of equal area. Hence, using a rectangle of 10 (or a multiple of 10) small squares as the unit, ½ + ⅕ = 7/10:



Note that because students are used to working with square area units, many will choose a 10 × 10 square as 1 unit and then combine ½ unit + ⅕ unit.

Students may bring up a variety of methods that work and some that don't. It is important to explore both. For example, the sum 2/7 is often given for ½ + ⅕. Sometimes this is because students think one simply "adds the numerators and adds the denominators" to get the sum. Or, sometimes students use different units to model each fraction, as shown here, and get the incorrect sum:



Standard algorithms grew out of a need for responses to certain types of problems. That is, agreement to interpret notation and use certain procedures came about historically because there were problems that required a particular solution. You may need to point out that, by agreement, ½ + ⅕ typically implies ½ unit plus ⅕ unit, for a single area unit. In the preceding example, the student used 3 different area units. You may also encounter students who interpret ½ + ⅕ as meaning "½ + ⅕ of ½." Although this is a reasonable interpretation of the symbols, it is not the conventional interpretation.

Some students may view ½ + ⅕ as illustrated at the left and, hence, may report ½ + ⅕ as the complex fraction $3^{1/2}/5$. This is a correct sum, although it is more common to see fractions written with whole numbers as the numerator and denominator.

Encourage use of both the take-away and difference models with linear and area representations.

Focus Teacher Activity (cont.)

ACTIONS

5 Repeat Action 4 for several computations selected from the following:

a) $\frac{5}{6} + \frac{3}{4}$

b) $2\frac{1}{3} + 1\frac{1}{7}$

c) $\frac{3}{4} + 1\frac{3}{5}$

d) $\frac{3}{5} - \frac{1}{3}$

e) $4 - 2\frac{2}{3}$

f) $1\frac{1}{8} - \frac{9}{10}$

g) $\frac{3}{5} \times \frac{1}{3}$

h) $\frac{3}{2} \times \frac{2}{3}$

i) $\frac{5}{6} \times 1$

j) $\frac{3}{4} \div \frac{1}{2}$

k) $\frac{1}{3} \div 3$

l) $2\frac{1}{7} \times 1\frac{1}{2}$

m) $2\frac{1}{7} \div 1\frac{1}{2}$

n) $\frac{2}{3} \div \frac{2}{3}$

o) $1\frac{2}{3} \times 1\frac{1}{8}$

p) $5\frac{1}{2} \div 2\frac{9}{10}$

q) $3 - 1\frac{7}{8}$

r) $2\frac{3}{4} \times 1\frac{5}{8}$

s) student choice.

COMMENTS

Encourage students to use the area and repeated addition interpretations of multiplication and the grouping, sharing, and area models for division.

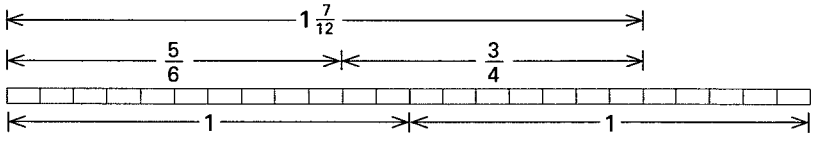
Rather than discussing algorithms for computing fractions now, keep students working at a conceptual level, emphasizing solving problems with concrete materials and verbalizing their understandings. In Lesson 44, students will invent algorithms that are representations of their actions with lengths and areas.

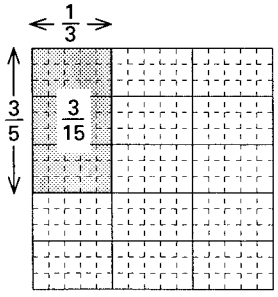
5 Some of these (or other) computations could be assigned as homework prior to assigning the Follow-up. In this case, be sure to provide segment strips and grids and ask students to use these materials (or sketches of them) to solve the problems. It may be helpful to have students work and discuss some of these computations one at a time, so they can observe and try new methods other students suggest. You might also invite students to pose other "challenge computations" for the class to discuss and solve using particular methods suggested by students.

To lay groundwork for generalizing methods, before having the students use the segment strips or grids to solve selected computations, ask them to imagine and predict the linear unit needed to solve that computation and to discuss in their groups the procedures they would use.

Note that there are many ways of interpreting and modeling each computation. Here are some examples for a), g), and j).

a) $\frac{5}{6} + \frac{3}{4}$ requires using a unit that can be divided into both sixths and fourths. Hence, a unit with 12 parts (or any multiple of 12) can be used, as shown here:



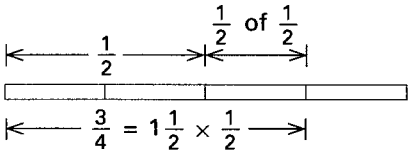


g) To show $\frac{3}{5} \times \frac{1}{3}$, students may choose a linear unit of 15 (or any multiple of 15) segments and form a rectangle with dimensions $\frac{3}{5}$ linear unit $\times \frac{1}{3}$ linear unit, and area $\frac{3}{15}$ square unit, as illustrated at the left. Or,

(Continued next page.)

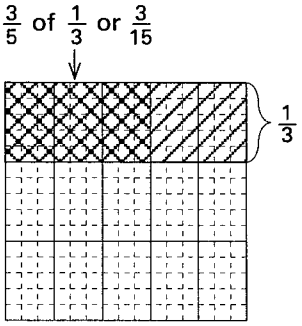
Focus Teacher Activity (cont.)

ACTIONS



COMMENTS

5 (continued.)
students may form an area of 1/3 square unit and then determine 3/5 of 1/3, as shown below.

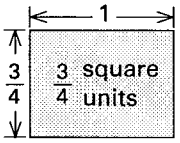


Yet another method is to find 3/5 of a length of 1/3 linear unit.

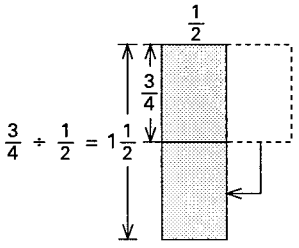
j) 3/4 ÷ 1/2 could be viewed, for example, as "How many groups of 1/2 are there in 3/4?" The diagram at the left shows there are 1 half and 1/2 of another half, or 1 1/2 halves in 3/4. Hence, 3/4 ÷ 1/2 = 1 1/2.

Another interpretation of 3/4 ÷ 1/2 is based on the area model: "Given an area of 3/4 and one dimension 1/2, what is the other dimension?" One way to determine this follows:

Start with a rectangle with area 3/4 and dimensions $1 \times 3/4$.



Cut apart and rearrange the rectangle to form a new rectangle with area 3/4, one dimension 1/2, and the other dimension 3/4 ÷ 1/2 = 1 1/2.



Note that for the area model for division, it is often easiest to start with a rectangle of the given area with a dimension of 1 linear unit. Then cut and rearrange the rectangle to obtain a new rectangle with the desired dimensions.

Follow-up Student Activity 42.2

NAME _____ DATE _____

1 On another sheet of paper, tape or glue a model of each of the following situations. Then, next to each model write at least 3 observations regarding mathematical relationships you can "see" in the model. Whenever possible write number statements or equations to represent your observations.

a) $2\frac{1}{8}$ copies of a length of $\frac{1}{2}$ linear unit are joined together end-to-end.

b) This rectangle has dimensions $\frac{5}{6}$ linear unit by $\frac{4}{5}$ linear unit.

c) These 2 rectangles both have area $1\frac{1}{8}$ square units, but their perimeters are different.

d) If $3\frac{1}{3}$ copies of a certain Mystery Length are placed end-to-end, their total length is 5 linear units.

2 Use your segment strips only to find the answer to each of the following. On another sheet, draw or tape the strips to illustrate your methods. Label each diagram and add a few comments to explain each step of your methods with the strips.

a) $\frac{3}{8} + \frac{1}{3}$ b) $\frac{4}{5} - \frac{1}{4}$ c) $\frac{3}{4} \times \frac{5}{7}$ d) $\frac{5}{6} \div \frac{3}{4}$

3 Repeat Problem 2 but this time using area representations of each problem. Cut out your models from $\frac{10}{12}$-cm grid paper and tape them in place on another sheet. Label your models and add comments to explain each step of your thought processes.

4 On the attached grid paper, write 4 fraction computations, 2 easy and 2 hard. (Don't use the problems from 2 and don't solve the problems yet!).

a) Label the 2 computations you think are easy and the 2 that are hard. Tell why you think each problem is easy or hard.

(Continued on back.)

Follow-up Student Activity (cont.)

b) Now show how to solve the 4 problems using segment strips or area representations on grid paper.

5 Suppose the length of 1 white segment is 1 linear unit. Cut out 3 noncongruent rectangles, each with area 1 square unit. Arrange these rectangles in order by size of their perimeters and tape them onto another sheet. Record the dimensions, area, and perimeter of each rectangle.

6 Suppose the length of 1 white segment is 1 linear unit. Cut out 3 noncongruent rectangles, each with perimeter 1 linear unit. Arrange these rectangles in order by the size of their areas and tape them onto another sheet. Record the dimensions, area, and perimeter of each rectangle.

7 Record at least 1 multiplication and at least 1 division equation represented by each rectangle in Problems 5 and 6. (Write these equations next to the rectangles.)

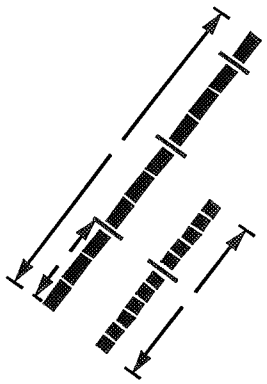
8 Write at least 3 observations and/or conjectures based on mathematical relationships you notice in the rectangles that you formed in Problems 5 and 6.

Linear & Area Measure

Lesson 43

THE BIG IDEA

Comparisons between a unit (linear and/or area) subdivided to show tenths and hundredths and the same unit subdivided to show other-sized portions provide a basis for determining decimal approximation of many fractions. These comparisons build intuitions about relationships between fractions and decimals and offer a natural way to introduce the metric and English systems of measurement. Students with a conceptual knowledge of fractions, decimals, and standard measurement systems can model situations and solve problems about those situations.



CONNECTOR

OVERVIEW

The lengths of segments subdivided to show hundredths and tenths are compared to the lengths of segments subdivided to show other fractional portions. Hence, students determine decimal approximations for the fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set per student.
- ✔ Red segment strips, 1 strip per student.
- ✔ 3" × 5" index cards, 1 per group.
- ✔ Blank paper, one 8½" × 11" sheet per group.
- ✔ Segment strips for the overhead.
- ✔ Scissors, 1 pair per student.

FOCUS

OVERVIEW

Segment strips and grids that are divided to show tenths and hundredths are used to model decimal approximations for fractions and to introduce the metric and English measurement systems. Students then use segment strips to model and investigate situations involving fraction and decimal measurements.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set per student.
- ✔ Red and purple segment strips, 1 of each per student.
- ✔ 3" × 5" index cards, 1 per group.
- ✔ Blank paper, one 8½" × 11" sheet per group.
- ✔ Scissors, 1 pair per student.
- ✔ Tape, 1 roll per group.
- ✔ 10-cm × 10-cm grids (see *Blackline Masters*), 101 grids per class (see Comment 2).
- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ 1-inch grid paper, (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ 1-cm grid paper, 1 sheet per group.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Segment strips for the overhead.

FOLLOW-UP

OVERVIEW

Students use segment strips and grids to determine lengths of other segments, to find decimal approximations of fractions, and to model situations involving fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 43.1, 1 copy per student.
- ✔ Segment strips, 1 standard set plus 1 red segment strip per student.
- ✔ 10/12-cm grid paper, 2 sheets per student.
- ✔ 1-cm grid paper, 1 sheet per student.

LESSON IDEAS

FOLLOW-UP

You could ask each student to have the adults who listen to their presentation of Problems 5 and 7 send back a note with 3 positive comments about the presentation (or 3 "I appreciate..." and 1 "I wish..." statements).

LOOKING AHEAD

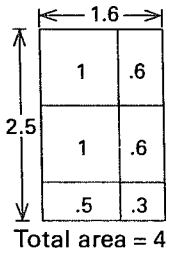
In Lesson 44 and *Visual Mathematics, Courses II-IV* students will continue to develop their understanding and facility with fraction operations and decimal/fraction/percentage relationships.

TEACHER NOTES:

SELECTED ANSWERS

1. a) $\frac{1}{6} \approx .16$ or 16%
   b) $\frac{5}{12} \approx .42$ or 42%
   c) $\frac{3}{2} = 1.5$ or 150%
   d) 3.0 or 300%
   e) $\frac{1}{4} = .25$ or 25%
   f) $\frac{5}{3} = 1.67$ or 167%

2. a) 6
   b) $2\frac{2}{5}$ or 2.4
   c) $\frac{2}{3}$ or $.\overline{6}$
   d) $\frac{1}{3}$ or $.\overline{3}$
   e) 4
   f) $\frac{3}{5}$ or .6

3. 

Total area = 4

Connector Teacher Activity

OVERVIEW & PURPOSE

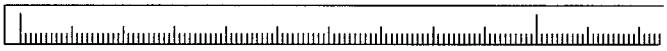
The lengths of segments subdivided to show hundredths and tenths are compared to the lengths of segments subdivided to show other fractional portions. Hence, students determine decimal approximations for the fractions.

MATERIALS

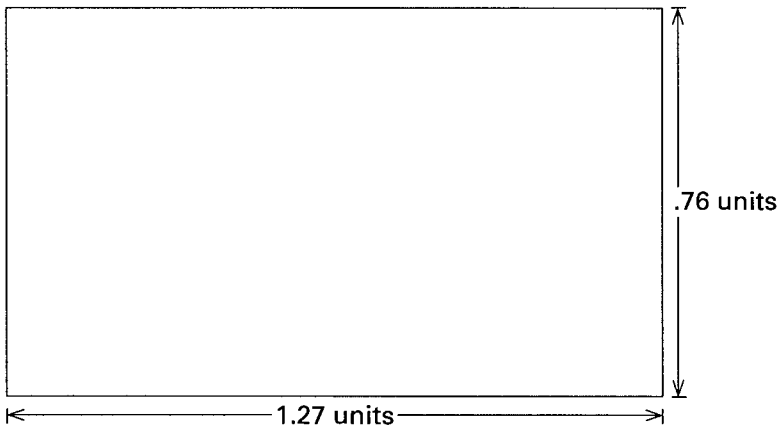
- ✔ Segment strips, 1 standard set per student.
- ✔ Red segment strips, 1 strip per student.
- ✔ 3" × 5" index cards, 1 per group.
- ✔ Blank paper, one 8½" × 11" sheet per group.
- ✔ Segment strips for the overhead.
- ✔ Scissors, 1 pair per student.

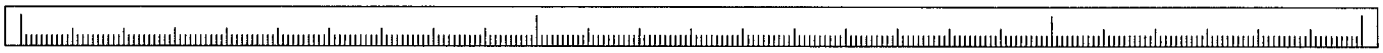
ACTIONS

1 Place the students in groups and give each student 1 red segment strip. Ask the students to make observations about relationships they notice among the markings on these strips. Discuss.



2 Give each group an index card and a blank sheet of paper. Tell them that for now the distance between 2 consecutive "tall" marks on the segment strips distributed in Action 1 is 1 linear unit. Ask them to use these strips as "rulers" and to determine, as accurately as possible, the dimensions of the index card and the sheet of blank paper. Have volunteers demonstrate their methods.





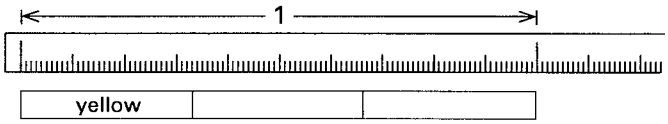
COMMENTS

1 Notice there are 3 types of markings ("short," "medium," and "tall") on each red strip. The distance between 2 consecutive tall marks is divided into 100 equal segments by the short and medium marks, and the distance between 2 consecutive medium marks is divided into 10 equal segments. Some students may notice similarities between these segment strips and rulers they have used. (Although these strips are marked to indicate millimeters, centimeters, and decimeters, unless students bring it up, we prefer to delay naming the units until Action 1 of the Focus activity.)

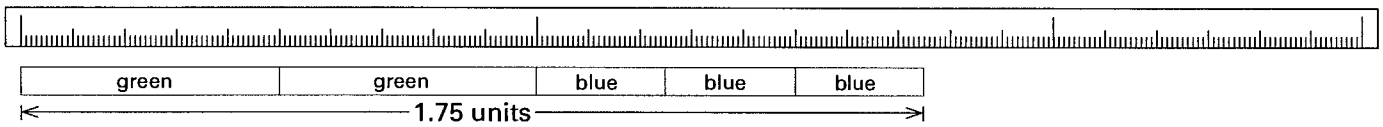
2 Some students may report the dimensions using fractions and some using decimals. Encourage both. A standard (3" × 5") index card has dimensions of approximately .76 linear units × 1.27 linear units. If students have difficulty rounding their measurements, you may wish to review ideas in Lesson 30.

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
3 Distribute a pair of scissors and 1 standard set of colored segment strips to each student. Using the distance between 2 consecutive tall marks on their rulers from Action 1 as 1 linear unit, have the students determine the length, as accurately as possible, of 1 segment of each of the different colored strips. Encourage the use of both fractions and decimals to represent lengths.	**3** Although a measurement made with a ruler is considered an approximation, students may rely on their knowledge of fraction and decimal relationships (see Lesson 28) to tell when a decimal is an approximation or an exact equivalent of a fraction. They should find: 1 white segment is 1 linear unit in length; 1 green has length $1/2 = .5$ linear unit; 1 yellow is $1/3 \approx .33$; 1 blue is $1/4 = .25$; 1 pink is $1/6 \approx .17$; and 1 orange is $1/12 \approx .08$.



4 Using the length of 1 white segment as the linear unit, have the groups determine the decimal approximation of the total length of the following combinations of segments (placed together end-to-end). Ask them to record the approximations as fractions and as decimals. Discuss the students' methods and results and their ideas about expressing each length as a percentage of 1 unit. a) 2 green segments and 3 blue segments b) 2 white and 1 yellow c) 6 pink and 5 blue d) 2 pink, 4 orange, 2 yellow, and 1 green	**4** Students may use prior knowledge about relationships among the segments to determine fractional values (e.g., 4 blue are equal in length to 1 white which is 1 linear unit, so 3 blue have length $3/4$ linear unit) and their prior knowledge about decimal-fraction relationships to determine decimal values. Or, students may use the red strip to determine decimal and fraction lengths (e.g., 3 blue have length $.75 = 75/100$ linear units). Note that using the red segment strips may yield other reasonable approximations such as $.73 = 73/100$ or $.76 = 76/100$ as the length of 3 blue. Here are the most common student responses given for a)-d). a) $1\,3/4 = 1.75$ c) $2\,1/4 = 2.25$ b) $2\,1/3 \approx 2.33$ d) $1\,10/12 \approx 1.83$ Since the "short" markings divide 1 unit into 100 equal parts, each of the above lengths can be written as the following percentages of 1 unit: a) 175%; b) ≈ 233%; c) 225%; d) ≈ 183%.



5 (Optional) Using the distance between 2 consecutive "medium" marks as the linear unit, have the students find the decimal length of a)-d) from Action 4.	**5** a) $17\,1/2 = 17.5$ c) $22\,1/2 = 22.5$ b) $23\,1/3 \approx 23.3$ d) $18\,1/3 \approx 18.3$ If, as you circulate, you notice a variety of measurements for a)-d) you might list them all on the overhead and ask groups to discuss and reach consensus on the measurements that they feel are most accurate.

Focus Teacher Activity

OVERVIEW & PURPOSE

Segment strips and grids that are divided to show tenths and hundredths are used to model decimal approximations for fractions and to introduce the metric and English measurement systems. Students then use segment strips to model and investigate situations involving fraction and decimal measurements.

MATERIALS

- ✔ Segment strips, 1 standard set per student.
- ✔ Red and purple segment strips, 1 of each per student.
- ✔ 3" × 5" index cards, 1 per group.
- ✔ Blank paper, one 8½" × 11" sheet per group.
- ✔ Scissors, 1 pair per student.
- ✔ Tape, 1 roll per group.
- ✔ 10-cm × 10-cm grids (see *Blackline Masters*), 101 grids per class (see Comment 2).
- ✔ 10/12-cm grid paper (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ 1-inch grid paper, (see *Blackline Masters*), 1 sheet per group and 1 transparency.
- ✔ 1-cm grid paper, 1 sheet per group.
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 1 copy per group and 1 transparency.
- ✔ Segment strips for the overhead.

ACTIONS

1 Place the students in groups and distribute a pair of scissors, a standard set of segment strips and a red segment strip to each student; and an index card and blank sheet of paper to each group. If it hasn't been suggested by a student, point out that the distance between 2 consecutive tall marks on the red strip is called 1 *decimeter* (abbreviated dm); the distance between 2 consecutive medium marks is 1 *centimeter* (cm); and the distance between 2 short marks is 1 *millimeter* (mm). A length of 100 centimeters (also 1000 millimeters and 10 decimeters) is 1 *meter* (m) and 1000 meters is 1 *kilometer* (km).

Ask the students to record the dimensions of their note card and sheet of paper in centimeters, millimeters, decimeters, and meters. Discuss their ideas about objects that would be appropriate to measure in kilometers.

2 Distribute a pair of scissors and 1 sheet of 1-cm grid paper to each group. Have each group cut out the following: a square centimeter, square millimeter, and square decimeter; and linear pieces for 1 cm, 1 mm, 1 dm, and 1 m. Distribute tape and 10-cm × 10-cm grids (see *Blackline Masters*) to the groups and ask them to help with the construction of a square meter. Label and post these pieces on the classroom wall. Discuss relationships they observe.

COMMENTS

1 Part of the intent of this activity is explore to some common standard metric and English units of measure. Encourage students to offer their ideas about the value of measuring with linear units that are recognized worldwide.

A standard (8½" × 11") sheet of copy paper has approximate dimensions 21.6 cm × 27.9 cm, 216 mm × 279 mm, 2.16 dm × 2.79 dm, and .216 m × .279 m.

If time and weather permit, or as an outside-of-class project, students could pace off the school track, for example, and determine the approximate number of times around the track in 1 kilometer.

2 To save time, cut apart the 10-cm × 10-cm grids before class. A square meter is a 10 × 10 grid of square decimeters (i.e., 100 cm × 100 cm) and can be formed by taping together 100 copies of the 10-cm × 10-cm grid. If the base ten mat-mat (see Lesson 25) is still posted on the wall, it is also 1 square meter.

Focus Teacher Activity (cont.)

ACTIONS

3 Have the groups determine the approximate area in square centimeters, millimeters, and meters of the index card and sheet of paper distributed in Action 1. Discuss their methods.

4 Distribute 1 sheet of $^{10}/_{12}$-cm grid paper to each group. Tell them that for now the length of 1 white segment is the linear unit. Ask them to cut out and label rectangles (from the $^{10}/_{12}$-cm grid paper) with the following areas:

a) $^{3}/_{4}$ square unit

b) $^{2}/_{3}$ square unit

c) $^{5}/_{8}$ square unit

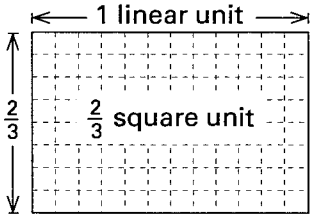
5 Distribute 1 sheet of 10-cm × 10-cm grids (see *Blackline masters*) to each group. Ask them to determine the area of each 10 × 10 square assuming still that the length of 1 white segment is 1 linear unit. Then have groups determine (as accurately as possible) the portion of the area of this square that can be covered by their rectangular region formed in Action 4a). Ask them to express their approximation as a fraction, as a decimal, and as a percent of a unit. Have volunteers show their methods at the overhead. Repeat for the regions formed in Action 4b) and c).

COMMENTS

3 Some students may place the index card, for example, on the centimeter grid paper and approximate the area by counting squares and parts of squares. Others may use the area model for multiplication and multiply the dimensions. If one method is used predominantly by the class, you may wish to have them verify their results using another method. Some students may be interested in extending this action by determining the dimensions and area of the card and/or sheet of paper in kilometers and square kilometers. Note that if students use scientific calculators, computing areas in kilometers will generate discussion about scientific notation. If they use nonscientific calculators, the product of the dimensions will show as 0. These ideas are explored in *Visual Mathematics, Course II.*

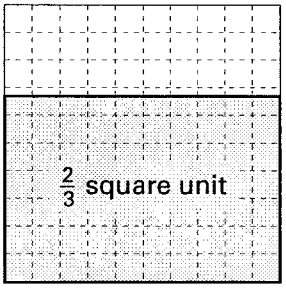
4 There are many possible rectangles that could be formed for each area given.

b) Here is one possible rectangle with area $^{2}/_{3}$ square unit:



5 Because 1 white segment is also 10 cm in length, one 10 × 10 grid has dimensions 1 linear unit by 1 linear unit and area 1 square unit. By superimposing or outlining their rectangles from Action 4 on a 10-cm × 10-cm square, students can determine a decimal approximation of the area of each rectangle.

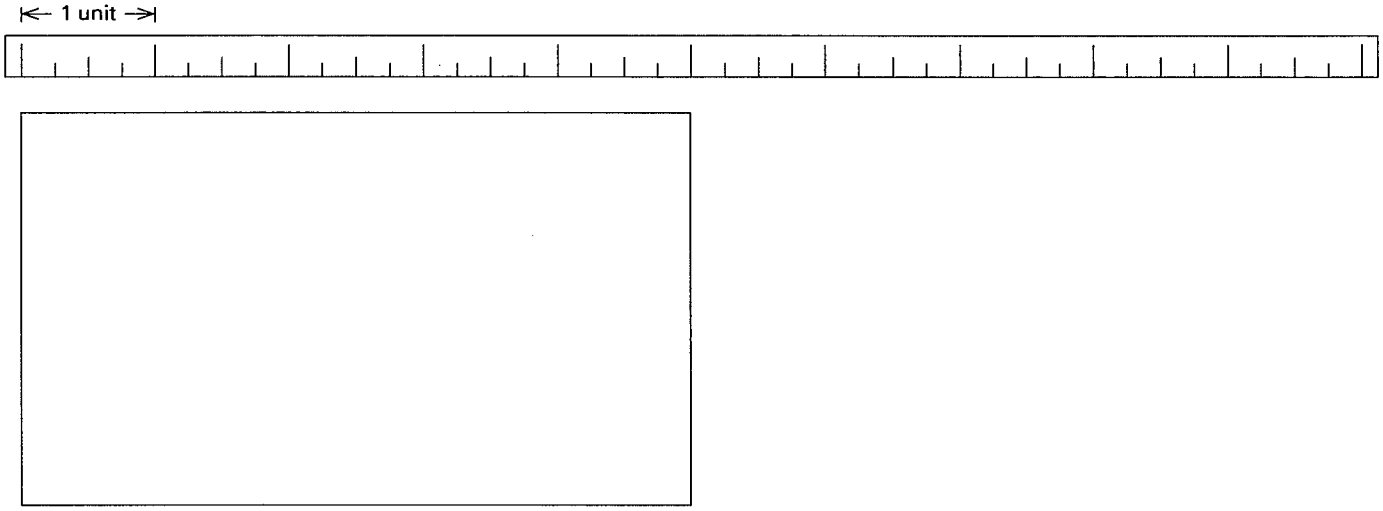
b) A rectangle with area $^{2}/_{3}$ square unit covers approximately 67% of the small squares or .67 area unit. Some students may determine this by counting squares, while others will use their prior knowledge of decimal and fraction relationships.



Focus Teacher Activity (cont.)

ACTIONS

6 Give each student a purple segment strip. Ask them to measure, as accurately as possible, the perimeters of their index card and blank paper, using the distance between 2 consecutive "tall" marks on these strips as the linear unit.



7 If it hasn't come up, point out that the linear unit used in Action 6 is commonly called an *inch*. Invite students to describe situations in which they have encountered inches, feet, and yards, and to tell what they know about relationships among these units.

8 Distribute 1 sheet of 1-inch grid paper to each group. Have the groups cut out linear pieces for 1 inch, 1 foot, and 1 yard. Have them also imagine and discuss what a square inch, a square foot, and a square yard would look like, including relationships among these squares and the metric squares formed in Action 2.

9 (Optional) Have the students measure (as accurately as possible) the dimensions and area of the index card and paper in inches, feet, and yards. Discuss.

COMMENTS

6 Using this segment strip and the designated linear unit, a standard small note card has dimension 3 linear units by 5 linear units and a standard-sized sheet of paper has dimensions $8\frac{1}{2}$ linear units by 11 linear units.

7 In the English system of measurement, there are 12 inches in 1 *foot*, 3 feet in 1 *yard*, and 5,280 feet in a *mile*.

8 To provide reinforcement and ease of reference, have students label and post one of each of these linear pieces in the classroom. You may also wish to have them assemble the square inch, square foot, and square yard to reinforce comparisons to metric area pieces. It isn't important that students be able to convert between the English and metric systems. Rather, encourage statements such as "there are about $2\frac{1}{2}$ cm in 1 inch and about 6 cm^2 in 1 $inch^2$," or "a meter is a little more than a yard," "a kilometer is about .6 mile," etc.

9 Some students may remeasure the index card and paper to determine the dimensions for each unit, while others may adapt their earlier measures (e.g., since a sheet of paper measures $8\frac{1}{2}" \times 11"$, some students have reported these dimensions as $8\frac{1}{2}/12$ foot $\times$ $11/12$ foot or $17/24$ foot $\times$ $11/12$ foot). If you wish to have students measure objects involving smaller parts than $\frac{1}{4}"$, you could have them use a standard ruler.

Focus Teacher Activity (cont.)

ACTIONS

10 Place a transparency of Focus Master A on the overhead and ask the groups to discuss and identify the metric and English unit they feel would be most appropriate for measuring each object. Discuss the students' thoughts about the advantages and disadvantages of measuring in the metric and English systems.

Linear and Area Measure Lesson **43**

Focus Master A

1. Decide which metric and which English unit you believe would be most appropriate to measure each of the following:

a) the size of the gym floor
b) the length of the football field
c) your teacher's height
d) the length of a newborn baby
e) the distance between San Francisco and New York City
f) the size of a farmer's cornfield
g) the size of a paperclip
h) the thickness of a dime
i) the amount of wallpaper for a bedroom wall
j) how far one runs in a marathon
k) the size of a belt
l) the amount of wrapping paper needed to cover a box

2. For each of the following units, describe an object (different from those listed above) you feel would be appropriate to measure using the unit. Be sure to tell what aspect of its size you would measure (e.g., it's area, length, width, etc.) with the unit.

a) centimeter
b) square foot
c) mile
d) millimeter
e) meter
f) square mile
g) inch
h) yard
i) square millimeter
j) kilometer
k) square kilometer
l) square centimeters
m) foot

11 Give each group a copy of Focus Master B and ask the students to use their segment strips and/or grid paper to model the relationships in Situation a). Select a model from those presented and ask the groups to generate a list of observations and conclusions that can be made about the situation by using that model. Discuss.

Repeat for one or more of Situations b)-l).

COMMENTS

10 Students' answers will vary. To measure the "size" of the gym floor, for example, students may suggest finding the dimensions and/or the area. Appropriate linear units could be feet, yards, or meters and appropriate area units could be square feet, square yards, or square meters. Encourage students to explain their rationale for selecting a particular unit. Accept all reasonable responses.

11 After discussing a) you may wish to have the students work on several problems before additional discussion. If so, ask them to make a drawing of each model they form (or glue or tape their models on a sheet of paper). If needed, remind students not to spend energy drawing details that don't contribute to understanding important mathematical relationships (e.g., elaborate fish, children, etc.) in each situation.

As students create models of each situation, be sure to have them note important mathematical information that can be gleaned from the models (i.e., questions that could be answered by observing relationships within each model). You may wish to pose some questions to the class for investigation.

Focus Teacher Activity (cont.)

ACTIONS

Lesson **43** Linear and Area Measure
Focus Master B

a) One package weighs $2\frac{2}{5}$ pounds and the other weighs $1\frac{1}{2}$ pounds.

b) For a community service project, 3 scouts each washed $\frac{1}{5}$ of the windows in a pre-school. Another 4 scouts completed the project by each washing an equal part of what was left.

c) Cassie and Angie prepared flyers announcing a school play. In $\frac{2}{3}$ of an hour Cassie folded 64 flyers. In 25 minutes Angie folded 40 flyers.

d) The length of the head of a fish is $\frac{1}{8}$ of its total length. The head is $\frac{1}{3}$ as long as its tail. When the head and tail of the fish are removed for cleaning 24 cm are left.

e) The dimensions of Anna's miniature painting are 2.7 cm × 3.1 cm. The dimensions of Christopher's painting are 2.3 cm × 3.5 cm.

f) Monica is $4\frac{1}{6}$ times as tall as her little brother, Peter, whose height is $1\frac{1}{2}$ feet.

g) The garden in Julie's yard covers $1\frac{7}{8}$ as much area as the garden in Jennie's yard. The area of Julie's garden is 120 square feet.

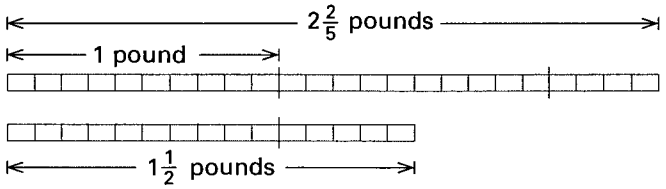
h) Farmer Bale's pasture covers $3\frac{3}{4}$ acres. This is 20% of the area of her farm.

i) Teresa rode her bike $3\frac{1}{3}$ miles to get to Linden's house, and Morgan rode $1\frac{5}{6}$ miles to get there.

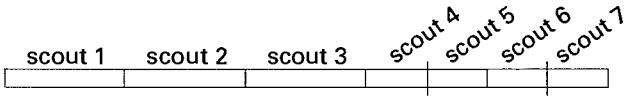
COMMENTS

Following are examples of models and a few questions that could be answered using each model:

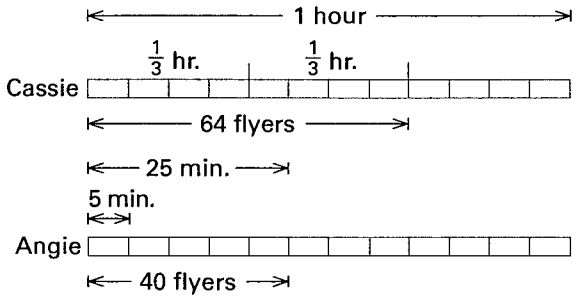
a) What's the difference in their weights? ($\frac{9}{10}$ pounds) How many times heavier than the small package is the large one? (1.6) Altogether, how much do they weigh? (3.9 pounds)



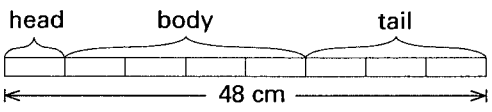
b) What part of the windows did the last 4 scouts wash altogether? ($\frac{2}{5}$) What part of the windows did each of the last 4 scouts wash? ($\frac{1}{10}$) How much more did scout 1 wash than scout 7? (twice as much)



c) How many flyers would each fold in 5 minutes? (8) In 1 hour? (96) Who is faster? (neither)



d) How long is the head? (6 cm) How long are the body and tail combined? (42 cm) How many times as long as the head is the tail? (3) What part of the length of the tail is the length of the body? ($1\frac{1}{3}$)



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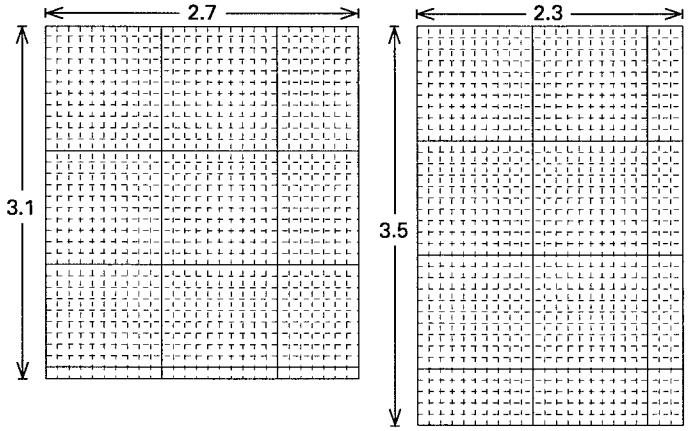
Focus Teacher Activity (cont.)

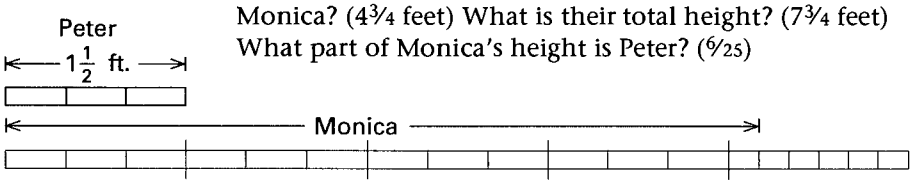
ACTIONS

COMMENTS

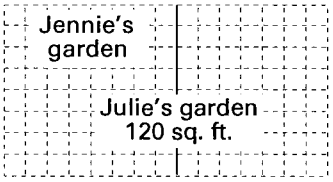
11 (continued.)
e) What's the difference in their areas? (.32 sq. cm) In their perimeters? (0)



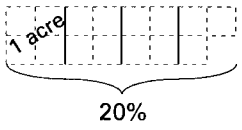
f) How tall is Monica? (6¼ feet) How much taller is Monica? (4¾ feet) What is their total height? (7¾ feet) What part of Monica's height is Peter? (6/25)

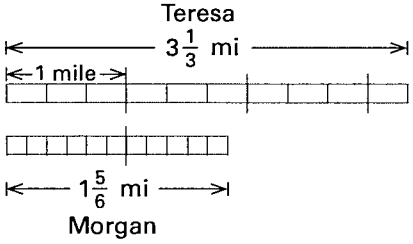


g) What is the area of Jennie's garden? (64 sq. feet) What's the difference in the areas of Julie's and Jennie's gardens? (56 sq. feet) How many copies of Jennie's garden would it take to exactly cover Julie's garden? (1 56/120) What part of Julie's garden could be covered by Jennie's? (64/120)



h) How many acres are in Farmer Bale's farm? (18¾) 1 acre is what percent of the farm? (5⅓%)





i) How many times as far Morgan rode did Teresa ride? (1 9/11) How much farther did Teresa ride? (1½) What is the total distance the two rode? (5⅙ miles)

Follow-up Student Activity 43.1

NAME _____ DATE _____

For this activity you need a set of segment strips, 2 sheets of $^{10}/_{12}$-cm grid paper, and 1 sheet of 1-cm grid paper.

1 Use the distance between 2 "tall" marks on the red segment strip as the linear unit. Find the length as accurately as possible of each of the following. Write each length as a fraction, a decimal, and a percent of 1 unit.

a) 1 pink c) 3 green e) 1 blue

b) 5 orange d) 3 white f) 5 yellow

g) Explain how you determined the fraction, decimal, and percent for f).

2 Find a close approximation of the distance between 2 "tall" marks on the red segment strip when each of the following is the linear unit.

a) 1 pink c) 3 green e) 1 blue

b) 5 orange d) 3 white f) 5 yellow

3 Using the length of 1 white segment as the linear unit, do the following on a sheet of 1-cm grid paper:

a) draw a rectangle whose dimensions are $1\frac{3}{5}$ linear units by $2\frac{1}{2}$ linear units;

b) explain how to use your diagram to find the area and perimeter of the rectangle;

c) write the area and perimeter as a fraction and as a decimal.

(Continued on back.)

Follow-up Student Activity (cont.)

4 Now shade 10% of your rectangle in Problem 3.

a) Next to the diagram tell how you determined what to shade.

b) Tell the total area of the shaded part and explain how you decided this.

5 Use segment strips to teach an adult why $\frac{3}{5} + \frac{1}{3} = \frac{14}{15}$. On another sheet of paper tell who you "taught" and describe (include diagrams) each step of your demonstration.

6 Model each of the following situations with segment strips and/or grids. On a separate sheet, draw a picture of each of your models (or tape your models to the sheet). Next to each model write at least three interesting mathematical questions *whose answers could be found by studying your model.* (Write the answers to your questions, too!)

a) The music teacher roped off an area in front of the stage that was equal to $\frac{1}{3}$ the area of the stage. The dimensions of the stage are 8 yards by 4 yards.

b) To get in shape for a track meet Travis ran the following distances last week: Monday, $1\frac{1}{2}$ miles; Tuesday, 2 miles; Wednesday, $1\frac{2}{5}$ miles; Thursday, 1.9 miles; and Friday, $2\frac{3}{10}$ miles.

c) Josh weighs 3.5 times as much as his little brother. Josh weighs 84.7 pounds.

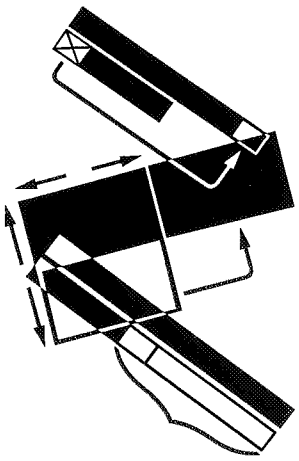
d) In the 3-mile Spring Fun Run, Erin had run 80% of the 3 miles when she stopped for a drink of water.

7 Pick one or more of the situations in Problem 6 to model (with segment strips or grids) for an adult. Explain to them how you were able to answer your questions by using the model. On another sheet, tell who you taught, which situation(s) you modeled, and the adult's reactions.

Inventing Algorithms For Fraction Operations Lesson 44

THE BIG IDEA

Methods of using linear and area models for computing fraction sums, differences, products, and quotients provide a meaningful basis for inventing algorithms. Efforts to apply such invented algorithms to other problems provide motivation to evaluate, refine, and generalize procedures.



CONNECTOR

OVERVIEW

Students recall mental strategies from Lessons 26, 37, and 38 and apply them to computations with fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set per student.
- ✔ 10/12-cm grid paper, 1 sheet per student.
- ✔ Connector Master A, 1 copy per student and 1 transparency.
- ✔ Connector Student Activity 44.1, 1 copy per student.

FOCUS

OVERVIEW

Students use segment strips and grids to carry out fraction computations involving the four basic operations. Then they create posters that illustrate their methods pictorially, verbally, and using math symbols. Finally, they generalize a variety of methods and discuss the convenience of selected methods.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Segment strips, 1 standard set per student.
- ✔ Scissors, 1 pair per student.
- ✔ 10/12-cm grid paper, 2 sheets per student.
- ✔ Focus Master A, 1 copy per class.
- ✔ Focus Master B, 1 copy per pair of students and 1 transparency.
- ✔ Focus Master C (optional), 1 copy per pair of students and 1 transparency.
- ✔ Butcher paper, 1 sheet per pair of students.
- ✔ Marking pens for each pair of students.
- ✔ Tape, 1 roll per pair of students.

FOLLOW-UP

OVERVIEW

Students use segment strips and grids to model the four operations. They write descriptions of situations involving the operations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 44.2, 1 copy per student.
- ✔ Segment strips, 1 standard set per student.
- ✔ 10/12-cm grid paper, 2-3 sheets per student.

LESSON IDEAS

JOURNALS

After this lesson is completed, ask the students to write the following in their journals: "My understanding of and confidence about computing with fractions have grown in the following ways..." Ask them to cite as many examples as possible to illustrate their "before and after" notions about fraction operations.

LOOKING AHEAD

In *Courses II* and *III* of *Visual Mathematics* students will continue conceptual explorations of fraction concepts and operations and will continue the process of developing algorithms that represent their thought processes.

SELECTED ANSWERS

2. For example: $7\frac{5}{8} - 5$, $6\frac{5}{8} - 4$, $10 - 7\frac{3}{8}$

3. Here are two possibilities:

   Add up method: $2\frac{3}{4} + \frac{1}{4} = 3$; $3 + \frac{1}{2} = 3\frac{1}{2}$, so $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ must be added to 2.

   Equal differences method: add $\frac{1}{4}$ to $2\frac{3}{4}$ to get 3 and add $\frac{1}{4}$ to $3\frac{1}{2}$ to get $3\frac{3}{4}$. Since $3 + \frac{3}{4} = 3\frac{3}{4}$, $2\frac{3}{4} + \frac{3}{4} = 3\frac{1}{2}$.

4. For example: 60 + 30 (add $6\frac{1}{2}$ to $53\frac{1}{2}$ and subtract $6\frac{1}{2}$ from $36\frac{1}{2}$); 50 + 40 (subtract $3\frac{1}{2}$ from $53\frac{1}{2}$ and add $3\frac{1}{2}$ to $36\frac{1}{2}$); 55 + 35 (add $1\frac{1}{2}$ to $53\frac{1}{2}$ and subtract $1\frac{1}{2}$ from $36\frac{1}{2}$).

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

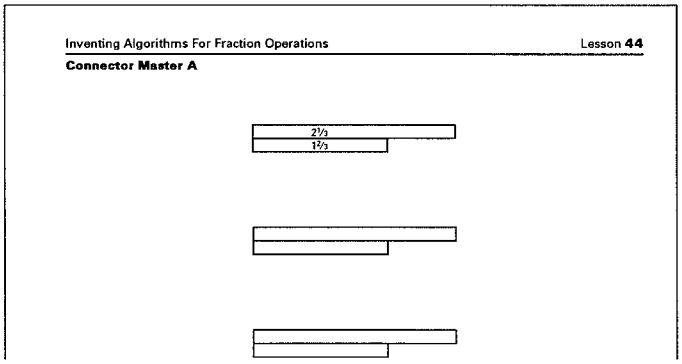
Students recall mental strategies from Lessons 26, 37, and 38 and apply them to computations with fractions. In the Focus, students invent algorithms that reflect these and other strategies of computing.

MATERIALS

- ✔ Segment strips, 1 standard set per student.
- ✔ 10/12-cm grid paper, 1 sheet per student.
- ✔ Connector Master A, 1 copy per student and 1 transparency.
- ✔ Connector Student Activity 44.1, 1 copy per student.

ACTIONS

1 Arrange the students in groups and give each student a copy of Connector Master A, a set of segment strips, and a sheet of 10/12-cm grid paper. On a transparency of Master A, place the numbers $2\frac{1}{3}$ and $1\frac{2}{3}$ in one of the diagrams, as shown below:

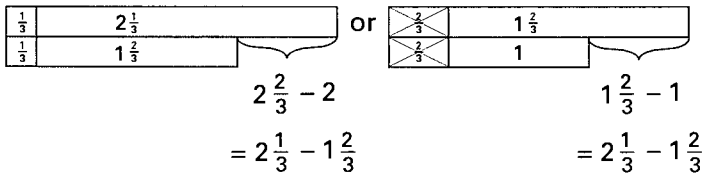


Ask the groups to imagine 4 other pairs of numbers whose difference is the same as the difference between $2\frac{1}{3}$ and $1\frac{2}{3}$, and 4 pairs whose sum is the same as the sum of $2\frac{1}{3}$ and $1\frac{2}{3}$. Discuss their methods of determining these pairs of numbers and ask them to identify which sums and differences they would prefer to compute.

2 Repeat Action 1 as needed for sums and differences of other pairs of numbers, such as: $7\frac{1}{8}$ and $4\frac{5}{8}$, $3\frac{1}{2}$ and $1\frac{5}{6}$, 3 and $1\frac{1}{5}$, $12\frac{7}{8}$ and $17\frac{1}{4}$. Have additional copies of Master A available for those who need them.

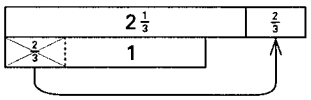
COMMENTS

1 At first, some students may focus on fractions equivalent to $2\frac{1}{3}$ and $1\frac{2}{3}$. After acknowledging these, suggest they look for nonequivalent pairs whose difference is $2\frac{1}{3} - 1\frac{2}{3}$. Some students will find it helpful to use their segment strips and/or grids to determine *equal sums* and *equal differences*. As first introduced in Lesson 26, equal differences can be formed by adding equal amounts to both numbers or by subtracting equal amounts from both:



$$2\frac{2}{3} - 2 = 2\frac{1}{3} - 1\frac{2}{3} \qquad 1\frac{2}{3} - 1 = 2\frac{1}{3} - 1\frac{2}{3}$$

Equal sums can be formed by adding an amount to one number and subtracting the same amount from the other:



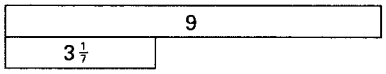
$$2\frac{1}{3} + 1\frac{2}{3} = (2\frac{1}{3} + \frac{2}{3}) + (1\frac{2}{3} - \frac{2}{3})$$
$$= 3 + 1$$

2 Note that the diagrams on Master A are not to scale since they are used to represent several differences. Most students prefer to compute differences that involve taking away a whole number and they prefer to compute sums that involve at least one whole number. Forming equal sums and differences is an especially useful strategy for computing mentally. Placing the numbers in the diagrams on Master A helps students see why the methods work and reinforces a visual model for thinking about equal sums and differences.

Connector Teacher Activity (cont.)

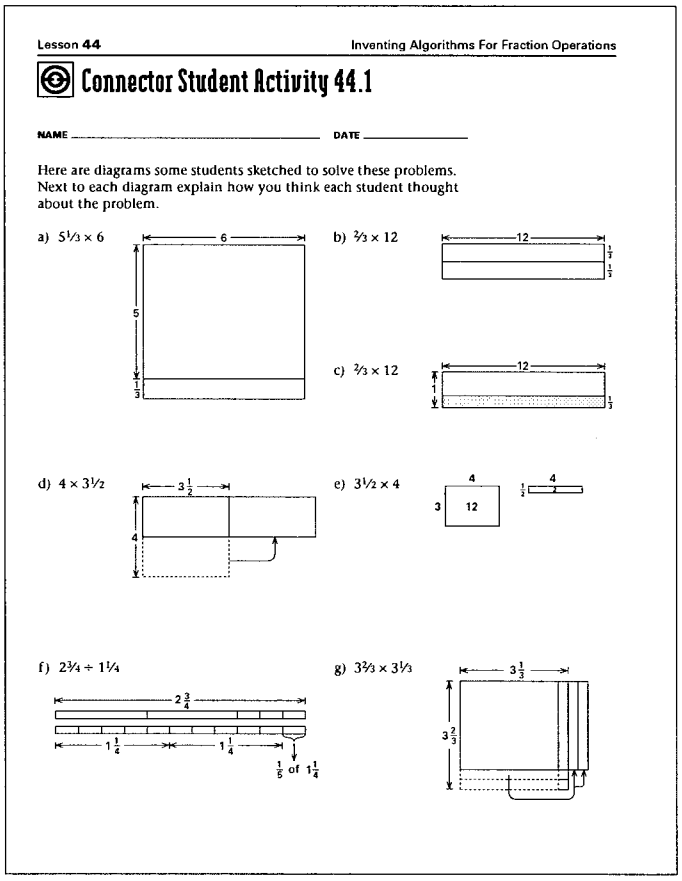
ACTIONS

3 Record the following on the overhead and ask the students to compute the sum and difference of 9 and $3\frac{1}{7}$. Discuss their strategies.



Repeat as needed for $6\frac{1}{8}$ and $2\frac{3}{8}$, 5 and $2\frac{1}{4}$, or $3\frac{1}{2}$ and $1\frac{5}{8}$.

4 Give each student a copy of Student Activity 44.1 to complete. Discuss.



Lesson 44 Inventing Algorithms For Fraction Operations

Connector Student Activity 44.1

NAME _____ DATE _____

Here are diagrams some students sketched to solve these problems. Next to each diagram explain how you think each student thought about the problem.

a) $5\frac{1}{3} \times 6$

b) $\frac{2}{3} \times 12$

c) $\frac{2}{3} \times 12$

d) $4 \times 3\frac{1}{2}$

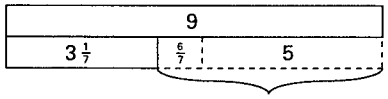
e) $3\frac{1}{2} \times 4$

f) $2\frac{3}{4} \div 1\frac{1}{4}$

g) $3\frac{2}{3} \times 3\frac{1}{3}$

COMMENTS

3 Some students will find it helpful to model this with segments or grids. In the following diagram, a student has determined the difference by using the add-up method (if this method isn't suggested you may wish to suggest it as an alternative): $3\frac{1}{7} + \frac{6}{7} = 4$ and $4 + 5 = 9$, so the difference is $\frac{6}{7} + 5 = 5\frac{6}{7}$.



$$9 - 3\frac{1}{7} = \frac{6}{7} + 5 = 5\frac{6}{7}$$

To build students' comfort with different strategies, you may wish to ask them to try specific strategies (e.g., add-up, equal differences, or equal sums on some problems).

4 These problems use methods that were explored in Lessons 37 and 38 for whole numbers and decimals. If students have difficulty, you may wish to have groups explore and discuss these one or two at a time. Encourage use of equations and number statements (see Lesson 3) to record their ideas. For example, students might view each diagram as follows:

a) $5\frac{1}{3} \times 6 = (5 \times 6) + (\frac{1}{3} \times 6)$

b) $\frac{2}{3} \times 12 = (\frac{1}{3} \times 12) + (\frac{1}{3} \times 12)$

c) $\frac{2}{3} \times 12 = (1 \times 12) - (\frac{1}{3} \times 12)$

d) $4 \times 3\frac{1}{2} = 2 \times 7$

e) $3\frac{1}{2} \times 4 = (3 \times 4) + (\frac{1}{2} \times 4) = 12 + 2$

f) $1\frac{1}{4} + 1\frac{1}{4} + \frac{1}{5}$ of $1\frac{1}{4}$ is $2\frac{3}{4}$, or $2\frac{1}{5} \times 1\frac{1}{4} = 2\frac{3}{4}$, so $2\frac{3}{4} \div 1\frac{1}{4} = 2\frac{1}{5}$

g) $3\frac{2}{3} \times 3\frac{1}{3} = (3 \times 4) + (\frac{1}{3} \times \frac{2}{3})$

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use segment strips and grids to carry out fraction computations involving the four basic operations. Then they create posters that illustrate their methods pictorially, verbally, and using math symbols. Finally, they generalize a variety of methods and discuss the convenience of selected methods.

MATERIALS

- ✔ Segment strips, 1 standard set per student.
- ✔ Scissors, 1 pair per student.
- ✔ $^{10}/_{12}$-cm grid paper, 2 sheets per student.
- ✔ Focus Master A, 1 copy per class.
- ✔ Focus Master B, 1 copy per pair of students and 1 transparency.
- ✔ Focus Master C (optional), 1 copy per pair of students and 1 transparency.
- ✔ Butcher paper, 1 sheet per pair of students.
- ✔ Marking pens for each pair of students.
- ✔ Tape, 1 roll per pair of students.

ACTIONS

1 Place the students in pairs and give each student a pair of scissors, a set of segment strips, and 2 sheets of $^{10}/_{12}$-cm grid paper. To each pair of students give 1 number card cut from Focus Master A (see Comment 1), a sheet of butcher paper, marking pens, tape, and a copy of Focus Master B. When the posters described on Master B are completed, have the students display them on the wall. Allow time for the pairs to study each other's posters and give feedback to one another. Discuss.

Lesson 44 Inventing Algorithms For Fraction Operations

Focus Master B

Use segment strips and/or $^{10}/_{12}$-cm grid paper to find the sum, difference, product, *and* quotient of the numbers on your card. Then in order to clearly communicate the "story" of your procedures with the strips and/or grids, create a poster with the following information.

1. For each computation that you did, tell the story of your actions using each of the following:

   a) a *pictorial representation* of your methods (a diagram or model that illustrates your methods—no words);

   b) a *verbal representation* of your methods (a description in words only);

Try to make each representation above so clear that your reader could duplicate your actions with the strips and/or grids simply by studying that representation. (You could test your ideas by having someone in another group study a representation and try to duplicate your actions with no other clues.)

2. For the sum and difference you computed invent a symbolic representation of your methods using numbers and arithmetic symbols only—no words or pictures.

Other Ideas

Here are some ways you could go "above and beyond" on your poster:

- explain and illustrate what happens if you reverse the order of the numbers in each of your 4 computations;
- determine whether any of your methods generalize (that is, would they work for *any* pair of fractions or mixed numbers?);
- make a symbolic (numbers and math symbols only) representation of your methods of using strips or grids to find the product and/or quotient of your numbers;
- add diagrams that show ways to solve each computation based on *all* the different meanings of each operation.

COMMENTS

1 To make the number cards, make 1 copy of Focus Master A and cut apart the cards.

Inventing Algorithms For Fraction Operations Lesson 44

Focus Master A

Number Cards

$1\frac{1}{2}$, $\frac{7}{8}$	$3\frac{3}{4}$, $1\frac{5}{6}$	$4\frac{1}{5}$, $3\frac{2}{3}$	$2\frac{1}{7}$, $\frac{3}{2}$	$1\frac{3}{8}$, $2\frac{5}{12}$
$4\frac{1}{8}$, $1\frac{13}{16}$	$\frac{7}{4}$, $\frac{24}{10}$	$\frac{7}{8}$, $1\frac{2}{3}$	$3\frac{4}{5}$, $4\frac{1}{3}$	$\frac{16}{5}$, 5

Note that to show the difference between the numbers on their card, some students may suggest taking away the lesser amount from the greater, while other students may suggest subtracting in whatever order the numbers appear on their card and invent ways of dealing with negative differences.

To save time you may wish to provide extra segment strips and grid paper for students to use in place of drawings on their posters (i.e., they can tape or glue their actual models to their posters).

As pairs study each other's posters, they could write two "We appreciate..." and two "We wish..." statements, which they attach to the back side of their classmates' posters. Encourage students to focus their comments on the effectiveness of the mathematical communication and evidence of mathematical understanding in the posters (rather than on the quality of the artwork, etc.). This feedback could be in response to oral presentations

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

2 Select one of the unused number cards (or make up a new one), record the numbers from the card on the overhead, choose a method from one of the posters, and ask the students to apply that method (both with the strips or $^{10}/_{12}$-cm grid paper and with symbols) to the numbers you have recorded.

Discuss their views about whether the method generalizes (i.e., will it work for any pair of mixed numbers or fractions).

3 (Optional) Repeat Action 2 for a variety of methods and operations.

4 (Optional) Give each pair of students a copy of Focus Master C and ask them to determine what they believe were the actions (with strips or grids) represented by each set of symbols. Have volunteers show their ideas at the overhead.

Inventing Algorithms For Fraction Operations Lesson **44**

Focus Master C

Each set of symbols below tells the "story" of a student's actions with segment strips or grids (or sketches of them) to solve a problem. Draw or cut out diagrams to show what you think were each student's actions with strips or grids and write a brief explanation of each diagram. Will each method work on other problems?

Keisha, $3^3/_5 - 2^2/_3$

$3^3/_5 = 3^9/_{15} = 2^{24}/_{15}$
$\underline{2^2/_3} = \underline{2^{10}/_{15}} = \underline{2^{10}/_{15}}$
$\underline{\underline{^{14}/_{15}}}$

John, $4^1/_5 - 1^3/_5$

$4^1/_5 \rightarrow 4^3/_5$
$-\underline{1^3/_5} \quad -\underline{2}$
$\underline{\underline{2^3/_5}}$

Lisa, $3^3/_5 + 2^2/_3$

$3^3/_5 = {}^{18}/_5 = {}^{54}/_{15}$
$\underline{2^2/_3} = \underline{^8/_3} = \underline{^{40}/_{15}}$
$\underline{\underline{^{94}/_{15}}}$

Colleen, $2^5/_7 + 1^1/_2$

$\left.\begin{matrix} 2^5/_7 \\ 1^1/_2 \end{matrix}\right\} \left.\begin{matrix} 2^{10}/_{14} + {}^4/_{14} \\ 1^7/_{14} - {}^4/_{14} \end{matrix}\right\} \begin{matrix} 3 \\ \underline{1^3/_{14}} \\ \underline{\underline{4^3/_{14}}} \end{matrix}$

David, $3^3/_5 \times 5^5/_6$

$(3 \times 5) + (3 \times {}^5/_6) + ({}^3/_5 \times 5) + ({}^3/_5 \times {}^5/_6)$
$= 15 + {}^{15}/_6 + 3 + {}^3/_6$
$= 18 + {}^{18}/_6$
$= 18 + 3 = \underline{\underline{21}}$

Juanita, $4^1/_2 - 3^1/_7$

$3^1/_7 + {}^6/_7 = 4$
$4 + {}^1/_2 = 4^1/_2$
${}^6/_7 + {}^1/_2 = {}^{12}/_{14} + {}^7/_{14} = \underline{\underline{^{19}/_{14}}}$

Daniel, $4^2/_5 \times 1^1/_2$

$2^1/_5 \times 3$
$= (2 \times 3) + ({}^1/_5 \times 3)$
$= 6 + {}^3/_5$
$= \underline{\underline{6^3/_5}}$

COMMENTS

1 (continued.)
of posters or be based on observations made as students rotate around the room studying each other's posters.

2 This action is intended to reinforce the following: many methods are possible, many methods can be generalized, and a student's choice of an algorithm is arbitrary. Encourage students to discuss their thoughts about the ideas shared. Are there methods that seem to be more convenient in some situations than others? Are there methods they can picture more easily in their minds? Are there symbolic recordings that suggest strong mental images to them? How would they feel if you restricted them to only one method for use in all situations?

3 Be sure to examine a variety of methods involving all four operations.

4 This action is listed as optional since algorithms created by students in Action 1 and explored in Actions 2 or 3 may closely resemble these.

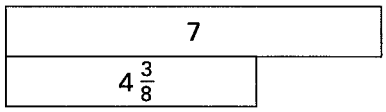
By exploring a variety of methods, students are able to choose (or adapt) the ones that make most sense to them. We prefer not to insist everyone use the same algorithm or to suggest that one method is better than another. It is most important that students be able to solve fraction computations based on concept understanding. More work with and use of algorithms is in *Visual Mathematics, Course II.*

Follow-up Student Activity 44.2

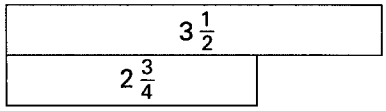
NAME _____ **DATE** _____

1 On another sheet of paper, show how to use segment strips and/or 10/12-cm grids to find the sum, difference, product, *and* quotient of 2½ and 1¾. You may either make a diagram of your segments or attach segments to your paper. Next to each diagram, write number statements or equations and as few words as possible to describe each step of your methods.

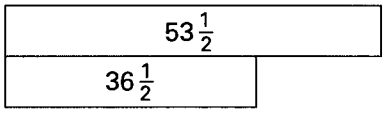
2 What are 3 pairs of numbers whose difference is the same as $7 - 4\frac{3}{8}$? Tell how you determined these numbers. Circle the pair whose difference you prefer to compute.



3 What must you add to 2¾ to equal 3½? Explain how you decided this.



4 What are 3 pairs of numbers whose sum is the same as 36½ + 53½. Tell how you determined these numbers. Circle the pair whose sum you prefer to compute.



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Follow-up Student Activity (cont.)

5 For each set of numbers and operation(s) listed below, describe an everyday situation that involves the numbers and operation(s).

a) $\frac{3}{4}$, $\frac{2}{3}$; multiplication

b) $2\frac{1}{2}$, $3\frac{5}{6}$, $1\frac{3}{4}$; addition

c) $7\frac{4}{5}$, $2\frac{1}{3}$; subtraction

d) $7\frac{1}{2}$, $1\frac{1}{3}$; division

e) $1\frac{1}{4}$, $2\frac{1}{3}$; multiplication and division

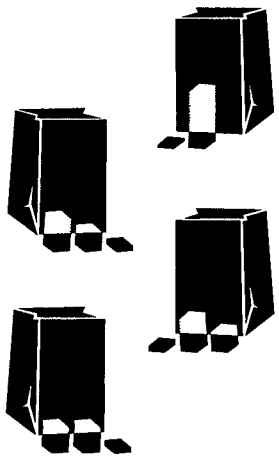
f) $2\frac{1}{8}$, $3\frac{1}{4}$, $1\frac{1}{6}$, and 4; addition, subtraction, multiplication, and division

Sampling, Confidence, & Probability

Lesson 45

THE BIG IDEA

Predictions, and one's confidence in those predictions, can be influenced by the quality and quantity of information that is uncovered about a situation. Hands-on activities that involve conjecturing about a situation, planning an experiment, gathering data, and analyzing the results help students develop a sense for what might reasonably be expected to happen in a chance situation.



CONNECTOR

OVERVIEW

Students use clues involving probability to determine collections of colored tile.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Colored square tile (in at least 3 colors), 30 per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Transparent colored tile.
- ✔ Calculators, 1 per group of students.
- ✔ Student journals (optional).

FOCUS

OVERVIEW

Students produce samples from sacks of tile in order to predict the contents of the sacks. They compare theoretical and experimental probabilities and discuss their confidence in their predictions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Secret Sacks (lunch-size paper bags filled with 12 red, 6 blue, and 2 green tile), 1 per group of students.
- ✔ Butcher paper, 2 poster-size sheets per group of students.
- ✔ 2-cm grid paper (optional), 1-2 sheets per group and 1 transparency.
- ✔ Ruler, tape, and scissors, 1 of each per group.
- ✔ Colored markers for each group.
- ✔ Focus Masters A-D, 1 transparency of each.
- ✔ Calculators, 1 for each group.

FOLLOW-UP

OVERVIEW

Students analyze data produced in an experiment and make predictions based on that analysis. They make visual representations of situations and make mathematical observations about the situations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 45.1, 1 copy per student.

LESSON IDEAS

LOOKING BACK

Have students re-read their mathographies from the Lesson 1 Follow-up and review their journals and growth-folios. Then ask them to complete one or more of the following end-of-the-year activities:

- Add a "chapter" to your mathography describing your development this year as a mathematician. Give specific examples to illustrate your growth.
- Write a letter to an incoming math student. Let them know what to expect from this course, special ways that you have grown as a mathematician this year, and tips for maximizing their mathematical learning.
- Based on your experiences this year, describe your goals for yourself as a mathematician next year. Tell why you have set these goals and some ways you plan to accomplish them.
- Tell what you liked best and least about this class and why you feel this way.
- If you could change one thing about this class, what would it be and why.

LOOKING AHEAD

Issues related to sampling, confidence, and probability are explored in more depth in *Visual Mathematics, Courses II-IV.*

SELECTED ANSWERS

2. Sack 1, likely; sack 2, impossible; sack 3, likely; sack 4, unlikely.

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students use clues involving probability to determine collections of colored tile. The intent is for students to relate their understanding of probability (see Lesson 12) to their knowledge of fractions, decimals, and percent. These ideas are used and extended in the Focus activity.

MATERIALS

- ✔ Colored square tile (in at least 3 colors—preferably red, blue, and green), 30 per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Transparent colored tile.
- ✔ Student journals (optional).
- ✔ Calculators, 1 per group of students.

ACTIONS

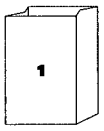
1 Ask each student to spend 3-5 minutes writing completions of the following statements. Discuss their ideas.

What the term probability means to me is ...
For example, ...

2 Place the students in groups of 3 or 4 and distribute tile to each student. Place a transparency of Connector Master A on the overhead and, revealing only the description of Sack 1, ask the groups to use their tile to form what they believe is the contents of the sack.

Have a volunteer form their group's collection of tile at the overhead and describe the reasoning they used to determine that collection. Discuss their ideas about the probability of selecting a blue tile from Sack 1.

Sampling, Confidence, and Probability Lesson 45
Connector Master A



Sack 1 This sack contains a total of 36 red, blue, and green square tile. If a tile is selected from the bag at random, the theoretical probability of selecting a red tile is 1/4, and the theoretical probability of selecting a green tile is .$\overline{3}$.

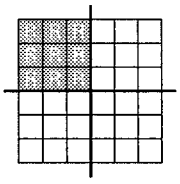
COMMENTS

1 Students' responses could be written in their journals, providing evidence of their views at this time of the meanings of probability. It is not important that students know formal definitions or have a solid understanding of probability at this point. The purpose of the Focus is to clarify and deepen their understanding.

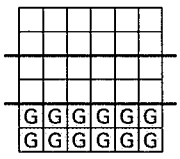
2 Students' earlier experiences with probability involved fractions. Rather than telling students how to use decimals to represent probabilities, encourage them to use their knowledge of the relationships among fractions and decimals (see Lesson 28) to interpret decimal representations.

Theoretical probability was introduced in Lessons 12 and 34 and represents what ought to occur when one knows the actual distribution of tile in the collection and when draws are truly random. Experimental probability, which is based on the outcomes of an experiment, was also introduced in Lessons 12 and 34. Both types of probability will be explored in depth in the Focus activity.

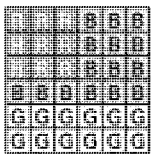
Since the theoretical probability of selecting a red square from Sack 1 is 1/4, then, 1/4 of 36 or 9 tile are red,



and 1/3 of 36 or 12 tile are green.



Combine the 1/4 and 1/3 to show that 36 – (9 + 12) = 15 tile must be blue.



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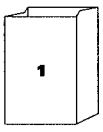
Connector Teacher Activity (cont.)

ACTIONS

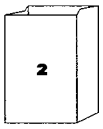
3 Repeat Action 2 for Sacks 2, 3, and 4. Discuss the probability of selecting a blue tile from each sack.

Sampling, Confidence, and Probability Lesson **45**

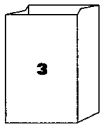
Connector Master A



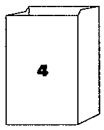
Sack 1 This sack contains a total of 36 red, blue, and green square tile. If a tile is selected from the bag at random, the theoretical probability of selecting a red tile is 1/4, and the theoretical probability of selecting a green tile is $.\overline{3}$.



Sack 2 There is a total of 40 tile in this sack. If one tile is drawn at random from the sack, the theoretical probability of selecting a green tile is .625. The probability of selecting a red is 0. The only other color of tile in the sack is blue.



Sack 3 A tile is to be drawn randomly from this sack of 56 red, blue, and green tile. The theoretical probability of selecting a green tile is 1/4. There are twice as many blue tile as red ones.



Sack 4 This sack contains the same percentage of red, blue, and green tile as Sack 3, but this sack contains exactly 38 blue tile.

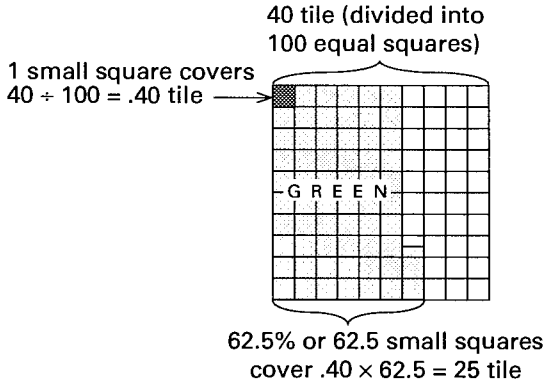
COMMENTS

2 (continued.)
The 15 blue tile can be viewed as $^{15}/_{36}$ or, in groups of 3, as $^{5}/_{12} = .41\overline{66}$ of the tile in Sack 1. Hence, the probability of selecting a blue tile is $^{15}/_{36} = {}^{5}/_{12} = .41\overline{66}$. Encourage the use of calculators to carry out conversions between fractions and decimals.

Some students may express probabilities as percentages. For example, in Sack 1 the probability of selecting a red tile is 1/4 or 25%, and of selecting a green tile is $^{1}/_{3} = 33\,^{1}/_{3}\%$. Hence, the probability of selecting a blue tile is $100 - (33\,^{1}/_{3} + 25) = 41\,^{2}/_{3}\%$. If this idea comes up, you may wish to discuss it. The intent in this activity, however, is to focus primarily on the probability issues of sampling and confidence rather than on fraction, decimal, and percent conversions.

3 The probabilities used to describe each sack are theoretical. Hence, students can use them to determine the exact contents of each sack. Following are examples of reasoning students have used.

Sack 2. Since the probability of drawing a green tile is .625, then if the group of 40 tile was divided into 100 equal parts, $^{62.5}/_{100}$, or 62.5% of 40 are green, as illustrated below:

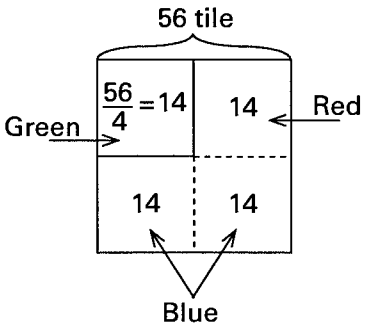


No tile are red. Therefore 40 – 25 = 15 are blue, and $^{15}/_{40}$ is the probability of drawing a blue. Using the previous diagram, notice that this could be expressed as a decimal, .375, or as a percent, 37.5%.

Sack 3. Since the probability of drawing a green is 1/4, 56 ÷ 4 = 14 tile are green. This means that 56 – 14 = 42 tile are either red or blue. These 42 tile can be divided into 3 equal groups of 14 tile. Since there are twice as many blue tile as red ones, 2 groups of 14 = 28 tile must

Connector Teacher Activity (cont.)

ACTIONS



COMMENTS

be blue and 1 group of 14 tile must be red. Hence, $^{28}/_{56} = \frac{1}{2}$ of the tile are blue, and the probability of selecting a blue is $\frac{1}{2} = .5 = 50\%$.

Sack 4. Since this sack contains the same percentage of each color as Sack 3, $\frac{1}{2}$ of the bag or 38 tile is blue. Therefore, the bag must contain $2 \times 38 = 76$ tile. Of the 38 tile that are not blue, 19 must be red and 19 green.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students produce samples from sacks of tile in order to predict the contents of the sacks. They compare theoretical and experimental probabilities and discuss their confidence in their predictions.

MATERIALS

- ✔ Secret Sacks (lunch-size paper bags filled with 12 red, 6 blue, and 2 green tile), 1 per group of students.
- ✔ Butcher paper, 2 poster-size sheets per group of students.
- ✔ 2-cm grid paper (optional), 1-2 sheets per group and 1 transparency.
- ✔ Ruler, tape, and scissors, 1 of each per group.
- ✔ Colored markers for each group.
- ✔ Focus Masters A-D, 1 transparency of each.
- ✔ Calculators, 1 for each group.

ACTIONS

1 Discuss the students' ideas about situations in which it would be necessary to make predictions based on a sampling of data rather than on theoretical evidence.

2 Place the students in groups of 3 or 4. Point out to the students that you have a Secret Sack for each group (don't distribute the sacks yet). Tell them that the Secret Sacks contain identical collections of 20 colored tile and their task is going to be to predict a reasonable approximation of the number of each color, without looking in the sacks. Their predictions are to be based *only* on information gathered from the sampling procedure described on the next page.

COMMENTS

1 Note that in the Connector activity, students could base their predictions about the contents of each sack on theoretical evidence. However, there are many times when it is either not possible or not reasonable to use theoretical relationships as a basis for making decisions or predictions. Instead, data that has been collected either formally or informally may influence one's decisions; for example, "It hasn't rained here on July 4th for the last 10 years, so let's plan a picnic for the 4th of July this year" or "Seattle's average daytime temperature during the month of December for the past 20 years has been 38°, so I need to buy a warm coat for my visit there at Christmas." Sometimes a situation requires collecting a sample in order to make a decision; for example, a candidate for an office wants to know where to spend the most energy campaigning, in California or in Texas, so she conducts a telephone survey in those states. Or, a running shoe company wants to advertise a new shoe, so they interview a sampling of participants in running competitions across the country to find out what magazines the runners read.

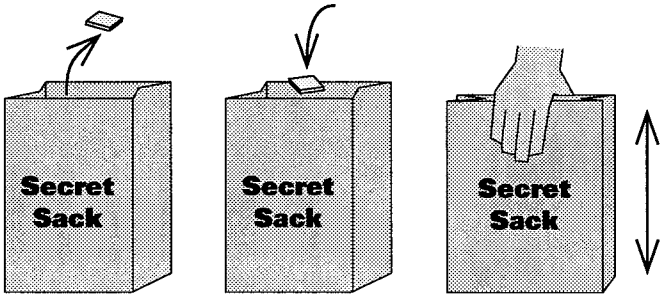
2 Lunch-size paper bags work well for the Secret Sacks. Prior to class place 12 red, 6 blue, and 2 green tile in each sack, fold over the top of each sack, and paper clip the fold in place (this will help prevent "peeks" when you pass out the sacks). There is nothing special about the distribution 12-6-2 for the colors. You may use other distributions; however, at least one color should be quite rare.

Be sure the class understands the sampling procedure. Each tile must be replaced before another is drawn. This

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



Without looking, draw 1 tile randomly from sack.

Return tile to sack.

Shake sack after each draw.

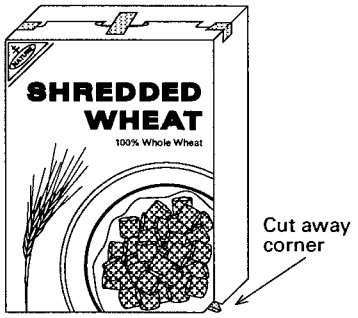
3 Ask each group to spend a few minutes devising and writing down a specific plan for using the sampling procedure to gather data and for using their data to predict the contents of the Secret Sack. Discuss.

COMMENTS

2 (continued).
ensures that draws are always made from the original contents of the sack. It is also very important to shake the sack to mix the tile after each draw.

Note that this procedure will be the *only* source of information about the contents of the sack. You may wish to let students know that at the very end of the activity they will have an opportunity to decide whether they want to check their predictions by looking in the sacks. In the meanwhile it is very important not to reveal the contents of the sack.

An alternative to paper sacks is to place the tile in a box (cereal box, shoe box, etc.) and tape the cover in place. Cut a corner off the box, as shown below:



Collect a random sample from the box as follows: shake the box and tip it to allow a tile to fall into the cut-off corner; record the color of the tile showing; shake the box and repeat. This method decreases the likelihood of peeking.

3 This is an example of a probability experiment. Each group is free to devise its own plan within the constraints of the sampling procedure described in Action 2. If a group has difficulty creating a plan, you may wish to suggest aspects for them to consider. For example, groups will need to decide how many draws to make, how to record and organize the data they collect, how to involve all group members in the process, and how they intend to use the data to predict the contents. Another way to provide groups with ideas is to have volunteers tell the class about their plans.

Some groups may decide on sample sizes that you feel are too small. However, rather than suggesting they change their plan, allow them time to recognize this on their own in the coming actions.

Focus Teacher Activity (cont.)

ACTIONS

4 When groups have completed their planning, give them each a Secret Sack and have them carry out their plans. Have calculators available for those who wish to use them. Let students know that it is okay to deviate from their plans if they feel adjustments will help them get better predictions.

5 After groups have completed their experiments, distribute 1 sheet of butcher paper and colored markers to each group. Have scissors, tape, and 2-cm grid paper available. Place a transparency of Focus Master A on the overhead and ask each group to make a poster that contains the information described on the transparency. When completed, ask each group to display their poster and briefly describe their process, results, and confidence level.

Sampling, Confidence, and Probability Lesson **45**
Focus Master A

a) Our original plan.

b) Adjustments we made in our plan, and our reasons for making those adjustments.

c) A visual display (graph) of our data.

d) Our prediction of the contents of our Secret Sack.

e) How we used our data to make our prediction.

f) How confident we feel about our prediction.

COMMENTS

4 It is very important that students not look inside the sacks, since the major goal of this activity is to use sampling to *predict* the contents.

5 Groups will likely display their results in different ways. Some may use tallies, bar graphs, area models, or circle graphs to represent the tile they drew, and some may make several types of graphs of their data. Following are excerpts from 4 groups' posters:

Group 1 (excerpt)

Our data:

Red	Blue	Green
𝍸 𝍸	𝍸 𝍸	||
𝍸 𝍸	𝍸 |||	

Our prediction and how we decided:

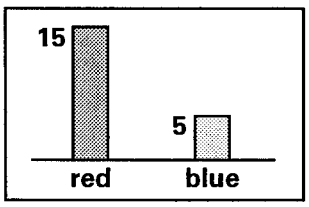
In 40 tries, we got red 20 times, blue 18 times, and green twice. We decided to make a prediction by cutting our results in half, so we think the sack contains 10 red, 9 blue, and 1 green.

Our confidence:

We feel pretty certain there are about as many red as blue, though we're not sure if there's more of one of these colors. We feel good about our guess for green–there can't be many green tile in the sack!

Group 2 (excerpt)

Here is a bar graph of our results. We got 3 times as many red as blue in 20 draws. So, our guess: 15 red and 5 blue tile are in the sack.



We're not sure, though. Maybe we should have drawn more times. We do feel good about predicting more red than blue.

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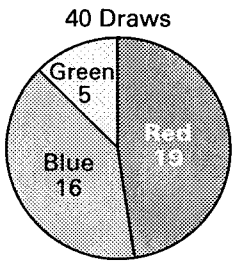
Focus Teacher Activity (cont.)

ACTIONS

Group 3 (excerpt)



We each made 10 draws for a total of 40 draws. Here is a pie graph of our data:

Using this information we predict the following for a bag of 20 tile:

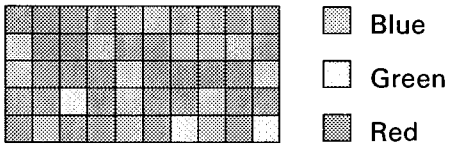
Red — about 8-10
Green — about 2-4
Blue — about 7-9

We think we would like to see what other groups learned to feel more confident about our prediction.

COMMENTS

5 (continued.)

Group 4 (excerpt)



We kept track of our 50 draws in order, so we colored squares of the grid in order to match our draws. Our group got mostly red. We feel pretty sure there are more than twice as many red tile as blue ones (note that this is true in 3 out of the 5 rows and 7 out of 10 columns in our chart) and there aren't very many green tile. Our prediction: 14 red, 5 blue, 1 green.

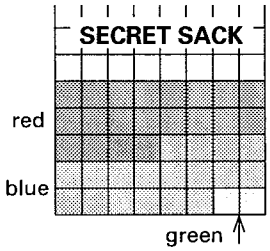
6 As needed, initiate a discussion about other possibilities for displaying data visually. Select one method for *all* groups to use to display their data. Ask them to include on this display the experimental probability (based on their group's data) associated with each color and to express these probabilities as fractions, decimals, and percentages. Have groups attach these displays to their posters.

Provide time for groups to study, compare, and question each other's results and methods of determining probabilities.

6 You may wish to explore students' ideas about ways to use methods such as bar graphs, area models, circle graphs, etc. that were not used by the groups. (See comment 5 for examples of several methods. See also Lessons 33 and 34.)

The groups' experimental results are easier to compare when they are presented in the same format. Since the area model is an especially useful way to show relationships among data, you may want to focus on this method. The following diagrams show area model and fraction/decimal/percent representations of data from Group 1 and Group 2 in Comment 5:

Group 1



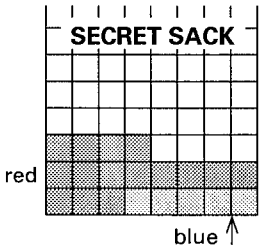
Probabilities:

$\text{red} = \frac{20}{40} = .50 = 50\%$

$\text{blue} = \frac{18}{40} = .45 = 45\%$

$\text{green} = \frac{2}{40} = .05 = 5\%$

Group 2



Probabilities:

$\text{red} = \frac{15}{20} = .75 = 75\%$

$\text{blue} = \frac{5}{20} = .25 = 25\%$

$\text{green} = \frac{0}{20} = 0 = 0\%$

Focus Teacher Activity (cont.)

ACTIONS

7 Place a transparency of Focus Master B on the overhead and ask the groups to discuss and write a response to each question. Ask for volunteers to describe their group's conclusions.

Lesson 45 Sampling, Confidence, and Probability
Focus Master B

a) What can you say *for sure* at this point about the contents of the Secret Sack?

b) What else can you say about the contents of the Secret Sack?

c) At this point, what do you think is a reasonable *range* of color mixtures likely to be in the Secret Sack?

d) How confident are you of your answer to Question c).

COMMENTS

Notice that in each of these examples, colors are shown contiguously, while in the example from Group 4 in Comment 5, the grid was colored to show the sequence in which colors were drawn. In the latter case it is possible to discuss questions such as "What was the longest run of red tile?" or "How many draws did it take before the green tile appeared?"

7 It is important that students have an opportunity to reflect about the confidence they have in their predictions. This confidence may be influenced by their observations of other groups' results. Variation in the results (which might be considerable if small numbers of draws were made) may cause groups to question their predictions. On the other hand, they may be very confident of some observations, such as "the sack contains more red tile than green ones" or "there are no yellow tile."

As groups share their conclusions it is helpful to record them on the transparency of Focus Master B.

The *range* of experimental probabilities for each color spans from the lowest to the highest probability obtained by the groups (such as 25% to 45% blue for groups 1-4 in Comment 5). This range can be expressed using fractions, decimals, or percents and can help to give an indication of likely and unlikely mixes of colors in the sack. For example, if every group obtained more than half red, it is likely that more than half the tile in the sack are red and unlikely that only 6 of the tile are red. Similar statements can be made about each color.

Based on their own results and observations of other groups' results, students may feel confident now about making statements such as: "There are probably more than 10 red tile." "There are only a few green tile." "I'd be surprised if someone opened the sack and found more green tile than red ones!"

If 7 out of 8 groups draw 0%-5% green tile, for example, and one group draws 20% green tile, some students may question that group's sampling procedures and predict 0%-5% as a reasonable range for green, treating 20% green as an *outlier* because they believe it is unreasonable. Thus, even though the exact number of each color is still unknown, one can begin to feel confident about a "range of reasonable" contents for the Secret Sack. This might be compared with the students' predictions and confidence levels in Action 5.

Focus Teacher Activity (cont.)

ACTIONS

8 Ask each group to list some ways they could improve their experiments so the class could get a better idea of what is in the Secret Sack, and feel more confident about that idea. Discuss their ideas and reasoning. Bring up important aspects of data collection and analysis that don't come up.

9 Tell the class that you want them to design and conduct an experiment, as a large group, that will enable them to predict with a high level of confidence the contents of the Secret Sack. Discuss their ideas and reach consensus on methods of data collection, display, and analysis for a whole-class experiment.

10 Have the students carry out the plans they devised in Action 9. Discuss the results, and the effect of these results on students' feelings about the questions on Focus Master B (see Action 7).

COMMENTS

8 Students may have many suggestions. Acknowledge all ideas that are reasonable. Important points to consider include: using larger samples; collecting same-sized samples; expressing probabilities in ways that are easy to compare; following sampling procedures uniformly; and displaying data in similar formats.

In the extreme, a sample of only 1 or 2 tile would not give much information about the contents of the sack; 20 or 30 give a better picture; and 50 to 100 tile should give an even better idea. The main point is that larger samples are less likely to misinform us about the contents of the sack.

Having all groups draw the same-sized sample gives a better basis of comparison across groups. Note that students may suggest finding the average of the experimental probabilities from Action 6. This would not give as useful information, however, unless the sample sizes are equal. If one group only drew 10 tile and another drew 50, the percentages of red, blue, and green tile in their samples may be quite different. For example, a group that draws only 10 tile may get 8 red tile and 2 blue and not discover a green one.

As discussed in Action 6, probabilities are easier to compare when they are all expressed as decimals, as percents, or as fractions with the same denominator. Similarly, when all displays use the same format it is easier to get a feel for overall trends and to compile data into a single display.

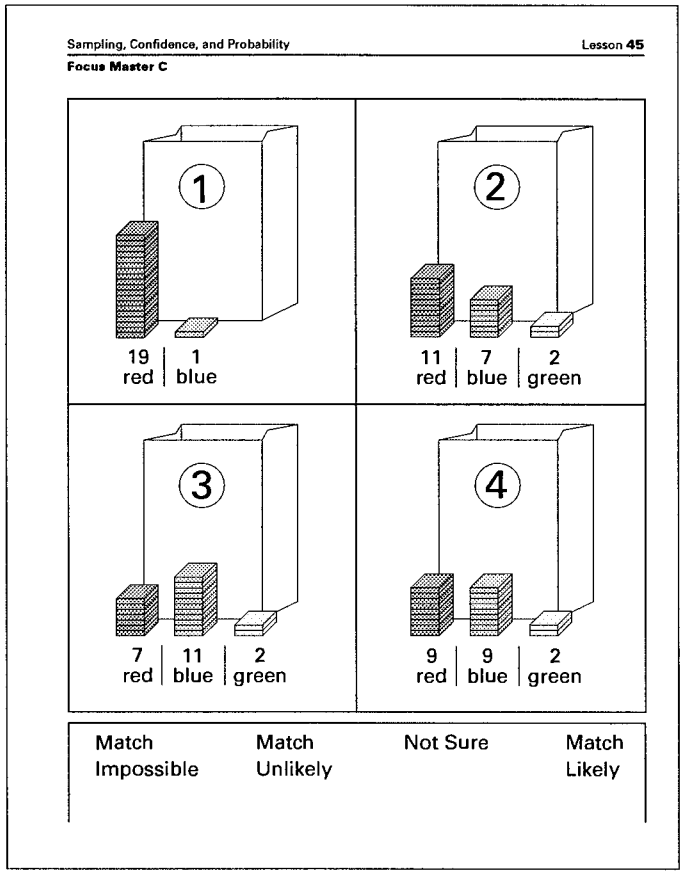
9 When designing a plan, remind students to take into consideration important issues related to data collection and analysis discussed in Action 8. You may find it helpful to make some suggestions. For example, it would be reasonable for each group to collect a sample of 50 and then compile their data, but it wouldn't be reasonable (because of time constraints) for each to collect a sample of 500 draws.

10 You may wish to place a transparency of Focus Master B on the overhead and see what new information or levels of confidence students have after the class experiment.

Focus Teacher Activity (cont.)

ACTIONS

11 Place a transparency of Focus Master C on the overhead. Have the groups determine whether they think the contents of each of the 4 sacks shown on the transparency is an impossible match, an unlikely match, a "not sure," or a likely match to the contents of the Secret Sack. Discuss the groups' reasoning and conclusions.



12 Place a transparency of Focus Master D (see next page) on the overhead and ask each group to discuss and write down several other sacks of 20 tile they feel have color distributions that fall into each of the categories. Have volunteers share their groups' ideas and reasoning. Record their suggestions on the transparency.

COMMENTS

11 You may wish to give each group a copy of Master C for easier viewing. The distinction between impossible, possible, and likely is an important idea. The students may want to carry out an experiment with one or more of these sacks to check how close the drawn samples come to the contents of the Secret Sack. Following is one possible placement of sacks 1-4 from Master C:

Match Impossible	Match Unlikely	Not Sure	Match Likely
(1)	(3)	(4)	(2)
19 R, 1 B	*7 R, 11 B, 2 G*	*9 R, 9 B, 2 G*	*11 R, 7 B, 2 G*

Note that all of the above predictions are based on data collected during previous actions. Hence, one can say that Sack 1 is an "impossible" match to the contents of the Secret Sack because at least 1 green has been drawn during the previous experiments.

12 Without knowledge of the Secret Sack, the samples drawn in the experiments in Actions 4 and 10 could have been drawn from a range of color distributions, some of which are more likely than others. It is important to be aware of these possibilities and to reflect about the likelihood of each.

One possible direction for the discussion is to invite students to express their tolerance for "possible" as opposed to "likely" and "impossible" contents for the Secret Sack. Where will they draw the line? For example, a composition of 1 red, 1 blue, and 18 green is possible, but highly unlikely. A composition of 10 blue and 10

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

12 (continued).
green is definitely impossible, since there is already evidence of red tile. Following are some sample suggestions from students:

Lesson 45 Sampling, Confidence, and Probability
Focus Master D

Propose some 20 tile sacks that you feel could fit in each of these categories. List the contents—number of red, blue, green tile—for each sack you propose under the category.

Match Impossible	Match Unlikely	Not Sure	Match Likely
19 R, 1 B	*7 R, 11 B, 2 G*	*9 R, 9 B, 2 G*	*11 R, 7 B, 2 G*
20 B	*5 R, 14 B, 1 G*	*10 R, 9 B, 1 G*	*13 R, 5 B, 2 G*
10 B, 10 G	*10 R, 5 B, 5 G*		*11 R, 6 B, 3 G*

13 Ask each group to study and discuss all the data collected so far, both by their group and by the class as a whole, and to make its final prediction of a Likely Sack.

List each group's Likely Sack on the overhead or chalkboard and discuss these predictions. How do probabilities (experimental and/or theoretical) associated with each Likely Sack compare to the range of experimental probabilities predicted for the Secret Sack (see Actions 7 and 10)? How confident (say on a scale of 1-10) are students with their Likely Sack? If an experiment like the one designed by the class in Action 9 were carried out on their Likely Sack, theoretically how should the results compare to the results of the class experiment in Action 10?

13 It is important to emphasize that there is a range of "right" answers for the Likely Sack, and there is not a "best" Likely Sack. Rather, you are looking for reasonably close matches to the Secret Sack.

14 Let the class vote to decide whether to reveal the contents of a Secret Sack (we've not had a group vote "no" yet). Discuss the students' feelings about whether their experiments and reasoning were successful. Do they feel they made reasonable predictions of the contents of the Secret Sack? What influenced their confidence levels throughout this lesson? Why is it useful to predict in terms of ranges rather than exact amounts? What are examples of situations in which one would need to predict or make decisions from a sampling? What are students' recommendations for others who need to make decisions based on a sampling of information?

14 As discussed in Action 1, there are times when it is not possible (or reasonable) to use theoretical relationships as a basis for making decisions or predictions. Here is one example of a situation that could require a sampling in order to make a decision: A soft drink company, wishing to increase its sales in the western United States, wants to know the soft drink preferred by 12- to 14-year-olds in the region. It isn't reasonable to gather data from every 12- to 14-year-old so it would be important to gather a reliable sample. This situation could generate an interesting discussion of students' ideas about what sampling procedures might produce an unbiased sample.

Follow-up Student Activity 45.1

NAME _____ DATE _____

1 There are 30 tile in a paper sack. The tile have the colors red, yellow, blue, and green. To predict the contents of the sack, 4 students each conduct an experiment. The procedure they use is: without looking, draw one tile; record its color; return the tile to the sack; and then repeat for a total of 30 draws each. The numbers of each color these students obtained is shown below.

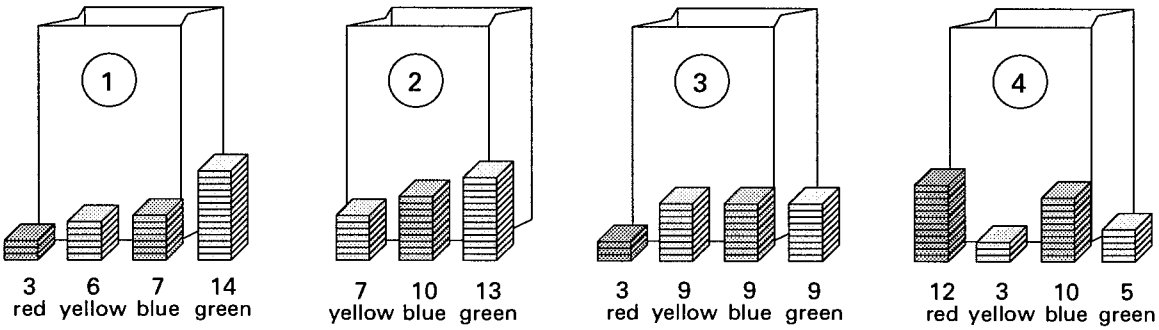
	Janine	Eric	Tyson	Rachelle
Red	1	2	0	4
Yellow	8	5	8	3
Blue	7	10	9	5
Green	14	13	13	18

Based on the information above, write what you think is likely to be the contents of the sack. Give valid mathematical arguments, supported by clear diagrams, to support your prediction.

(Continued on back.)

Follow-up Student Activity (cont.)

2 Compare the following sacks to what you think might be in the original sack in Problem 1. Label each sack as either Impossible, Unlikely, Not Sure, or Likely, and write a reason for each of your decisions.



3 Explain what you believe are the most important points to consider when you need to use sampling to make a confident prediction.

4 On another sheet of paper, make a diagram to show the relationship described in each of these situations. Next to each diagram write 3 or more interesting mathematical observations or predictions about the situation.

a) A radio station placed contest entry forms in a box with the names of 50 people of age 9-12, 70 people of age 13-19, and 80 people of age 20 or older. The winner will be selected by a random draw from the box.

b) There are 60 pieces of bubble gum with colors of blue, yellow, and green in a machine. The theoretical probability of selecting a blue is $\frac{3}{10}$, and the probability of selecting a green is 50%.

Materials

NECESSARY MANIPULATIVES AND MATERIALS

INSTRUCTIONAL MATERIALS

Visual Mathematics, Course I (Lessons 1-45)
Starting Points for Implementing Visual Mathematics
Visual Mathematics, Course I Blackline Masters
Blackline Masters include Student Activities Grids.
Segment Strips Print Masters

MANIPULATIVES—MAKE FROM BLACKLINE MASTERS OR PURCHASE FROM MLC

Base Five Area Pieces
1 sheet per student

Base Five Linear Pieces
1 set per 36 students

Base Ten Area Pieces
1 sheet per student

Base Ten Linear Pieces
1 set per 36 students

OVERHEAD ITEMS—MAKE FROM BLACKLINE MASTERS OR PURCHASE FROM MLC

Base Five Area/Linear Pieces, Overhead
1 set of each per teacher

Base Ten Area/Linear Pieces, Overhead
1 set of each per teacher

Segment Strips, Overhead
1 set per teacher

Pattern Blocks, Overhead
Standard set of 120 blocks

Colored Tiles, Overhead
40 1-inch tile in 4 colors

MANIPULATIVES AVAILABLE FROM MLC

Cubes, Wood (¾" or 2-cm)
120 cubes

Dice, 6-sided
8 dice

Linear Pieces (1-inch red)
About 120 pieces

Large Linear Unit Pieces (2-cm black)
1 set per teacher

Game Markers
1 bag per 15 students

Pattern Blocks
Standard set of 250 blocks

Colored Tile
About 1,200 1-inch tiles in 4 colors

Geoboards, Clear
1 per student and teacher

Coffee Stirrers
About 60 stirrers

Overhead Pens
1 set per teacher

Rulers
1 per student

Visit catalog.mathlearningcenter.org to order these and other materials.

MATERIALS NOT AVAILABLE FROM MLC

Blank Transparencies
Scissors
Yarn or String
Poster Paper
Tape, Glue, Glue Sticks
Colored Markers
Pom-Poms
About 48 small pom-poms

OPTIONAL MATERIALS

ADDITIONAL VISUAL MATHEMATICS TEXTS

Visual Mathematics, Course I Student Activities Packet
Visual Mathematics Student Journal

VIDEOS FROM MLC

Math and the Mind's Eye Video
A Change of Course: Implementing Visual Mathematics in the Classroom Video

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- Visual reasoning, and problem solving are integral parts of every *Visual Mathematics* lesson, as are expectations for student behaviors such as cooperation and communication.

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