math alive!

Linda Cooper Foreman and Albert B. Bennett, Jr.

visual mathematics

COURSE III

Math Alive!
Visual Mathematics, Course III
by Linda Cooper Foreman and Albert B. Bennett Jr.

Math Alive! Visual Mathematics, Course III is preceded by:
Visual Mathematics, Course I
Visual Mathematics, Course II

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Contents

Introduction

Math Alive! is a series of four comprehensive, NCTM Standards-based, one-year courses for students in the middle grades. This curriculum is in development with support from the National Science Foundation and is the grades 5-8 portion of a new Math Learning Center seamless K-8 curriculum.

The first two courses in the *Math Alive!* middle grades series were originally published under the names *Visual Mathematics, Course* I and *Course II*. Subsequent publications of those courses will be renamed *Math Alive! Course I* and *Course II*. This book, *Math Alive! Course III*, is the third course in this series. The fourth course is now in development. *Math Alive! Course III*, is designed for use by teachers in grades 7 or 8 whose students have completed *Course II*, or by teachers in grades 7-9 whose students are exploring *Math Alive!* for the first time. To support the teacher whose students may need additional background, there is extensive cross referencing to *Courses I* and *II*.

Throughout the 17 *Math Alive! Course III* lessons, each averaging about two weeks of class time, there are many implementation suggestions. In addition, the teachers' resource book, *Starting Points for Implementing Math Alive!*, provides an overview of the philosophy and goals of the *Math Alive!* courses, together with extensive suggestions for: organizing materials; planning, pacing, and assessing lesson activities; working with parents and the community outside your classroom; finding support as you seek changes in your teaching practices; and creating a classroom climate that invites risk-taking and nourishes the mathematician within each student. The ideas in *Starting Points* are based on our own classroom experiences and comments we have received from many other teachers field testing *Math Alive!* courses.

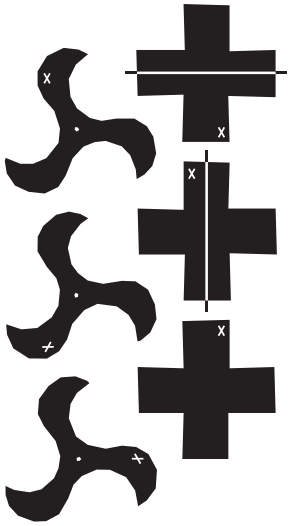
Teaching *Math Alive!* ourselves has affirmed our beliefs in the potential within each student, enriched our views of mathematics and the art of teaching mathematics, and reinforced our commitment to support teachers in their efforts to change the way mathematics is learned and taught. It is our hope that teaching *Math Alive!* will be equally fulfilling for you.

Exploring Symmetry

Lesson 1

THE BIG IDEA

Actions such as drawing, cutting, tracing, framing, rearranging, flipping, turning, imagining, and discussing shapes build spatial sense and promote insights and intuitions about symmetry concepts. Investigations that involve forming polygons which satisfy certain symmetry conditions prompt conjectures and generalizations. Such activities provide a rich context for experiencing the mathematical process.



CONNECTOR

OVERVIEW

Students draw frames for 2-dimensional figures to identify the figures' reflectional and rotational symmetries and order of symmetry.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Masters A-C, 1 copy per group and 1 transparency.
- ✔ Connector Master D, 2 copies per student and 2 transparencies.
- ✔ Note (file) cards, 1 unlined card per student.
- ✔ Plain (unlined) paper, 1 sheet per student.
- ✔ Butcher paper, 1 large sheet per class.
- ✔ Butcher paper strips (4-5" long), 8-10 per group.
- ✔ Scissors, 1 pair per student.
- ✔ Tape and marking pens for each group.
- ✔ Quartered blank transparencies (optional) for use at the overhead.

FOCUS

OVERVIEW

Students draw symmetric figures on square grids and on triangular grids and note the different symmetry types possible. They investigate all orders of symmetry possible for 3-sided to 8-sided polygons and make conjectures and generalizations based on their observations. Finally, students reflect on how these activities relate to the goals of the class.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Student Activities 1.1-1.2, 1 copy of each per student and 1 transparency of each.
- ✔ "We conjecture…/We wonder…" poster from the Connector activity, 1 for each class.
- ✔ 1-cm squared grid paper, 4 sheets per student and 1 transparency.
- ✔ 1-cm triangular grid paper, 4 sheets per student and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Butcher paper strips, 8 for the teacher, 16 for each group, and several for use by the class, as needed during the lesson.
- ✔ Marking pens and tape for each group.

FOLLOW-UP

OVERVIEW

Students form conjectures and generalizations about the order of symmetry for regular *n*-gons. They identify the symmetries of flags and logos and create a logo with symmetry. They investigate and generalize about situations involving symmetry.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 1.3, 1 copy per student.

LESSON IDEAS

STUDENT INVESTIGATIONS
Investigations in this lesson provide a rich context for "doing mathematics." You might take time during the Focus activity to discuss the fact that, like for professional mathematicians and scientists, "successful" investigations may leave the students with more questions than answers.

FOLLOW-UP
Keep in mind that Follow-ups require extended time for students to investigate and communicate their ideas. They are not designed to be "due the next day." It is not necessary to assign every problem, and it is reasonable to move on to the next lesson while students complete a Follow-up at home. Rather than discussing and showing "how-to" solve problems, during class discussions of the Follow-up, suggest that students seek and share "clues" to jump start each other's thinking. Some teachers select problems from Follow-ups for in-class assessment activities.

In our classroom, we ask that students revise incorrect work on Follow-ups before a grade is entered in the grade book and provide opportunities before and after school to discuss students' questions. Grades on Follow-ups are determined according to criteria on the Follow-up Assessment Guide given in the teacher resource book, *Starting Points for Implementing Math Alive!*

PACING
On average, each lesson in this course is designed to take about 2 weeks. This may vary according to your students' backgrounds, your familiarity with the curriculum, your school schedule, and student- or teacher-generated extensions you choose to explore.

QUOTE
Symmetry in two and three dimensions provides rich opportunities for students to see geometry in the world of art, nature, construction, and so on. Butterflies, faces, flowers, arrangements of windows, reflections in water, and some pottery designs involve symmetry. Turning symmetry is illustrated by bicycle gears. Pattern symmetry can be observed in the multiplication table, in numbers arrayed in charts, and in Pascal's triangle.
NCTM Standards

SELECTED ANSWERS

1. triangle: 3 reflectional symmetries, 3 rotational symmetries

   square: 4 reflective, 4 rotational

   pentagon: 5 reflective, 5 rotational

   hexagon: 6 reflective, 6 rotational

   heptagon: 7 reflective, 7 rotational

   octagon: 8 reflective, 8 rotational

   Measures of the angles of rotational symmetry for the following regular polygons:

   triangle: 120°, 240°, 360°

   square: 90°, 180°, 270°, 360°

   pentagon: 72°, 144°, 216°, 288°, 360°

   hexagon: 60°, 120°, 180°, 240°, 300°, 360°

   heptagon: $51\frac{3}{7}$°, $102\frac{6}{7}$°, $154\frac{2}{7}$°, $205\frac{5}{7}$°, $257\frac{1}{7}$°, $308\frac{4}{7}$°, 360°

   octagon: 45°, 90°, 135°, 180°, 225°, 270°, 315°, 360°

3. Here are possible expressions (others are also possible), listed in the order of the chart: $2n$; n; n; $k\left(\frac{360°}{n}\right)$ for $k = 1, 2, 3\ldots n$; $\frac{(n-2)180°}{n}$.

9. a) True
   b) True

Connector Teacher Activity

OVERVIEW & PURPOSE

Students draw frames for 2-dimensional figures to identify the figures' reflectional and rotational symmetries and order of symmetry.

MATERIALS

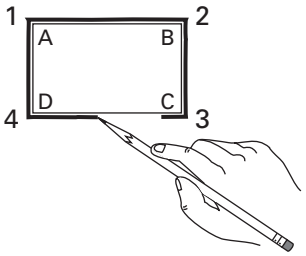
- ✔ Connector Masters A-C, 1 copy per group and 1 transparency.
- ✔ Connector Master D, 2 copies per student and 2 transparencies.
- ✔ Note (file) cards, 1 unlined card per student.
- ✔ Plain (unlined) paper, 1 sheet per student.
- ✔ Butcher paper, 1 large sheet per class.
- ✔ Butcher paper strips (4-5" long), 8-10 per group.
- ✔ Scissors, 1 pair per student.
- ✔ Tape and marking pens for each group.
- ✔ Quartered blank transparencies (optional) for use at the overhead.
- ✔ Coffee stirrers (optional), 1 per student.

ACTIONS

1 Arrange the students in groups and give each student a plain (unlined) rectangular note (file) card and a plain sheet of paper. Ask them to label the corners of their note cards A, B, C, and D. Hold a note card against a sheet of plain paper mounted on the wall, and draw a frame around the card. Label the corners of the frame 1, 2, 3, and 4. Ask the students to also draw and label frames for their note cards. Then give each group a copy of Connector Master A (see next page) and have the students carry out the instructions. Discuss, inviting volunteers to demonstrate their methods and observations at the overhead.

COMMENTS

1 Leave as little space as possible between the card and its frame:



Note that ready to copy masters for all Connector and Focus Masters and Student Activities are contained in *Blackline Masters*. In addition, *Student Activity Packets* are available from The Math Learning Center (MLC); each packet contains a one-student supply of masters and student activities needed by individual students for this course. A one-student supply of grid paper, *Student Activity Grids*, is also available from MLC.

Students who have difficulty reading may need some help here. Encourage students to support their groupmates as they interpret the instructions in a)-c).

There are an infinite number of points around which the card can be rotated 360° (or 0°) to exactly fit back into its frame, since a 360° (or 0°) rotation about any point on the card replaces the card in its original position. A point about which a shape is rotated to fit back

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Exploring Symmetry Lesson **1**

Connector Master A

ROTATIONS/TURNS

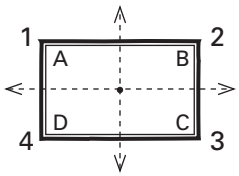
a) Complete this procedure:

- Position your note card so that it fits in its frame with no gaps or overlaps.
- Mark a point anywhere on your card with a dot, and label this point P.
- Place a pencil point on your point P and hold the pencil firmly in a vertical position at P.
- Rotate the card about P until the card fits back into its frame with no gaps or overlaps.

b) How many *different* rotations of the card about your point P are possible so that the card fits back in its frame with no gaps or overlaps? Assume that rotations are different if they result in different placements of the card in its frame.

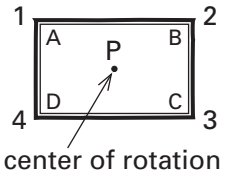
c) If only a 360° (or 0°) rotation about your point P brings the card back into its frame, find another position for P on the card so that *more than one different rotation* about this point is possible. What are the measures of the rotations and how did you determine them?

2 Give each group a copy of Connector Master B and ask them to carry out the instructions. Discuss their results.

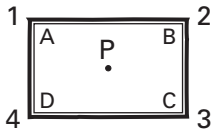


COMMENTS

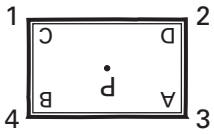
1 (continued.)
into its frame is the *center* of the rotation. The point P illustrated below is the only point about which a rotation other than a 360° (or 0°) rotation is possible.



The card can be rotated 180° and 360° (or 0°) about the point P shown above, and each of these rotations produces a different position of the card in its frame. Note that *different positions* refers to the placements of the card, not the methods to reach those placements. Lettering the corners of the card and the corners of the frame helps distinguish the positions:



Position after 360° (or 0°) rotation about P.



Position after 180° rotation about P.

Students may notice that the result of a 180° counter clockwise rotation about point P leaves the card in the same position as a 180° clockwise rotation about the point. They may wonder if this is true for all rotations (it is not).

2 The result of flipping the card over a line *l* is equivalent to the image observed when a mirror is placed along the line (with the exception of labels). If you have access to mirrors, you could distribute them and have students confirm this fact. Because of this relationship, the terms *flip* and *reflect* are used synonymously, and when the card fits back into its frame the line *l* is referred to as a *line,* or *axis, of reflection.*

Another test for a line of reflection is to determine whether the card can be folded along the line to form two halves that exactly match one another.

There are only 2 axes/lines over which the rectangular card can be reflected to fit back into its frame, as shown by the dashed lines at the left.

Connector Teacher Activity (cont.)

ACTIONS

Lesson **1** Exploring Symmetry

Connector Master B

REFLECTIONS/FLIPS

Figure 1 below shows the frame for a rectangular card with a line *l* drawn across the frame. In Figure 2, the card has been placed in the frame. Figure 3 shows the result of *reflecting*, or *flipping*, the card over line *l*. Notice that after the reflection over line *l*, the card does not fit back in its frame.

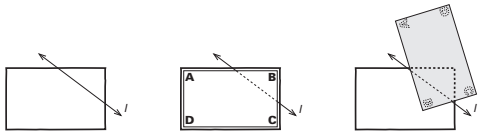
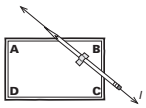


Figure 1 Figure 2 Figure 3

Determine all the *different* possible placements of line *l* so that when you flip your card once over *l*, the card fits back in its frame with no gaps or overlaps.

HINT: As a guide for flipping the card about a line, you could tape a pencil or coffee stirrer to the card along the path of line *l*, as shown below. Then keep the pencil or coffee stirrer aligned with line *l* as you flip the card.



3 Give each group a copy of Connector Master C. When the groups have carried out the instructions, lead a discussion of their ideas, clarifying terminology as needed.

Exploring Symmetry Lesson **1**

Connector Master C

a) Discuss your group's ideas and questions about the meanings of the following terms. Talk about ways these terms relate to a nonsquare rectangle such as your note card. Record important ideas and questions to share with the class.

i) reflectional symmetry
ii) axis of reflection (also called *line of reflection*)
iii) rotational symmetry
iv) center of rotation
v) frame test for symmetry

b) If a shape is symmetrical, its *order of symmetry* is the number of different positions for the shape in its frame, where different means the sides of the shape and the sides of the frame match in distinctly different ways. Develop a convincing argument that your rectangular note card has symmetry of order four.

COMMENTS

3 Although terminology comes up here for discussion, the intent is that students gain comfort with terms throughout the lesson. Rather than memorizing terms, it is important that students develop a general feel for the motions that take a shape from one position to another in its frame. It is helpful if you model appropriate usage of terms by using informal and formal terms interchangeably. For example, you might reinforce the meanings of reflection and rotation by using the terms *flip* and *turn* along with *reflect* and *rotate*.

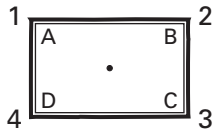
a) In *Math Alive! Course I* (Lesson 16), students learned that a shape which fits in its frame in more than one position is said to have *symmetry* or to be *symmetrical*.

A 2-dimensional shape has *reflectional symmetry* if it can be moved from one position in its frame to another (i.e., so it fits exactly back into the frame) by flipping it about a line called an *axis of symmetry*, or *line of symmetry*. (Note: reflectional symmetry for a plane figure is also referred to as *reflection symmetry*, *reflective symmetry*, and

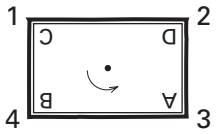
(Continued next page.)

Connector Teacher Activity (cont.)

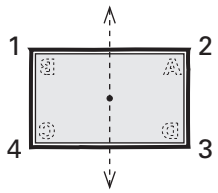
ACTIONS



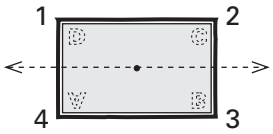
Position 1
ABCD→1234
same after 360°
or 0° rotation



Position 2
ABCD→3412
rotate Position 1 180°



Position 3
ABCD→2143
reflect Position 1
over vertical axis



Position 4
ABCD→4321
reflect Position 1
across horizontal axis

COMMENTS

3 (continued.)
bilateral symmetry.) A nonsquare rectangle, for example, has 2 reflectional symmetries because it has 2 lines, or axes, of symmetry. Although a mirror provides a method of checking for reflectional symmetry, the mirror is not useful for checking for rotational symmetry. Thus, an advantage of using a frame is that it can be used to check for both reflectional and rotational symmetry.

A shape has *rotational symmetry* if it can be moved from one position in its frame to another position in its frame by rotating, or turning, it less than 360° about a point, called the *center of rotation* so that the shape fits exactly into its frame. For example, a rectangular note card has 2 rotational symmetries because both a 180° and 360° (or 0°) rotation move the shape to fit exactly into its frame and each rotation produces a different relationship between the sides of the card and the sides of the frame. Note: rotational symmetry is also referred to as *rotation symmetry*.

It is standard to say that, if a figure can be rotated *only* through a full turn (360°) to fit exactly back into its frame, it has *no rotational symmetry*. If other rotations are possible for the shape or, if there are one or more reflectional symmetries, then the 360° rotation is included when counting the shape's number of rotational symmetries.

In general, using a "frame test" to determine a shape's rotational and reflectional symmetries involves checking for rotations (other than 360°) and reflections that lead to different final positions of the shape in its frame.

b) Any nonsquare rectangle has symmetry of order 4 because there are exactly 4 different positionings of the rectangle in its frame, as illustrated at the left. There are other methods one can use to reposition the card in its frame, such as to reflect and then rotate the card, or vice versa, but the final placement of the card will be one of the 4 shown at the left.

Students may recall from *Course I* that the *order of the rotational symmetry* of a shape is the number of different rotations that take the shape back into its frame, including a 360° rotation. To avoid confusion over usage of the term order, throughout this lesson reference is made only to the overall order of symmetry of a shape (i.e., the total number of different positionings of a shape in its frame due to rotations and reflections).

Connector Teacher Activity (cont.)

ACTIONS

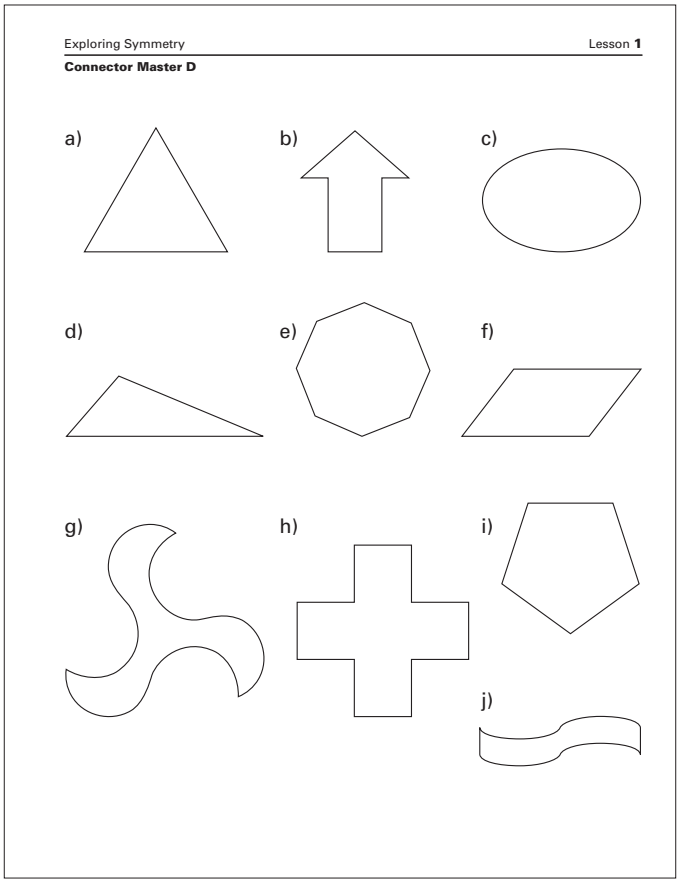
4 Give each student 2 copies of Connector Master D and a pair of scissors. Have the students each cut out Shape a) from one copy of Connector Master D and using Shape a) on the other copy as a frame, do the following, *if possible*:

i) draw all axes of reflection; record the number of reflectional symmetries,

ii) mark the shape's center of rotation and list the measures of all angles of rotation,

iii) determine and record the shape's order of symmetry.

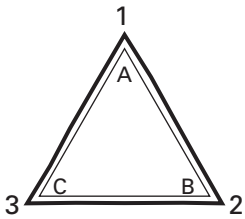
Discuss the students' results, inviting volunteers to demonstrate their conclusions and reasoning at the overhead.



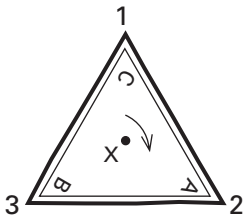
COMMENTS

4 Copying the shaded page 2 on the back of page 1 of Connector Master D enables students to distinguish shapes from their reflections.

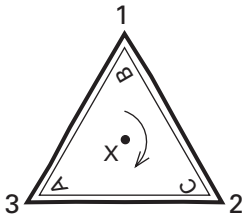
Shape a) is an equilateral triangle. Since there are 3 lines it can be flipped across to fit exactly into its frame, Shape a) has 3 reflectional symmetries. Since it can be rotated 120°, 240°, and 360° to fit into its frame, it has 3 rotational symmetries. Further, because shape a) has 6 different positionings in its frame (i.e., with distinctly different pairings of the sides of the card and sides of the frame), it has symmetry of order 6.



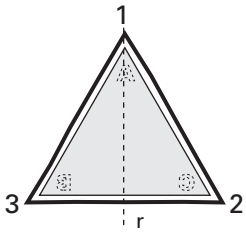
Position 1
ABC→123
same after 360°
or 0° rotation



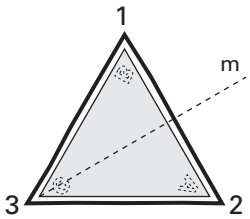
Position 2
ABC→231
rotate Position 1
120° about X



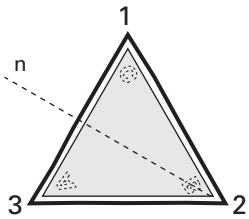
Position 3
ABC→312
rotate Position 1
240° about X



Position 4
ABC→132
flip Position 1
across line r



Position 5
ABC→213
flip Position 1
across line m



Position 6
ABC→321
flip Position 1
across line n

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

5 Ask the students to repeat Action 4 for the remaining shapes on Connector Master D and suggest they confer with their groupmates as they work. Post a large sheet of poster paper with the heading, "We conjecture…/We wonder… ." Distribute several strips of butcher paper, tape, and marking pens to each group. Ask that, as students work on Connector Master D, they record their ideas, conjectures, and questions on strips of butcher paper and attach them to the poster.

When the students have examined all of the shapes on Connector Master D, discuss their results, including conjectures and questions that have been posted.

COMMENTS

4 (continued.)
Note that students may find other methods of moving Shape a) so that it fits in its frame, but such motions will place the shape in one of the above final positions in its frame.

Students may conjecture about relationships they notice. Encourage such discussion, but rather than confirming or correcting their ideas, suggest that they look for verification or contradictions during the remaining actions of this lesson.

5 It is helpful to have a large supply of 4-5" long butcher paper strips on hand throughout this and subsequent lessons, so that students can record their conjectures and questions before posting them. During this activity, as you circulate while students work, you might collect and post their statements, delaying discussion until after students have completed their work on Connector Master D. You might suggest that it is okay if groups post similar conjectures or questions; differences in wordings may be useful to examine.

This activity, including writing "We conjecture … " and "We wonder …" statements, could also be completed as homework and then discussed in class.

b) 1 reflectional and no rotational symmetries; order 2

c) 2 reflectional and 2 rotational (180° and 360°) symmetries; order 4

d) no symmetry

e) 8 reflectional and 8 rotational (45°, 90°, 135°, 180°, 225°, 270°, 315°, and 360°) symmetries; order 16

f) no reflectional and 2 rotational (180° and 360°) symmetries; order 2

g) no reflectional and 3 rotational (120°, 240°, and 360°) symmetries; order 3

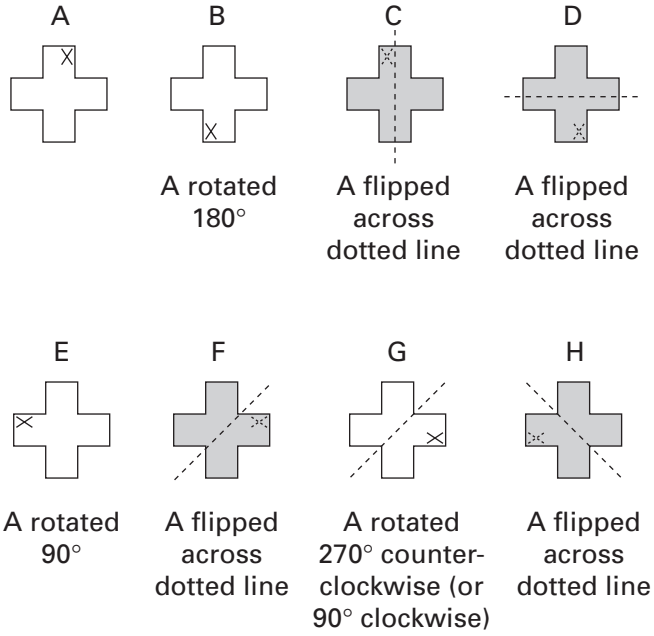
h) 4 reflectional and 4 rotational (90°, 180°, 270°, and 360°) symmetries; order 8

i) 5 reflectional and 5 rotational (72°, 144°, 216°, 288°, 360°) symmetries; order 10.

j) no reflectional and 2 rotational (180°, 360°); order 2.

Connector Teacher Activity (cont.)

ACTIONS



COMMENTS

Rather than labeling vertices or several points on a shape, another way to count the positionings of a shape in its frame is to place an X on the shape, and then to determine the different positions that X can take inside the frame. For example, one can mark an X in the upper right corner of Shape h) from Connector Master D and see there are 8 different positions (see diagram at the left) possible for the X.

It is possible to use other rotations (turns) and/or reflections (flips), or combinations of rotations and reflections, to reach the positions. However, no other positions are possible.

For some students, the motions they use to change the position of a shape in its frame may be intuitive. To encourage thought about the properties of the motions, you might ask questions such as the following: Can you describe that action in more detail? How did you decide the direction to flip the shape? What line are you flipping the shape over? How did you decide the size of the turn? Where is the pivot/turning point? If a student uses a pencil point, for example, to hold a shape in place while rotating it, ask whether they can hold the pencil point in other places.

Allow plenty of time for groups to investigate and conjecture. You might suggest that they form new shapes to test their conjectures. A particular observation can often be prompted into a generalization by asking, "When else will it be true?"

Following are some conjectures that students have given. If these do not come up now, there will be opportunities during the next action and during the Focus activity.

Shapes with just 1 line of symmetry have 2 different final positions in their frames.

Shapes with no symmetry fit in their frames in only 1 way.

Any shape with at least 1 line of symmetry has an even number of different final positions in its frame.

If a shape has an odd number of final positions in its frame then it only has rotational symmetry.

Two possible final positions implies 1 line of symmetry and no rotational symmetry, or 2 rotational symmetries and no reflectional symmetry.

Three final positions implies 3 rotational symmetries.

Connector Teacher Activity (cont.)

ACTIONS

6 Ask the groups to determine ways to sort and classify shapes from Connector Master D according to the symmetries of the shapes. Have the groups label each classification, and draw another shape that fits in each classification. Discuss, asking the groups to share new conjectures and "We wonder… " statements that surface.

7 (Optional) Ask the groups to sketch, if possible, a shape (different from those on Connector Master D) for one or more of the following conditions. Invite volunteers to sketch their shapes at the overhead for verification by the other students. Record any new conjectures or "We wonder…" statements that are suggested.

a) This shape fits back into its frame in exactly 1 way.

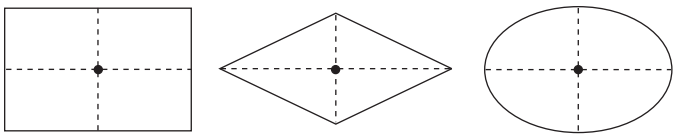
b) This shape fits back into its frame in exactly 2 ways.

c) This shape fits back into its frame in exactly 3 ways.

COMMENTS

6 Rather than responding to the correctness of the students' "We conjecture…" or "We wonder…" statements at this time, you might suggest that students keep them in mind as the lesson proceeds. If there is debate over an idea listed on the poster, you might write a "?" next to it as a reminder to discuss it again later. If students' questions or conjectures are limited at this point, note there will be opportunities to refine, confirm, respond, and add to the conjectures and questions on this poster throughout the lesson.

Classifying shapes according to the number of different positions possible in their frames may prompt conjectures such as those listed in Comment 5. Notice, for example, that a rhombus and an oval, like a nonsquare rectangle, can be classified as shapes with 2 lines of symmetry and 2 rotational symmetries:



2 reflectional and 2 rotational symmetries

7 This action may elicit additional conjectures. To save time verifying symmetry at the overhead, you might have volunteers draw 2 copies of their shapes on quartered sheets of transparencies prior to coming to the overhead.

a) Notice that a shape that fits in its frame in exactly one way has no symmetry. A symmetrical shape must fit in its frame in 2 or more ways; hence, the order of a symmetry for a symmetrical shape is always greater than or equal to 2.

b) Students may notice that all shapes that fit in their frames in exactly 2 ways (i.e., have symmetry of order 2) have exactly one line of symmetry and no rotational symmetry or 2 rotational and no reflectional.

c) One method that students frequently use to create shapes with symmetry of order 3 is to start with an equilateral triangle and alter each side so the shape has 3 rotational symmetries and no reflectional symmetry.

Focus Teacher Activity

OVERVIEW & PURPOSE

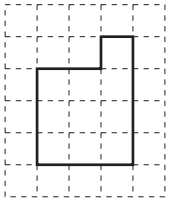
Students draw symmetric figures on square grids and on triangular grids and note the different symmetry types possible. They investigate all orders of symmetry possible for 3-sided to 8-sided polygons and make conjectures and generalizations based on their observations. Finally, students reflect on how these activities relate to the goals of the class.

MATERIALS

- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Student Activities 1.1-1.2, 1 copy of each per student and 1 transparency of each.
- ✔ "We conjecture…/We wonder…" poster from the Connector activity, 1 for each class.
- ✔ 1-cm squared grid paper, 4 sheets per student and 1 transparency.
- ✔ 1-cm triangular grid paper, 4 sheets per student and 1 transparency.
- ✔ Scissors, 1 pair per student.
- ✔ Butcher paper strips, 8 strips for the teacher, 16 small strips for each group, and several for use by the class, as needed during the lesson.
- ✔ Marking pens and tape for each group.

ACTIONS

1 Arrange the students in groups. Draw the following figure on a transparency of 1-cm grid paper.



a) Ask for volunteers to describe several ways the figure could be made symmetrical.

b) Ask each student to determine mentally the number of different ways one square of the grid can be added to the figure to make it symmetrical. Invite volunteers to report their conclusions and, without revealing the possible positions of the square, to describe the system they used to arrive at their answer. Discuss.

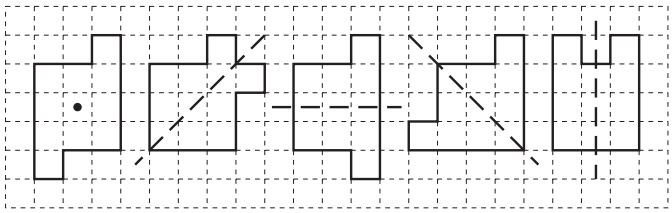
c) When there is some agreement about the number of possible locations for the square, distribute a sheet of 1-cm grid paper to each student and ask them to draw the symmetric shapes. Have volunteers sketch these at the overhead and discuss the types of symmetry the resulting shapes have.

d) Ask the students for any conjectures they wish to add to the "We Conjecture…/We wonder…" poster from the Connector. Suggest that they add to and/or edit the list throughout the remainder of this lesson.

COMMENTS

1 a) Some students may suggest that the small square in the upper right corner could be removed. Others may suggest "adding" to the figure (e.g., adding a 2 by 3 rectangle to the right-hand side). There are many possibilities.

b) One purpose here is to encourage mental geometry—to enhance the students' powers of imagery. Another is to encourage a systematic approach to the problem. One system is to mentally move the small square round the figure, stopping at each position to consider whether a symmetric figure results. There are 5 possibilities:



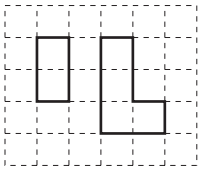
The first of the above shapes has 180° rotational symmetry, the others have reflectional symmetry. Students can use the frame test to verify the symmetry of a shape.

d) Have a supply of butcher paper strips and marking pens available so that students/groups can record their questions and conjectures as they come up. Periodically throughout this lesson, take time to review the list and discuss students' additions or edits.

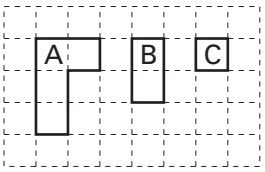
Focus Teacher Activity (cont.)

ACTIONS

2 Place cutouts of the following 2 shapes on a transparency of grid paper. Ask the students to determine mentally the number of noncongruent symmetrical shapes that can be made by joining these two shapes and determine the symmetries of the resulting shapes. Discuss strategies the students use and have volunteers sketch the symmetrical shapes at the overhead.

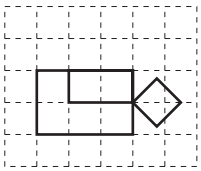


3 Place cutouts of the following 3 shapes on a transparency of grid paper.

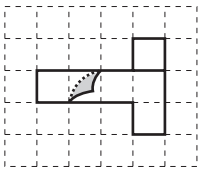


a) Ask for a few volunteers to show ways the 3 shapes can be put together to form a symmetrical shape.

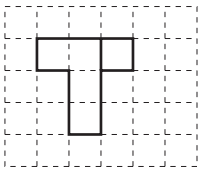
b) Give each student a pair of scissors and 1 sheet of 1-cm grid paper. Ask the groups to investigate the number of noncongruent symmetrical shapes that can be formed by these 3 shapes. Have them sketch each symmetrical shape and describe its symmetry. Discuss.



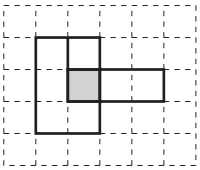
Join shapes at the corners



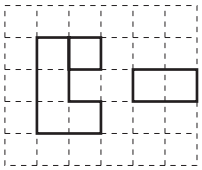
Let shapes overlap



Use 2 shapes



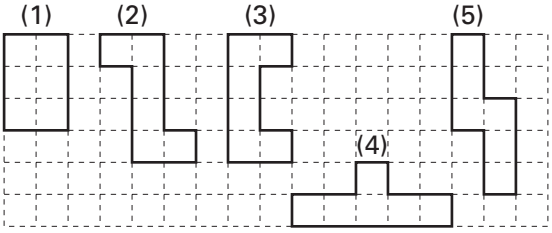
Have gaps



Have shapes that don't touch

COMMENTS

2 One strategy is to take the rectangular piece and mentally place it in different positions around the L-shaped piece. Several figures that can be obtained without overlapping the original two shapes are shown below. Figure (1) has 2 lines of symmetry and 2 rotational symmetries; Figures (3) and (4) have 1 line of symmetry, and Figures (2) and (5) have 2 rotational symmetries.



3 Have additional grid paper available for use as needed.

a) If there is disagreement over the symmetry of a shape, students could see if the shape passes the frame test, i.e., can it be positioned in its frame in more than one way? If so, it is symmetrical; if not, it has no symmetry.

b) This may take some time; hence, after groups have worked for an adequate period in class, you might proceed with the lesson and ask the students to continue the investigation as homework.

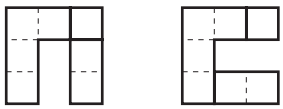
Students may ask about the "rules;" for example, they may ask if they are allowed to form shapes such as those shown at left. As questions such as these arise, let the students make their own decisions. Encourage them to consider the consequences. For example, if overlaps are allowed, the amount of the overlap may be varied infinitely and the problem becomes unmanageable. Deciding under what conditions an investigation becomes manageable, and determining how the choice of conditions affects the conclusions, are integral parts of carrying out a mathematical investigation.

There are 17 noncongruent shapes that can be formed by using shapes A, B, and C and assuming that squares always match edge to edge with no overlaps or gaps. Some groups may make their own cutouts and keep a record of their results on grid paper. Others may only sketch on the grid paper.

Students may begin in a random way. As you observe them at work, asking them about their plan of attack may encourage systematic approaches.

Focus Teacher Activity (cont.)

ACTIONS



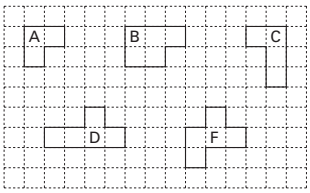
4 Distribute one copy of Focus Student Activity 1.1 to each student for completion. Discuss their results and methods.

Exploring Symmetry Lesson **1**

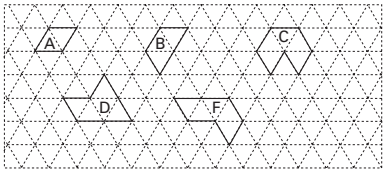
Focus Student Activity 1.1

NAME _____ DATE _____

1 For each shape below, determine mentally how many ways one square of the grid can be added to the shape to make it symmetrical. Assume no gaps or overlaps and that squares meet edge-to-edge.



2 For each shape below, determine mentally how many ways one triangle of the grid can be added to the shape to make it symmetrical. Assume no gaps or overlaps and that triangles meet edge-to-edge.



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COMMENTS

When forming these 17 noncongruent shapes, students may notice that a given shape may formed by different arrangements of shapes A, B, and C, as shown at the left. Assuming that squares always match edge-to-edge and no gaps or overlaps are allowed, there are 33 total symmetrical arrangements—even more if one counts rotations and reflections of the individual pieces in an arrangement!

4 This could be assigned as a homework activity.

1)-2) One strategy is to mentally move the additional square or triangle from one possible location to the next, checking each time whether or not the resulting shape is symmetrical.

3)-4) Students may find that using more complicated shapes tends to limit the number of ways a square or triangle can be placed to form a shape that is symmetrical.

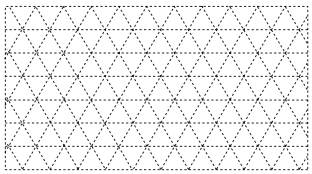
One way to conduct sharing of students' results for 3) and 4) is to have the students sketch their shapes on squared and triangular grid paper and exchange with classmates who verify that the shapes fit the criteria of Problems 3 and 4.

Lesson **1** Exploring Symmetry

Focus Student Activity (cont.)

3 Create a shape that is made of squares joined edge-to-edge (no overlaps) and has exactly 3 ways of adding one additional square to make the shape symmetrical.

4 Create a shape that is made of triangles joined edge-to-edge (no overlaps) and has exactly 4 ways of adding one additional triangle to make the shape symmetrical.

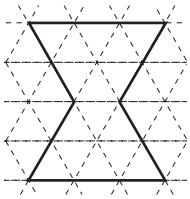


Focus Teacher Activity (cont.)

ACTIONS

5 Give each student a sheet of 1-cm triangular grid paper.

a) Sketch the following hexagon on a transparency of triangular grid paper. Ask the groups to determine its symmetries. Then ask them to draw other noncongruent hexagons which have the same symmetries. Discuss.



b) Ask the groups to construct examples of hexagons which have:

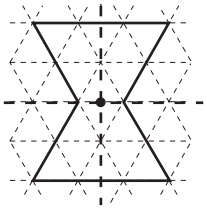
i) reflectional but no rotational symmetry,
ii) rotational but no reflectional symmetry.

As needed, have volunteers show examples at the overhead. Discuss the students' observations.

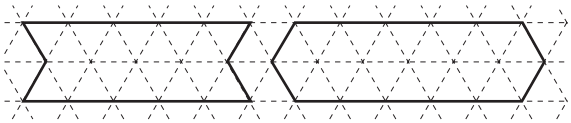
6 Ask the groups to each prepare a chart showing the different types of symmetry a hexagon can have and an example of each type. Discuss their results. Encourage the students to add to the "We conjecture…/We wonder…" poster. Discuss as appropriate.

COMMENTS

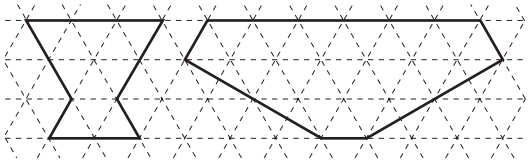
5 a) This hexagon has 2 reflectional and 2 rotational symmetries. Students may ask if the hexagons they create must follow grid lines—moving off the grid lines will not affect the outcomes of the investigation, but the grids are useful in making accurate sketches.



Shown below are two other hexagons, one concave and one convex, with the same symmetries as above.

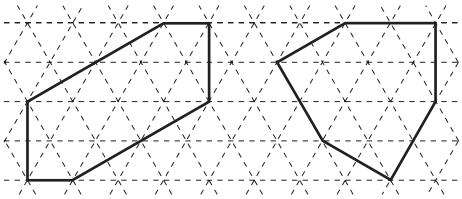


b) i) These hexagons have reflectional symmetry and no rotational symmetry:



The students may notice that it is not possible to have 2 axes of symmetry and no rotational symmetry (see Comment 6).

ii) The hexagons shown below have 2 and 3 rotational symmetries, respectively, but no reflectional symmetry.



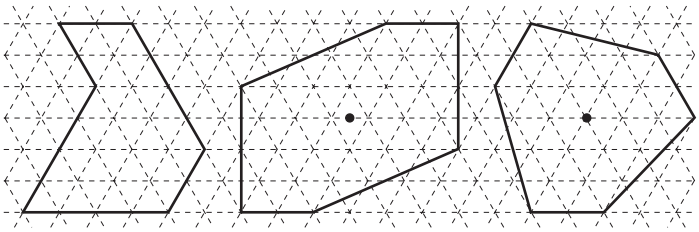
6 Following are examples of the seven possible types of symmetries that a hexagon can have.

Focus Teacher Activity (cont.)

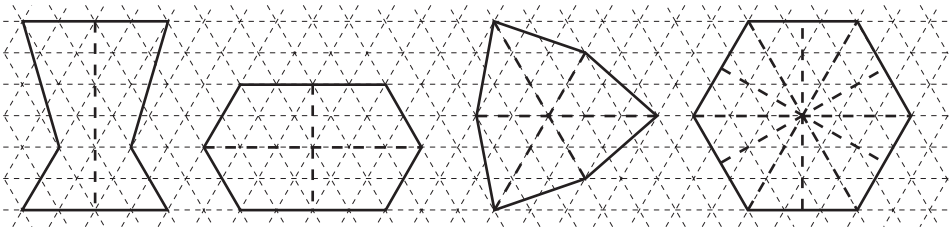
ACTIONS

COMMENTS

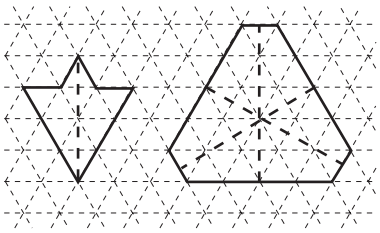
Hexagons Classified by Symmetry Type



A. no reflectional no rotational | B. no reflectional 2 rotational | C. no reflectional 3 rotational



D. 1 reflectional no rotational | E. 2 reflectional 2 rotational | F. 3 reflectional 3 rotational | G. 6 reflectional 6 rotational



i) ii)

As you circulate while students work, you might note questions, observations, and conjectures you overhear. You could record these on the "We conjecture…/We wonder…" poster, or ask the students to do so. For example, students may make observations about the axes of symmetry of a hexagon. Notice that hexagon D above has 1 reflectional and no rotational symmetry, and its axis of symmetry connects midpoints of opposite sides. Hexagon i) at the left also has one reflectional and no rotational symmetry, but the axis of symmetry connects opposite vertices. Hexagon ii) has 3 reflectional and 3 rotational symmetries and has axes of symmetry connecting midpoints of opposite sides, as compared to hexagon F above which also has 3 reflectional and 3 rotational symmetries but has axes of symmetry which connect opposite vertices. Note: these "types" of lines of symmetry of polygons are investigated in Action 11.

Students may make observations about the possible orders of symmetry for hexagons; if so, note that the orders of symmetry for hexagons B-G respectively, are 2, 3, 2, 4, 6, and 12; figure A has no symmetry.

The students may observe that a shape which has 2 or more axes of symmetry also has rotational symmetry. If so, you might ask them to investigate now why they think this is so, or you could make note of it as a topic

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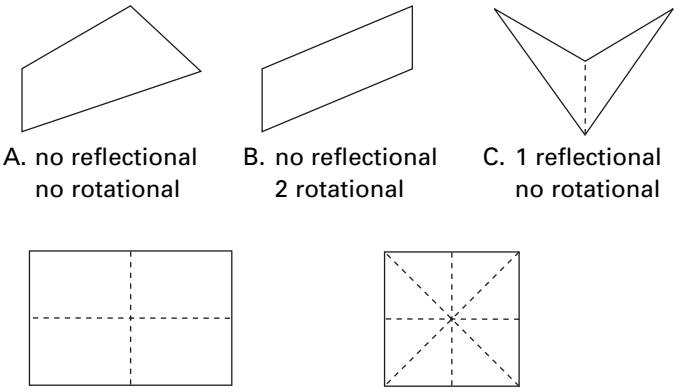
Focus Teacher Activity (cont.)

ACTIONS

7 Repeat Action 6 for quadrilaterals.

Quadrilaterals Classified by Symmetry Type



A. no reflectional no rotational

B. no reflectional 2 rotational

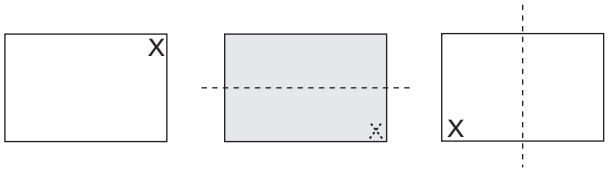
C. 1 reflectional no rotational

D. 2 reflectional 2 rotational

E. 4 reflectional 4 rotational

COMMENTS

6 (continued.)
for investigation later. In general, changing the position of a shape by first flipping it about one of its axes of symmetry, and then flipping it about a second axis, has the same effect as rotating the shape through an angle twice that of the angle between the axes. Notice, for example, reflecting a rectangle about its horizontal axis and then about its vertical axis results in the same final position as rotating the rectangle 180° (i.e., double the 90° angle of intersection of the axes).

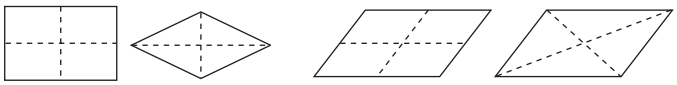


Step 1 Step 2 Step 3

7 There are 5 symmetry types for quadrilaterals, as illustrated at the left.

Squares are the only quadrilaterals which have 4 reflectional and 4 rotational symmetries. All nonsquare rectangles and nonsquare rhombuses have 2 reflectional and 2 rotational symmetries, although the axes of symmetry of rectangles connect midpoints of opposite sides and those of rhombi connect pairs of opposite vertices. Kites, both convex (other than rhombi) and concave, and isosceles trapezoids have one reflectional and no rotational symmetry. Parallelograms (other than rectangles and rhombi) have no reflectional and 2 rotational symmetries.

The symmetry of a parallelogram that is not a rectangle or rhombus sometimes causes problems. Students often think it has 2 axes of symmetry, like the rectangle or like the rhombus (see diagram below). This can be disproved by using the frame test, by cutting the parallelogram out and folding it along one of its diagonals, or by placing a mirror along a diagonal.



2 lines of symmetry No lines of symmetry

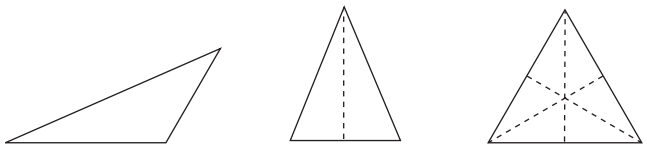
Notice that quadrilateral A at the left has no symmetry; B has symmetry of order 2; C, order 2; D, order 4; and E, order 8.

Focus Teacher Activity (cont.)

ACTIONS

8 Ask the students to classify triangles according to their symmetries. Compare this classification with other classifications of triangles.

Triangles Classified by Symmetry Type



Scalene:	Isosceles:	Equilateral:
no reflectional	1 reflectional	3 reflectional
no rotational	no rotational	3 rotational

9 (Optional) Repeat Action 6 for pentagons and/or octagons. Encourage conjectures and generalizations.

10 Ask the groups to examine the class "We conjecture… We wonder… " poster, to add new ideas and edit existing ones. Discuss, clarifying as needed.

11 Give each student a copy of Focus Student Activity 1.2 (see next page) and ask them to select one of the given problems for investigation (or formulate other questions). Explain your expectations for their written work, including your timeline and methods of evaluation.

COMMENTS

8 There are 3 symmetry types for triangles, as shown at the left. Notice the classification of triangles by symmetry type corresponds to the classification of triangles as *scalene* (no equal sides), *isosceles* (2 equal sides), or *equilateral* (3 equal sides). Note: an equilateral triangle is also isosceles.

You might want to discuss with the students their ideas about types of symmetry a right triangle could have.

9 There are only 3 symmetry types for pentagons: no reflectional and no rotational; 1 reflectional and no rotational; 5 reflectional and 5 rotational. Octagons, like hexagons, have 7 symmetry types.

Students may be interested here (or in Action 11) to investigate the symmetries possible for polygons with a prime number of sides. Any polygon with n sides, where n is a prime number has one of the following 3 symmetry types: no reflectional and no rotational; 1 reflectional and no rotational; or n reflectional and n rotational.

10 You might use this as an opportunity to remind the students that, while they examined many important symmetry relationships and discovered many new ideas during this lesson, a "big idea" of the lesson is to engage them in the process of conjecturing, questioning, and generalizing about mathematical ideas. Rather than memorizing or mimicking the work of mathematicians, the students are the mathematicians.

11 Students may need additional grid paper. Sketching symmetric shapes is facilitated by using the square or triangular grids.

If the questions on Focus Student Activity 1.2 were examined by your students during other actions, if you feel the students need to examine another idea, or if there are areas of particular interest that emerged during Action 10, you could pose other questions for investigation. If students have difficulty engaging in a problem, you might encourage them to explore related ideas that surface during their investigation.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Exploring Symmetry Lesson **1**

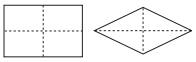
Focus Student Activity 1.2

NAME _____ **DATE** _____

Write a well-organized, sequential summary of your investigation of *one* of Problems 1 or 2. Include the following in your summary:

- a statement of the problem you investigate
- the steps of what you do, including any false starts and dead-ends
- relationships you notice (small details are important)
- questions that occur to you
- places you get stuck and things you do to get unstuck
- your AHA!s and important discoveries
- conjectures that you make—include what sparked and ways you tested each conjecture
- evidence to support your conclusions.

1 A nonsquare rectangle and a nonsquare rhombus each have 2 reflectional symmetries. However, the 2 lines of symmetry are of 2 different types—the lines of symmetry of a rectangle connect the midpoints of opposite sides and the lines of symmetry of a rhombus connect opposite vertices. Investigate other polygons with exactly 2 lines of symmetry of these 2 types. Generalize, if possible.



2 What, if any, is the minimum number of sides for a polygon with 3 rotational symmetries and no reflectional symmetry? What, if any, is the maximum number of sides? What, if any, is the minimum number of sides for polygons with 4 rotational symmetries and no reflectional symmetries? 5 rotational and no reflectional symmetries? *n* rotational and no reflectional symmetries? Investigate.

COMMENTS

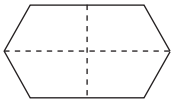
11 (continued.)
After a specified time for investigation, you might hold a large group discussion of progress so far, adding any new conjectures or questions to the class poster. This can give new "momentum" to students having difficulty.

Because these investigations involve extended thought and exploration, you may want to assign this activity as an individual or group assessment, to be started in class and completed outside of class. You and the students could develop a scoring guide for use in evaluating their work. See the Assessment chapter of *Starting Points for Implementing Math Alive!* for suggestions regarding extended projects and assessment guides.

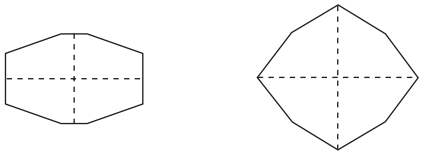
As you think about assessing the students' development, keep in mind that, while the students should by this time have a solid conceptual sense about the meanings of rotational and reflectional symmetry and should be comfortable using the frame test to determine the symmetries of a polygon, it isn't reasonable to expect that they memorize all the symmetry types of a hexagon, for example, or that they memorize definitions of terms. A meaningful assessment activity should engage students in further investigations involving symmetry ideas and should ask them to discuss their methods, reasoning, and generalizations. The students' written work for such investigations should reveal their knowledge of the concepts of symmetry and their understanding of the mathematical process. The problems on Focus Student Activity 1.2 and Follow-up Student Activity 1.3 are designed to provide such information.

Following are examples of observations that may come up related to Problem 1 on Focus Student Activity 1.2:

For hexagons with exactly 2 lines of reflection, one line must connect 2 opposite vertices and the other must connect midpoints of 2 sides, as illustrated below.

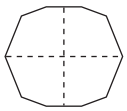


Octagons may have 2 lines of symmetry that connect opposite vertices or 2 lines that connect midpoints of opposite sides, as shown below.



Focus Teacher Activity (cont.)

ACTIONS



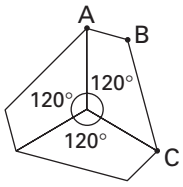
COMMENTS

For a decagon with exactly 2 lines of symmetry, there must be one connecting opposite vertices and one connecting midpoints of opposite sides, as shown at the left.

If the number of sides of a nonregular polygon is an even number which is a multiple of 4, and if it has exactly 2 lines of symmetry, it must have one type (vertex-to-vertex or side-to-side) or the other, but not both. If the number of sides of a nonregular polygon is an even number which is not a multiple of 4, and if there are exactly 2 lines of symmetry, there must be one of each kind.

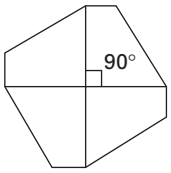
Some observations related to Problem 2 from Focus Student Activity 1.2 follow.

The minimum number of sides for a polygon with 3 rotational symmetries and no reflectional symmetries is 6. This can be obtained by drawing 3 line segments from a central point which form 120° angles and then drawing 2 noncongruent sides of the polygon AB and BC. These sides can then be rotated 120° and 240° to obtain the remaining sides of the hexagon.

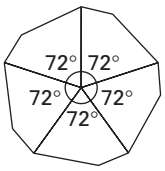


Hexagon - 3 rotational symmetries

In a similar manner, the minimum number of sides for a polygon with 4, 5, or *n* rotational symmetries and no reflectional symmetries is 8, 10, and 2*n* as suggested by the following figures.



Octagon
4 rotational symmetries



Decagon
5 rotational symmetries

Some students may reason that shapes with the minimum number of sides for 3, 4, or 5 rotational symmetries can be obtained by building triangular "arms" that are congruent to each other off a "base" that is an equilateral triangle, square, or regular pentagon, as shown

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

6-sided polygon
3 rotational symmetries

12 (Optional) Write each of the goals a)-h) from Focus Master A (see next page) in large print on a separate strip of butcher paper. Post these strips about the classroom. Distribute a copy of Focus Master A to each student. Arrange the students in groups and give each group 8 blank strips of butcher paper and marking pens. Ask the groups to complete i) below:

> **i) Describe in your own words what you believe is the meaning of each of the goals listed on Focus Master A. Write each description on a separate blank strip. Post each completed strip under its corresponding goal on the wall.**

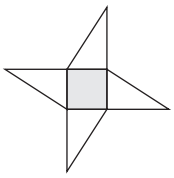
Next have the groups discuss their ideas from i). Then give each group 8 additional strips of butcher paper and have them complete ii):

> **ii) For each of goals a)-h), list one or more specific examples from the class activities during this lesson that illustrate our class' successful work on that goal. Post each completed strip under its corresponding goal on the wall.**

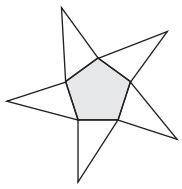
Discuss the groups' results ii).

COMMENTS

11 (continued.)
below. Notice, the 2 exposed sides of each arm must be noncongruent to assure no reflective symmetry. Forming arms that are other shapes generates polygons with more sides and with the same number of rotational symmetries as their "base" regular polygon.



8-sided polygon
4 rotational symmetries



10-sided polygon
5 rotational symmetries

12 You may wish to amend the list of goals on Focus Master A to include other goals you have for the class. Students may also have ideas they wish to add to the list. The goals listed are the goals considered by the authors during the development of each *Math Alive!* lesson.

As you circulate while groups are working on this task, it may be helpful to pose some questions, or if a group is stuck trying to articulate an idea, you might share some of your ideas. Following are some thoughts about goals a) and b) that may be useful for discussion.

Goal a): In *Math Alive!* visual thinking refers to a 3-part process: perceiving, imaging, and portraying. In the article, *Mathematics and Visual Thinking* by Eugene Maier (see the Appendix of *Starting Points*) perceiving is described as "becoming informed through the senses… and through kinesthesia, the sensation of body movement and position." In *Math Alive!* perceiving occurs through the use of manipulatives and hands-on activities to investigate mathematical ideas and through discussions about those ideas. Such experiences are multisensory. That is, *Math Alive!* activities involve the senses of sight, touch, and hearing, and they are kinesthetic in nature.

The second aspect of the visual thinking process is imaging. One's sensory experiences can be recreated as mental images which provide a basis for further thought and discussion. That is, the sensory experiences themselves often do not need to be *physically* recreated in order to reconsider or extend the idea. Rather, the experiences can be recreated in the mind's eye. For example, after drawing and cutting out a variety of shapes, and

Focus Teacher Activity (cont.)

ACTIONS

Exploring Symmetry Lesson **1**

Focus Master A

Our Goals as Mathematicians

We are a community of mathematicians working together to develop our:

a) visual thinking,

b) concept understanding,

c) reasoning and problem solving,

d) ability to invent procedures and make generalizations,

e) mathematical communication,

f) openness to new ideas and varied approaches,

g) self-esteem and self-confidence,

h) joy in learning and doing mathematics.

COMMENTS

then flipping and turning the shapes to fit back into their frames, many students can look at a shape and "see" those motions in their mind's eye.

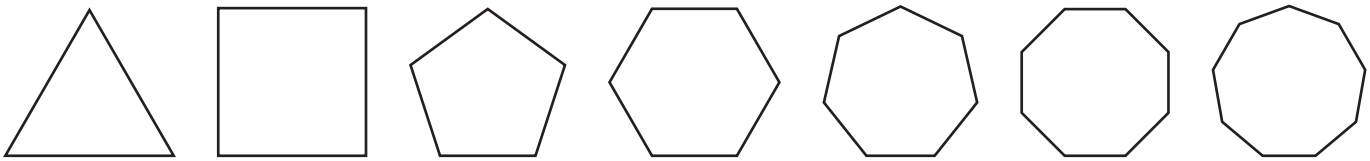
The third part of the visual thinking process, portraying, is representing a perception using sketches, diagrams, models, or other symbolic forms. These representations can be used as tools for solving problems or investigating conceptual relationships.

Goal b): Understanding concepts is different from knowing definitions and procedures, although many people mix these ideas. One who understands the meaning of a concept is usually capable of inventing procedures for solving problems involving the concept. On the other hand, one who has memorized definitions or procedures without understanding conceptual relationships may have difficulty solving problems involving the idea or procedure.

TEACHER NOTES:

Follow-up Student Activity 1.3

NAME _____ **DATE** _____



1 Trace and cut out a copy of each of the above regular polygons. Use the copies and original polygons, but no measuring tools (no rulers, protractors, etc.), to help you complete the following chart:

								
No. of different positions in frame								
No. of reflectional symmetries								
No. of rotational symmetries								
Measures of *all* angles of rotation								
Measure of each interior angle*								

*Interior angles are the angles "inside" the polygon and are formed by intersections of the sides of the polygon.

Complete the following problems on separate paper. Be sure to write about any AHA!s, conjectures, or generalizations that you make.

2 Explain the methods that you used to determine the angles of rotation and the interior angle measures for the chart above. Remember, no protractors.

3 Label the last column of the chart in Problem 1 "Regular *n*-gon" and then complete that column. For each expression that you write in the last column, draw a diagram (on a separate sheet) to show "why" the expression is correct.

(Continued on back.)

Follow-up Student Activity (cont.)

4 Discuss the symmetries of a circle. Explain your reasoning.

5 Locate a resource that shows flags of the countries of the world. For each of the following, if possible, sketch and color a copy of a different flag (label each flag by its country's name) and cite your resource.

a) rotational symmetry but no reflectional symmetry,

b) reflectional symmetry across a horizontal axis only,

c) no symmetry,

d) both rotational and reflectional symmetry,

e) 180° rotational symmetry.

6 Sort and classify the capital letters of the alphabet according to their types of symmetry.

7 Attach pictures of 2 different company logos that have different types of symmetry. Describe the symmetry of each logo.

8 Create your personal logo so that it has symmetry. Record the order of symmetry for your logo, show the location of its line(s) of symmetry, and/or record the measures of its rotational symmetries.

9 Jamaal made conjectures a) and b) below. Determine whether you think each conjecture is always/sometimes/never true. Give evidence to show how you decided and to show why your conclusion is correct. If you think a conjecture is not true, edit it so that it is true.

If a shape has exactly 2 axes of reflection, then

a) those axes must be at right angles to each other.

b) the shape also must have 2 rotational symmetries.

ROTATIONS/TURNS

a) Complete this procedure:

- Position your note card so that it fits in its frame with no gaps or overlaps.
- Mark a point anywhere on your card with a dot, and label this point P.
- Place a pencil point on your point P and hold the pencil firmly in a vertical position at P.
- Rotate the card about P until the card fits back into its frame with no gaps or overlaps.

b) How many *different* rotations of the card about your point P are possible so that the card fits back in its frame with no gaps or overlaps? Assume that rotations are different if they result in different placements of the card in its frame.

c) If only a 360° (or 0°) rotation about your point P brings the card back into its frame, find another position for P on the card so that *more than one different rotation* about this point is possible. What are the measures of the rotations and how did you determine them?

REFLECTIONS/FLIPS

Figure 1 below shows the frame for a rectangular card with a line *l* drawn across the frame. In Figure 2, the card has been placed in the frame. Figure 3 shows the result of *reflecting*, or *flipping*, the card over line *l*. Notice that after the reflection over line *l*, the card does not fit back in its frame.

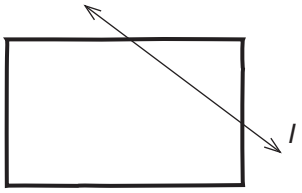


Figure 1

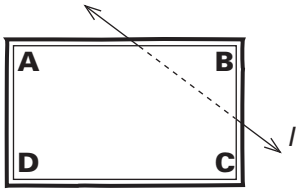


Figure 2

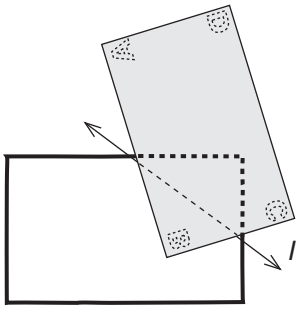
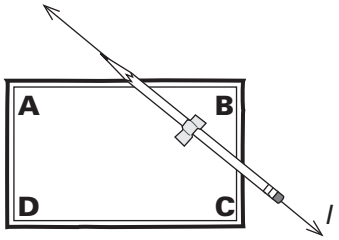


Figure 3

Determine all the *different* possible placements of line *l* so that when you flip your card once over *l*, the card fits back in its frame with no gaps or overlaps.

HINT: As a guide for flipping the card about a line, you could tape a pencil or coffee stirrer to the card along the path of line *l*, as shown below. Then keep the pencil or coffee stirrer aligned with line *l* as you flip the card.



Connector Master C

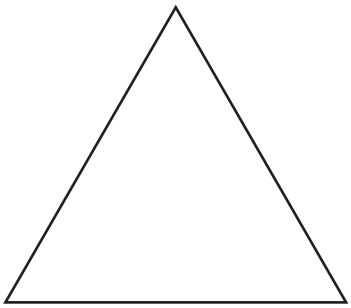
a) Discuss your group's ideas and questions about the meanings of the following terms. Talk about ways these terms relate to a nonsquare rectangle such as your note card. Record important ideas and questions to share with the class.

i) reflectional symmetry
ii) axis of reflection (also called *line of reflection*)
iii) rotational symmetry
iv) center of rotation
v) frame test for symmetry

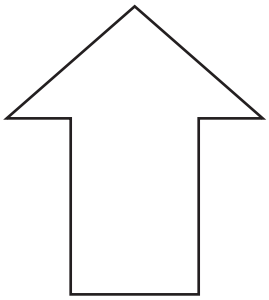
b) If a shape is symmetrical, its *order of symmetry* is the number of different positions for the shape in its frame, where different means the sides of the shape and the sides of the frame match in distinctly different ways. Develop a convincing argument that your rectangular note card has symmetry of order four.

Connector Master D

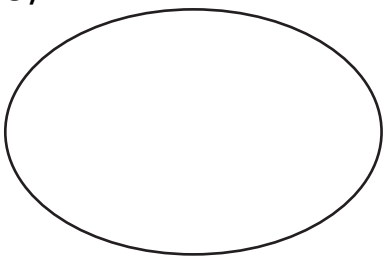
a)



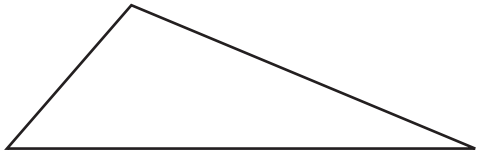
b)



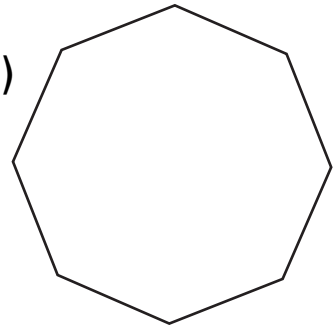
c)



d)

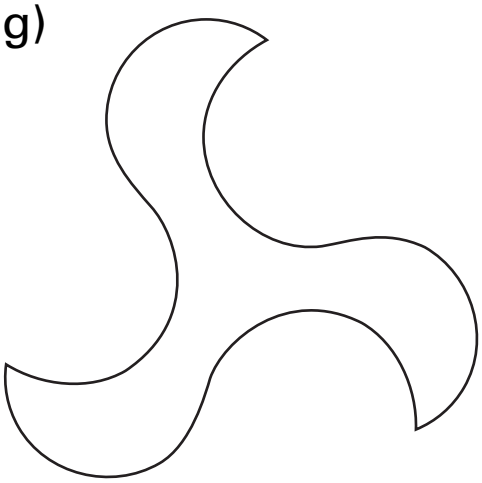


e)

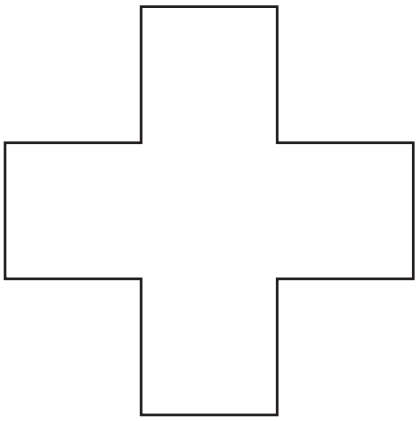


f)

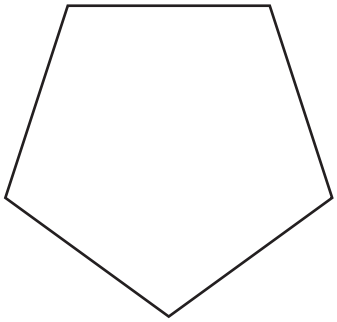
g)



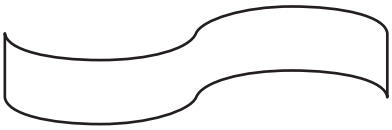
h)



i)



j)



Connector Master D (page 2)

Focus Master A

Our Goals as Mathematicians

We are a community of mathematicians working together to develop our:

a) visual thinking,

b) concept understanding,

c) reasoning and problem solving,

d) ability to invent procedures and make generalizations,

e) mathematical communication,

f) openness to new ideas and varied approaches,

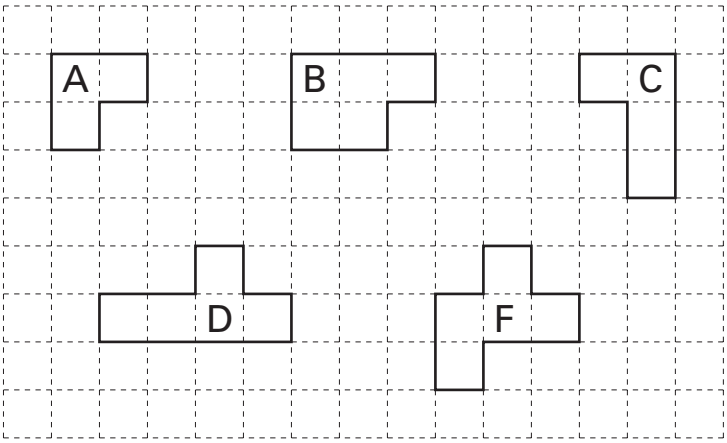
g) self-esteem and self-confidence,

h) joy in learning and doing mathematics.

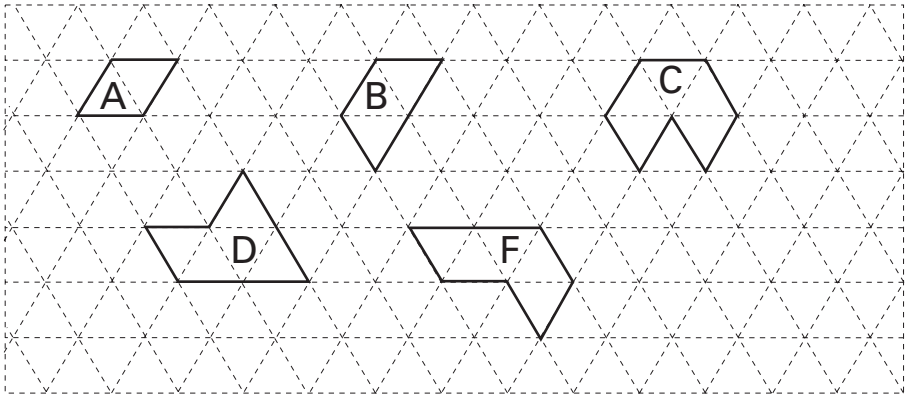
Focus Student Activity 1.1

NAME _____ **DATE** _____

1 For each shape below, determine mentally how many ways one square of the grid can be added to the shape to make it symmetrical. Assume no gaps or overlaps and that squares meet edge-to-edge.



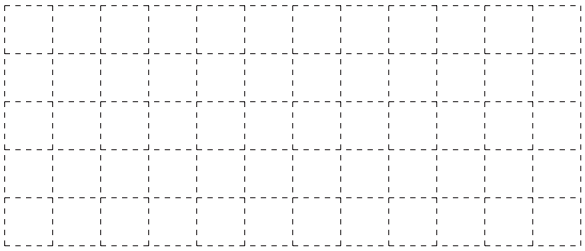
2 For each shape below, determine mentally how many ways one triangle of the grid can be added to the shape to make it symmetrical. Assume no gaps or overlaps and that triangles meet edge-to-edge.



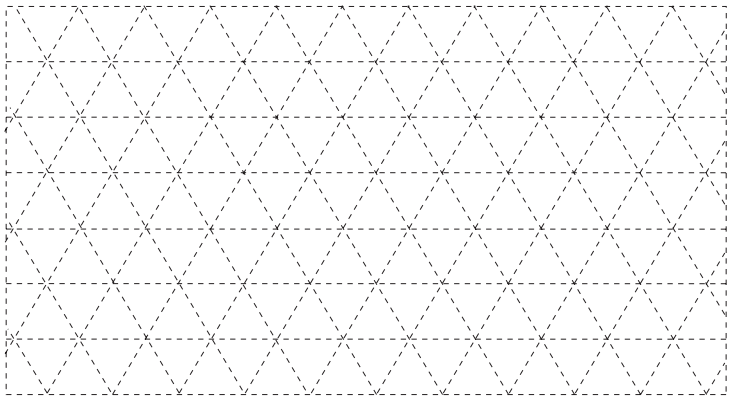
(Continued on back.)

Focus Student Activity (cont.)

3 Create a shape that is made of squares joined edge-to-edge (no overlaps) and has exactly 3 ways of adding one additional square to make the shape symmetrical.



4 Create a shape that is made of triangles joined edge-to-edge (no overlaps) and has exactly 4 ways of adding one additional triangle to make the shape symmetrical.



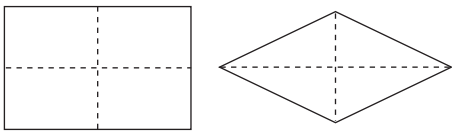
Focus Student Activity 1.2

NAME _____ **DATE** _____

Write a well-organized, sequential summary of your investigation of *one* of Problems 1 or 2. Include the following in your summary:

- a statement of the problem you investigate
- the steps of what you do, including any false starts and dead-ends
- relationships you notice (small details are important)
- questions that occur to you
- places you get stuck and things you do to get unstuck
- your AHA!s and important discoveries
- conjectures that you make—include what sparked and ways you tested each conjecture
- evidence to support your conclusions.

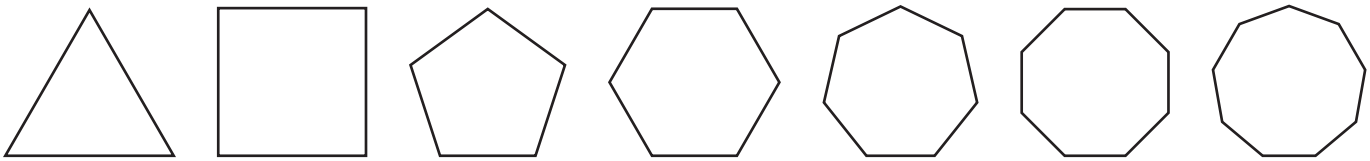
1 A nonsquare rectangle and a nonsquare rhombus each have 2 reflectional symmetries. However, the 2 lines of symmetry are of 2 different types—the lines of symmetry of a rectangle connect the midpoints of opposite sides and the lines of symmetry of a rhombus connect opposite vertices. Investigate other polygons with exactly 2 lines of symmetry of these 2 types. Generalize, if possible.



2 What, if any, is the minimum number of sides for a polygon with 3 rotational symmetries and no reflectional symmetry? What, if any, is the maximum number of sides? What, if any, is the minimum number of sides for polygons with 4 rotational symmetries and no reflectional symmetries? 5 rotational and no reflectional symmetries? *n* rotational and no reflectional symmetries? Investigate.

Follow-up Student Activity 1.3

NAME _____ **DATE** _____



1 Trace and cut out a copy of each of the above regular polygons. Use the copies and original polygons, but no measuring tools (no rulers, protractors, etc.), to help you complete the following chart:

								
No. of different positions in frame								
No. of reflectional symmetries								
No. of rotational symmetries								
Measures of *all* angles of rotation								
Measure of each interior angle*								

*Interior angles are the angles "inside" the polygon and are formed by intersections of the sides of the polygon.

Complete the following problems on separate paper. Be sure to write about any AHA!s, conjectures, or generalizations that you make.

2 Explain the methods that you used to determine the angles of rotation and the interior angle measures for the chart above. Remember, no protractors.

3 Label the last column of the chart in Problem 1 "Regular *n*-gon" and then complete that column. For each expression that you write in the last column, draw a diagram (on a separate sheet) to show "why" the expression is correct.

(Continued on back.)

Follow-up Student Activity (cont.)

4 Discuss the symmetries of a circle. Explain your reasoning.

5 Locate a resource that shows flags of the countries of the world. For each of the following, if possible, sketch and color a copy of a different flag (label each flag by its country's name) and cite your resource.

a) rotational symmetry but no reflectional symmetry,

b) reflectional symmetry across a horizontal axis only,

c) no symmetry,

d) both rotational and reflectional symmetry,

e) 180° rotational symmetry.

6 Sort and classify the capital letters of the alphabet according to their types of symmetry.

7 Attach pictures of 2 different company logos that have different types of symmetry. Describe the symmetry of each logo.

8 Create your personal logo so that it has symmetry. Record the order of symmetry for your logo, show the location of its line(s) of symmetry, and/or record the measures of its rotational symmetries.

9 Jamaal made conjectures a) and b) below. Determine whether you think each conjecture is always/sometimes/never true. Give evidence to show how you decided and to show why your conclusion is correct. If you think a conjecture is not true, edit it so that it is true.

If a shape has exactly 2 axes of reflection, then

a) those axes must be at right angles to each other.

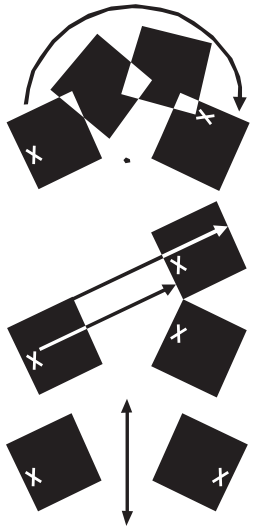
b) the shape also must have 2 rotational symmetries.

Introduction to Isometries

Lesson 2

THE BIG IDEA

Viewing congruence in terms of isometries—motions that preserve size and shape—provides insights about relationships among congruent shapes. Hands on investigations of the four basic isometries—translations, rotations, reflections, and glide reflections—promote spatial awareness and provide a dynamic context for experiencing geometry. Isometries provide a mathematical basis for understanding and creating frieze patterns and tessellations used in wallpaper designs.



CONNECTOR

OVERVIEW

Students construct and analyze the symmetry of various polyominoes—shapes made by joining together squares.

MATERIALS FOR TEACHER ACTIVITY

- ✔ 1-cm grid paper, 1 sheet per student and 1 transparency.
- ✔ Scissors for each student.

FOCUS

OVERVIEW

Students explore different ways of moving a shape onto a congruent shape. These movements strengthen students' awareness of congruence and motivate introductory experiences with isometries (reflections, translations, rotations, and glide reflections) and provide the basis for creating friezes and tessellations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activities 2.1, 2.2, and 2.3, 1 copy of each per student and 1 transparency of each. (Note: the last 2 pages of 2.3 should be run back-to-back on cardstock.)
- ✔ Focus Masters A, B, and J, 1 copy of each per student and 1 transparency.
- ✔ Focus Master C, 2 copies per student and 1 transparency.
- ✔ Focus Masters D, E, and I, 1 transparency of each.
- ✔ Focus Masters F, G, H, 1 copy of each per student and 2 transparencies of each.
- ✔ Patty paper, 8 sheets of hamburger "patty paper" (or ¼-sheets of tracing paper) per student.
- ✔ Scissors and straightedges, 1 of each per student.
- ✔ Protractors (see *Blackline Masters,* copy on transparencies), 1 per student.
- ✔ 2-cm triangular grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ 2-cm squared grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Marking pens for each group.
- ✔ Blank transparencies (¼-sheets) for use at the overhead.
- ✔ Butcher paper strips, several for each group.

FOLLOW-UP

OVERVIEW

Students create and determine the symmetry of polyamonds. They investigate and build arguments supporting or refuting several conjectures involving isometries. They perform transformations of shapes in the plane and create frieze patterns and tessellations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 2.4, 1 copy per student.
- ✔ 1-cm squared grid paper, 1 sheet per student.
- ✔ 1-cm triangular grid paper, 1 sheet per student.
- ✔ 2-cm triangular grid paper, 3 sheets per student.

LESSON IDEAS

FOLLOW-UP
Students may find it helpful to cut out triangles to explore arrangements that form different polyamonds for Problem 1. Some students may be especially challenged by finding all the hexamonds and may benefit by periodic opportunities to compare results with classmates.

BULLETIN BOARDS
You might create a bulletin board display of the tessellations from Actions 19 and 20 of the Focus activity and a display of the friezes from Problem 9 of the Follow-up. If you let students know that this is your plan, they may put more care into their designs.

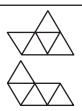
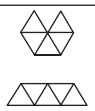
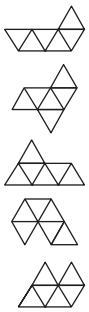
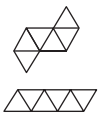
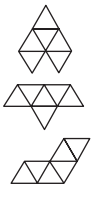
JOURNALS
If students are keeping journals (see *Starting Points* for ideas), throughout this lesson you might have them begin or add to a glossary in which they represent the meanings of terms using diagrams and minimal numbers of words. Periodically, you might select a term and ask for volunteers to show their pictorial explanations from their glossaries. Note that in the last action of the Focus activity, the students summarize their understanding of terminology.

QUOTE
Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing, and testing hypotheses precede the development of more formal summary statements. In the process, definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas in informal arguments.
NCTM Standards

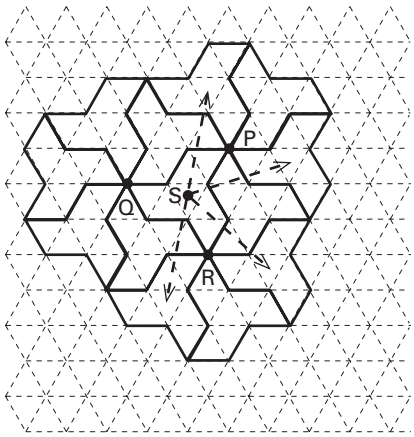
SELECTED ANSWERS

1.

Polyamond Symmetries

	no symmetry	no reflective 2 rotational	1 reflective no rotational	2 reflective 2 rotational	3 reflective 3 rotational	6 reflective 6 rotational
triamond and diamond						
tetramonds						
pentamonds						
hexamonds						

2-3. Rotations about S: 120°, 240°, 360°; about P, Q, R: 60°, 120°, 180°, 240°, 300°, 360°. Translations along the vectors shown, their opposites, and their multiples.



5. a) T b) T c) T d) T
e) F f) T g) F

Connector Teacher Activity

OVERVIEW & PURPOSE

Students construct and analyze the symmetry of various polyominoes—shapes made by joining together squares. During the Focus activity, the students determine which of these shapes tessellate the plane.

MATERIALS

- ✔ 1-cm grid paper, 1 sheet per student and 1 transparency.
- ✔ Scissors for each student.

ACTIONS

1 Arrange the students in groups and give each student a pair of scissors and a sheet of 1-cm grid paper. Ask the students to each cut out a *tetromino*, an arrangement of 4 squares joined edgewise. Then have the groups determine how many noncongruent tetrominoes are possible. Discuss their conclusions and the methods they use to decide they have found all the tetrominoes.

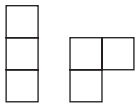
COMMENTS

1 The expression "joined edgewise" implies that squares must be joined edge to edge, not corner to corner, and squares should meet full edge to full edge without overlaps.

Students could silhouette the different cutout tetrominoes on the overhead. Note that, if one tetromino can be moved to coincide exactly with another, then the tetrominoes are congruent. In the Focus activity students will explore in depth the motions, or transformations, that move a shape so that it coincides with a congruent shape.

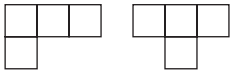
Following are two methods students' have suggested for determining the 5 possible noncongruent tetrominoes.

Method 1: Determine the ways in which one square can be attached to these 2 different arrangements of 3 squares:

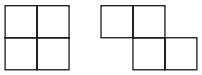


Method 2: Consider the case in which 4 squares are in a row:

Then the cases where the maximum number of squares in a row is 3:



And, finally, the cases where the maximum number of squares in a row is 2:



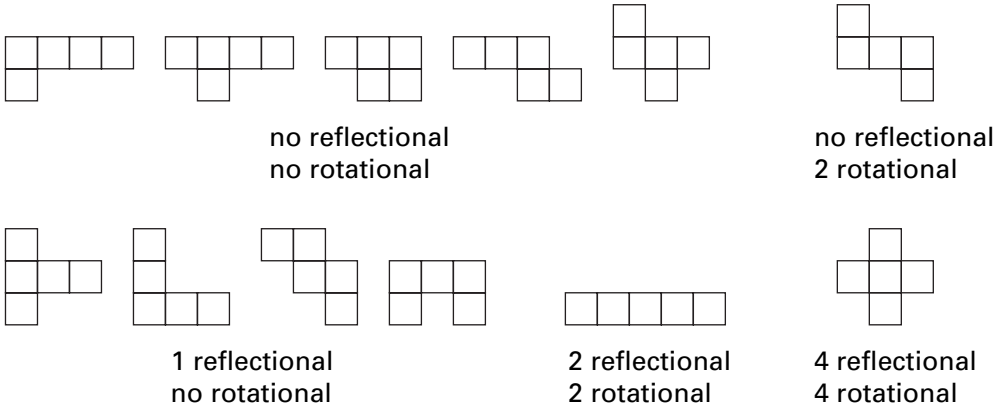
Connector Teacher Activity (cont.)

ACTIONS

2 Ask the groups to determine and cut out all the possible noncongruent pentominoes, arrangements of 5 squares joined edgewise. Then have them sort and classify the pentominoes according to their symmetry types, and to label each classification. Discuss.

COMMENTS

2 Students formed all the different pentominoes in Lesson 1 of *Math Alive! Course II*. The 12 different pentominoes fall into 5 different symmetry types. Note that half of the pentominoes have no reflectional symmetry and half do.

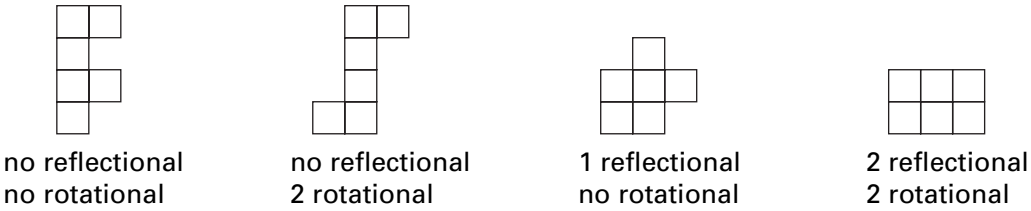


3 Ask the groups to classify the different tetrominoes formed in Action 1 according to their symmetry types. Discuss.

3 The 5 tetrominoes each have one of the 5 types of symmetry listed above.

4 Ask the groups to see if they can construct a hexomino, an arrangement of 6 squares joined edgewise, for each of the 5 symmetry types found in Action 2.

4 Shown below are hexominoes for 4 of the above types of symmetry. No hexomino has 4 reflectional and 4 rotational symmetries.



Focus Teacher Activity

OVERVIEW & PURPOSE

Students explore different ways of moving a shape onto a congruent shape. These movements strengthen students' awareness of congruence, motivate introductory experiences with isometries (reflections, translations, rotations, and glide reflections) and provide the basis for creating friezes and tessellations.

MATERIALS

- ✔ Focus Student Activities 2.1, 2.2, and 2.3, 1 copy of each per student and 1 transparency of each. (Note: the last 2 pages of 2.3 should be run back-to-back on cardstock.)
- ✔ Focus Masters A, B, and J, 1 copy of each per student and 1 transparency.
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- ✔ Focus Masters D, E, and I, 1 transparency of each.
- ✔ Focus Masters F, G, H, 1 copy of each per student and 2 transparencies of each.
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- ✔ 2-cm triangular grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ 2-cm squared grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Marking pens for each group.
- ✔ Blank transparencies (¼-sheets) for use at the overhead.
- ✔ Butcher paper strips, several for each group.

ACTIONS

1 Arrange the students in groups and give each student a copy of Focus Master A (see next page), a sheet of patty paper, a straightedge, and a pair of scissors. Ask the students to investigate the problem posed on Master A, discussing their ideas with their groupmates as they work. Discuss. As groups share their methods, encourage them to explain their motions as fully as possible.

Use the students' descriptions of motions they use during Actions 1-3 as a context for introducing one or more of these terms: *transformation, reflection*, *rotation, mapping, translation vector, magnitude, pre-image, image, isometry, one-to-one correspondence,* and *composite,* noting that learning terminology is not the primary purpose here and it isn't expected that all the terms be introduced in this action.

COMMENTS

1 Patty paper is available in some supermarkets, from restaurant suppliers, and from the MLC catalog. Standard sheets of tracing paper (generally more expensive) cut into ¼-sheets can be used in place of patty paper.

Students are likely to use a variety of strategies. Some may trace and cut out a copy of Square F and actually move it to coincide with Square D. Others may only trace a copy and move the tracing (patty) paper. Still others may imagine moves.

To facilitate student sharing at the overhead, you could have volunteers trace a shape on a quarter sheet of transparency and demonstrate the transformations that move the shape onto its image.

The intent here is for students to informally describe motions (several examples are shown on the next page)

(Continued next page.)

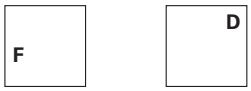
Focus Teacher Activity (cont.)

ACTIONS

Introduction to Isometries Lesson **2**

Focus Master A

Investigate ways to use slides, flips, and/or turns to move Square F exactly onto Square D. Use words and/or mark diagrams to explain the movements that you use.



COMMENTS

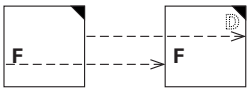
1 (continued.)
that involve the use of *transformations* called *rotations, reflections*, and *translations*, and for the teacher to attach more formal language as appropriate. Students will become familiar with technical language through usage over time. The explanations that follow are teacher background information for use in clarifying student comments throughout Actions 1-3. Not all this information will necessarily be discussed during this action.

In general, a *transformation* is a *mapping* of one set of points, called the *pre-image*, onto another set of points, called the *image*. There is a *one-to-one correspondence* between the points on a pre-image and the points on its image because, for every point on the pre-image there is a corresponding point on the image and, conversely, for every point on the image there is a corresponding point on the pre-image.

In this lesson, students explore transformations that map shapes onto congruent shapes. Dilations—enlargements and reductions—that students constructed in Lesson 30 of *Math Alive! Course II*, are transformations that map shapes onto similar shapes.

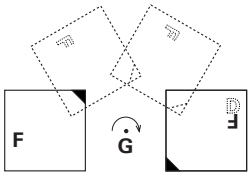
Since Square F and Square D are congruent, a transformation of Square F onto Square D must not change the size or shape of Square F. Any transformation that preserves the size and shape of a figure is called an *isometry*. Hence, two figures are *congruent* if there is an isometry that maps one exactly onto the other.

Method 1. Slide Square F directly on top of Square D.



A *translation* is an isometry that can be modeled by a *slide*. Under a translation, all of the points on a figure slide the same distance along parallel paths to its image (see Method 1 at the left). A translation is defined by its direction and distance. A *translation vector* (e.g., an arrow in the diagram at the left) is used on a drawing to show the direction and distance of the translation. The length of the vector from its starting point to the tip of the arrowhead tells how far each point slides. The distance is called the *magnitude* of the translation. The direction the vector points indicates the *direction* of the translation. All points slide along paths that are parallel and equal in length to the translation vector.

Method 2. Trace Square F and, using point G as the turning (or pivot) point, rotate the tracing until it fits on Square D.



A *rotation* is an isometry that can be modeled by *turning* a figure about a fixed point, called the *center* of the rotation, as illustrated in Method 2 at the left. Every point on the original figure must rotate about the same point and through the same angle (180° in this example). The amount and direction of a rotation is indi-

Focus Teacher Activity (cont.)

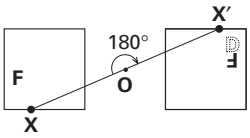
ACTIONS

COMMENTS

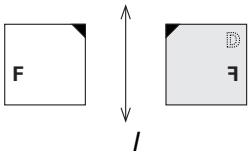
cated by a *rotation vector*, which is an arc (with an arrowhead on one end) of a circle whose center is the center of the rotation. The degree measure of the arc is the angle of rotation and is called the *magnitude* of the rotation. The arrowhead indicates the direction of the rotation.

Note that identifying centers and angles of rotation may challenge students. Throughout this lesson, provide time and encouragement for students to invent and discover informal methods of their own. To approximate the center of rotation of Square F onto Square D, for example, one can trace Square F onto tracing paper, predict a center of rotation and place a pencil point at that point on the tracing paper. Then, keeping Focus Master A in place and using the pencil point to hold the tracing paper, rotate the tracing of Square F to see if it will map exactly onto Square D. If it does, then the position of the pencil point is a center of rotation (a very close approximation). If not, then the process can be repeated until a close approximation of the center is identified.

The following method works to determine the angle of rotation of one shape onto another congruent shape: mark a point X on the pre-image; trace the pre-image; locate the center of rotation and label it point O; rotate the tracing about point O onto the image; mark the image of point X and label it X′; draw a line segment from point O to point X on the pre-image and another segment from point O to X′. The measure of the angle X0X′ is the measure of the angle of rotation (180° in this case).



Method 3. Flip Square F over line *l* onto Square D.



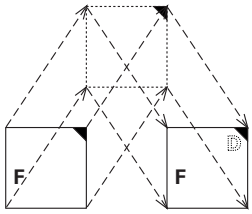
A *reflection* is an isometry that can be modeled by a *flip*, as illustrated in Method 3 at the left. In this isometry, every point on a figure is flipped, or reflected, across a line to a corresponding point on the image of the figure. A line of reflection is the perpendicular bisector of every segment that connects a point on the original figure with its corresponding point on the image of the figure (some students may notice this now, and if not, it will come up later). A figure and its reflection are *mirror images* of one another across the *line,* or *axis, of reflection*. Folding along the line of reflection should map Square F onto Square D.

(Continued next page.)

Focus Teacher Activity (cont.)

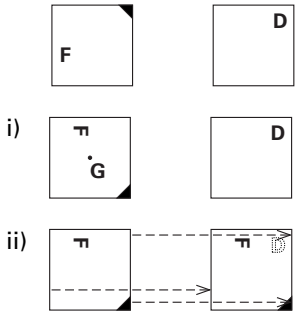
ACTIONS

Method 4. Slide Square F, without turning, along the paths of the arrows until Square F fits onto Square D.



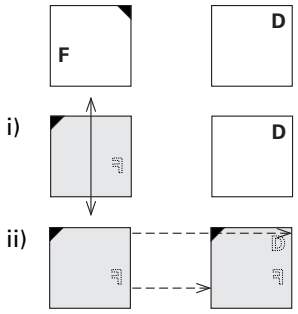
Method 5.
i) Rotate Square F clockwise a ¼-turn (90°) about point G and then,

ii) slide Square F onto Square D.



Method 6.
i) Flip Square F over a line of symmetry and then,

ii) slide Square F onto Square D.



2 Distribute a copy of Focus Master B to each student and ask them to find all the *different* mappings of Square F onto Square D, assuming different mappings means the sides of the pre-image and image match in distinctly different ways. Ask the students to identify ways they locate axes of reflection and centers of rotation. Discuss.

COMMENTS

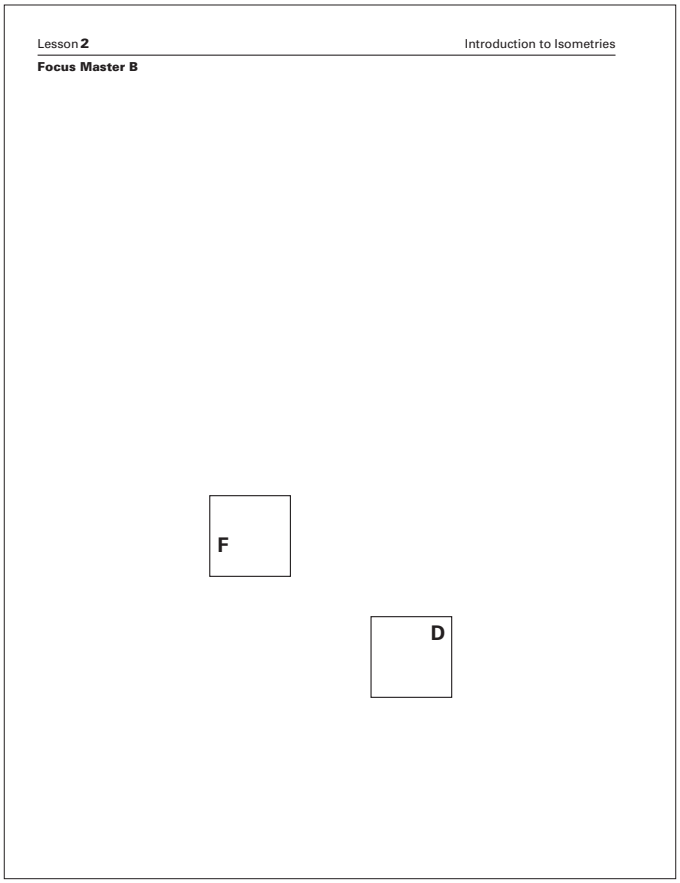
1 (continued.)
The students will likely also describe several transformations that are combinations of translations, rotations, and/or reflections. Such combinations are called *composites* of the transformations. Methods 4, 5, and 6 shown in the left-hand column are examples. Notice that Method 4 uses a combination of 2 translations, Method 5 uses a rotation followed by a translation, and Method 6 uses a reflection followed by a translation.

Some groups may conclude there are an infinite number of movements. Others may restrict their count in some way. For example, some may not count two movements as "different" unless the sides of the pre-image and image match in distinctly different ways. Under such restrictions, Methods 1, 2, 3, 5, and 6 describe different movements; however, Methods 1 and 4 are the "same." One of the problems that may come up is to mark Square F so that the different movements can be recognized. Methods used in the Lesson 1 Connector may be helpful.

2 Students can use their tracings of Square F from Action 1 to explore this problem. Keep in mind that methods students use to locate centers of rotation and axes of reflection will be informal (e.g., paperfolding to locate lines and experimenting with pencil points about which tracings can be rotated). Following are examples of 4 of the 8 different possible mappings of Square F onto Square D. Note that students may use different combinations of movements to achieve the 8 different mappings. However, just as there are 8 different positions for a square in its frame, there are exactly 8 different ways the sides of the pre-image can map to the sides of the image. Labeling the vertices of the pre-image and vertices of the image as in the Lesson 1 Connector may be helpful.

Focus Teacher Activity (cont.)

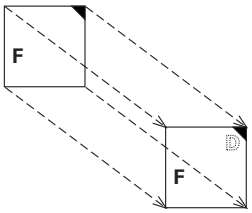
ACTIONS



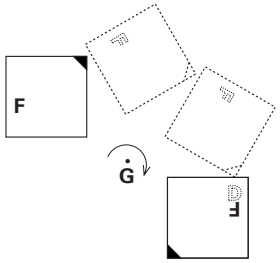
3 Give each group several strips of butcher paper and marking pens. Give each student two additional sheets of patty paper, a protractor, and a copy of Focus Student Activity 2.1 (see next page) to complete. Ask them to use their protractors to determine the measures of angles of rotation. Encourage the students to confer with their groupmates as they work and record on butcher paper strips any conjectures or generalizations that come up. Post these on the "We conjecture… We wonder… " poster from Lesson 1 (or on a new poster). Discuss as a large group as needed.

COMMENTS

Method 1:
Translate Square F onto Square D.

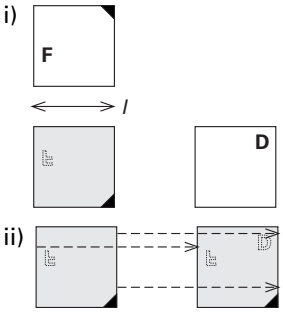


Method 3:
Rotate Square F 180° clockwise about point G.

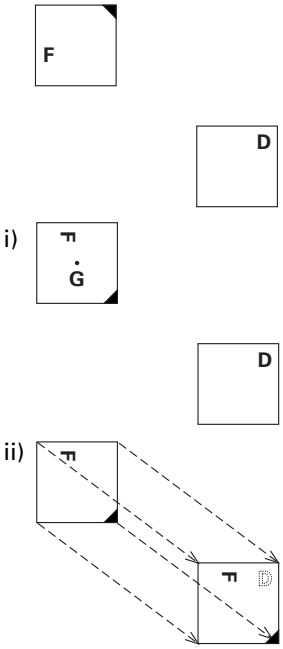


Method 2:
i) Reflect Square F across line l and then,

ii) translate Square F onto Square D.



Method 4:
i) Rotate Square F clockwise 90° about point G and then,
ii) translate Square F onto Square D.



3 It is helpful to keep a supply of butcher paper strips on hand throughout the lesson, so that students can record conjectures, questions, and generalizations and attach them to a poster for consideration by the class.

See Comment 1 for an example of a method for determining the measure an angle of rotation.

(Continued next page.)

Focus Teacher Activity (cont.)

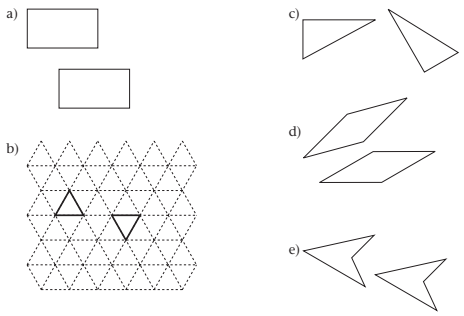
ACTIONS

Introduction to Isometries Lesson **2**

Focus Student Activity 2.1

NAME _____ DATE _____

1 Shown below are several pairs of congruent shapes. Investigate ways to use one or more translations, reflections, rotations, or combinations of them, to move each first shape exactly onto the second. For each pair of shapes, write an explanation in words only of your "favorite" motion or combination of motions; explain in enough detail that a reader would be able to duplicate your motions without additional information.



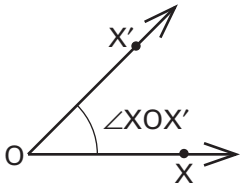
2 Challenge. Each motion or combination of motions that you determined for Problem 1 produces a *mapping* of the first shape (the *pre-image*) exactly onto the second (the *image*). How many different mappings are there for each of a)-e), if different means the sides of the pre-image and the sides of the image match in distinctly different ways.

3 Record your "I wonder…" statements, conjectures, or conclusions.

4 Remind the students that reflections, translations, and rotations are isometries because they are motions that *do not change the size or shape of an object*. Place a transparency of Focus Master C on the overhead and ask the students to describe ways to move Shape A to Position 1 (shown by the shape numbered 1) using one or more reflection, translation, and/or rotation. Have volunteers demonstrate their ideas at the overhead.

COMMENTS

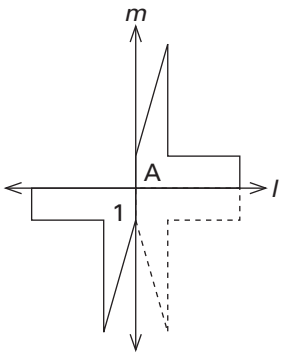
3 (continued.)
Note: in general, angles are referred to by their vertex or by 3 points—the vertex and a point on each side of the angle. The notation ∠XOX′, for example, denotes the angle with vertex O, one ray extending through the point X, and the other ray extending through the point X′. The small arc inside the angle suggests that ∠XOX′ is the acute angle formed by rays $\overrightarrow{OX}$ and $\overrightarrow{OX'}$ rather than the reflex angle.



Problem 3) Some students may correctly conjecture that the number of different mappings of any shape onto another congruent shape is equal to the number of symmetries of the shape. For example, in Lesson 1 students determined that a nonsquare rectangle has symmetry of order 4 (2 reflectional and 2 rotational symmetries). Notice in a) there are 4 different mappings of a nonsquare rectangle onto a congruent rectangle: a translation followed by a translation; a horizontal translation followed by a reflection; a vertical translation followed by a reflection; and a rotation.

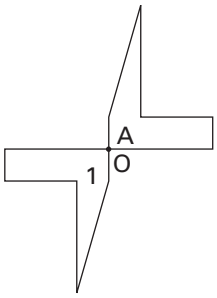
4 Here are some possibilities for moving Shape A to Position 1:

Method 1



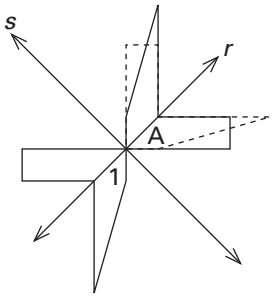
Flip Shape A over line *l* and then over line *m*.

Method 2



Rotate Shape A 180° about point O.

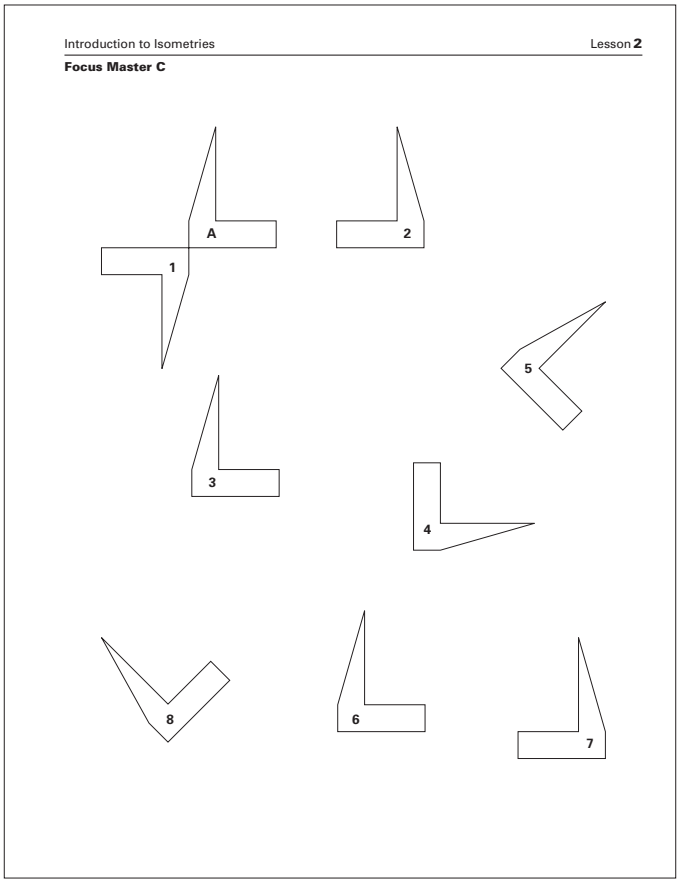
Method 3



Reflect Shape A over line *r* and then over line *s*.

Focus Teacher Activity (cont.)

ACTIONS



5 Give each student another sheet of patty paper and a copy of Focus Master C (see above). Place a transparency of Focus Master D on the overhead revealing Part I only for the students to complete. When Part I is completed, ask the students to discuss their results and observations with their groupmates. Discuss as a large group, using the students' observations as a context for introducing the terms *direct transformation* and *indirect transformation* (see Comment 5). Encourage conjectures and generalizations.

Lesson **2** Introduction to Isometries

Focus Master D

Part I

It is possible to move Shape A directly to several of the numbered positions using exactly one of these isometries only once: translation, reflection, or rotation. Find each position for which this is possible, and tell the single motion that moves Shape A to that position.

COMMENTS

5 It is possible to move Shape A directly to each of the given positions except 7 by using only one of the 3 given isometries.

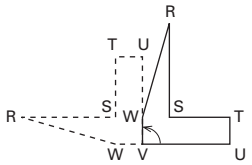
Here are some questions you might use to encourage students to explain their thinking: How did you decide what motions might be possible? How did you determine where to put lines of reflection? How did you locate centers of rotation?

The intent of this action is to informally strengthen students' awareness about these motions. It is likely that students will experiment when locating axes of reflection or centers of rotations; however, conjectures about methods of identifying axes and centers may begin to surface. For example, if it hasn't come up previously, students may notice that a line of reflection is the perpendicular bisector of the segments connecting a point on the pre-image to its corresponding point on the image.

(Continued next page.)

Focus Teacher Activity (cont.)

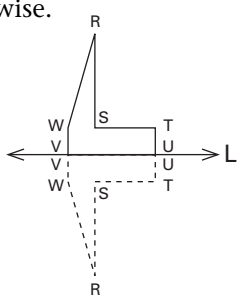
ACTIONS



COMMENTS

5 (continued.)
Translations and rotations are called *direct transformations*, because they *preserve the orientation* of a shape. For example, if the vertices of a polygon are labeled R, S, T, U, etc., in a clockwise direction, after a direct transformation they still read clockwise, as shown in the example at the left. Other rotations, regardless of their magnitude or direction, also preserve the clockwise orientation of the letters.

Similarly, a translation preserves the orientation of the letters. However, notice in the following example that after a reflection, letters on vertices switch from clockwise to counterclockwise.



Reflections are called *indirect* transformations because they *reverse the orientation* of a shape and thus change the direction in which labeled vertices read.

6 Give each student another sheet of patty paper and another copy of Focus Master C. Reveal Part II on the transparency of Focus Master D. When the students have completed their work, discuss their methods and conclusions. Introduce the term *glide reflection*.

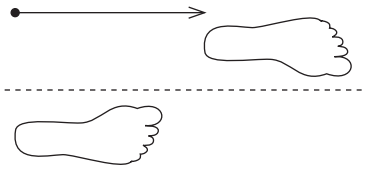
Lesson **2** Introduction to Isometries
Focus Master D

Part I
It is possible to move Shape A directly to several of the numbered positions using exactly one of these isometries only once: translation, reflection, or rotation. Find each position for which this is possible, and tell the single motion that moves Shape A to that position.

Part II
Describe ways to move Shape A from its starting position to each numbered position using a combination of exactly two reflections, rotations, and/or translations. Note: combinations of more than one type of motion are allowed as long as no more than two motions are used.

6 Groups may have discovered some solutions to this problem while investigating Part I in Action 5.

A *glide reflection* is a two-step isometry involving: 1) a translation and 2) a reflection across a line parallel to the translation vector. A pair of footprints is an example of a glide reflection—notice it is possible to slide and then flip, or flip and then slide (the order of the motions does not matter):



Shape A can be moved onto position 7 by a glide reflection. A glide reflection is an indirect isometry because the orientation of the image is reversed with respect to the pre-image.

Focus Teacher Activity (cont.)

ACTIONS

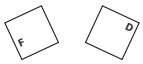
7 Distribute one copy of Focus Student Activity 2.2 to each student for completion. Encourage "I conjecture…" and "I wonder…" statements. Discuss their results.

Lesson **2** Introduction to Isometries

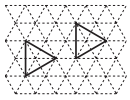
Focus Student Activity 2.2

NAME _____ **DATE** _____

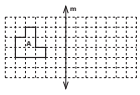
1 Shown at the right are 2 congruent squares. Determine ways to use exactly one isometry (translation, reflection, rotation, or glide reflection) to move Square F exactly onto Square D.



2 Repeat Problem 1 for the 2 equilateral triangles shown here:

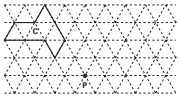


3 Sketch the reflected image of Shape A across line *m*. Next to your sketch write several mathematical observations about relationships you notice. Then explain how you verified that the image is a reflection of Shape A across line *m*.



4 Challenge. Develop a method of accurately reflecting Shape B across line *n*. Show and describe your method of locating the reflected image of Shape B and tell how you verified that your method was correct. Can you generalize?

5 Sketch the image of Shape C after a 120° clockwise rotation about point P. Next to your sketch write several mathematical observations about relationships that you notice. Then explain how you verified that the image is a 120° rotation about point P.

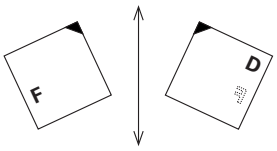


6 Challenge. Invent a method of rotating Shape D 170° clockwise about point P, without using a grid. Show and describe your method of locating the rotated image of Shape D and tell how you verified that your method was correct.

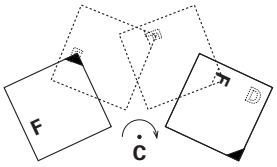
COMMENTS

7 Students may need additional patty paper. Here are two methods for Problem 1:

Method 1: Reflection

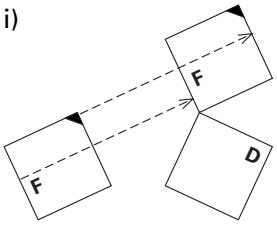


Method 2: Rotation about point C. Note there are other possible placements of the center of rotation.

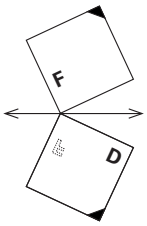


Notice that the composite of the translation and reflection shown below move Square F onto Square D. However, note this is not a glide reflection because the line of reflection is not parallel to the translation vector.

i)



ii)

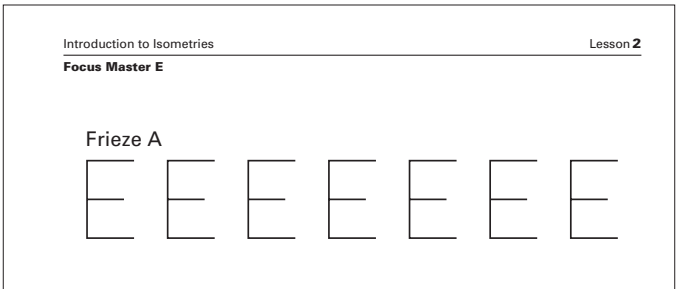


For Problems 1 and 2 students may be interested in determining all the different possible ways, where different implies different pairings of the sides of the preimage and sides of the image. The students may correctly conjecture that the number of possibilities equals the number of ways the shapes fit in their frames (see Lesson 1).

Focus Teacher Activity (cont.)

ACTIONS

8 Place a transparency of the top half (cut apart from the bottom half) of Focus Master E on the overhead.



Ask the students to suppose this *strip pattern*, or *frieze*, extends indefinitely to the left and the right. Allow a few moments for the students to examine the pattern. Then place the bottom half of Focus Master E over the top half so the patterns coincide.

a) Ask the students to mentally determine ways the top transparency can be moved so that the patterns on the two transparencies again coincide, keeping in mind the pattern extends indefinitely in both directions. Discuss their ideas.

b) Have the groups discuss their ideas about the *symmetry of the strip pattern*, identifying any centers of rotation, translation vectors, and lines of reflection. Discuss as a large group. Clarify as needed.

9 Give each student a copy of Focus Master F (see next page) and ask the groups to determine the symmetries of the friezes shown and methods of moving the frieze onto itself. Have them label centers of rotation, translation vectors, and lines of reflection. Invite volunteers to show their group's ideas at the overhead using two transparencies of Master F.

COMMENTS

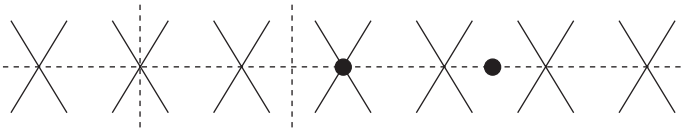
8 a) The top transparency of the pattern can be flipped along a horizontal line down the centerline of the E's, or it can be translated in the direction of the strip, or both.

b) An isometry transformation of a strip pattern or frieze onto itself is called a *symmetry of the strip pattern*. A frieze can have reflectional, rotational, translational, and/or glide reflectional symmetry.

The symmetries of the frieze shown are: reflectional about a horizontal line through the middle of the strip, translational in the direction of the strip; and glide reflectional. This frieze has no rotational symmetry.

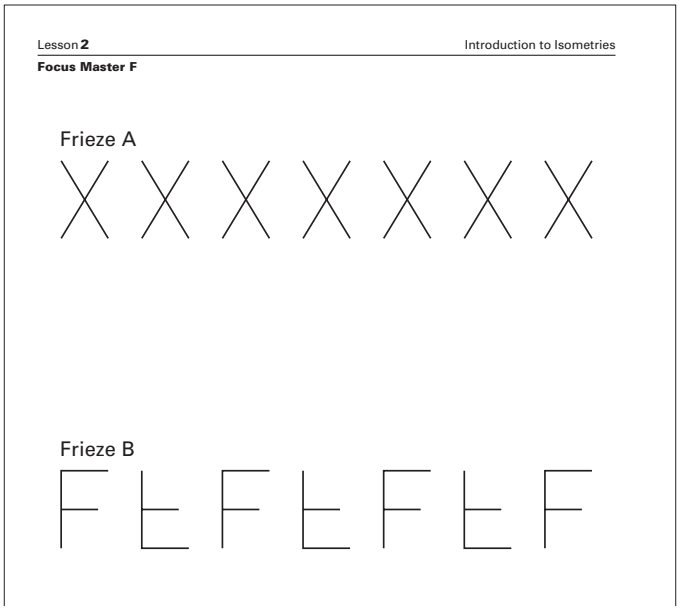
9 Students can work with a partner, superimposing their copies of the friezes so tracing is unnecessary. If students have difficulty seeing through one copy, they could hold the superimposed copies up to a window.

A symmetry of Frieze A is either a) a translation in the direction of the strip, b) a reflection about a horizontal line through the center of the X's, c) a reflection about a vertical line through the center of any X, d) a reflection about a vertical line midway between adjacent X's, e) a half turn about the center of any X, f) a half turn about the midpoint of the line segment connecting the centers of adjacent X's, or g) any combination of the above six types. Representative axes of symmetry and centers of rotation are shown below.

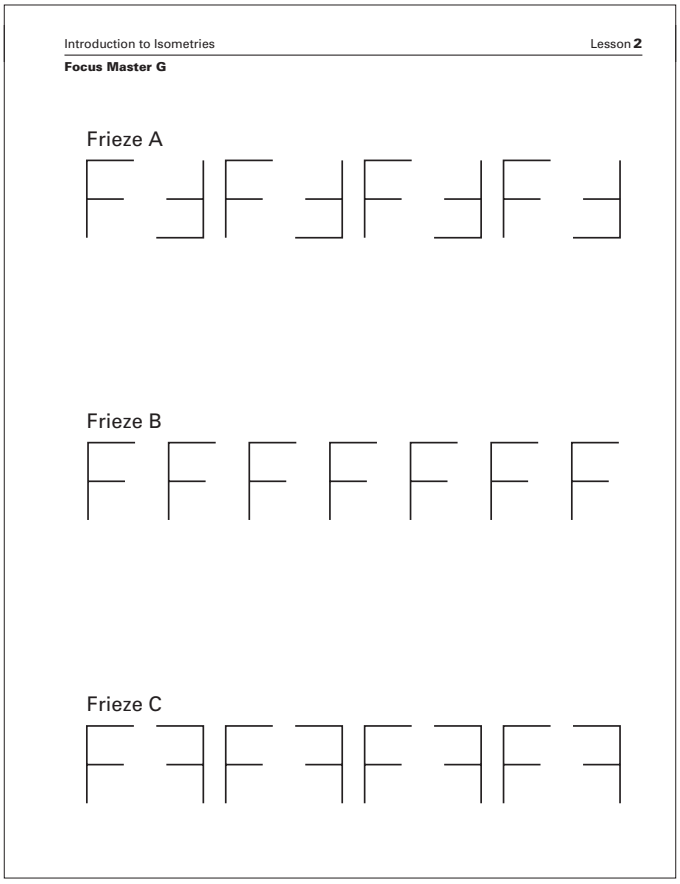


Focus Teacher Activity (cont.)

ACTIONS



10 Give each student a copy of Focus Master G and repeat Action 9 for the friezes on the master.



COMMENTS

The symmetries of Frieze B are: a) a translation in the direction of the strip, b) a glide reflection (e.g., a translation in the direction of the strip followed by a reflection about a horizontal line through the center of the pattern), or c) combinations of a) and b).

You might ask the students to create notation for identifying the types of symmetry for the friezes. Or, you could introduce the following (this notation is used throughout the remaining actions and comments):

T–translational
H–reflectional about a horizontal line
V–reflectional about a vertical line
R–rotational through a half turn
G–glide reflectional

Frieze A on Focus Master F has all of the above types. Frieze B has types **T** and **G**.

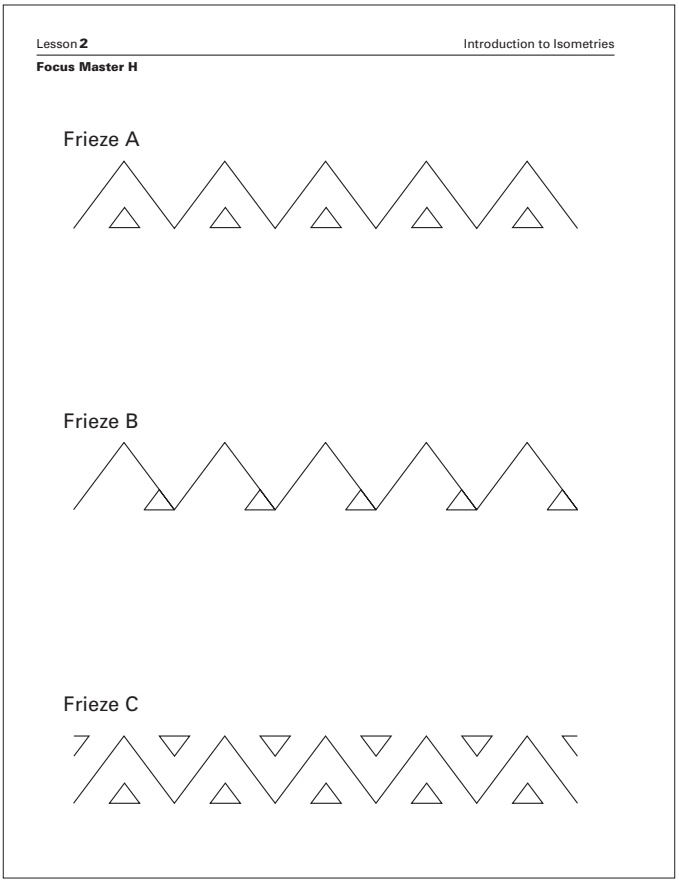
10 Using the notation defined above, the three strip patterns have, respectively, the following types of symmetry: **T** and **R**; **T** only; and **T** and **V**.

Focus Teacher Activity (cont.)

ACTIONS

11 Give each group several marking pens and a sheet of 2-cm squared grid paper. Ask the groups to design a frieze which has **T**, **G**, and **R** type symmetries. Invite volunteers to show their group's patterns to the class for verification. Discuss the students' conjectures and generalizations about friezes with **T**, **G**, and **R** type symmetries, and the reasoning behind their conjectures and generalizations.

12 Give each student a copy of Focus Master H and ask the groups to determine the set of symmetries of each pattern. Discuss.


Lesson **2** Introduction to Isometries

Focus Master H

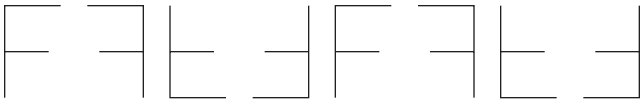
Frieze A

Frieze B

Frieze C

COMMENTS

11 One example of a frieze with **T**, **G**, and **R** type symmetries is the following:



Students may notice that the above pattern also has type **V** symmetry. To draw out conjectures and generalizations about frieze patterns with **T**, **G**, and **R** type symmetries, you could post several groups' friezes and ask the class to make observations about similarities and differences they notice. For, example, every frieze which has **G** and **R** type symmetries also has **V** type symmetry, since the combination of a glide reflection followed by a half turn is the same as a reflection about a vertical line.

12 The friezes have, respectively, the following sets of symmetries:

Frieze A: **T** and **V**;

Frieze B: **T** only;

Frieze C: **T**, **R**, **V**, and **G**.

Focus Teacher Activity (cont.)

ACTIONS

13 Ask the groups to determine all the possible sets of symmetries for a frieze and to design a frieze for each possible set, using a grid of choice. Have the groups exchange their completed patterns for verification. Discuss.

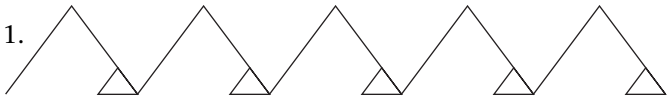
COMMENTS

13 It is helpful to have available a supply of 2-cm squared grid paper and 2-cm triangular grid paper (see *Blackline Masters*).

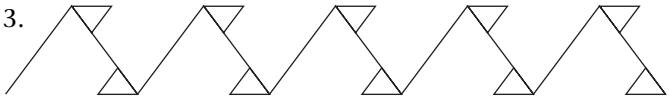
Some groups may take some time identifying all the possible sets of symmetries for a strip pattern. There are 7 possibilities:

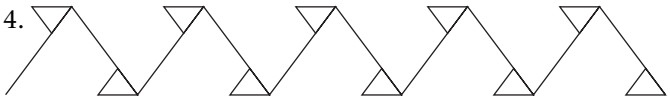
1. **T** only
2. **T** and **V**
3. **T** and **R**
4. **T** and **G**
5. **T**, **H**, and **G**
6. **T**, **R**, **V**, and **G**
7. **T**, **R**, **H**, **V**, and **G**

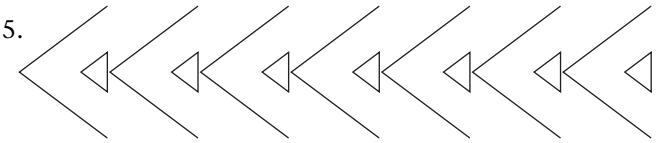
Shown below are examples of strip patterns which have, respectively, the above seven sets of symmetries.

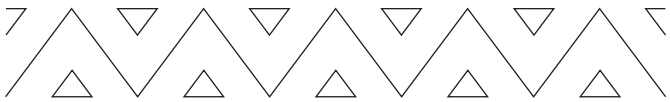
1.


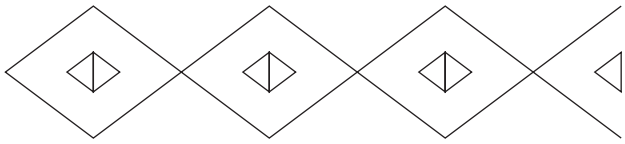
2.


3.


4.


5.


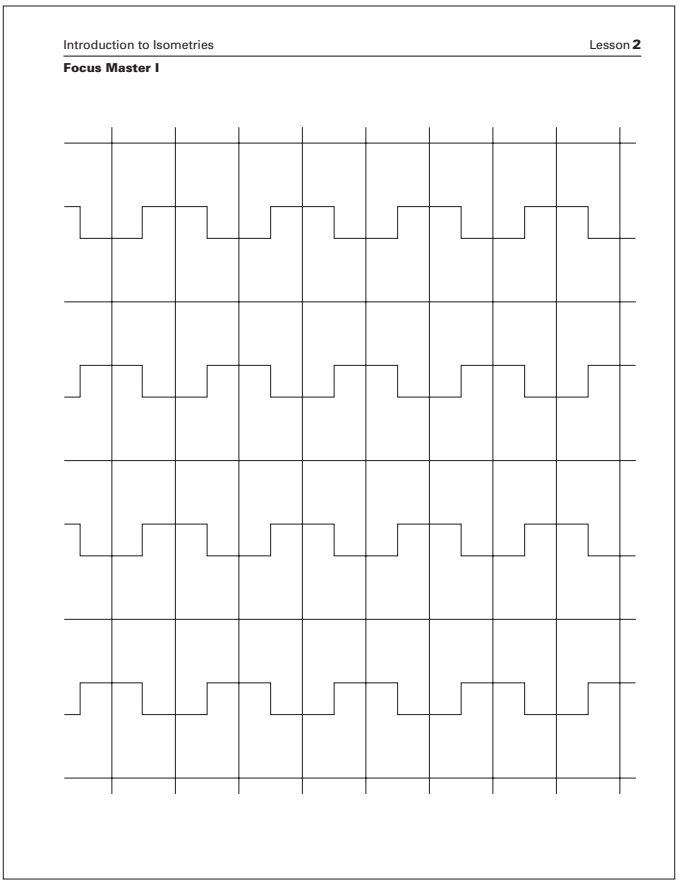
6.


7.


Focus Teacher Activity (cont.)

ACTIONS

14 Place a transparency of Focus Master I on the overhead. Ask the students to imagine that the pattern shown continues indefinitely in all directions and tell them it is an example of a *tessellation*, i.e., a *tiling* by a set of one or more congruent figures that cover the *plane* without gaps or overlaps. Ask the students for their observations about this tessellation, including its symmetries. Discuss. Have students use a second transparency of Focus Master I to illustrate their ideas about the symmetries of the tessellation at the overhead.



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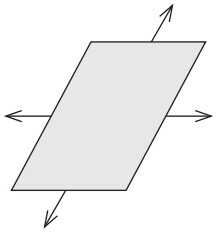
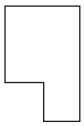
COMMENTS

14 It may be helpful to discuss students' ideas about the geometric meaning of the term *plane* and ask them to identify surfaces that are portions of a plane. For example, a sheet of paper, a wall, the floor, a table top, the ceiling, windows, etc., are all portions of a plane, since they are all bounded by edges. A plane extends indefinitely and has no boundaries:



A tessellation is symmetrical if there is an isometry which maps it back onto itself, i.e., a tracing of the tessellation which can be rotated, translated, reflected and/or glide reflected from one position in which it is coincident with the tessellation to another position coincident with it. Notice there is a pattern to the way the tessellating figures fit together at vertices of the tessellation.

Here are examples of observations students have made about the tessellation on Focus Master I:



The tessellation is made by repeating the same L-shape over and over, but changing its orientation.

We see a tiling made up of rectangles formed by 2 L-shapes.

The tessellation has rotational symmetry. If a copy of the tessellation is positioned so that it coincides with the original, and then the copy is rotated 180° about either of the points shown in Figure 1, it will again coincide with the tessellation.

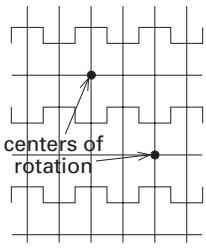


Figure 1

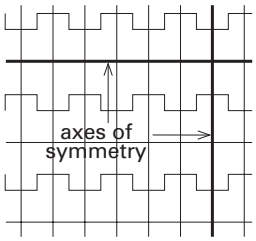


Figure 2

The tessellation has reflective symmetry. A copy that coincides with the tessellation can be flipped across any of the horizontal or vertical lines of the tessellation and the copy will again coincide with the tessellation (see Figure 2 above).

Focus Teacher Activity (cont.)

ACTIONS

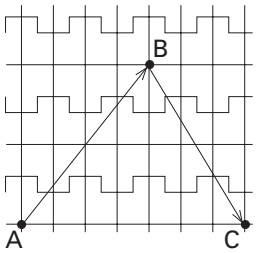


Figure 3

COMMENTS

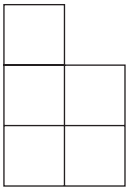
The tessellation has translational symmetry, with many translation vectors. For example, if we place a copy so it coincides with the tessellation, we can slide the copy the distance and direction of either of vectors $\overrightarrow{AB}$ *or* $\overrightarrow{BC}$ *(see Figure 3), and the copy will again coincide with the original. Slides that are along the same translation vectors but twice as long also work. The vectors can also go in the opposite direction. We also see lots of different horizontal and vertical slides that are possible. And there are more!*

Many glide reflections are possible. For example, translate along a horizontal translation vector and then reflect across one of the horizontal lines of symmetry; or translate along a vertical translation vector, and then reflect across a vertical line of symmetry. We could also reverse the process by flipping and then sliding. Note: the translation vector and line of reflection must be parallel.

Use two reflections: flip across one of the vertical lines of symmetry, then flip across one of the horizontal lines of symmetry.

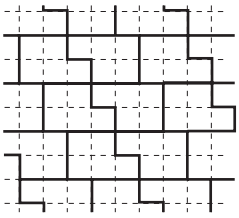
We notice that there is a pattern to the ways the corners of the L-shapes meet and there are two different types of intersections.

15 Give each student one sheet of 2-cm squared grid paper. Ask them to cut the sheet in half. Point out that the pentomino shown below is the basic tile for the tessellation on Focus Master I.



Ask each group to draw on half sheets of grid paper (using marking pens, and completely filling each half sheet with one tessellation) several other tessellations based on the above tile, including one that has no reflective symmetry. Invite volunteers from the groups to hold up their tessellations for the class to see. Discuss the symmetry types, centers of rotation, axes of reflection, translation vectors, etc.

15 Some groups may prefer to cut out copies of the pentomino and fit the pieces together to create tessellations. Then they can copy the tessellation on grid paper. Here is one possibility that has no reflective symmetry:



Notice the tessellation above has 180° and 360° rotational symmetry.

You may need to remind the students that a tessellation covers all points of the plane so it has no gaps. Also each point is covered once only, so there are no overlaps. To tessellate a half sheet, students should fill the sheet, including partial pentominoes that fit along the edges.

Focus Teacher Activity (cont.)

ACTIONS

16 Give each student another sheet of 2-cm squared grid paper and ask them to cut the sheet in half. Ask the groups to, if possible, draw two different tessellations which are based on other pentominoes and so the tessellations have symmetry as described in a) below. Post some of these tessellations for verification and discussion by the class. Repeat for one or more of b)-d).

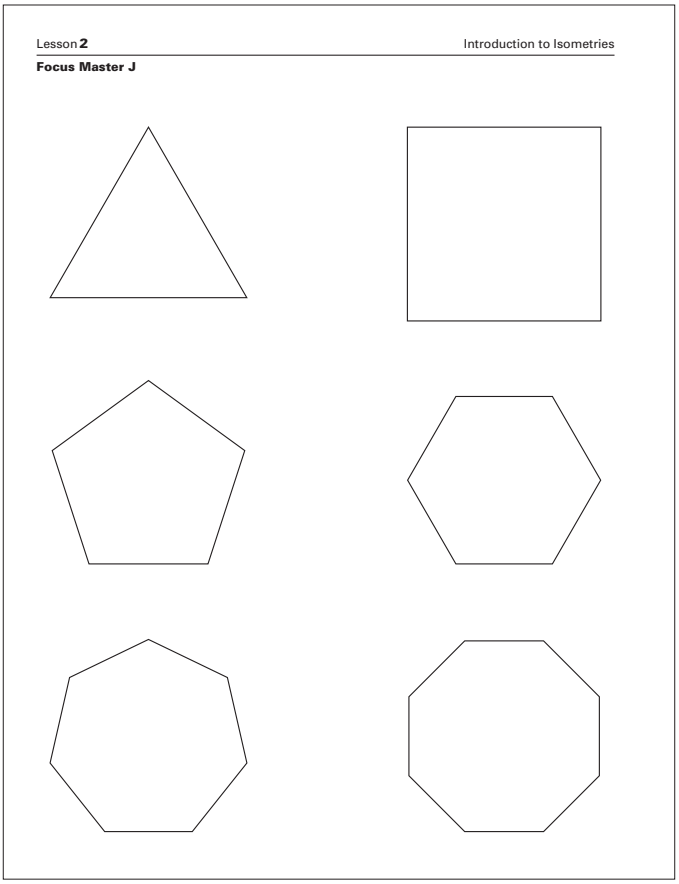
a) translational symmetry only,

b) only translational and 2 rotational symmetries,

c) only translational and 4 rotational symmetries,

d) only translational and reflectional symmetry, with axes of symmetry that extend in one direction only.

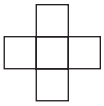
17 Give each student a copy of Focus Master J and ask them to determine which, if any, of the regular polygons shown on Master J tessellate the plane. For any that do not tessellate, have them explain why they think this is so. Then ask them to determine whether there are other regular polygons that tessellate. Discuss.



COMMENTS

16 Students drew the 12 different pentominoes in the Connector activity. You might remind the groups to use marking pens so their tessellations will be easy to see when you post them. To facilitate sharing, for each of parts a)-e), you could label a sheet of butcher paper headed with the given symmetry type, then groups could attach their tessellations to the poster.

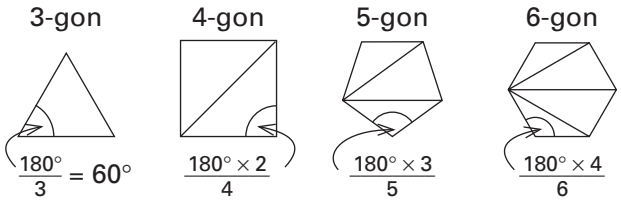
c) There is only one possibility:



17 To test for tessellations students could trace copies of the shapes on blank paper, or they could cut out the shapes and combine them with their groupmates' cutout shapes.

The equilateral triangle, square, and hexagon are the only regular polygons that tessellate the plane. Since the measures of the angles of an equilateral triangle are 60°, 6 triangles can be arranged about a point to exactly fill out 360°, with no gaps or overlaps. Similarly, because the angles of a square all measure 90° and because the angles of a regular hexagon are each 120°, 4 copies of a square can be arranged about a point to exactly fill out 360°, as can 3 copies of a regular hexagon. However, the remaining regular polygons do not have angle measures that are factors of 360°.

Note: the students computed interior angle measures for these regular polygons on the Lesson 1 Follow-up Student Activity. The following diagram together with the fact that the sum of the measure of the angles in any triangle is 180°, provides the basis for one line of reasoning that the measure of the interior angle of a regular n-gon is $\frac{(n-2)180°}{n}$.



Students may discover they can tile the plane with certain combinations of two or more noncongruent

Focus Teacher Activity (cont.)

ACTIONS

18 Ask the groups to complete a)-c) below. Invite volunteers to share their group's conclusions and reasoning. Repeat for d) through f).

a) Use a straightedge to draw a scalene triangle.

b) Label the angles of the triangle a, b, and c.

c) Determine whether the triangle will tessellate. Why or why not?

d) Determine all the *different* possible edge-to-edge tessellations of the triangle, where two tessellations are different if the arrangement of the triangles at each vertex in one tessellation differs from the arrangement at the vertices in the other.

e) Determine whether all triangles tessellate. Why or why not?

f) Determine whether all quadrilaterals will tessellate.

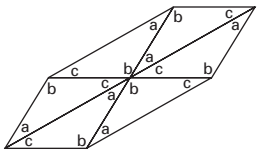
19 Give each student a copy of Focus Student Activity 2.3 (see next 3 pages) and several half-sheets of blank paper. Ask them to carefully read Procedure A and Project A. Discuss their questions. If needed to clarify the students' questions, you might demonstrate Procedure A at the overhead before having the students begin Project A.

Once the students have completed Project A, allow time for them to examine one another's tessellations, to discuss the symmetries of the tessellations, and to point out conjectures or questions that surfaced while they worked.

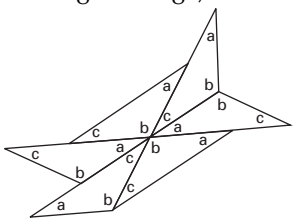
COMMENTS

shapes at each vertex. These are called *semi-pure tessellations*, whereas those that involve only one shape are *pure tessellations*. The focus of this lesson is on pure tessellations for which there is a pattern in the order the shapes meet at vertices.

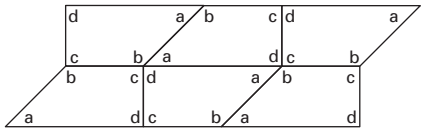
18 All triangles tessellate. This is because the sum of the measures of the angles of any triangle is 180°, and hence, 6 copies of a triangle can be arranged around a point to exactly cover 360°, as illustrated here:



Notice that, although any combination of the angles a, a, b, b, c, and c fills the space around a vertex, not all combinations meet edge-to-edge, as shown here:



Similarly, all quadrilaterals tessellate since the sum of the angles of a quadrilateral always total 360°. Four copies of any quadrilateral can be arranged about a point to exactly fill out 360°. However, not all arrangements form edge-to-edge tessellations. Here is an edge-to-edge arrangement that tessellates the plane:



19 Focus Student Activity 2.3 contains 8 pages. It is important to copy page 7 of the activity on cardstock or lightweight tagboard (old file folders also work), and to copy the shading sheet (page 8) on the back side of page 7 (this enables students to distinguish between the front and back of their patterns when tessellating.) Note: if students each have a Student Activities Packet, there is a cardstock copy of page 7 (with the backside shaded) included.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Introduction to Isometries Lesson **2**

Focus Student Activity 2.3

NAME _____ **DATE** _____

The following procedures create tessellations similar to the type created by the Dutch artist, Maurice C. Escher, who is famous for his tessellations of birds and animals. Escher's first inspirations came from the Alhambra, which was built in the 13th century in Granada, Spain, and is famous for its variety of tessellations. Procedure A below gives a method of creating a tessellation based on rectangles.

Procedure A—a translation tessellation based on rectangles.

Step 1: Beginning at one corner and ending at an adjacent corner, cut out a portion of the rectangle.

Step 2: Slide the cutout portion across the rectangle to the opposite side, matching the straight edges and the corners, as shown at the right. Then tape the pieces in place.

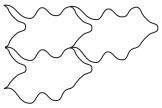
Hint: be careful not to flip the pieces over and don't let the tape extend beyond the edge of the figure.

Step 3: Beginning at an endpoint of one of the remaining two unaltered sides and ending at the other endpoint, cut another portion from the rectangle. An example is shown here.

Step 4: Slide this cutout portion to the opposite side, matching edges and endpoints, as illustrated at the right. Tape the pieces in place.

Hint: It is important that nothing be trimmed or altered to fit!

Step 5: Tile a page with this new shape by repeatedly tracing the shape so the tracings fit together, with no gaps or overlaps (a portion of a tiled page is shown at the right). Describe the symmetries of the tessellation.



Hint: Remember not to flip the piece over when you tile with it.

(Continued on back.)

COMMENTS

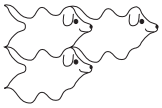
19 (continued.)
Circulate while students are working. Problems can occur if students trim pieces inaccurately to fit, if they flip or turn a piece before sliding, or if tape extends beyond the edges of their pieces. Students may wish to trace extra rectangle patterns to experiment with different cuts before deciding on a shape to tessellate and decorate for Project A.

It is helpful if students use a sharp pencil and lightly trace around the shape so they can easily make any needed erasures or adjustments. After the page is tessellated, they may darken their tracings and add color and details to the figures. They may find it necessary to "fudge" slightly to make up for slight gaps or overlaps caused by inaccuracies in tracing.

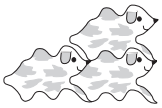
Lesson **2** Introduction to Isometries

Focus Student Activity 2.3 (page 2)

Step 6. Add color details to the figure to produce an interesting and creative work of art.



It could be puppy dogs on the run…



…or, puppy dogs at rest!

Project A

Part I Using a rectangle from page 7 of this activity and Procedure A, create an *original* tessellating shape. On another sheet of paper, trace enough copies of your figure to show that it forms a translation tessellation.

Part II Challenge. Using a shape that is not a rectangle, adapt Procedure A and create another figure that can form a translation tessellation. On another sheet of paper, trace enough copies of the figure to verify it forms a translation tessellation.

(Continued.)

Focus Teacher Activity (cont.)

ACTIONS

20 Discuss your guidelines and expectations regarding the completion of the remainder of Focus Student Activity 2.3. Then have the students complete the activity accordingly. Display their results in the classroom.

COMMENTS

20 If students had difficulty with Project A, or are confused by Procedure B, you might work through Procedure B together.

To motivate high quality work, you could show examples from students in other classes or illustrations from art books or other resources. You might also have the students help you develop a guide for assessing the completed projects, or you could provide them with a guide created by you (see the Assessment chapter of *Starting Points* for ideas).

Completed tessellations can be used for a bulletin board or hall display. Laminating helps to preserve them. Some teachers make slides of student work to show to future classes.

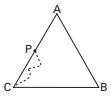
This activity could be carried out as an extended project outside of class, while you continue in class with the next lesson (see the Assessment chapter of *Starting Points* for a discussion of Extended Projects).

Introduction to Isometries Lesson **2**

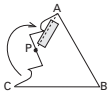
Focus Student Activity 2.3 (page 3)

Procedure B—a rotation tessellation that is based on equilateral triangles.

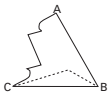
Step 1. Mark the midpoint, P, of side AC on equilateral triangle ABC, as shown at the right. Beginning at vertex C and ending at P, cut a portion from the triangle.



Step 2. Place a finger on P and rotate the cutout portion 180° about P. Then tape the rotated portion in place.



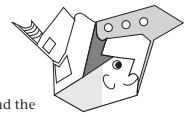
Step 3. Beginning at vertex C and ending at vertex B, cut out a portion of the triangle.



Step 4. Place a finger on vertex B and rotate the new cutout portion clockwise 60°. Tape the rotated portion in place.

Step 5. Tessellate the page by repeatedly tracing and fitting together (with no gaps or overlaps) the shape. Add detail and color to produce a creative piece of art that fills the page.

Hint: To tessellate with this shape it is necessary to rotate the figure.



Project B

Part I Use an equilateral triangle from page 7 of this activity and the method of Procedure B, or another procedure that you invent, to create an *original* shape that can be copied to form a rotation tessellation. On another sheet of paper, trace enough copies of your shape to verify that it forms a rotation tessellation.

Part II Challenge. Invent a way to alter a regular hexagon from page 7 to create a shape that forms a rotation tessellation of the plane. Verify. Show a tracing of several copies to illustrate the beginning of a tessellation.

(Continued on back.)

Lesson **2** Introduction to Isometries

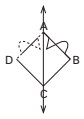
Focus Student Activity 2.3 (page 4)

Procedure C—a reflection tessellation that is based on rhombuses.

Before you read on, trace a rhombus from page 7 and explore your ideas regarding ways to create a reflection tessellation based on nonsquare rhombuses. Then compare your ideas to the following procedure.

Step 1. Lightly trace the rhombus from page 7 on a blank sheet and label its vertices A, B, C, and D. Draw a curve from A to B (notice the curve can extend outside the rhombus). Note: in the remainder of this activity, the notation c (A-B) means *the curve from point A to point B*.

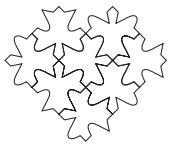
Step 2. Reflect c (A-B) about line AC. Hint: use tracing paper to increase accuracy of copied curves.



Step 3. Rotate c (A-D) 90° about point D to form c (D-C) so that A maps to C.

Step 4. Reflect c (D-C) about line AC so that D maps to B.

Step 5. Erase the lines of the original rhombus that are not part of the curve, and tessellate! Notice the lines of reflection in this tessellation.



Project C

Use a rhombus from page 7 of this activity and the method of Procedure C, or another method that you invent, to create an original shape that can be used to form a reflection tessellation. On another sheet of paper, trace enough copies of your shape to verify that it forms a reflection tessellation.

(Continued.)

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS

20 (continued.)

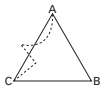
Introduction to Lsometries Lesson **2**

Focus Student Activity 2.3 (page 5)

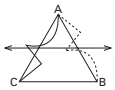
Procedure D—a glide reflection tessellation that is based on equilateral triangles.

Before you read on, trace an equilateral triangle from page 7 and explore your ideas regarding ways to create a glide reflection tessellation based on equilateral triangles. Then compare your ideas to the following procedure.

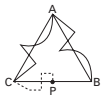
Step 1. Lightly trace the equilateral triangle from page 7 on a blank sheet and label its vertices A, B, and C. Draw a curve from A to C (as in Procedure C, the curve can extend outside the triangle).



Step 2. Reflect c (A-C) about the line parallel to the base BC and passing through the midpoint of AC. Then translate this reflected curve to connect from A to B.

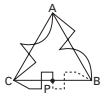


Step 3. Locate point P, the midpoint of side BC. Draw c (C-P).



Step 4. Rotate c (C-P) 180° about point P to form c (P-B).

Step 5. Tessellate!



Project D

Use Procedure D, or another method that you invented, to create an original shape that can be used to form a glide reflection tessellation. Verify by showing a tracing of several copies of your shape.

(Continued on back.)

Lesson **2** Introduction to Lsometries

Focus Student Activity 2.3 (page 6)

Project E

Part I

Pick your favorite tessellating shape from those you created for Projects A-D, or create a new tessellating shape based on Procedures A-D, or combinations of them, and:

a) *completely* tessellate a sheet of paper with the shape,

b) be creative in ways that you *color* and fill in the *details* of this tessellating figure,

c) mount the completed tessellation on a sheet of colored paper, or frame it creatively.

Note: to make a poster-sized tessellation you could enlarge the polygon that is the basis for your tessellation (e.g., double each edge of the hexagon pattern from page 7), create a new tessellating shape based on that enlarged polygon, and then tessellate a sheet of poster paper.

Part II

On the back of your tessellation (after you have mounted it):

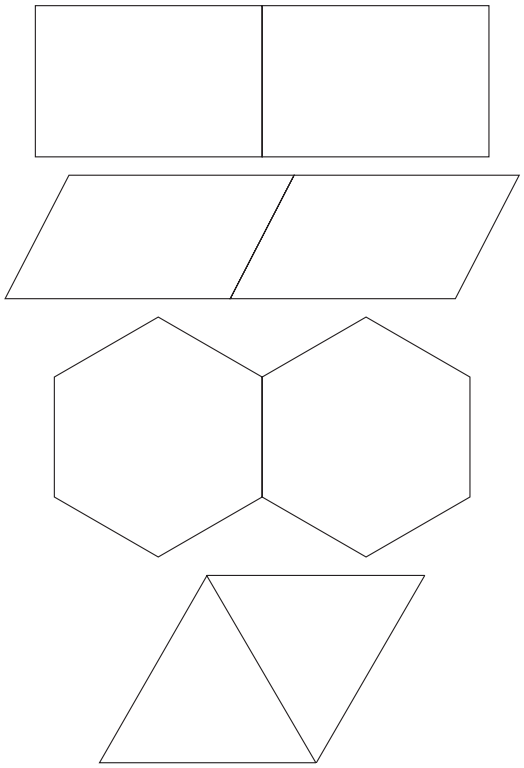
a) Trace the original polygon on which you based your tessellation.

b) Trace the tessellating shape.

c) Describe all symmetries, if any, of your tessellating shape, and all symmetries of your tessellation. Describe translation vectors and the locations of lines of reflection and centers of rotation for your tessellation.

Introduction to Lsometries Lesson **2**

Focus Student Activity 2.3 (page 7)



Focus Teacher Activity (cont.)

ACTIONS

21 (Optional) Give the groups each a sheet of butcher paper and a marking pen. Ask the groups to each spend several minutes brainstorming terminology related to symmetry and isometries. Then give each group a sheet of butcher paper and ask them to create a poster showing a glossary of symmetry terms, so that each term is explained using a carefully labeled diagram and as few words as possible.

COMMENTS

21 You might have the students work on this privately, perhaps as homework, prior to beginning work on a group or class poster. If students are keeping glossaries in their journals, they might use the information from the posters as a basis for editing or adding to their glossaries.

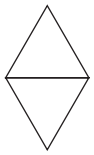
TEACHER NOTES:

Follow-up Student Activity 2.4

NAME _____ **DATE** _____

Complete all of your work for this assignment on separate paper. Include a statement of each problem next to your work about the problem.

1 A shape made from 2 equilateral triangles joined edge-to-edge is called a *diamond*. In general, a shape made of 2 or more equilateral triangles joined edge-to-edge with no gaps or overlaps is called a *polyamond*. Use triangular grid paper to determine and make a chart that shows all the different diamonds, triamonds, tetramonds, pentamonds, and hexamonds and their symmetry types.



Diamond

2 Draw the polyamond with the least number of equilateral triangles that has exactly 3 rotational symmetries but no reflective symmetry. Label its center and angles of rotation.

3 Completely fill a ½-sheet of 2-cm triangular grid paper with a tessellation of your polyamond from Problem 2. Describe the symmetries of your tessellation. Label centers of rotations, lines of reflection, and/or translation vectors.

4 Explain in your own words the meanings of the term *isometry*. Describe the important features of each isometry explored in class.

5 Sherrill made conjectures a)-g) below. Examine several examples for each conjecture and decide whether you agree or disagree. For each conjecture, state your conclusion, show the examples you examine, and give solid mathematical evidence to support your conclusion. If you disagree, give a counter-example and tell how you would change her conjecture so it is true.

a) *If I draw 2 intersecting lines and reflect a shape, first over one of the lines and then over the other, I think the end result is the same as the result of a single rotation. The center and the measure of the rotation have a special relationship to the angle of intersection of the lines.*

b) *An enlargement is a transformation, but it is not an isometry.*

c) *For any translation, I can always locate 2 parallel lines so that the result of the translation is the same as the result of a reflection first across one of the parallel lines and then across the other.*

(Continued on back.)

Follow-up Student Activity (cont.)

d) *Every rotation about a point can be replaced by a reflection across first one line and then another.*

e) *It is not possible that a shape and its reflection can overlap. The same is true for a shape and its rotation, translation, or glide reflection.*

f) *A line of reflection is the perpendicular bisector of the line segments that connect each point on the pre-image to its corresponding point on the image.*

g) *There is no isometry that maps a shape back onto itself.*

6 Use a protractor and ruler to complete a)-c).

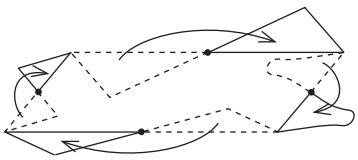
a) Draw a nonregular hexagon. Label one vertex of the hexagon H.
Draw a line M that does not intersect the hexagon.
Draw the reflection of the hexagon across line M.
Label the image of point H as H′.
Tell how you verify that your method is correct.

b) Draw a nonregular pentagon. Label one vertex P.
Mark a point Q that is not on the pentagon.
Draw a 45° rotation of the pentagon about point Q.
Label the image of point P as P′. Explain how you verify that point P rotated 45°.

c) Draw a scalene triangle. Label a point on the triangle C.
Label a point R that is not on the triangle. Draw a ray so that point R is the endpoint of the ray and so the ray does not intersect the triangle. Label a point S on your ray.
Sketch a translation of the triangle, using your ray with endpoint R as a translation vector with magnitude RS.
Label the image of point C as C′.

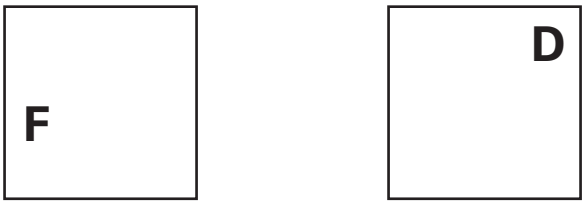
7 Create an *original* frieze pattern and describe its symmetries. Then adapt your design to create 6 new friezes that each illustrate a different possible symmetry type for friezes. Record the symmetries of each of your 7 friezes.

8 Study the following method of forming a tessellating figure that is based on a parallelogram. Investigate to see whether the method works for a trapezoid. If it works, complete such a tessellation of a trapezoid. If the method does not work, explain why not.

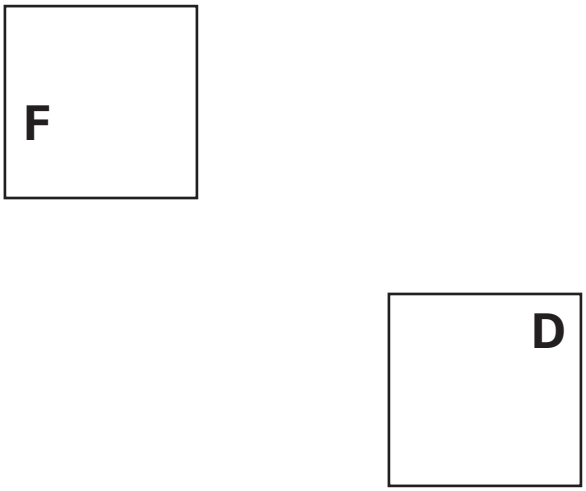


Focus Master A

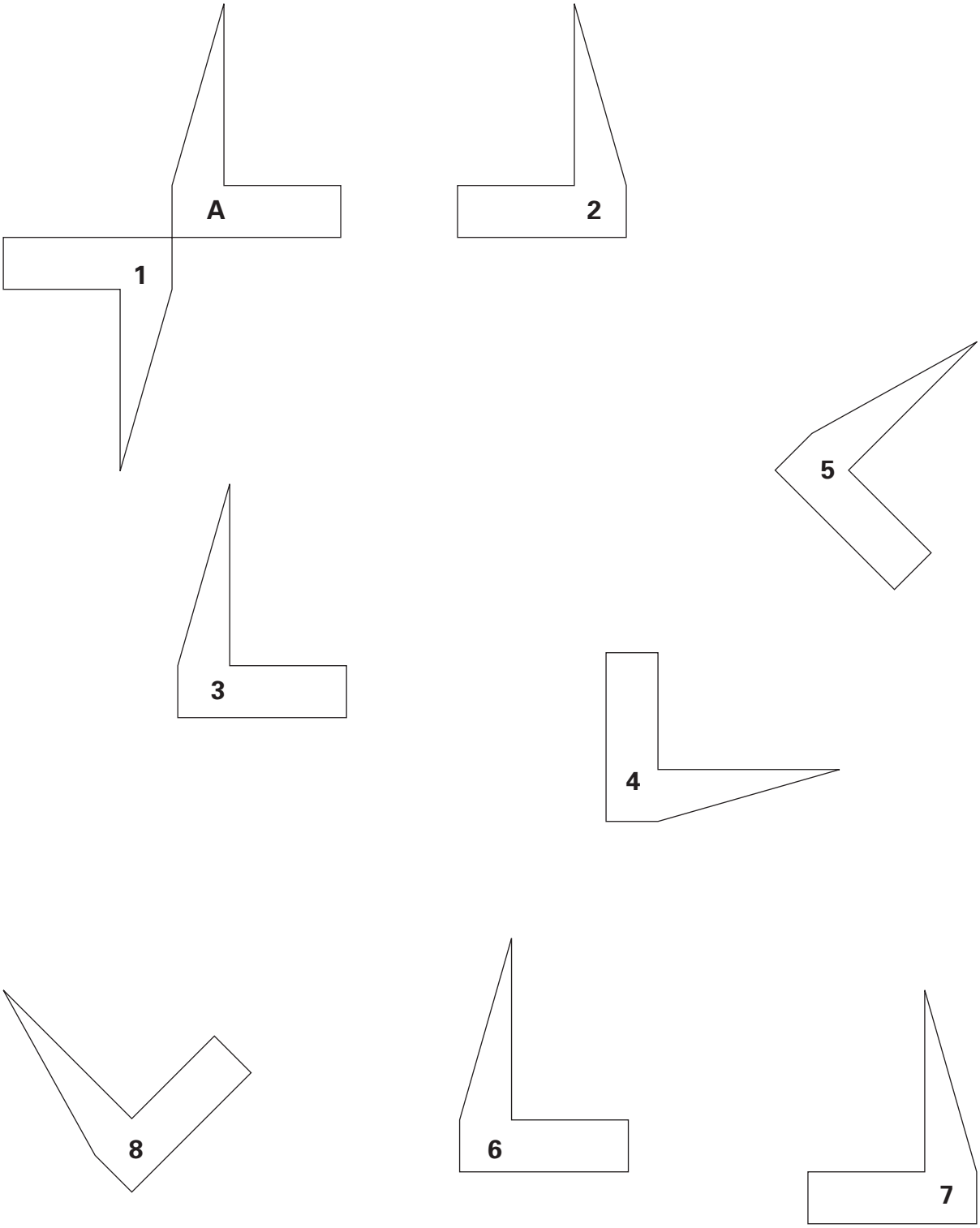
Investigate ways to use slides, flips, and/or turns to move Square F exactly onto Square D. Use words and/or mark diagrams to explain the movements that you use.



Focus Master B



Focus Master C



Focus Master D

Part I

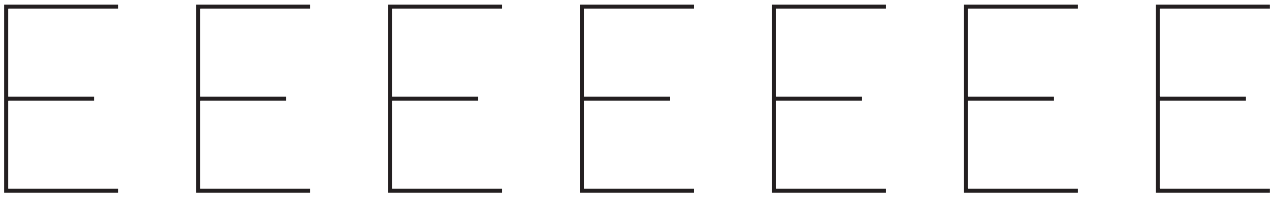
It is possible to move Shape A directly to several of the numbered positions using exactly one of these isometries only once: translation, reflection, or rotation. Find each position for which this is possible, and tell the single motion that moves Shape A to that position.

Part II

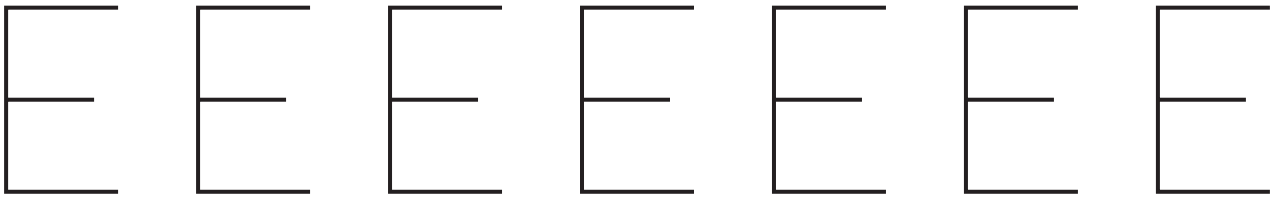
Describe ways to move Shape A from its starting position to each numbered position using a combination of exactly two reflections, rotations, and/or translations. Note: combinations of more than one type of motion are allowed as long as no more than two motions are used.

Focus Master E

Frieze A



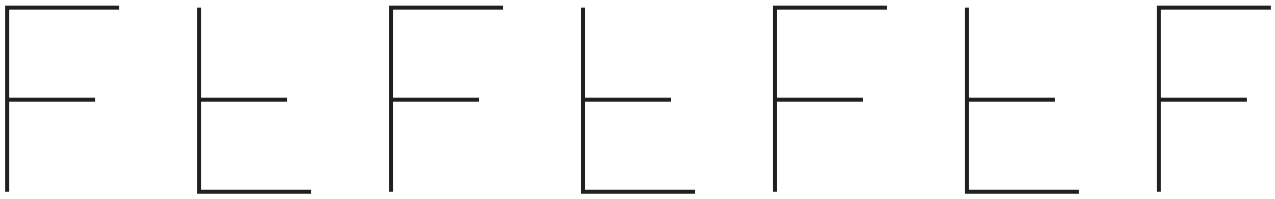
Frieze B



Focus Master F

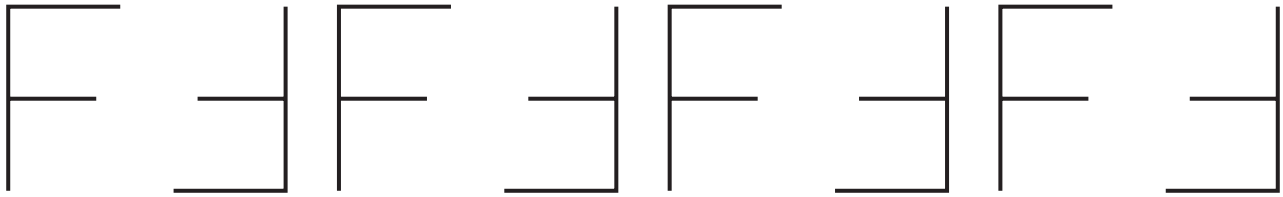
Frieze A

Frieze B

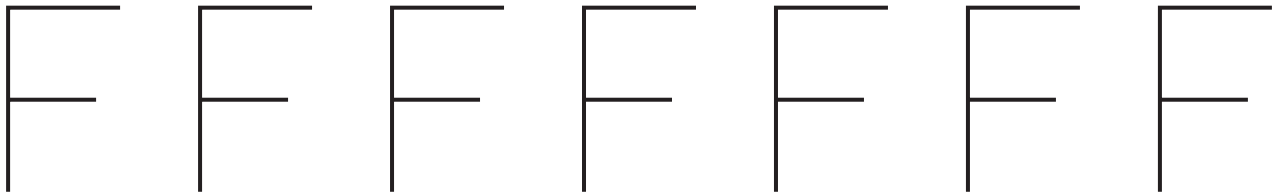


Focus Master G

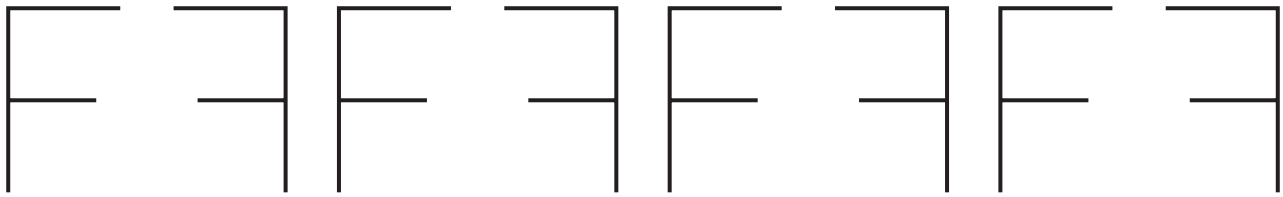
Frieze A



Frieze B

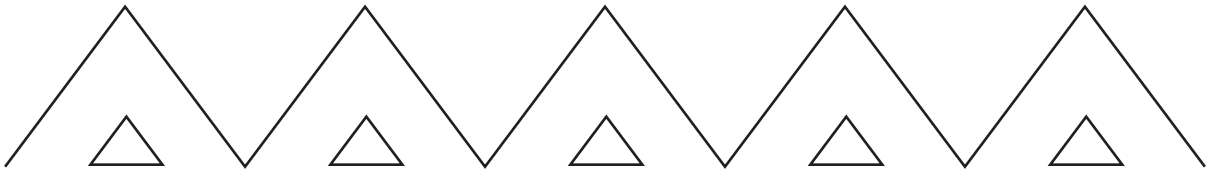


Frieze C

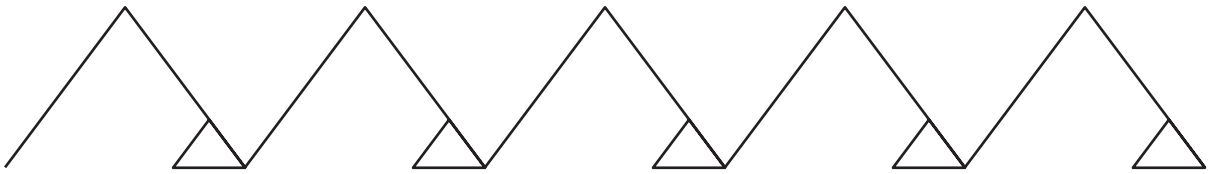


Focus Master H

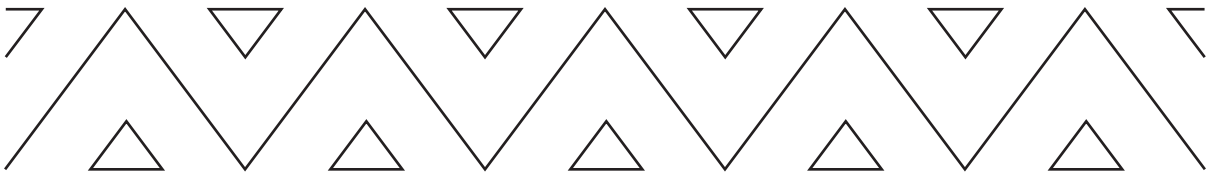
Frieze A



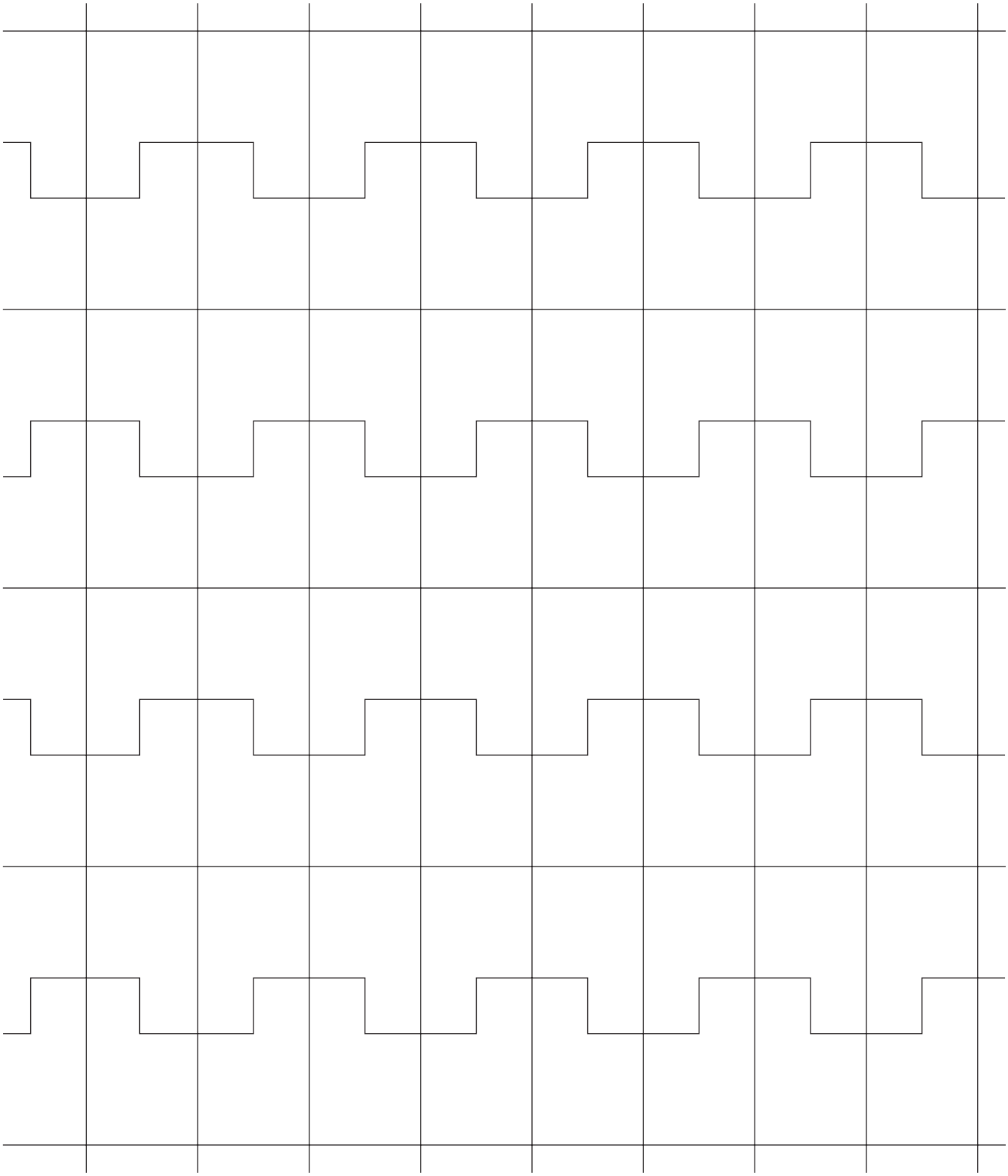
Frieze B



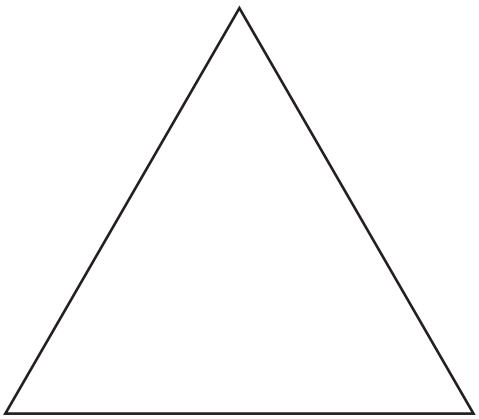
Frieze C

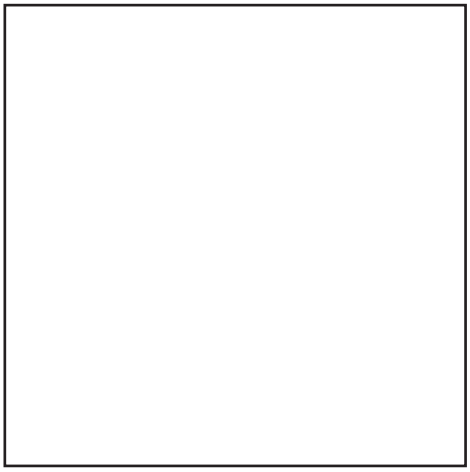


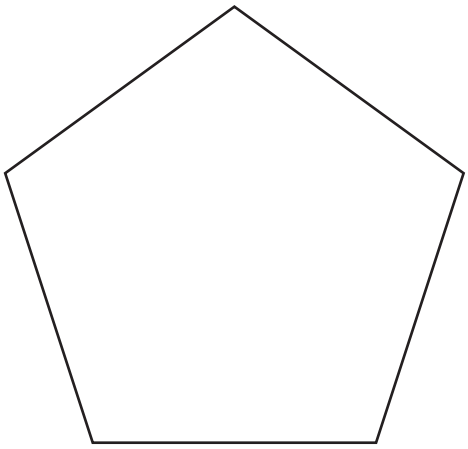
Focus Master I

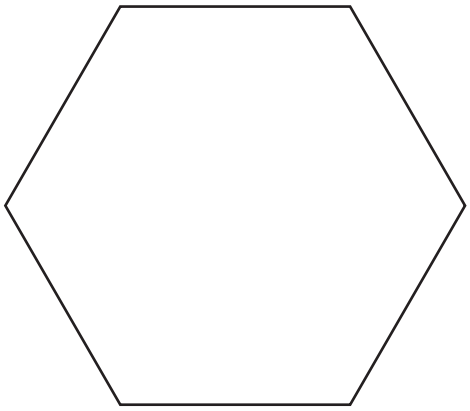


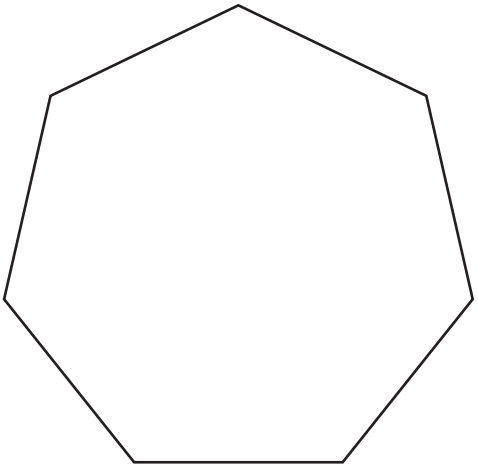
Focus Master J

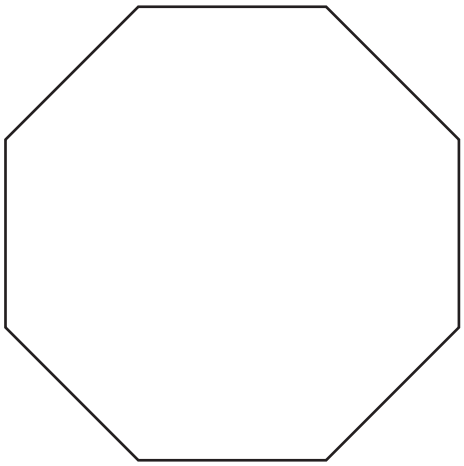










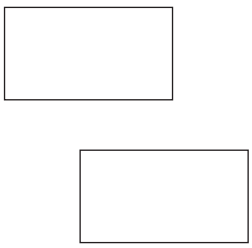


Focus Student Activity 2.1

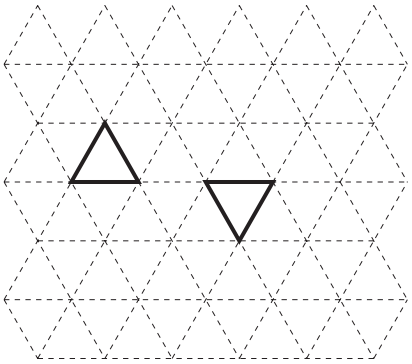
NAME _____ **DATE** _____

1 Shown below are several pairs of congruent shapes. Investigate ways to use one or more translations, reflections, rotations, or combinations of them, to move each first shape exactly onto the second. For each pair of shapes, write an explanation in words only of your "favorite" motion or combination of motions; explain in enough detail that a reader would be able to duplicate your motions without additional information.

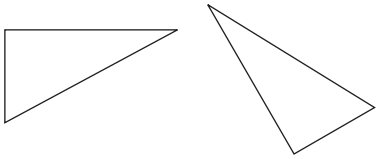
a)



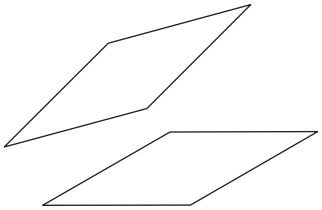
b)



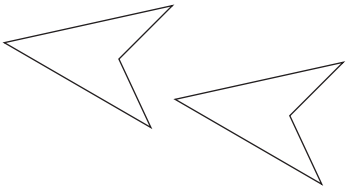
c)



d)



e)



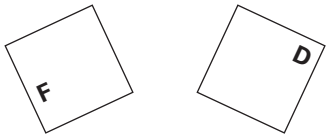
2 Challenge. Each motion or combination of motions that you determined for Problem 1 produces a *mapping* of the first shape (the *pre-image*) exactly onto the second (the *image*). How many different mappings are there for each of a)-e), if different means the sides of the pre-image and the sides of the image match in distinctly different ways.

3 Record your "I wonder…" statements, conjectures, or conclusions.

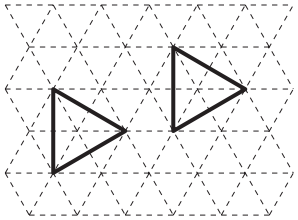
Focus Student Activity 2.2

NAME _____ **DATE** _____

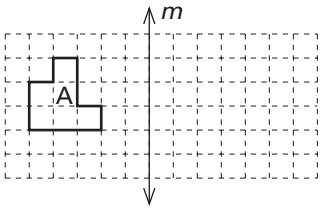
1 Shown at the right are 2 congruent squares. Determine ways to use exactly one isometry (translation, reflection, rotation, or glide reflection) to move Square F exactly onto Square D.



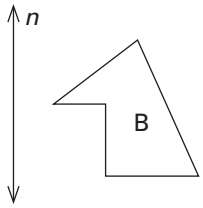
2 Repeat Problem 1 for the 2 equilateral triangles shown here:



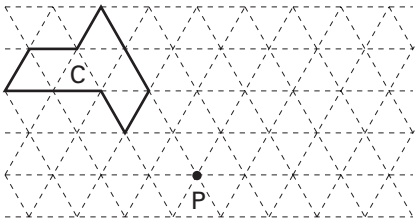
3 Sketch the reflected image of Shape A across line *m*. Next to your sketch write several mathematical observations about relationships you notice. Then explain how you verified that the image is a reflection of Shape A across line *m*.



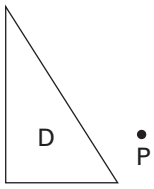
4 Challenge. Develop a method of accurately reflecting Shape B across line *n*. Show and describe your method of locating the reflected image of Shape B and tell how you verified that your method was correct. Can you generalize?



5 Sketch the image of Shape C after a 120° clockwise rotation about point P. Next to your sketch write several mathematical observations about relationships that you notice. Then explain how you verified that the image is a 120° rotation about point P.



6 Challenge. Invent a method of rotating Shape D 170° clockwise about point P, without using a grid. Show and describe your method of locating the rotated image of Shape D and tell how you verified that your method was correct.



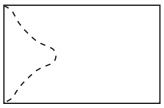
Focus Student Activity 2.3

NAME _____ **DATE** _____

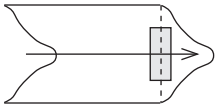
The following procedures create tessellations similar to the type created by the Dutch artist, Maurice C. Escher, who is famous for his tessellations of birds and animals. Escher's first inspirations came from the Alhambra, which was built in the 13th century in Granada, Spain, and is famous for its variety of tessellations. Procedure A below gives a method of creating a tessellation based on rectangles.

Procedure A—a translation tessellation based on rectangles.

Step 1: Beginning at one corner and ending at an adjacent corner, cut out a portion of the rectangle.

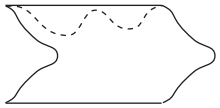


Step 2: Slide the cutout portion across the rectangle to the opposite side, matching the straight edges and the corners, as shown at the right. Then tape the pieces in place.

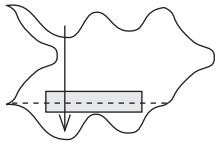


Hint: be careful not to flip the pieces over and don't let the tape extend beyond the edge of the figure.

Step 3: Beginning at an endpoint of one of the remaining two unaltered sides and ending at the other endpoint, cut another portion from the rectangle. An example is shown here.

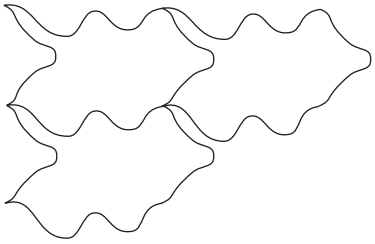


Step 4: Slide this cutout portion to the opposite side, matching edges and endpoints, as illustrated at the right. Tape the pieces in place.



Hint: It is important that nothing be trimmed or altered to fit!

Step 5: Tile a page with this new shape by repeatedly tracing the shape so the tracings fit together, with no gaps or overlaps (a portion of a tiled page is shown at the right). Describe the symmetries of the tessellation.



Hint: Remember not to flip the piece over when you tile with it.

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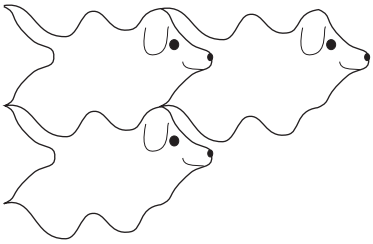
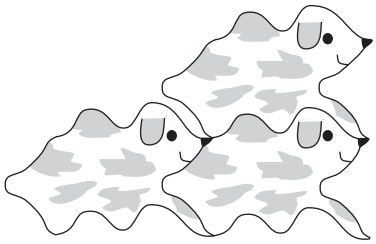
Focus Student Activity 2.3 (page 2)

Step 6. Add color details to the figure to produce an interesting and creative work of art.



It could be puppy dogs on the run…



…or, puppy dogs at rest!

Project A

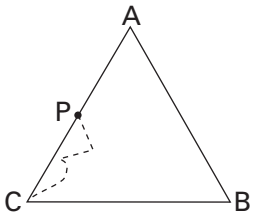
Part I Using a rectangle from page 7 of this activity and Procedure A, create an *original* tessellating shape. On another sheet of paper, trace enough copies of your figure to show that it forms a translation tessellation.

Part II Challenge. Using a shape that is not a rectangle, adapt Procedure A and create another figure that can form a translation tessellation. On another sheet of paper, trace enough copies of the figure to verify it forms a translation tessellation.

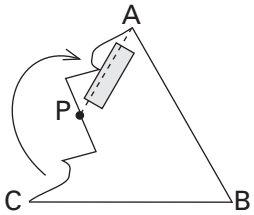
(Continued.)

Focus Student Activity 2.3 (page 3)

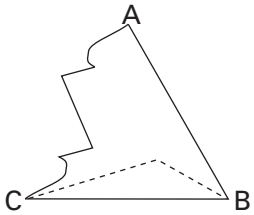
Procedure B—a rotation tessellation that is based on equilateral triangles.



Step 1. Mark the midpoint, P, of side AC on equilateral triangle ABC, as shown at the right. Beginning at vertex C and ending at P, cut a portion from the triangle.

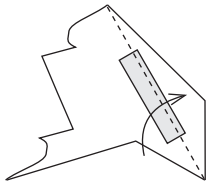


Step 2. Place a finger on P and rotate the cutout portion 180° about P. Then tape the rotated portion in place.



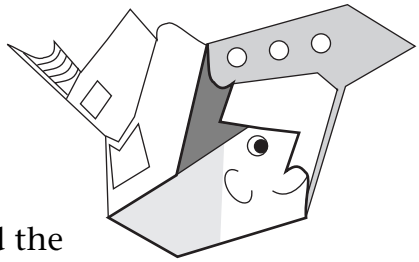
Step 3. Beginning at vertex C and ending at vertex B, cut out a portion of the triangle.

Step 4. Place a finger on vertex B and rotate the new cutout portion clockwise 60°. Tape the rotated portion in place.



Step 5. Tessellate the page by repeatedly tracing and fitting together (with no gaps or overlaps) the shape. Add detail and color to produce a creative piece of art that fills the page.

Hint: To tessellate with this shape it is necessary to rotate the figure.



Project B

Part I Use an equilateral triangle from page 7 of this activity and the method of Procedure B, or another procedure that you invent, to create an *original* shape that can be copied to form a rotation tessellation. On another sheet of paper, trace enough copies of your shape to verify that it forms a rotation tessellation.

Part II Challenge. Invent a way to alter a regular hexagon from page 7 to create a shape that forms a rotation tessellation of the plane. Verify. Show a tracing of several copies to illustrate the beginning of a tessellation.

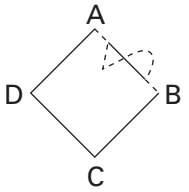
(Continued on back.)

Focus Student Activity 2.3 (page 4)

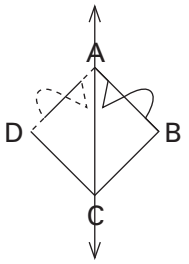
Procedure C—a reflection tessellation that is based on rhombuses.

Before you read on, trace a rhombus from page 7 and explore your ideas regarding ways to create a reflection tessellation based on nonsquare rhombuses. Then compare your ideas to the following procedure.

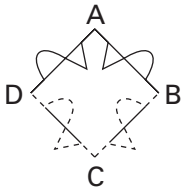
Step 1. Lightly trace the rhombus from page 7 on a blank sheet and label its vertices A, B, C, and D. Draw a curve from A to B (notice the curve can extend outside the rhombus). Note: in the remainder of this activity, the notation c (A-B) means *the curve from point A to point B*.



Step 2. Reflect c (A-B) about line AC. Hint: use tracing paper to increase accuracy of copied curves.

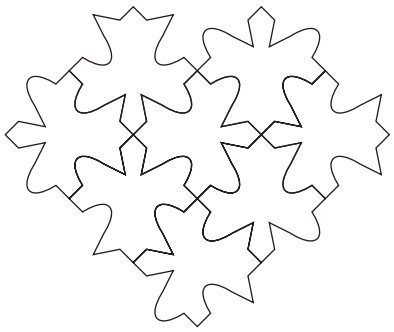


Step 3. Rotate c (A-D) 90° about point D to form c (D-C) so that A maps to C.



Step 4. Reflect c (D-C) about line AC so that D maps to B.

Step 5. Erase the lines of the original rhombus that are not part of the curve, and tessellate! Notice the lines of reflection in this tessellation.



Project C

Use a rhombus from page 7 of this activity and the method of Procedure C, or another method that you invent, to create an original shape that can be used to form a reflection tessellation. On another sheet of paper, trace enough copies of your shape to verify that it forms a reflection tessellation.

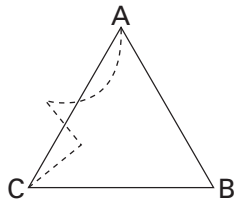
(Continued.)

Focus Student Activity 2.3 (page 5)

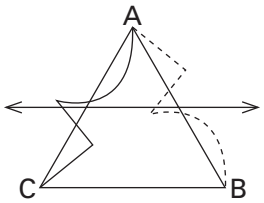
Procedure D—a glide reflection tessellation that is based on equilateral triangles.

Before you read on, trace an equilateral triangle from page 7 and explore your ideas regarding ways to create a glide reflection tessellation based on equilateral triangles. Then compare your ideas to the following procedure.

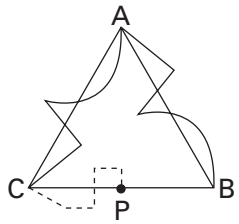
Step 1. Lightly trace the equilateral triangle from page 7 on a blank sheet and label its vertices A, B, and C. Draw a curve from A to C (as in Procedure C, the curve can extend outside the triangle).



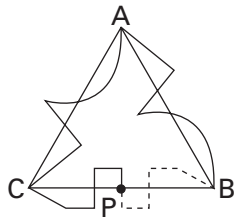
Step 2. Reflect c (A-C) about the line parallel to the base BC and passing through the midpoint of AC. Then translate this reflected curve to connect from A to B.



Step 3. Locate point P, the midpoint of side BC. Draw c (C-P).



Step 4. Rotate c (C-P) 180° about point P to form c (P-B).



Step 5. Tessellate!

Project D

Use Procedure D, or another method that you invented, to create an original shape that can be used to form a glide reflection tessellation. Verify by showing a tracing of several copies of your shape.

(Continued on back.)

Focus Student Activity 2.3 (page 6)

Project E

Part I

Pick your favorite tessellating shape from those you created for Projects A-D, or create a new tessellating shape based on Procedures A-D, or combinations of them, and:

a) *completely* tessellate a sheet of paper with the shape,

b) be creative in ways that you *color* and fill in the *details* of this tessellating figure,

c) mount the completed tessellation on a sheet of colored paper, or frame it creatively.

Note: to make a poster-sized tessellation you could enlarge the polygon that is the basis for your tessellation (e.g., double each edge of the hexagon pattern from page 7), create a new tessellating shape based on that enlarged polygon, and then tessellate a sheet of poster paper.

Part II

On the back of your tessellation (after you have mounted it):

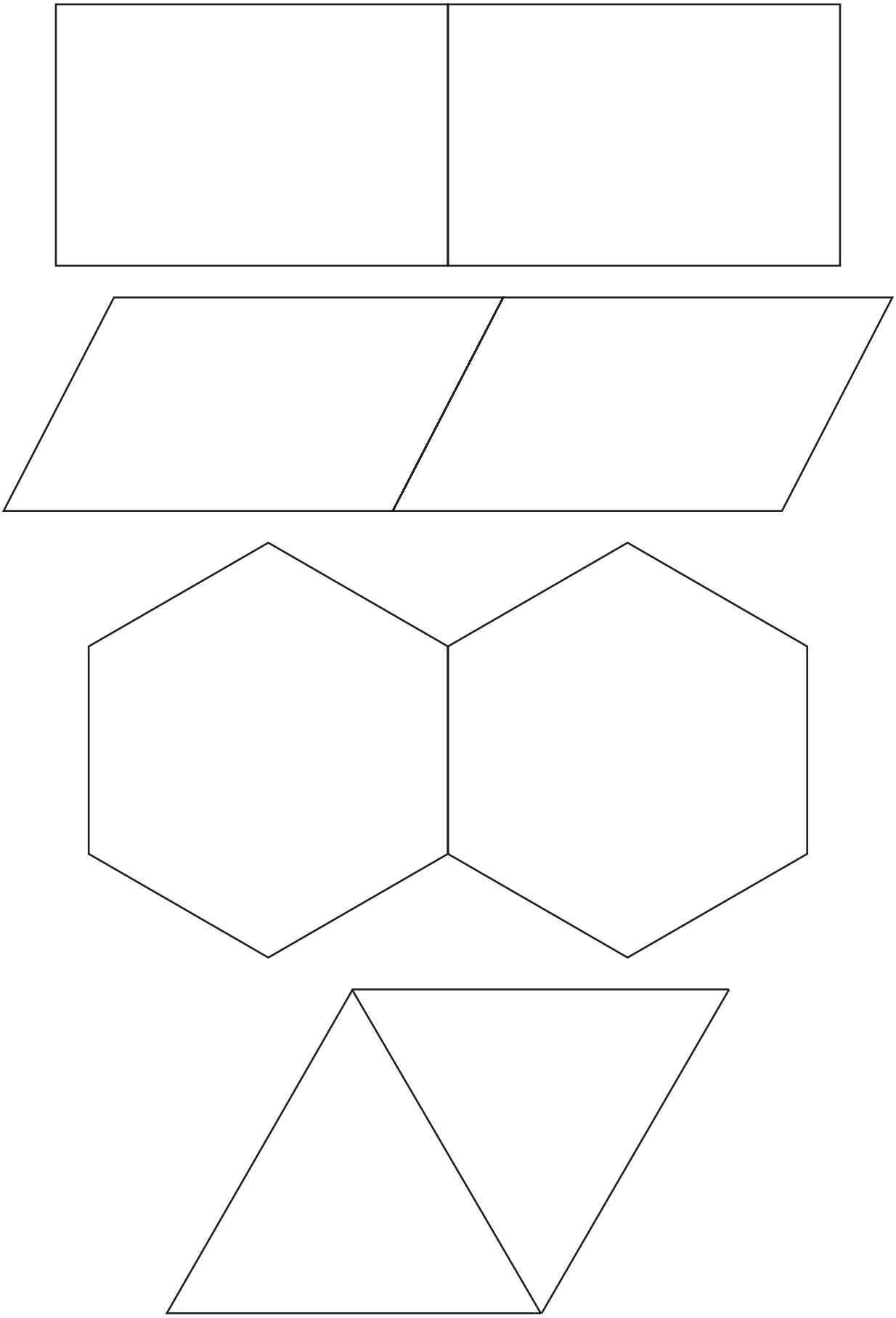
a) Trace the original polygon on which you based your tessellation.

b) Trace the tessellating shape.

c) Describe all symmetries, if any, of your tessellating shape, and all symmetries of your tessellation. Describe translation vectors and the locations of lines of reflection and centers of rotation for your tessellation.

(Continued.)

Focus Student Activity 2.3 (page 7)



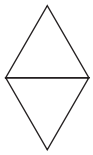
Focus Student Activity 2.3 (page 8)

Follow-up Student Activity 2.4

NAME _____ **DATE** _____

Complete all of your work for this assignment on separate paper. Include a statement of each problem next to your work about the problem.

1 A shape made from 2 equilateral triangles joined edge-to-edge is called a *diamond*. In general, a shape made of 2 or more equilateral triangles joined edge-to-edge with no gaps or overlaps is called a *polyamond*. Use triangular grid paper to determine and make a chart that shows all the different diamonds, triamonds, tetramonds, pentamonds, and hexamonds and their symmetry types.



Diamond

2 Draw the polyamond with the least number of equilateral triangles that has exactly 3 rotational symmetries but no reflective symmetry. Label its center and angles of rotation.

3 Completely fill a ½-sheet of 2-cm triangular grid paper with a tessellation of your polyamond from Problem 2. Describe the symmetries of your tessellation. Label centers of rotations, lines of reflection, and/or translation vectors.

4 Explain in your own words the meanings of the term *isometry*. Describe the important features of each isometry explored in class.

5 Sherrill made conjectures a)-g) below. Examine several examples for each conjecture and decide whether you agree or disagree. For each conjecture, state your conclusion, show the examples you examine, and give solid mathematical evidence to support your conclusion. If you disagree, give a counter-example and tell how you would change her conjecture so it is true.

a) *If I draw 2 intersecting lines and reflect a shape, first over one of the lines and then over the other, I think the end result is the same as the result of a single rotation. The center and the measure of the rotation have a special relationship to the angle of intersection of the lines.*

b) *An enlargement is a transformation, but it is not an isometry.*

c) *For any translation, I can always locate 2 parallel lines so that the result of the translation is the same as the result of a reflection first across one of the parallel lines and then across the other.*

(Continued on back.)

Follow-up Student Activity (cont.)

d) *Every rotation about a point can be replaced by a reflection across first one line and then another.*

e) *It is not possible that a shape and its reflection can overlap. The same is true for a shape and its rotation, translation, or glide reflection.*

f) *A line of reflection is the perpendicular bisector of the line segments that connect each point on the pre-image to its corresponding point on the image.*

g) *There is no isometry that maps a shape back onto itself.*

6 Use a protractor and ruler to complete a)-c).

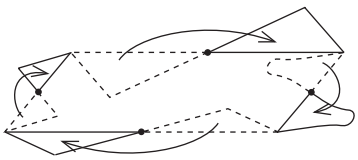
a) Draw a nonregular hexagon. Label one vertex of the hexagon H.
Draw a line M that does not intersect the hexagon.
Draw the reflection of the hexagon across line M.
Label the image of point H as H′.
Tell how you verify that your method is correct.

b) Draw a nonregular pentagon. Label one vertex P.
Mark a point Q that is not on the pentagon.
Draw a 45° rotation of the pentagon about point Q.
Label the image of point P as P′. Explain how you verify that point P rotated 45°.

c) Draw a scalene triangle. Label a point on the triangle C.
Label a point R that is not on the triangle. Draw a ray so that point R is the endpoint of the ray and so the ray does not intersect the triangle. Label a point S on your ray.
Sketch a translation of the triangle, using your ray with endpoint R as a translation vector with magnitude RS.
Label the image of point C as C′.

7 Create an *original* frieze pattern and describe its symmetries. Then adapt your design to create 6 new friezes that each illustrate a different possible symmetry type for friezes. Record the symmetries of each of your 7 friezes.

8 Study the following method of forming a tessellating figure that is based on a parallelogram. Investigate to see whether the method works for a trapezoid. If it works, complete such a tessellation of a trapezoid. If the method does not work, explain why not.



Measurement–Inventing Formulas

Lesson **3**

THE BIG IDEA

Experiences such as cutting, moving, and reshaping parallelograms, trapezoids, triangles, and circles enable one to "see" relationships, to invent area and perimeter formulas, and, hence, to retain, recall, adapt, and apply those formulas. Such experiences also enable one to make sense of formulas invented by others.

CONNECTOR

OVERVIEW

Students recall the meanings of several terms related to circles, polygons, and measurements of plane figures.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 copy per group and 1 transparency.
- ✔ Connector Master B, 1 copy per student and 1 transparency.
- ✔ Scissors, 1 pair per student.

FOCUS

OVERVIEW

Students invent area and perimeter formulas for parallelograms, trapezoids, triangles, and circles. They also invent surface area and volume formulas for prisms and cylinders, and solve problems involving area, perimeter, surface area, and volume.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 4 copies for each pair of students and 1 transparency.
- ✔ Focus Master C (4 pages) copied on cardstock, 1 copy per student.
- ✔ Focus Master D, 1 copy per pair of students and 1 transparency.
- ✔ Focus Student Activities 3.1-3.4, 1 copy of each per student and 1 transparency of each.
- ✔ Geoboard and rubber bands (or geoboard recording paper, see *Blackline Masters*), 1 per pair of students.
- ✔ Butcher paper, marking pens, tape, and scissors for each pair of students.
- ✔ String, 1 piece (approx. 75 cm) per pair of students.
- ✔ Metric ruler or meter stick, 1 per pair of students.
- ✔ Round or cylindrical objects (e.g., soup cans, potato chip containers, coffee cups, plates, etc.), 1 set of 5 objects per group of 4 students.
- ✔ Calculators, 1 per student.
- ✔ 1-cm grid paper, 2 sheets per pair of students.
- ✔ Grid paper of various sizes (¼", 1-cm, etc.; optional), available as needed (see Comment 12).
- ✔ Protractor (see *Blackline Masters*), 1 per student.
- ✔ Cubes, 30 per student.

FOLLOW-UP

OVERVIEW

Students solve problems involving area, perimeter, arc length, surface area and volume. They investigate and form conjectures regarding the types of figures which have minimum perimeter for given areas, and the relationship between areas and volumes of figures whose linear dimensions are doubled, tripled, etc.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 3.5, 1 copy per student.

LESSON IDEAS

QUOTE

Students who can use the relationship between the shape of the "parallelogram" and its area and the circumference of the circle to develop the formula for the area of the circle are demonstrating plausible and deductive reasoning. The argument is plausible if it makes common sense and is mathematically correct.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

1. a) 1490.6 m^2 (If the 2 outer ends of the track are combined, they form a circular track with outer radius of 26 m and inner radius 20 m. The remaining inner sections form two 6 m $\times$ 52 m rectangles.)

   b) 37.7 m

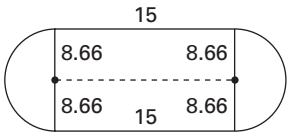
2. a) The circumference of the circle to 3 decimal places is 35.449, the perimeter of the square is 40. Selecting lengths and widths of a rectangle whose product is 100, produces a perimeter greater than 40. For example, if the dimensions are 9 and $11\frac{1}{9}$, the perimeter is $40\frac{2}{9}$. So the circle, square, and rectangle in this order have the least to greatest perimeters.

   b) The circle has the greatest area, 796.2 cm^2 in this example; the square has the next greatest area, 625 cm^2, and the rectangle has the least area.

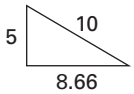
3. a) Each new perimeter is twice the preceding perimeter and each new area is 4 times the preceding area. In general, if the dimensions are multiplied by n, the new perimeter is n times the preceding perimeter and each new area is n^2 times the preceding area.

   b) In general, if the dimensions of a 3-D figure are multiplied by n, the new surface area is n^2 times the surface area of the original figure, and the volume is n^3 times the volume of the original figure.

4. area: 495.3 ft^2
   perimeter: 84.4 ft



   With a 10 foot chain, the dog can move approximately 8.66 feet away from the center line (dotted line) under the clothesline.



5. 80 cents

7. The height of the string above the earth is 1.3 feet to the nearest tenth of a foot. Increasing the circumference by 8 feet will increase the diameter by x feet, where $C + 8 = \pi(d + x)$ and $x = 8/\pi$. Thus, the diameter of the new circle is increased by $8/\pi$ and so the radius is increased by $\frac{1}{2} \times 8/\pi \approx 1.3$ feet. This height is large enough so that many people could crawl under it.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students recall the meanings of several terms related to circles, polygons, and measurements of plane figures.

MATERIALS

- ✔ Connector Master A, 1 copy per group and 1 transparency.
- ✔ Connector Master B, 1 copy per student and 1 transparency.
- ✔ Scissors, 1 pair per student.

ACTIONS

1 Arrange the students in groups and ask each group to write an explanation of the meanings of the terms *area* and *perimeter*, and to give an example that illustrates each explanation. Discuss the students' explanations and their ideas about relationships between area and linear units.

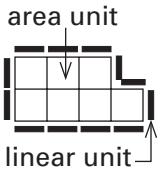
2 Give each group a copy of Connector Master A (see next page) and a pair of scissors. Ask the groups to solve the problem posed on the master, and then invite volunteers to demonstrate their group's methods.

Next, write the following terms on the overhead and ask the groups to identify which, if any, of pieces 1-8, are illustrations of the terms. Discuss, as needed.

> **triangle, equilateral triangle, isosceles triangle, scalene triangle, nonequilateral triangle, right triangle, parallelogram, rectangle, rhombus, square, isosceles trapezoid, nonisosceles trapezoid, polygon, quadrilateral, pentagon, regular pentagon, hexagon, regular hexagon.**

COMMENTS

1 In general, *perimeter* is the distance, in linear units, around a figure. For example, assuming 1 linear unit is the length of the edge of 1 small square in the figure shown at the right, then this figure has perimeter 12 linear units.

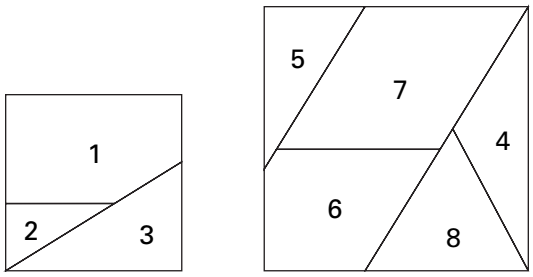


The *area* of a figure is the number of area units required to exactly cover the figure, with no gaps or overlaps. Squares are most commonly used as area units. For example, the area of the figure above is 7 square units, assuming 1 of the small squares is the area unit.

If  is the area unit, then the area of the above figure is 1¾ square units. If  is the area unit, then the area of the above figure is 14 area units.

Although not required, it is common practice to choose the linear unit as the length of the edge of a square area unit. Without this convention, the area model for multiplication does not hold.

2 The pieces of the rectangle and hexagon can be reassembled to form the following squares:



Since the rectangle has area 4 square units, so does the square formed by rearranging the pieces of the rectangle.

(Continued next page.)

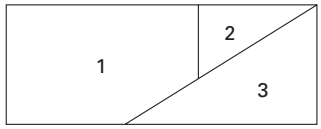
Connector Teacher Activity (cont.)

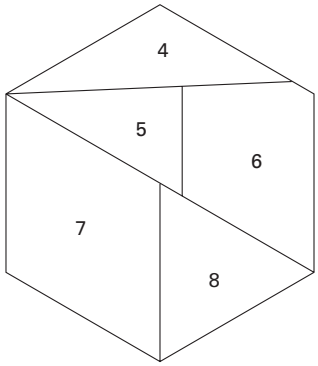
ACTIONS

Measurement—Inventing Formulas Lesson **3**

Connector Master A

The rectangle below has area 4 square units. When cut along the given lines, the rectangle can be reassembled to form a square. Similarly, the hexagon can be cut along the given lines and reassembled to form another square. Use this information to help you find a close approximation of the area of the hexagon. Then write a brief explanation of your methods. Note: use the same area and linear units for the rectangle and the hexagon.





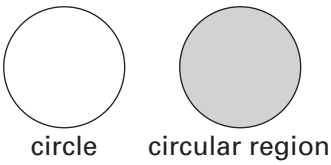
3 Distribute 1 copy of Connector Master B to each student. Ask the students to cut out the circle and to locate and label the *center*, a *diameter*, a *radius*, the *circumference*, a *central angle*, a *sector*, and an *arc* of the circle. Discuss, clarifying as needed. Introduce students to the meaning of the term *chord*.

COMMENTS

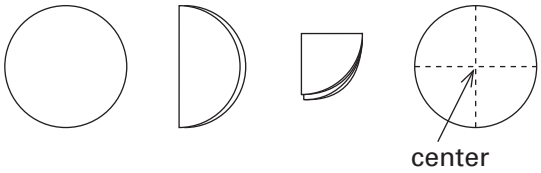
2 (continued.)
Hence, the square formed by pieces 1, 2, and 3 has side length 2 linear units. This can be used to show that a close approximation of the length of the side of the square made from the hexagon is 3 linear units. Thus, a close approximation of the area of the hexagon is 9 square units. The solution of this problem is based on understanding conservation of area—i.e., area is preserved when a region is dissected and rearranged.

Having students identify the names of the pieces provides an opportunity to review the meanings of some terminology (e.g., see *Math Alive! Course II*, Lessons 11-14). Rather than giving definitions, you might allow time for students to discuss and clarify their ideas and questions among themselves. If a group needs clarification, you might suggest they ask for input from the class before you enter into the discussion. A dictionary can also be helpful. The following terms are illustrated by the cut apart pieces: triangle, equilateral triangle, isosceles triangle, scalene triangle, nonequilateral triangle, right triangle, parallelogram, rhombus, nonisosceles trapezoid, polygon, quadrilateral, pentagon, and hexagon.

3 Note that, technically, students cut out a *circular region* (also called a disc) and its boundary is a circle. As is common practice, however, throughout this lesson we will refer to the region as a circle. (Note: A circle is defined in Comment 5 of the Focus.)



The center of the circle can be located by folding the circle into quarters, as illustrated below:

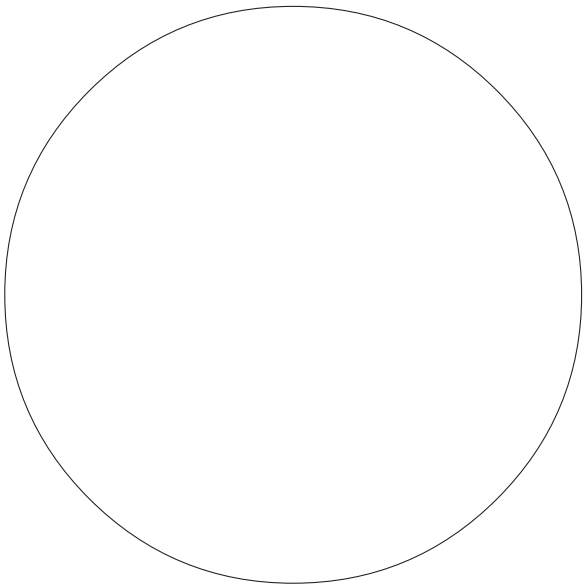


Connector Teacher Activity (cont.)

ACTIONS

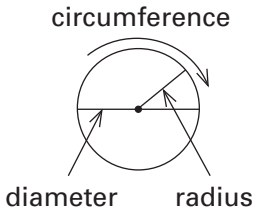
Lesson **3** Measurement—Inventing Formulas

Connector Master B

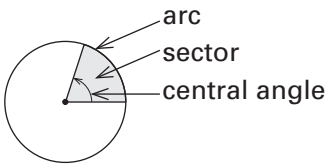


COMMENTS

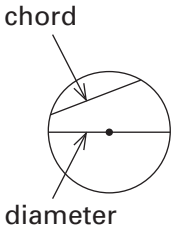
A *radius* of a circle is a segment that extends from the center to the edge of the circle. The length of this segment is also called the radius. A *diameter* of a circle is a segment that extends from the edge of the circle through the center to the opposite edge of the circle. The length of this segment is also called the diameter and it is twice the radius. The edge of the circle is called the *circumference*, and its length is also called the circumference.



A *central angle* of a circle (i.e., an angle whose vertex is the center of the circle and whose sides intersect the circle) determines a region that is called a *sector*, as illustrated below. The section of the circumference determined by the central angle is called an *arc*.



Any line segment whose endpoints are on the circumference of a circle is called a *chord* of the circle. Hence, a diameter is also a chord. Not all chords are diameters since it is not necessarily the case that a chord will pass through the center of the circle.



TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students invent area and perimeter formulas for parallelograms, trapezoids, triangles, and circles. They also invent surface area and volume formulas for prisms and cylinders, and solve problems involving area, perimeter, surface area, and volume.

MATERIALS

- ✔ Focus Master A, 1 transparency.
- ✔ Focus Master B, 4 copies for each pair of students and 1 transparency.
- ✔ Focus Master C (4 pages) copied on cardstock, 1 copy per student.
- ✔ Focus Master D, 1 copy per pair of students and 1 transparency.
- ✔ Focus Student Activities 3.1-3.4, 1 copy of each per student and 1 transparency of each.
- ✔ Geoboard and rubber bands (or geoboard recording paper, see *Blackline Masters*), 1 per pair of students.
- ✔ Butcher paper, marking pens, tape, and scissors for each pair of students.
- ✔ String, 1 piece (approx. 75 cm) per pair of students.
- ✔ Metric ruler or meter stick, 1 per pair of students.
- ✔ Round or cylindrical objects (e.g., soup cans, potato chip containers, coffee cups, plates, etc.), 1 set of 5 objects per group of 4 students.
- ✔ Calculators, 1 per student.
- ✔ 1-cm grid paper, 2 sheets per pair of students.
- ✔ Grid paper of various sizes (¼", 1-cm, etc.; optional), available as needed (see Comment 12).
- ✔ Protractor (see *Blackline Masters*), 1 per student.
- ✔ Cubes, 30 per student.

ACTIONS

1 Arrange the students in pairs and place a transparency of Focus Master A (see next page) on the overhead. Give each pair a sheet of butcher paper, marking pens, 2 sheets of 1-cm grid paper, a geoboard and rubber bands (or geoboard recording paper), tape, and scissors. Ask the pairs to do the following for each polygon on Focus Master A:

a) invent a formula for the area of each polygon shown, using only the given variables,

b) create a visual proof to show why and how each formula works,

c) give an example to illustrate each formula.

When completed, have the pairs share and compare their formulas and proofs.

COMMENTS

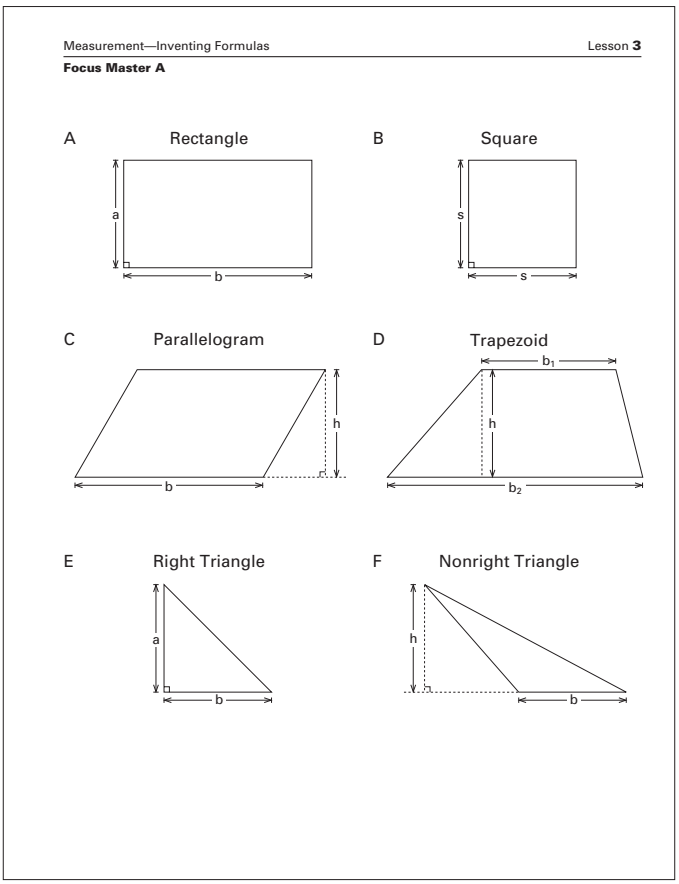
1 If geoboards are not available, you could distribute sheets of geoboard paper (see *Blackline Masters*). If students ask for clarification, the trapezoid on Focus Master A is not isosceles. It may be helpful to discuss the meaning of a variable and the fact that the drawings on Focus Master A represent "generic" polygons of each type.

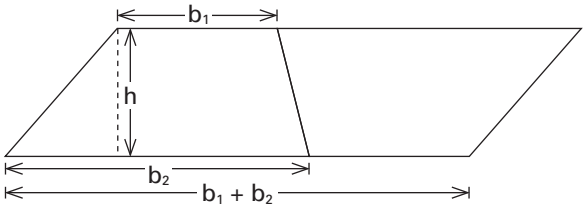
You may need to discuss properties of an *altitude* (e.g., see segments labeled *h* on Focus Master A) of a parallelogram, trapezoid, or triangle. The length of an altitude (also called the *height*) of a parallelogram or trapezoid is the perpendicular distance between opposite parallel sides. Notice in figure C on Focus Master A, the altitude is drawn outside the parallelogram. It could also be shown in other positions, such as shown in the diagram:

(Continued next page.)

Focus Teacher Activity (cont.)

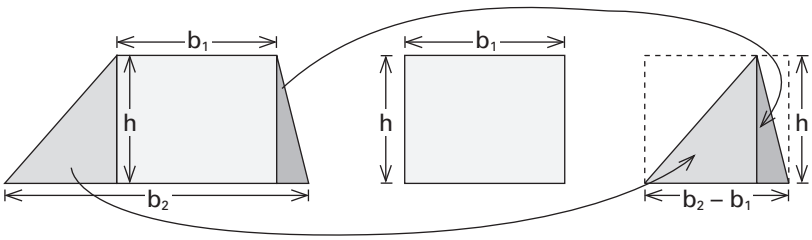
ACTIONS





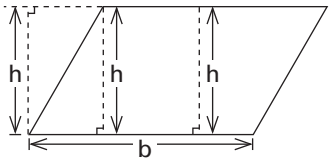
Total Area = $(b_1 + b_2)h$

Area of Trapezoid = $\frac{(b_1 + b_2)h}{2}$



COMMENTS

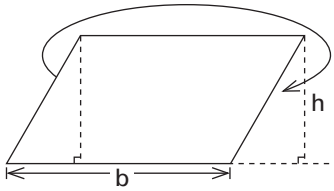
1 (continued.)



An *altitude* of a triangle is a perpendicular segment that connects a *vertex* to the opposite side (also called the *base*) of the triangle. The length of this segment is also called the altitude. Sometimes the altitude lies outside the triangle and connects to an extension of the base, as illustrated in figure F on Focus Master A.

A), B) Based on the area model for multiplication, if a rectangle has sides of length a and b, then its area is the product of $a \times b$. Students may use examples on grid paper to illustrate this. Since a square is a rectangle whose sides are equal in length, the area of a square with side length s is $s \times s = s^2$.

C) A parallelogram with base b and height h can be dissected and rearranged to form a rectangle with the same base and height, as illustrated below (other dissections are also possible). Hence, the area of a parallelogram is $b \times h$.



D) Students may invent a variety of formulas and proofs for the area of a trapezoid. Following are four examples students have given:

Method 1. *We combined 2 copies of the trapezoid to form a parallelogram, found the area of the parallelogram, and then divided by 2* (see diagram at the left).

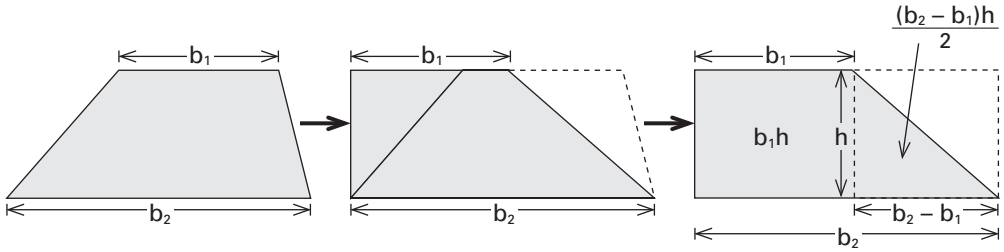
Method 2. *We dissected the trapezoid into 2 triangles and a rectangle. Then we rearranged the pieces of the trapezoid to form a rectangle and one triangle. The rectangle has area b_1h. The triangle can be enclosed in a rectangle with area $(b_2 - b_1)h$, and we can see that the triangle covers half of the rectangle, so the area of the triangle is $\frac{1}{2}(b_2 - b_1)h$. Therefore, the total area of the trapezoid is $b_1h + \frac{1}{2}(b_2 - b_1)h$.*

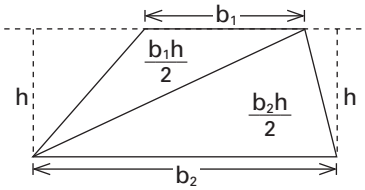
Focus Teacher Activity (cont.)

ACTIONS

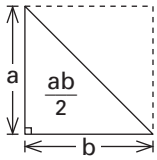
COMMENTS

Method 3. *We tested several trapezoids on the geoboard and conjecture that we can "slide" the top of a trapezoid to form a new trapezoid with the same area and one side that is perpendicular to the bases. We subdivided that new trapezoid into a rectangle with area b_1h, and a right triangle that is half of a rectangle with dimensions h by $(b_2 - b_1)$. Therefore, the trapezoid has area $b_1h + \frac{1}{2}(b_2 - b_1)h$. We aren't sure if this always works, but we think so!*

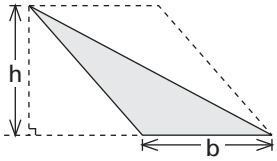




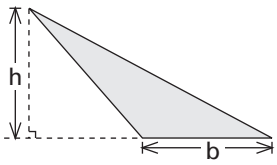
Method 4. *We got stuck on the trapezoid, so we worked on a formula for the area of a triangle first. Once we realized the area of a triangle is one half the length of the base times the height, we divided the trapezoid into 2 triangles. Then we added the areas of the triangles together to get the area of the trapezoid, or $\frac{b_1h}{2} + \frac{b_2h}{2}$.*

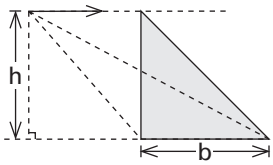


E) A right triangle with legs *a* and *b* can be enclosed in a rectangle with area *ab*. Since the triangle covers ½ of the rectangle the area of the right triangle is ½ *ab*.



F) One method is to put 2 copies of a triangle together to form a parallelogram with the same base and height as the triangle. Find the area of the parallelogram, *bh*, and divide by 2 to get the area of the triangle, $\frac{bh}{2}$.





As another method, some students may recall from experiences in earlier *Math Alive!* courses that triangles with equal bases and equal heights have the same area. Hence, they may reason that they can "slide" the top vertex of the triangle over to form a right triangle with base *b* and height *h*. Since this right triangle has area ½ *bh*, so must the given nonright triangle.

2 Ask the pairs to determine, if possible, formulas for the perimeters of the polygons on Focus Master A using only the given variables. If the information given is insufficient to determine the perimeter, have the students indicate "not enough information." Discuss.

2 The perimeters are as follows: square, 4*s*; rectangle, 2*a* + 2*b*; parallelogram, not enough information; trapezoid, not enough information; right triangle, $a + b + \sqrt{a^2 + b^2}$; nonright triangle, not enough information. Note: finding the length of the hypotenuse of a right triangle requires use of the Pythagorean Theorem which was introduced in Lesson 28 of *Math Alive! Course II* and is investigated further in Lesson 9 of this course.

Focus Teacher Activity (cont.)

ACTIONS

3 Give each student a copy of Focus Student Activity 3.1 to complete. Discuss their results and reasoning.

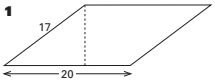
Measurement—Inventing Formulas Lesson **3**

Focus Student Activity 3.1

NAME _____ **DATE** _____

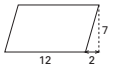
For each of the following polygons, use the given information to help you find the missing information. Mark each diagram or write equations or brief comments that communicate the steps of your thought processes. If it is not possible to find some of the missing information write NP and explain why. Note: diagrams are not drawn to scale; the dotted lines in the diagrams are altitudes; *a* represents area; *p*, perimeter; *h*, height (the length of the altitude); and *x*, *s*, and *b* are lengths that are marked on the diagrams.

1



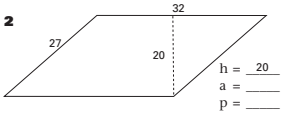
h = ____
a = 300
p = ____

4



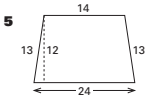
h = 7
a = ____
p = ____

2



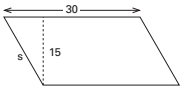
h = 20
a = ____
p = ____

5



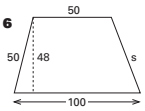
h = 12
a = ____
p = ____

3



h = ____
a = ____
p = 100
s = ____

6



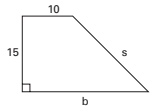
h = ____
a = ____
p = 260
s = ____

(Continued on back.)

Lesson **3** Measurement—Inventing Formulas

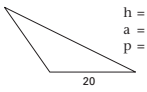
Focus Student Activity 3.1 (cont.)

7



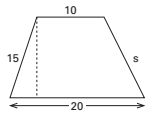
h = ____
a = 300
p = ____
s = ____
b = ____

10



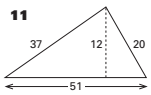
h = ____
a = 150
p = ____

8



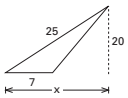
h = ____
a = 204
p = 62
s = ____

11



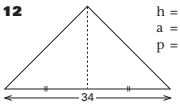
h = 12
a = ____
p = ____

9



h = ____
a = ____
p = ____
x = ____

12



h = ____
a = 216.75
p = ____

COMMENTS

3 Some students may rely on visual strategies such as cutting and rearranging the polygons. Others may rely on formulas. To discourage "mindless" plugging into formulas, you might suggest that students be prepared to show how and why their methods work.

Note that the Pythagorean Theorem is required for determining some values on Focus Student Activity 3.1 as well as Focus Student Activity 3.3 (see Action 14) and Follow-up Student Activity 3.5. If students are unfamiliar with the Pythagorean Theorem you could explore Lesson 28 of *Math Alive! Course II* or the Connector of Lesson 9 of this course before assigning this activity.

Answers for Focus Student Activity 3.1:

1. h = 15, p = 74
2. a = 640, p = 118
3. a = 450, s = 20
4. a = 84, p = 38.6 (rounded)
5. a = 228, p = 64
6. a = 3600, s = 60
7. h = 15, b = 30, s = 25, p = 80
8. h = 13.6, s = 17
9. h = 20, a = 70, p = 53.6 (rounded), x = 15
10. h = 15, there is not enough information to find p
11. a = 306, p = 108
12. h = 12.75, p = 76.5

Focus Teacher Activity (cont.)

ACTIONS

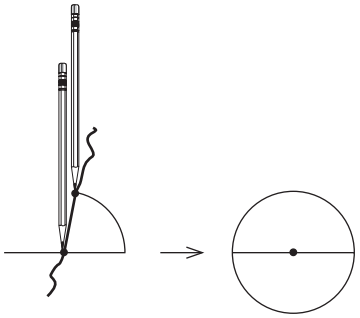
4 Provide each pair of students a blank sheet of paper, a straightedge, and a piece of string about 75 cm long. Ask the pairs to draw a line segment across the middle of the blank sheet, so that the length of the segment is at least half the shorter dimension of the paper.

5 Ask the pairs to devise a method of drawing, as accurately as possible, a circle whose diameter is the segment drawn for Action 4. Ask them to use no tools other than string, scissors, and pencils or pens to draw their circles. Tell them they may cut the string if they wish. Allow time for students to experiment with different approaches. Ask for volunteers to demonstrate their methods at the overhead. Then discuss the students' ideas about how to define a circle.

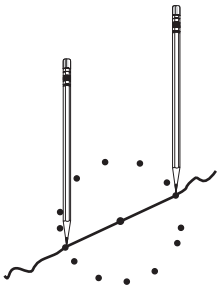
COMMENTS

4 It is helpful if the lengths of the pairs' segments differ. Segments should be drawn using a straightedge. To encourage students to use informal techniques for measuring and comparing lengths during Actions 5 and 6, do not distribute rulers yet.

5 One method of using string to construct a circle is the following: fold the segment to locate the midpoint of the diameter, i.e., the center of the circle; tie 2 knots in a piece of string so that the distance between the knots is half of the length of the diameter; place a pencil point through each knot; hold one knot at the center of the diameter; holding the string taut, trace a circular path with the pencil at the other knot. This method is illustrated in the diagram below:



Another method is to mark a string length equal to the diameter and tie knots at the endpoints; mark the center of the string length; line up the center of the diameter and the center of the string length, holding the string taut and in a straight line; mark the positions of the knots; repeat the preceding 2 steps several times; connect the "knot points" to form a polygon. As the number of knot points gets very large, the shape of the polygon more closely resembles a circle.



By definition, a circle is the set of all points in a plane at a given distance, called the *radius*, from a given point, called the *center*. Note that if the points are not restricted to a plane, but allowed to be any points in space, then a *sphere* is defined.

Focus Teacher Activity (cont.)

ACTIONS

6 Ask each pair of students to, without measuring with a ruler, make observations about relationships between the length of the diameter and the measure of the circumference of the circle they formed in Action 5. Ask for volunteers to share their observations.

7 Give each pair of students a metric ruler, and give 5 round or cylindrical objects to each group of 4 students (2 pairs will share the objects). Ask the pairs to use their string and rulers to measure, as accurately as possible, the diameter and circumference of each object. On the overhead or chalkboard, make a chart listing each of the objects students measure. Ask the students for assistance in recording the circumference and diameter of each object on the chart.

Object	C (Circumference)	D (Diameter)

8 Ask each pair to use a calculator to approximate the relationship between the circumference of each object measured in Action 7 and the diameter of the object. Ask for volunteers to record this information on the class chart. Discuss, encouraging generalizations.

Object	C (Circumference)	D (Diameter)	Relationship between C and D

COMMENTS

6 Students will use a variety of approaches to determine relationships. Some may notice that a piece of string the length of the diameter will fit along the circumference about 3 times. Others may lay a piece of string along the circumference of the circle, cut off that length, and compare it to the length of the diameter, noting that the circumference is a little more than 3 times the length of the diameter. Some students may estimate by folding a diameter into fourths to show that the circumference is a little less than 3¼ diameter lengths. Folding again shows that the circumference is about half way between 3 and 3.25 diameter lengths. Students may note that, while their circles are of varying sizes, the relationship between the measures of the diameter and the circumference of each circle seems to be the same.

7 It isn't necessary that all groups have the same 5 objects. In fact, more conjectures may emerge from objects with varied diameters.

Some students may remember a formula for the circumference of a circle. If so, ask them to use the tools provided to find the diameter and circumference of each object, to verify the formula works, and to try to determine the reasoning behind the formula.

To measure the diameter of a circular object, students may trace its outline and then paperfold to form a diameter. Or, they may lay a ruler across the end of a cylinder and move it to obtain the maximum distance across the cylinder. To measure the circumference of an object, they may wrap a piece of string tightly once around the object and then measure the string length.

8 Most students will probably observe that each circumference is about 3 times the length of the corresponding diameter, and some may make more exact approximations. Some students may want to "check out" this relationship on various other circular objects in the room, or they may need to remeasure objects if some measurements seem unreasonable.

Note: it is assumed that students have free access to a calculator during any *Math Alive!* lesson (unless they are specifically asked to work mentally).

Focus Teacher Activity (cont.)

ACTIONS

9 Tell the students that, for any circle, the ratio of its circumference to its diameter is the constant number called *pi* and is written π, a letter from the Greek alphabet. Hence, if c represents the circumference of a circle, and d represents the diameter, $c/d = \pi$. Many approximations of π are commonly used, such as 3, 3.1, 3.14, $22/7$, and 3.1416. Here is an approximation of π with 15 decimal places: 3.141592653589793.

No matter how many decimal places are used, π cannot be expressed exactly as a decimal, because the decimal part never repeats and never terminates. If they haven't already done so, ask the pairs to use their measurements from Action 7 to compute the ratio of each circumference to its corresponding diameter and to compare these ratios to the value of π. Discuss.

10 Give each student a copy of Focus Student Activity 3.2 (see below and next page) to complete. Discuss their results and methods, as needed.

Measurement—Inventing Formulas Lesson **3**

Focus Student Activity 3.2

NAME _____ DATE _____

Draw a diagram or sketch to solve each of the following problems. Use your calculator to carry out computations, and record all of the calculations that you do. Express each answer to the nearest tenth.

1 A circular lid has a diameter of 7 cm. What is its circumference, to the nearest tenth of a centimeter? Remember to show a sketch, the keys you press to compute the answer, and your answer.

2 A circular fish pond has a circumference of 5.8 meters. What is its diameter?

3 The diameter of one circle is twice as long as the diameter of another circle. How do the circumferences of the circles compare?

COMMENTS

9 A decimal such as π that never terminates and never repeats is an *irrational* number. If the students' calculators have a π button, you might ask them to press it to see what approximation their calculator uses for π.

For hundreds of years, approximating the value of the ratio of the circumference of a circle to its diameter was a challenge. The ancient Egyptians approximated the ratio of the circumference to the diameter as $3\frac{1}{6}$; the estimate in the ancient Orient was 3. Archimedes (about 287-212 B.C.) computed π to the equivalent of 2 decimal places. In 1596 Ludolph van Ceulen published a value of π to 20-places. Later he determined a 35-place approximation, which his wife had engraved on his tombstone! In 1841 Zacharaias Dase computed π to 200 places. By 1947, it was computed correctly to 710 places. In the present age of electronic computers, it can be computed quickly to thousands of decimal places and beyond. In *An Introduction to the History of Mathematics* by H. W. Eves, several mnemonics for remembering the first few decimal places of π are suggested. For example, in "May I have a large container of coffee?" the number of letters in each word is a digit in π.

10 Note that it isn't expected that students discuss circumference formulas before completing this activity. Rather, formulas may naturally surface as the students work on these problems and in Action 11 students discuss/invent formulas.

This activity could also be completed as homework. Solutions to Problems 1-3 are as follows:

1. 22.0 cm, to the nearest tenth
2. 1.8 m, to the nearest tenth
3. The circumference of the larger circle is twice the circumference of the smaller circle.

Students' answers may differ slightly based on the value used to approximate π. If preferred, you could assign a value, such as 3.14, for students to use.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

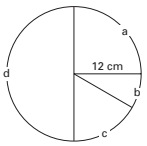
Lesson **3** Measurement—Inventing Formulas

Focus Student Activity 3.2 (cont.)

4 A circular patio has a circumference of 18.7 meters. What is its radius?

5 For a pony ride at the fair, Pete Pony is tied to the end of a 10 foot pole. The other end of the pole is connected to a "hub" that rotates 360°. One day at the fair, Pete completely circled the hub 220 times. How many miles did Pete travel on that day? 5,280 feet = 1 mile.

6 For the circle shown at the right, find a close approximation of the length of each of the indicated arcs (a, b, c, and d). Use math symbols to communicate your method of determining each arc length.



a = _____ Method:

b = _____ Method:

c = _____ Method:

d = _____ Method:

COMMENTS

10 (continued.)
Solutions to Problems 4-6 are as follows:

4. 3.0 m, to the nearest tenth
5. 2.6 miles, to the nearest tenth; (the 10 foot pole represents the radius of the circular path)
6a. $^{90}/_{360}$ of $24\pi \approx 18.8$
6b. $^{30}/_{360}$ of $24\pi \approx 6.3$ or $^{1}/_{3}$ of the length of part a
6c. $^{60}/_{360}$ of $24\pi \approx 12.6$
6d. $^{180}/_{360}$ of $24\pi \approx 37.7$

11 Ask the pairs to complete the following thought starter. Discuss their results.

> **Here are several observations and/or generalizations about a circle with diameter *d*, radius *r*, and circumference *c*:**

11 Students' observations may vary. If they don't suggest formulas, you might ask that, wherever possible, they rewrite their generalizations in the form of formulas. Here are a few formulas that students have suggested: $c = \pi d$; $d = 2r$; $c = 2\pi r$; $^{d}/_{2} = r$; $^{c}/_{d} = \pi$; $^{c}/_{2r} = \pi$.

12 Distribute 4 copies of Focus Master B to each pair of students. Ask the students to devise 3 or 4 methods of approximating the area of the circle. Ask them to:

a) illustrate each method on a separate copy of Focus Master B,

b) indicate whether they think each approximation is high or low, and

c) if possible, write a general procedure or formula that represents each method.

Allow plenty of time for pairs to explore and discuss possibilities. Then invite volunteers to share several methods at the overhead.

12 Focus Master B contains one large circle with a radius labeled *r*.

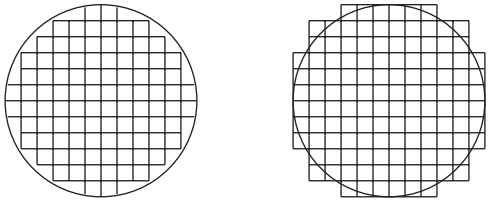
It is helpful to have a supply of variously sized grids available. If students recall a formula, suggest they devise methods of approximating the area in order to offer evidence for the formula.

There are many possibilities. Acknowledge all that come up. Following are several responses frequently offered by students. Notice that Method 7 provides a general method of finding the exact area of any circle.

Method 1: *Cover the circle with a centimeter grid, count the number of complete squares in an "inner covering" (I) and the number in an "outer covering" (O) and then average those two numbers. So area* $\approx \frac{I+O}{2}$. *See diagram on next page.*

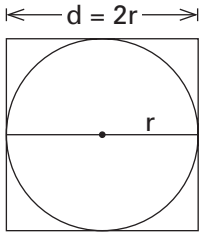
Focus Teacher Activity (cont.)

ACTIONS

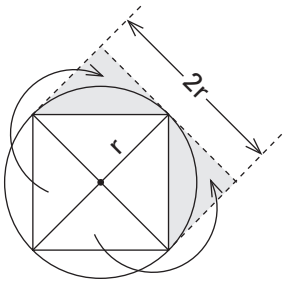


"inner" (I) covering "outer" (O) covering

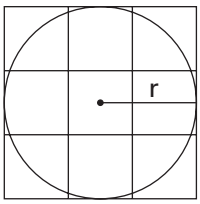
Method 1: Area of circle $\approx \frac{I+O}{2}$



Method 2: Area of circle $\approx d \times d = 4r^2$

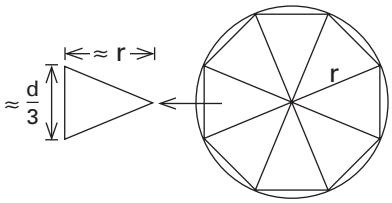


Method 3: Area of circle $\approx 2r \times r = 2r^2$



Area of large square $= 4r^2$

Method 5: Area of circle $\approx \frac{7}{9}(4r^2) = (\frac{28}{9})r^2$



Area of one triangle $\approx \frac{1}{2} \times \frac{d}{3} \times r$

Method 6: Area of circle $\approx 8 \times (\frac{1}{2} \times \frac{2r}{3} \times r) = (\frac{8}{3})r^2$

COMMENTS

Some students will suggest improving this estimate by splitting the centimeter squares into smaller squares and then repeating the process of counting and averaging.

Method 2: *Enclose the circle in a square. The sides of the square have length equal to the length of the diameter, or twice the length of the radius. The area of this square, $4r^2$, is a high approximation of the area of the circle.* This method is illustrated at the left. Students' methods of constructing the square will be informal; it is not necessary to develop or discuss formal construction methods here. However, you might use this as a context for introducing terminology such as *tangent*, *circumscribe*, and *inscribe*. Note that all the sides of the square are *tangent* to the circle because they intersect the circle in exactly one point. That point is called the *point of tangency*. When all sides of a polygon are tangent to a circle, the circle is *inscribed* in the polygon, and the polygon is *circumscribed* about the circle.

Method 3: *Draw a square inside the circle so that each of the corners lies on the circle. Draw the diameters that connect the vertices of the square, forming 4 congruent right triangles whose legs are the length of the radius. Move 2 of those triangles to form a rectangle with dimensions r by 2r. The area of this rectangle, $2r^2$, is a low approximation of the area of the circle.* This method is illustrated at the left. Note that, since the vertices of the polygon lie on the circle, the polygon is *inscribed* in the circle and the circle is *circumscribed* about the polygon. Notice each side of the square is a chord of the circle.

Method 4: *Average the high and the low approximations from the previous two methods to get an average of $3r^2$, a closer approximation of the area of the circle.*

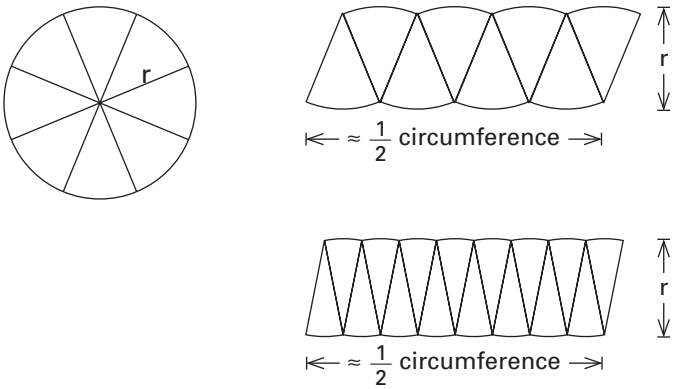
Method 5: *Circumscribe a square about the circle. Mark off each side of the square in thirds and cover the circle with a 3 by 3 grid. The area of the circle is approximately $\frac{7}{9}$ of the area of the square.* This method is shown at the left.

Method 6: *Cut the circle out and paperfold to locate the vertices of an octagon. Draw the octagon and 8 congruent triangles formed by the radii drawn to the vertices. The base of each triangle is about $\frac{1}{3}$ of the diameter and the height of each triangle is approximated by the radius. The area of the circle is approximated by the total area of the 8 triangles, or $(\frac{8}{3})r^2$.* See diagram at the left.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



Method 7: Area of circle = $\left(\frac{2\pi r}{2}\right)r = \pi r^2$

13 If students didn't suggest Method 7 from Comment 12, read aloud through part b) below. Allow time for the pairs to explore and conjecture about Malia and Jennie's thinking. Discuss. Then, if needed, read parts c)-f), allowing exploration time after each part.

Malia and Jennie, a pair of students in another class, determined the area of a circle to be πr^2. Following is how they described their methods:

a) We subdivided the circle into an even number of congruent sectors.

b) Next we rearranged the sectors, alternating point up and point down, to approximate a parallelogram.

c) Then we cut each sector in half and rearranged these parts, alternating point up and point down, and noticed the shape was closer to a parallelogram.

d) We think we could do the previous step over and over. When the sectors get smaller, the edge formed by the curved parts gets flatter and the shape gets closer to a rectangle with base equal to ½ of the circumference and height equal to the radius.

e) We can "see" the area of the circle is equal to the area of the rectangle or $\pi r \times r = \pi r^2$.

COMMENTS

12 (continued.)
Method 7: *Fold the circle to form 8 (or some other even number) congruent sectors. Then cut these sectors apart and rearrange them to approximate a parallelogram. The base of the "parallelogram" is approximately half the circumference; the height is approximately equal to the radius.*

Each of the 8 sectors can be cut in half and rearranged to form a figure that more closely resembles a parallelogram. As more "triangular" regions are formed their bases become "straighter" and the shape of the rearranged figure more closely approximates a rectangle whose area is $\left(\frac{2\pi r}{2}\right)r = \pi r^2$.

If students don't come up with Method 7, notice you can bring it up for discussion in the next action.

13 Malia and Jennie's method provides a visual demonstration of the fact that the area of a circle is πr^2. Their approach is illustrated in Method 7 of Comment 12. Note: If an odd number of sectors are formed in Step a) the figure approximated in b)-d) is a trapezoid, rather than a parallelogram. The formula for the area of a circle is then found by determining the area of a trapezoid.

Focus Teacher Activity (cont.)

ACTIONS

14 Give each student a protractor (see *Blackline Masters*) and a copy of Focus Student Activity 3.3 to complete. Discuss their results and strategies, as needed.

Measurement—Inventing Formulas Lesson **3**

Focus Student Activity 3.3

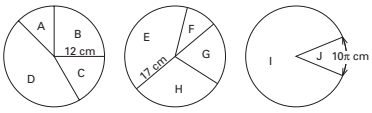
NAME _____ DATE _____

Write your responses to problems 1-7 on another sheet of paper. Be sure to put a copy of the problem next to your responses.

1 Write and illustrate two methods that you could use to show a younger student how to approximate the area of a circle with a radius of 7 cm.

2 Explain and sketch a diagram to show how a parallelogram can be used to find the area of a circle with a radius of 17 cm.

3 For each of sectors A-J shown below, without using a protractor, predict the area of the sector and the arc length along the edge of the sector. Record your predictions and reasoning. Then use a protractor to verify your predictions. Record your measurements and calculations.



4 Papa Paulo's Pizza Crust Company sells a crust with a 13 inch diameter for $2.29. They have decided to add a square crust to their product line. Assuming Papa Paulo wants to price all of his crusts at the same rate, what should he charge for a square crust with side length 13 inches?

5 On Carmen's clock, the second hand is 15 cm long and the hour hand is 10 cm long. Explain your methods of solving each of the following problems about her clock. Give answers to the nearest tenth of a centimeter.

a) How far does the tip of the second hand travel in 45 minutes?

b) How far does the tip of the hour hand travel in 45 minutes?

(Continued on back.)

COMMENTS

14 Protractors were introduced in *Math Alive! Course II*, Lesson 12. This activity could also be completed as homework. Selected answers are shown below.

3) Sector A: area ≈ 56.5 cm^2; arc length ≈ 9.4 cm
Sector B: area ≈ 113.1 cm^2; arc length ≈ 18.8 cm
Sector C: area ≈ 75.4 cm^2; arc length ≈ 12.6 cm
Sector D: area ≈ 207.3 cm^2; arc length ≈ 34.6 cm
Sector E: area ≈ 363.2 cm^2; arc length ≈ 42.7 cm
Sector F: area ≈ 90.8 cm^2; arc length ≈ 10.7 cm
Sector G: area ≈ 181.6 cm^2; arc length ≈ 21.4 cm
Sector H: area ≈ 272.4 cm^2; arc length ≈ 32.0 cm
Sector I: area ≈ 4398.2 cm^2; arc length ≈ 219.9 cm
Sector J: area ≈ 628.3 cm^2; arc length ≈ 31.4 cm

4) $2.92

5) a) 4241.2 cm
b) 47.1 cm

6) 114.5 inches. Note: this problem requires use of the Pythagorean Theorem (see Comment 3).

7) 4185.4 cm^2

Lesson **3** Measurement—Inventing Formulas

Focus Student Activity 3.3 (cont.)

6 In order to get a sign post that rises 8 feet above the ground to stand at right angles, Zane fastened one end of a wire to the top of the post and the other end to the ground 3 feet from the base of the post. If she needs 6 extra inches of wire on each end for fastening, how long (to the nearest inch) must she cut the wire? Make a diagram and show your reasoning.

7 What is the area (to the nearest square centimeter) of the largest circle that can be cut from a square piece of sheet metal 73 cm on a side? Explain your reasoning.

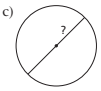
8 The following figures show the radius, diameter, circumference, or area of a circle. Use the information given to *mentally approximate* the requested measures. Use math symbols to communicate your *mental* calculations.

a) 

Circumference =_____
Mental calculations:
_____

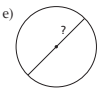
b) 

Area =_____
Mental calculations:
_____

c) 

Area = 75 cm^2
Radius =_____
Mental calculations:
_____

d) 
Circumference = 21 cm
Area =_____
Mental calculations:
_____

e) 

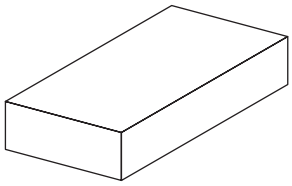
Circumference = 15 cm
Radius =_____
Mental calculations:
_____

f) 
Area = 48 cm^2
Circumference =_____
Mental calculations:
_____

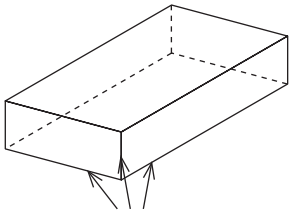
Focus Teacher Activity (cont.)

ACTIONS

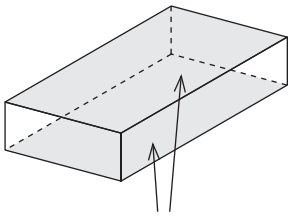
15 Distribute about 30 cubes to each student. Discuss the students' ideas about the meanings of the terms *rectangular right prism*, *surface area,* and *volume*. Include discussion of relationships among standard units of measure for length, area, and volume.



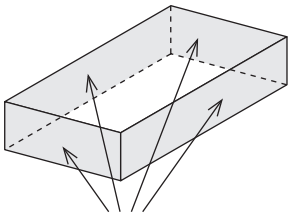
Rectangular Right Prism
Rectangular Parallelepiped
Box



3 of the 12 edges



2 parallel bases (upper and lower) that are rectangles



4 lateral faces that are rectangles

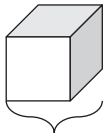
COMMENTS

15 You might encourage students to use the cubes to illustrate their ideas about the meanings of the given terms and relationships between units of measure. See Lessons 1, 2, and 29 of *Math Alive! Course II* for other discussion ideas.

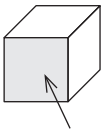
A *prism,* is a *polyhedron* (a 3-dimensional figure formed by flat surfaces that are enclosed by polygons) with 2 parallel *bases*, upper and lower, which are congruent polygons. A prism is named according to the shape of its bases. The *lateral faces* of all prisms are parallelograms. A square prism, for example, has square bases and lateral faces that are parallelograms. Note: a prism whose bases and lateral faces are parallelograms is also called a *parallelepiped*.

The lateral faces of a *rectangular right prism*—sometimes called a *box* or a *rectangular parallelepiped*—are rectangles which are perpendicular to the bases. Beginning in Action 18, students explore other types of prisms.

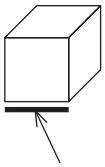
The *surface area* of a polyhedron is the total area, generally in square units, of all of its faces. The *volume* of a polyhedron is its capacity, usually in cubic units. It is standard practice to relate units of volume, area, and linear measure as follows:



1 cubic unit



1 square unit (a side or face of 1 cubic unit)



1 linear unit (the length of an edge of 1 cubic unit, also the length of the edge of 1 square unit)

16 Write the following on the overhead:

Rectangular prisms with dimensions, in linear units:

a) $3 \times 2 \times 1$ b) $4 \times 2 \times 3$ c) $5 \times 3 \times 2$

Ask the pairs to build the above rectangular prisms and to determine the surface area and volume of each, assuming that one of the small cubes that you distributed is the volume unit, and assuming that volume, area, and linear units are related as illustrated in Comment 15. Discuss the strategies students use to compute surface area and volume.

16 a) volume, 6 cubic units; surface area, 22 square units

b) volume, 24 cubic units; surface area, 52 square units

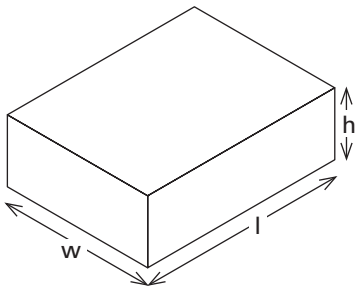
c) volume, 30 cubic units; surface area, 62 square units

Students may make generalizations about their strategies. If so, see Action and Comment 17 for discussion ideas.

Focus Teacher Activity (cont.)

ACTIONS

17 If it didn't come up during discussion for Action 16, ask the pairs to invent formulas for the surface area and volume of a rectangular solid with the following dimensions, in linear units: length, *l*; width, *w*; and height, *h*. Discuss.

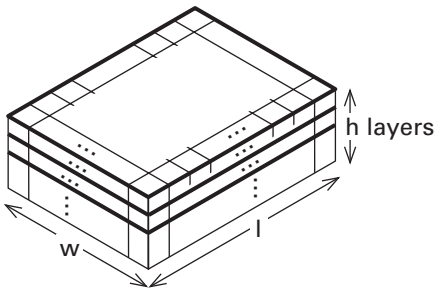


Rectangular prism with dimensions, in linear units *l*, *w*, and *h*.

COMMENTS

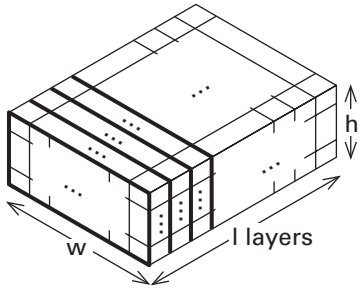
17 Some students may recall formulas they have learned in other classes. If so, encourage them to demonstrate why the formulas work.

One way of "seeing" the volume of a rectangular prism with dimensions *l*, *w*, and *h* is to slice it into *h* layers of thickness 1 unit, as shown here:

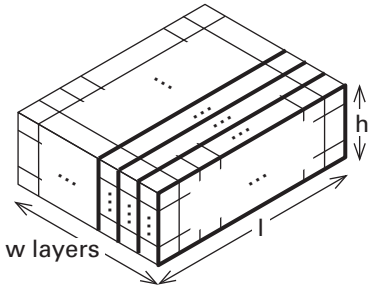


Slice the prism into *h* layers, each with $l \times w$ cubes. The volume, *V*, equals the product of the number of cubes per layer times the number of layers, or $V = (l \times w) \times h$.

Notice that one can also slice the rectangular prism vertically to get *l* layers each with $w \times h$ cubes, or to get *w* layers each with $l \times h$ cubes:



l layers of $(w \times h)$ cubes
$V = l \times (w \times h)$



w layers of $(l \times h)$ cubes
$V = w \times (l \times h)$

Each of the three slicing methods shown above demonstrates the fact that the volume, *V*, of a rectangular prism with dimensions *l*, *w*, and *h*, is $V = l \times w \times h$. The three expressions given above are equivalent due to order of operations and the commutative property for multiplication.

Focus Teacher Activity (cont.)

ACTIONS

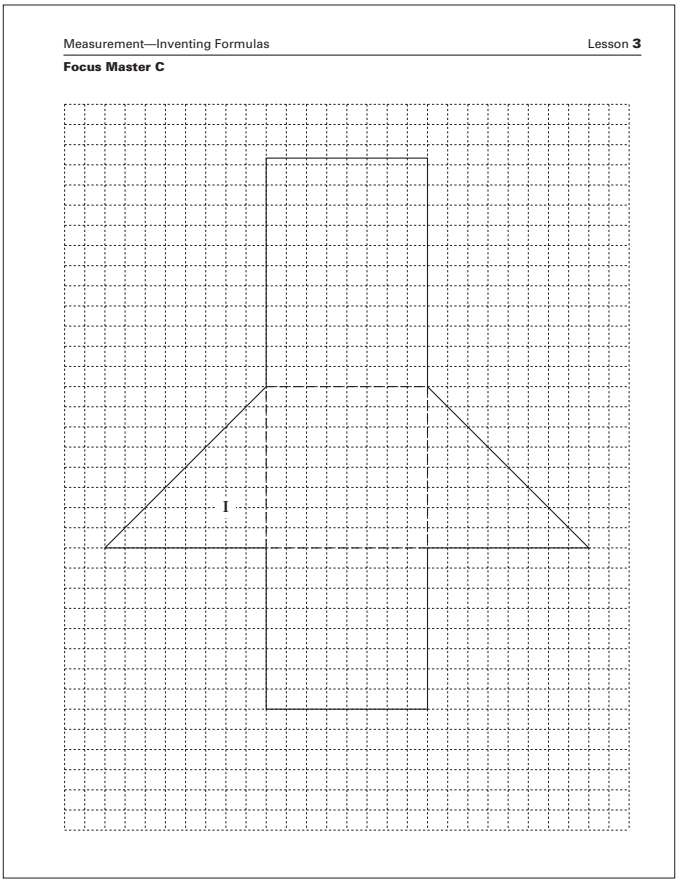
18 Provide each student a copy of Focus Master C (4 pages), a pair of scissors, and tape. Tell the students that each of figures I-V is a "net" for a 3-dimensional geometric figure. Ask the pairs to do a)-d) below for Net I on Focus Master C. Discuss, clarifying terminology as needed and encouraging efforts to generalize. Then repeat for Nets II–V.

a) *Predict* what 3-D figure will be formed by cutting along solid lines, folding along dotted lines, and taping the net together.

b) Cut out and assemble to form a 3-D figure.

c) Name the geometric figure formed in b) and make mathematical observations about it.

d) Determine the surface area and volume of the figure, investigating more than one strategy if possible.



COMMENTS

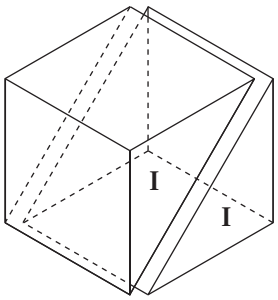
18 To promote generalizations, you might invite students to post "We conjecture…/We wonder…" statements on a class poster.

Cutting, assembling, and manipulating the 3-D figures are simplified if Focus Master C is copied on cardstock. Be sure that pages 1-4 of Focus Master C are not copied back-to-back. Each of Nets I-V forms either a prism or a cylinder. You might point out that all edges should meet without overlaps or gaps (i.e., full edge to full edge) and that each face of a figure should be a flat region bounded by a polygon. Further, each polygonal region that is outlined by dotted and/or solid lines on Focus Master C is a face of the figure.

It is not expected that students know formulas for determining the surface area and volume of 3-D figures. Rather, the intent is for students to use their understanding of the concepts of surface area and volume, together with their knowledge about the area of 2-D figures, as a basis for reasoning about these 3-D figures.

Net I forms a triangular right prism, with right triangles as bases, and rectangular lateral faces. The surface area is 282.5 square units, to the nearest tenth.

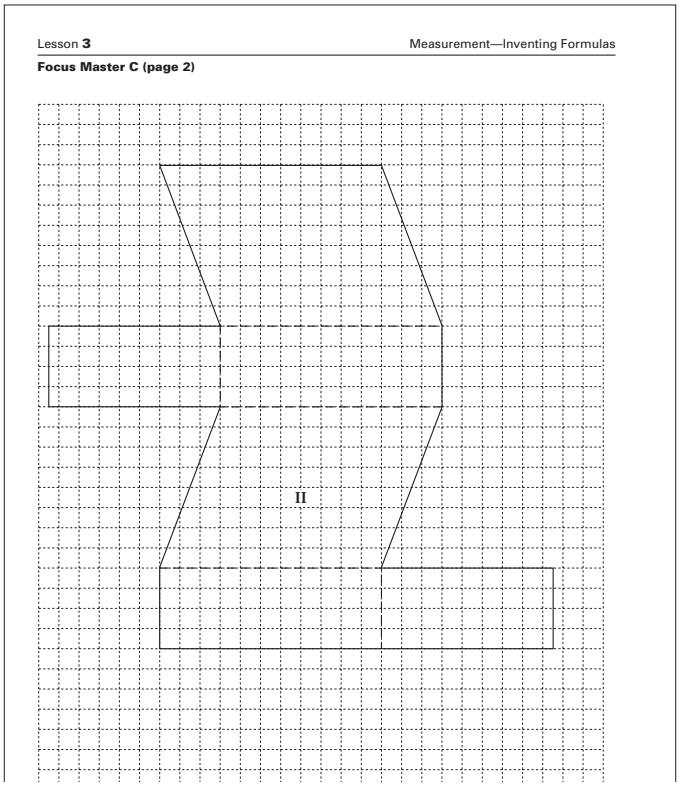
Some students may point out that 2 copies of this prism fit together to form a rectangular prism with volume $8 \times 8 \times 8 = 512$ cubic units. Hence, the triangular prism has volume $512/2 = 256$ cubic units. Others may "see" 8 layers of $64/2$ cubic units, or $8 \times 64/2 = 256$ cubic units. And, there are other possibilities.

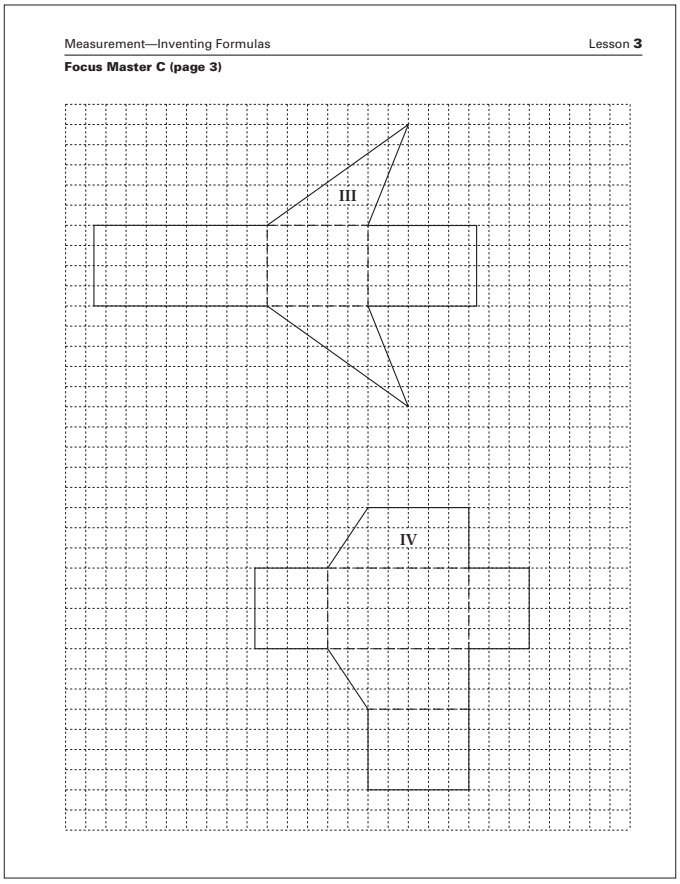


Net II forms a prism that may be viewed in two different ways: as a right prism with nonrectangular parallelograms as bases and rectangles as lateral faces (see Diagram A on the next page); or as an *oblique prism* with rectangular bases, 2 lateral faces that are rectangles and 2 that are nonrectangular parallelograms (see Diagram B on the next page). Note: a prism that has two or more lateral faces that are not rectangular is called an oblique prism.

Focus Teacher Activity (cont.)

ACTIONS





COMMENTS

To compute the surface area of the prism formed by Net II, notice there are 2 congruent parallelograms each with area 88, 2 congruent rectangles each with area 44, and 2 congruent rectangles each with area 34.2 to the nearest tenth, for a total surface area of $2(88 + 44 + 34.2) = 332.4$ square units.

Students who view this prism as in Diagram A below, may "see" 4 layers, each with (8×11) cubes. Notice that the number of cubes in each layer is equal to the number of square units in the parallelogram base; hence, the volume is equal to the product of the area of the base of the prism times the altitude of the prism, or $(8 \times 11) \times 4 = 352$ cubic units.

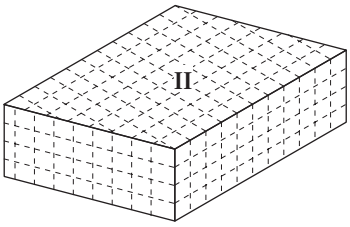


Diagram A
Right prism with nonrectangular parallelograms as bases

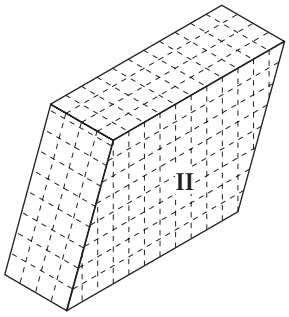
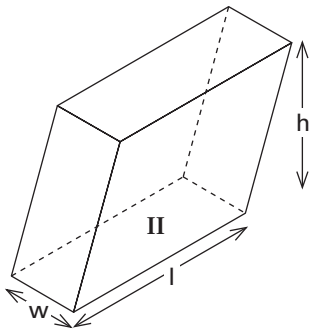
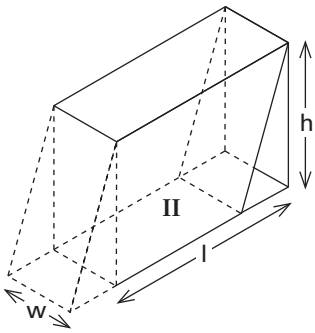


Diagram B
Oblique prism with rectangular bases

Students who view the prism as in Diagram B above, may suggest a "dissect and rearrange" strategy similar to that used to determine the area of a parallelogram in Action 1. Hence, the volume of the rectangular oblique prism with length, *l*, width, *w*, and height, *h*, is equal to the volume of the rectangular right prism with the same length, width, and height. Note: the height, or altitude, of an oblique prism is the perpendicular distance between its upper and lower bases.





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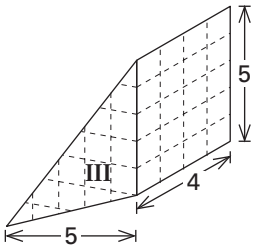
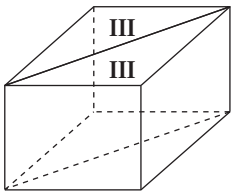
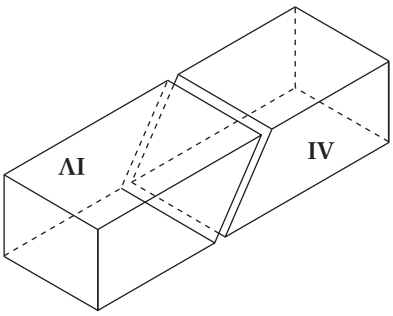
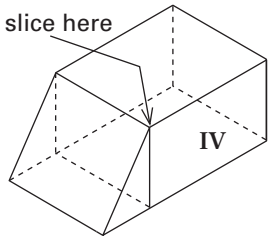
Focus Teacher Activity (cont.)

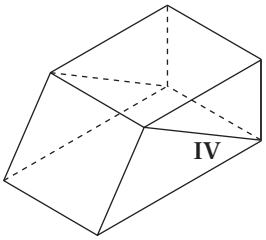
ACTIONS











COMMENTS

18 (continued.)
Net III forms a triangular right prism with scalene triangles as bases. This prism has volume 50 cubic units. One way of viewing the volume is as 4 layers of $\frac{(5 \times 5)}{2}$ cubes (see diagram at the left); one layer contains as many cubes as there are square units in the base, and the number of 1-unit layers is equal to the height of the prism. Hence, the volume of the prism is equal to the product of the height times the area of the base.

Here is another strategy for finding the volume of the triangular prism formed by Net III: combine 2 copies of the triangular prism to form a prism which has a parallelogram base (see diagram at the left); use strategies such as those described for Net II to find the volume of this prism; finally, divide the area of the prism by 2.

Some students may approximate the surface area of this prism by counting the whole squares in the gridded net and approximating partial squares. Others may use Pythagorean Theorem to determine that the lengths of the sides of the triangular base are $\sqrt{29} \approx 5.4$, 5, and $\sqrt{74} \approx 8.6$ linear units. Hence, the lateral faces of the prism have area $(4 \times 5) + 4\sqrt{29} + 4\sqrt{74} = 75.9$ square units to the nearest tenth. The area of the bases is $^{25}/_{2} \times 2 = 25$ square units. So, the surface area of this triangular prism to the nearest tenth is $75.9 + 25 = 100.9$ square units.

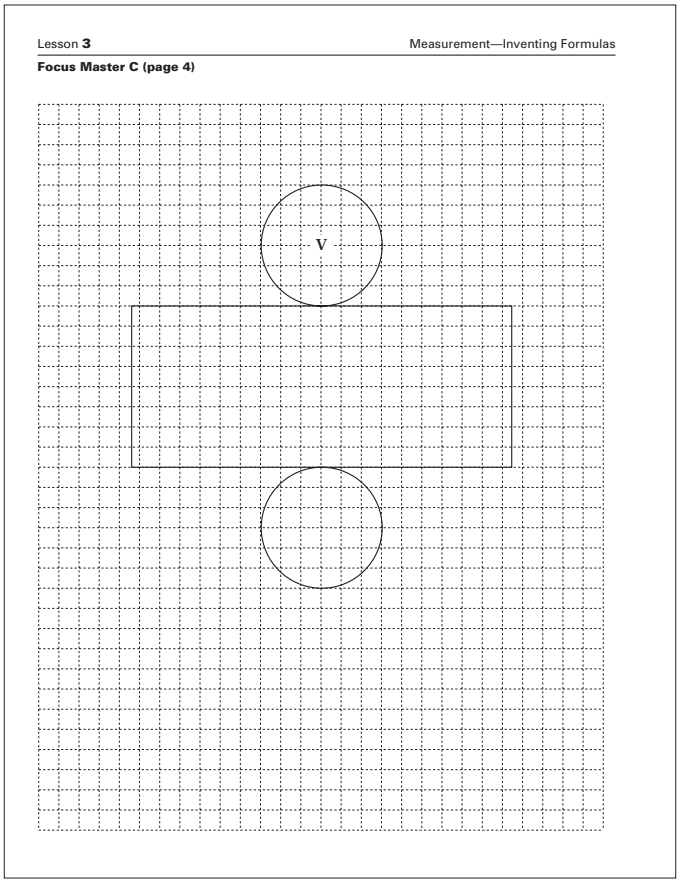
Net IV forms a trapezoidal right prism with surface area 110.4 square units. The volume, 72 cubic units, can be determined in several ways. For example, one can combine 2 copies of this prism to form a right rectangular prism, as shown at the left; find the volume of the rectangular prism; and then, divide the volume of the rectangular prism by 2 to get the volume of the trapezoidal prism.

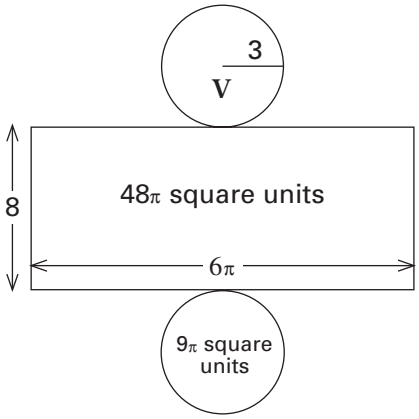
Or, one could find the volume by slicing the trapezoidal prism into a triangular prism and a rectangular prism, as shown at the left, and then find the sum of their volumes.

Yet another method is to divide the trapezoidal prism into two triangular prisms, as illustrated at the left, and find the sum of their volumes, using methods similar to those for Net III.

Focus Teacher Activity (cont.)

ACTIONS





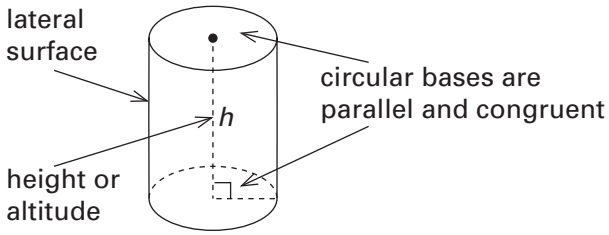
COMMENTS

Two general strategies for computing the volume of prisms have been illustrated thus far:

Method 1. Dissect and rearrange the figure to form a new figure with the same volume.

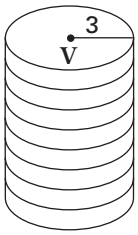
Method 2. Slice the prism into 1-unit layers that are parallel to the bases. Each layer contains as many cubes as the number of square units that cover the base (i.e., the area of the base). Hence, since the height in linear units indicates the number of 1-unit layers, the volume of the prism is the product of the area of the base times the height. This is often written, $V = Bh$, where B is the area of the base, and h is the height or altitude.

Net V forms a right circular cylinder, the circular counterpart of a prism. It has 2 parallel and congruent circular bases and a lateral surface that rises from one base to the other. If the segment that connects the centers of the bases is perpendicular to the bases, then the cylinder is a right cylinder. The altitude, or height, of a cylinder is the perpendicular distance between the bases. The radius of a cylinder is the radius of its base.



The surface area of a cylinder is the sum of the areas of the bases plus the area of the lateral surface. Each base of the cylinder formed from Net V has area $\pi(3)^2 = 9\pi$ square units. The lateral surface is a rectangle with length equal to the circumference of the base, 6π, and width equal to the height, 8, of the cylinder; hence, the area of the lateral surface is $(6\pi)(8) = 48\pi$ square units. Therefore, the surface area of the cylinder is $18\pi + 48\pi = 66\pi$ square units.

To find the volume of the Net V cylinder slice the cylinder into 8 circular layers, as shown here, with 9π cubes per layer. The volume is 72π cubic units. In general, like for a prism, the volume of a cylinder is the area of the base times the height or $\pi r^2 h$.



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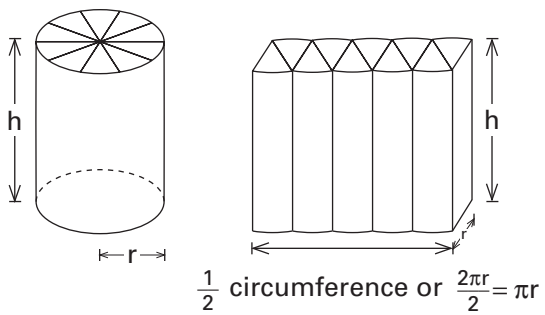
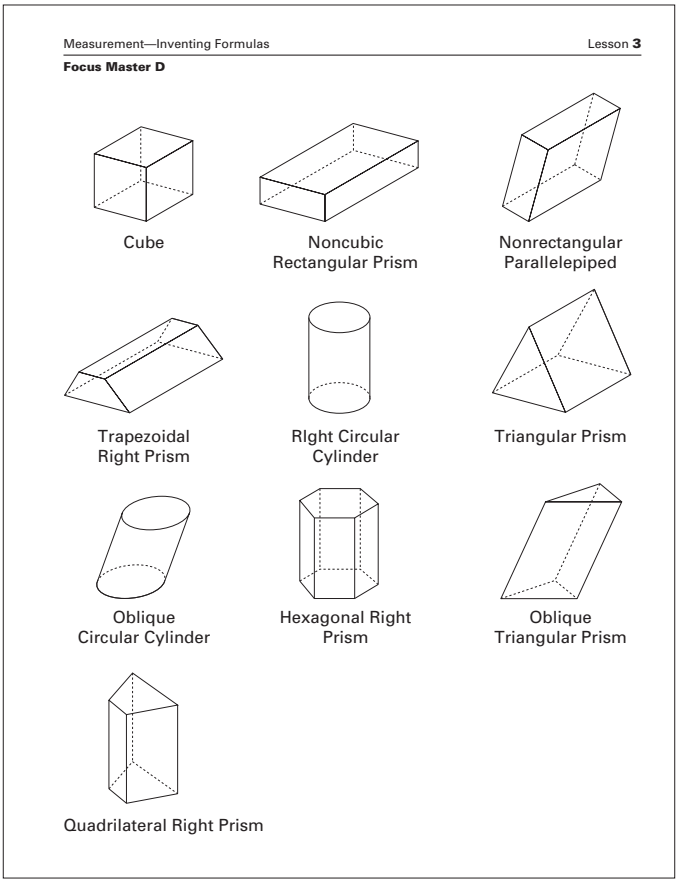
Focus Teacher Activity (cont.)

ACTIONS



19 Give each pair of students a copy of Focus Master D, a sheet of butcher paper, and marking pens. Ask the pairs to generate a list of "We conjecture…/We wonder…" statements regarding surface area and volume, basing their statements on the figures on Focus Master D and/or other 3-D figures. Discuss their results.



COMMENTS

18 (continued.)
In Action 12, students subdivided a circle into sectors and rearranged the sectors to approximate a parallelogram, noting that as the number of sectors increased, the shape of the "parallelogram" formed by the rearranged sectors approached a rectangle with one dimension equal to ½ the circumference of the circle, and the other dimension equal to the radius of the circle. Similarly, a right circular cylinder can be subdivided, as shown at the left, and rearranged to approximate a right rectangular prism whose volume is $\pi r \times r \times h = \pi r^2 h$, where h is the height and r the radius of the cylinder.

19 The diagrams on Focus Master D are intended to serve as "thought starters" for investigations and dialogue among the students. Dimensions and other measurements have been left off in order to prompt conjectures, what-if questions, and generalizations. Students may add variables to indicate dimensions and for use in formulas.

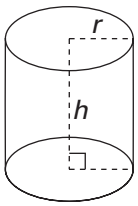
If students have difficulty getting started, you might initiate a class discussion of conjectures, questions, and generalizations that one could make about a cube. For example, if the length of an edge of a cube is s linear units, then the volume is s^3 cubic units and the surface area is $6s^2$ square units. If the volume is V, then the length of a side is $\sqrt[3]{V}$ linear units and the area of a face is $(\sqrt[3]{V})^2$ square units. Some possible what-if questions include: What if the length of each edge is enlarged by a factor of 2—what is the effect on surface area and volume? What if the length of each edge is enlarged/reduced by a factor of n? What if only one dimension is enlarged/reduced? What types of prisms are formed if a cube is sliced along a diagonal of one of its faces? What types of prisms can be formed using one cut of a cube? two cuts?

Rather than selecting specific figures or questions for students to investigate, you might allot a certain amount of time for this activity. Some pairs may examine one or two types of figures in depth, while others may investigate several types and their relationships to each other.

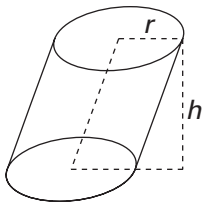
This action provides an opportunity for you to "eavesdrop" on student discussions, and thus to assess students' understanding of the concepts of surface area and volume, their ability to see and generalize relationships in 2-D and 3-D figures, and their comfort with inventing formulas. It is intended that emphasis be placed on the

Focus Teacher Activity (cont.)

ACTIONS



Right Circular Cylinder



Oblique Circular Cylinder

20 Give each student a copy of Focus Student Activity 3.4 (see below and next page) and ask the students to solve selected problems from this sheet. Discuss their results and reasoning.

Measurement—Inventing Formulas Lesson **3**

Focus Student Activity 3.4

NAME _____ DATE _____

1 Suppose you need to fill ½ of a right cylindrical can with water. Assuming that you have no measuring tools and no other containers, how can you determine when the can is half full?

2 When an object is placed in a container of water it changes the level of the water. The amount of water that is *displaced* is equal to the volume of the object. This fact is useful for finding the volume of irregularly shaped objects. Suppose that:

a) You drop a rock into a 5" × 5" × 12" square prism that is filled with water to a height of 7". The rock displaces the water level by 4". What is the volume of the rock?

b) You drop some coins in a right circular cylinder with radius 7 cm and altitude 15 cm. The coins displace the water level from 6 cm to 13 cm. What is the volume of the coins to the nearest cubic centimeter?

c) Gilly Goldfish and his sisters all jump out of their "filled-to-the-brim" rectangular-prism-shaped fish tank. The water level in this 12" by 14" by 18" tank drops ½" below the 18" height. What is the total volume of Gilly and his sisters?

3 The perimeter of a face of Cube X is $4(\sqrt[3]{125})$ inches. Determine the surface area and volume of Cube X. Show your methods and reasoning.

4 The following questions relate to a rectangular parallelepiped with volume 36 cubic units:

a) What are all the possible whole number dimensions this parallelepiped could have?

b) What is the minimum possible surface area for this parallelepiped if the dimensions are only whole numbers? if the dimensions are not whole numbers?

(Continued on back.)

COMMENTS

mathematical process and the invention of formulas, rather than on memorization of formulas. (Note: even on the SAT exam, standard formulas are provided to students.)

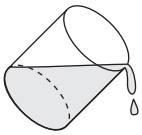
A cylinder whose altitude is not perpendicular to the plane of the bases is called oblique, as illustrated at the left.

The volume of an oblique cylinder is equal to the volume of a right cylinder with the same bases and same altitude. One way to illustrate this idea is to arrange a stack of coins to form a right cylinder. Then, "shear" the stack to approximate an oblique cylinder with the same base, height, and volume as the original (see diagram at the left).

20 Focus Student Activity 3.4 could also be assigned as homework. Answers may vary based on the value used for π.

Note; the cube root notation in Problems 3 and 4 may need some discussion. A cube whose volume is 125 cubic inches has an edge of length $\sqrt[3]{125}$ inches, and this equals 5 because $5^3 = 125$. In Problem 4 students may try to evaluate $\sqrt[3]{36}$ by using trial and error methods to find a number which, when raised to the third power, is close to 36. Or, this number can be found by using a calculator to compute $36^{1/3} \approx 3.301927249$.

1) One method is to tip the can until the bottom just becomes visible.



2a) 100 cubic inches
b) 1078 cubic centimeters
c) 84 cubic inches

3) surface area, 150 square inches; volume, 125 cubic inches

4a) $1 \times 1 \times 36$, $1 \times 2 \times 18$, $2 \times 2 \times 9$, $1 \times 4 \times 9$, $4 \times 3 \times 3$, $1 \times 12 \times 3$, $2 \times 6 \times 3$, $6 \times 6 \times 1$

b) The minimum surface area, if the dimensions are whole numbers, is 66 square units for $4 \times 3 \times 3$ parallelepiped. The minimum surface area, if the dimensions are not whole numbers, is $6(\sqrt[3]{36})^2 = 65.4$ square units (to the nearest tenth).

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **3** Measurement—Inventing Formulas

Focus Student Activity 3.4 (cont.)

c) What is the maximum possible surface area for this parallelepiped? Explain.

5 A box (without a lid) can be formed by cutting squares out of corners of a rectangular sheet of paper, folding the "flaps" that remain after the cuts, and taping . Assume that you can cut out squares whose sides have only whole number lengths.

a) Determine the dimensions of the squares to cut from the corners of a 40" by 50" sheet in order to to maximize the volume of the box.

b) Repeat a) for a 36" by 36" square sheet.

c) What are the dimensions of the box with minimum surface area that can be formed from a 40" × 50" sheet? the box with maximum surface area?

6 Generalize about the dimensions, surface area, and volume of all pairs of trapezoidal prisms for which:

a) one prism is a reduction of the other by a scale factor of $\frac{1}{2}$.

b) one prism is a reduction of the other by a scale factor of $\frac{2}{3}$.

c) one prism is an enlargement of the other by a factor of $\frac{7}{3}$.

d) one prism is an enlargement of the other by a factor of n.

COMMENTS

20 (continued.)
c) There is no maximum surface area, as the surface area can be arbitrarily large. For example, a parallelepiped whose dimensions are .001 × .001 × 36,000,000 has a volume of 36 cubic units and a surface area of 144,000 square units.

5a) Cutting out 7 × 7 inch squares produces a 26 × 36 × 7 inch open-top box with maximum volume 6552 cubic inches.

b) Cutting out 12 × 12 inch squares produces a 12 × 12 × 12 inch box with maximum volume 1728 cubic inches.

c) Cutting out 19 × 19 inch square corners produces a 12 × 2 × 19 inch box with minimum surface area 518 square inches, and cutting out 1 × 1 inch square corners produces a 48 × 38 × 1 inch box with maximum surface area of 1996 square inches.

6a) The smaller prism has a surface area which is $\frac{1}{4}$ that of the larger prism and a volume which $\frac{1}{8}$ that of the larger prism.

b) The smaller prism has a surface area which is $\frac{4}{9}$ that of the larger prism and a volume which is $\frac{8}{27}$ that of the larger prism.

c) The larger prism has a surface area which is $\frac{49}{9}$ times that of the smaller prism and a volume which is $\frac{343}{27}$ times that of the smaller prism.

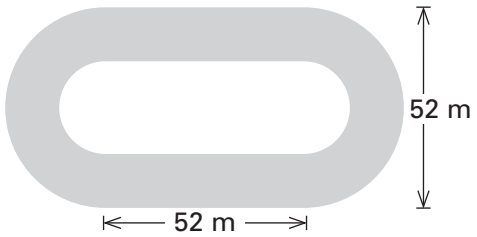
d) The larger prism has a surface area which is n^2 times that of the smaller prism and a volume which is n^3 times that of the larger prism.

Follow-up Student Activity 3.5

NAME _____ **DATE** _____

For this activity use $\pi \approx 3.14$. Record your responses on separate paper, and include a statement of each problem next to your work.

1 The outer edge of the racetrack shown below encloses a square region and 2 half circles.



a) If the racetrack is 6 meters wide, what is the area, to the nearest tenth of a square meter, of *only* the racetrack (shown by the shaded part of the diagram)? Explain your reasoning.

b) How much farther (to the nearest tenth of a meter) would a person running along the *outside* edge of the racetrack run than a person running along the *inside* edge? Explain.

2 For a square, a nonsquare rectangle, and a circle:

a) Suppose each encloses a region with area 100 cm^2. Order these figures from least to greatest perimeter. Explain your reasoning.

b) Suppose each has perimeter 100 cm. Compare their areas. Explain.

3 Given: a nonsquare rectangle with dimensions l and w; a square with side length s; a circle with radius r; a rectangular prism with dimension l, w, and h; and a right circular cylinder with radius r and height h. Support your conclusions about the following:

a) Suppose the sides of the square, both dimensions of the rectangle, and the radius of the circle are all doubled; tripled; multiplied by 5; by ¼; by n. What is the effect of each factor on the perimeter and area of each figure?

(Continued on back.)

Follow-up Student Activity (cont.)

b) What is the effect on surface area and volume if the 3 dimensions of the prism, and the radius and height of the cylinder are doubled? all tripled? multiplied by 5? by ¼? by *n*?

4 A guard dog is chained to a clothesline that is 15 feet long and 5 feet high. The dog's chain is 10 feet long and can slide from one end of the clothesline to the other. Determine the area and perimeter of the dog's territory. Show your reasoning and conclusions.

5 The Big Wheeler ferris wheel has a radius of 25 feet, and the Little Wheeler has a radius of 20 feet. Both ferris wheels makes 4 complete revolutions in one ride. A ride on Big Wheeler costs $1.00. What is a fair price for a ride on Little Wheeler? Explain your reasoning.

6 Outline a presentation about a) and b) below. Then give your presentation to an adult. Turn in your outline, an explanation of anything you did differently from your outline, and 3 "I appreciate…" and 2 "I wish…" statements from the adult.

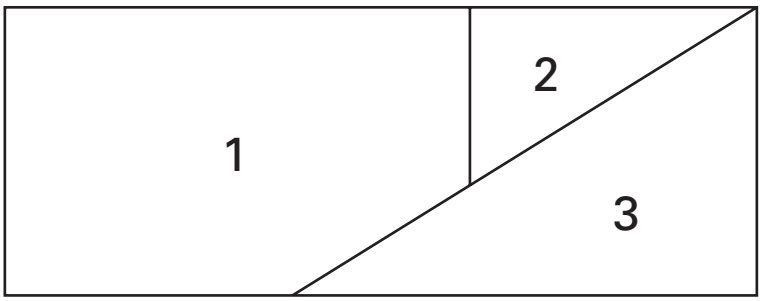
a) The relationships between the circumference, diameter, and radius of a circle.

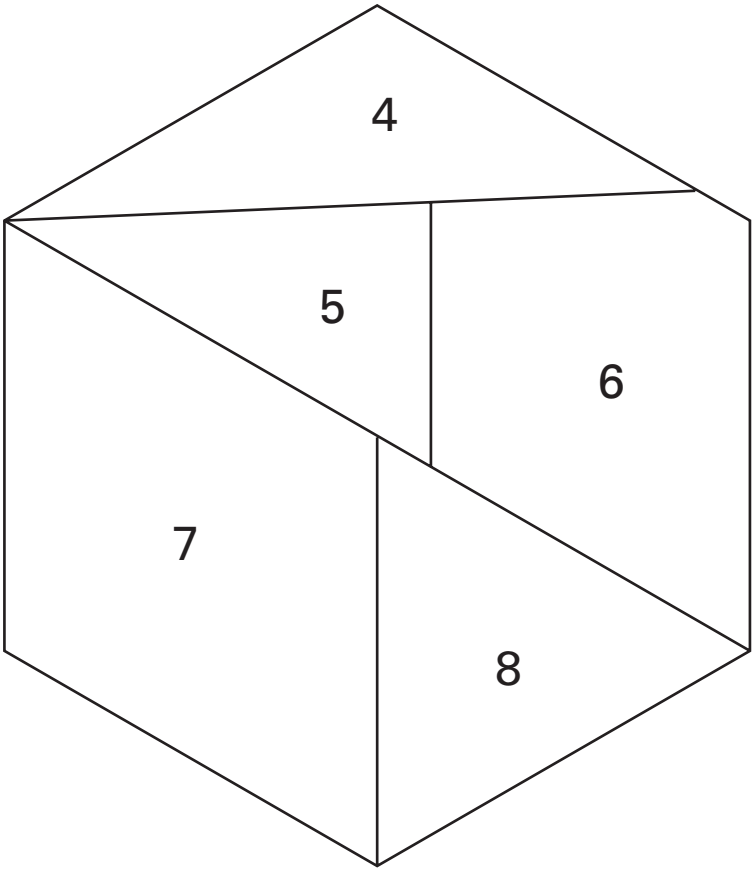
b) Visual "proofs" of formulas for: the area of a square, nonsquare rectangle, parallelogram, trapezoid, right triangle, nonright triangle, and circle; and the volume and surface area of a rectangular prism and a right circular cylinder. For each formula that you prove, give a specific example.

7 Challenge. Suppose the earth is a smooth sphere and a piece of string is wrapped tightly around it at the equator. At the point where the string meets end-to-end with no overlap, 8 feet of string is added. Then, at the equator, the earth is rewrapped by this longer piece so that the distance from the equator to the string is equal all around the earth. Which do you think is more likely to fit between the earth and the lengthened piece of string (predict before problem solving): a) a piece of paper, b) your fist, c) you crawling, or d) you walking? Finally, determine which of a)-d) is actually most likely to fit, and support your conclusion with mathematical reasoning. (Hint: you do not need to know the measure of the earth's radius.)

Connector Master A

The rectangle below has area 4 square units. When cut along the given lines, the rectangle can be reassembled to form a square. Similarly, the hexagon can be cut along the given lines and reassembled to form another square. Use this information to help you find a close approximation of the area of the hexagon. Then write a brief explanation of your methods. Note: use the same area and linear units for the rectangle and the hexagon.

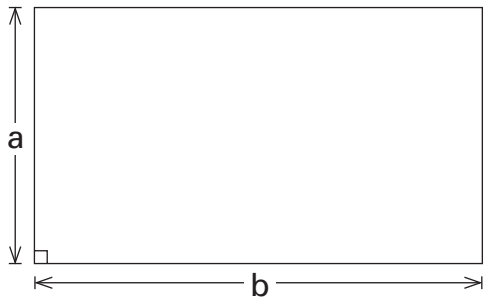




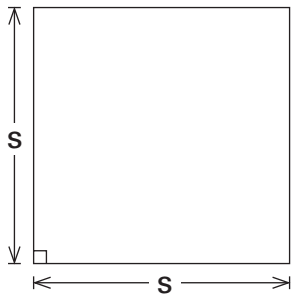
Connector Master B

Focus Master A

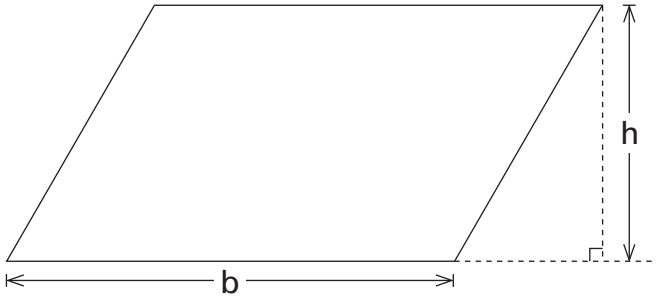
A Rectangle



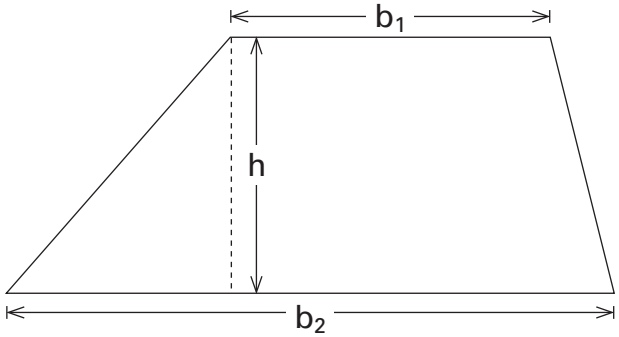
B Square



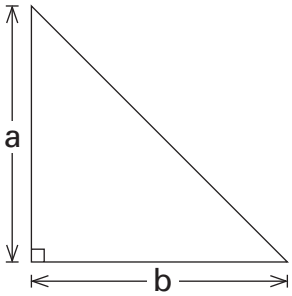
C Parallelogram



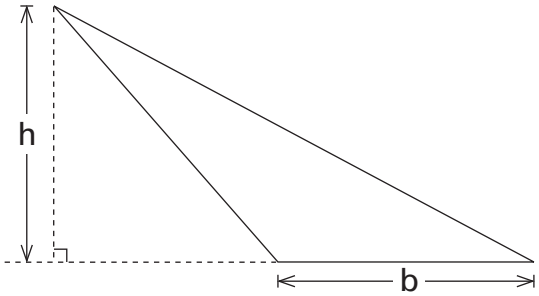
D Trapezoid



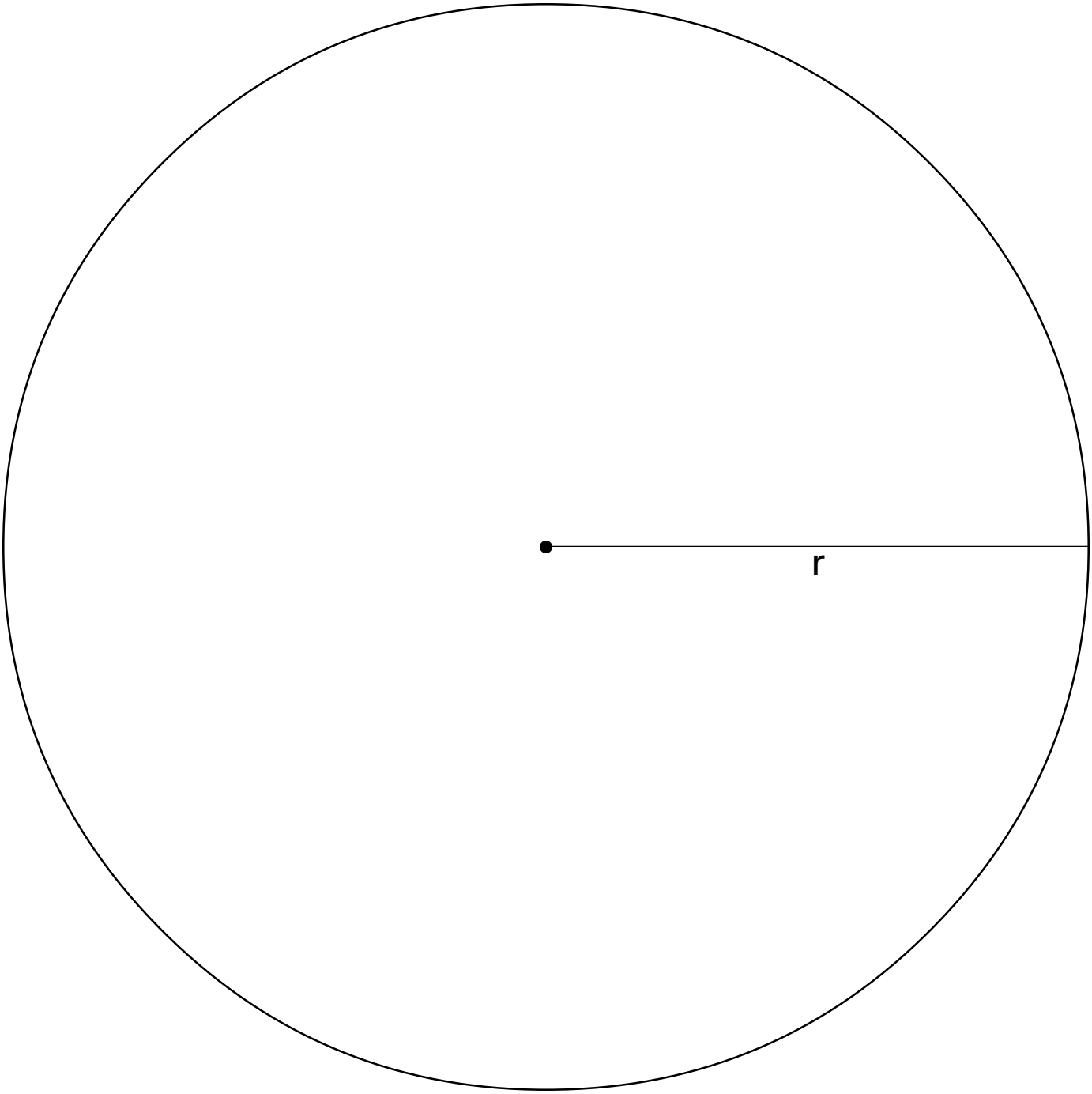
E Right Triangle



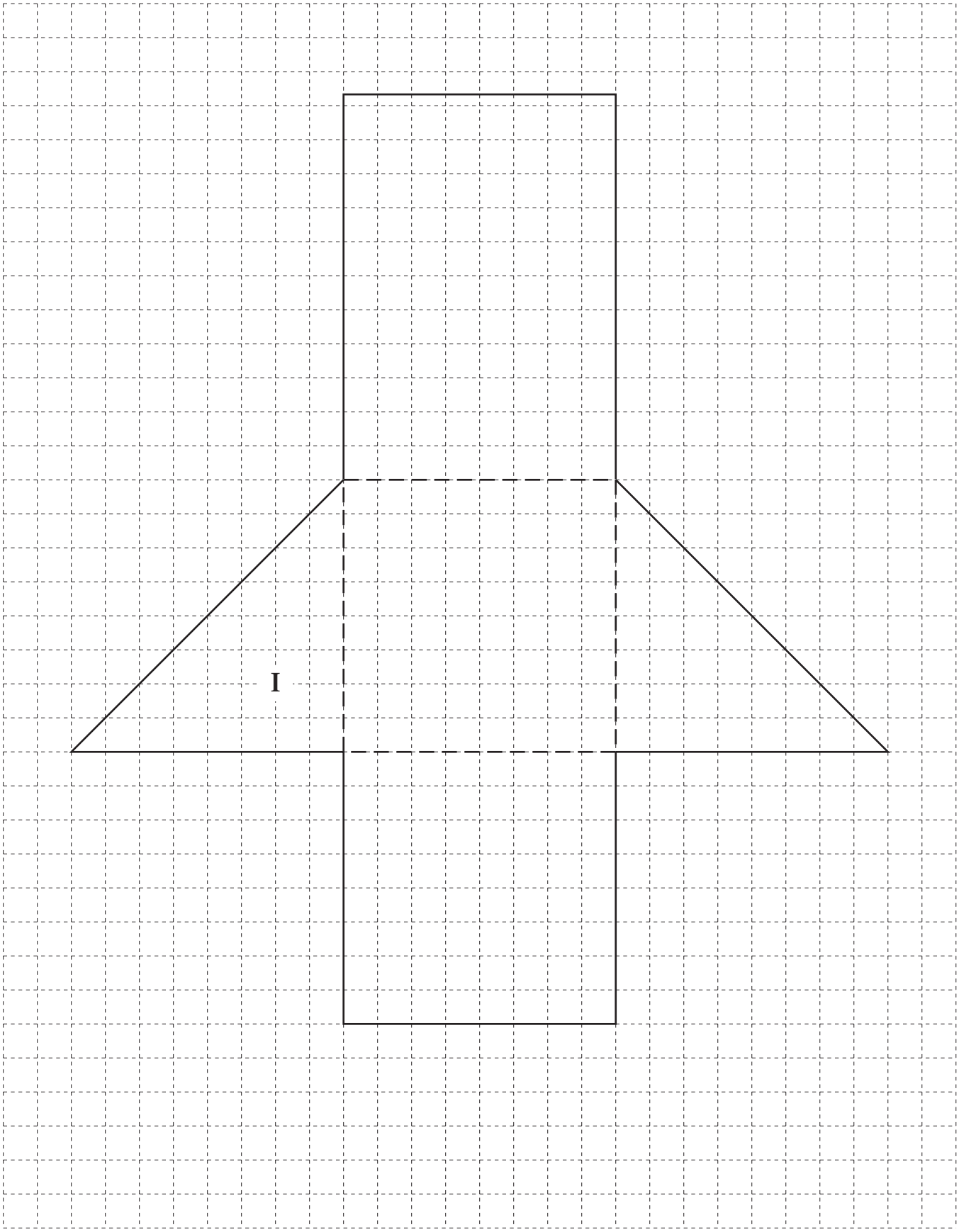
F Nonright Triangle



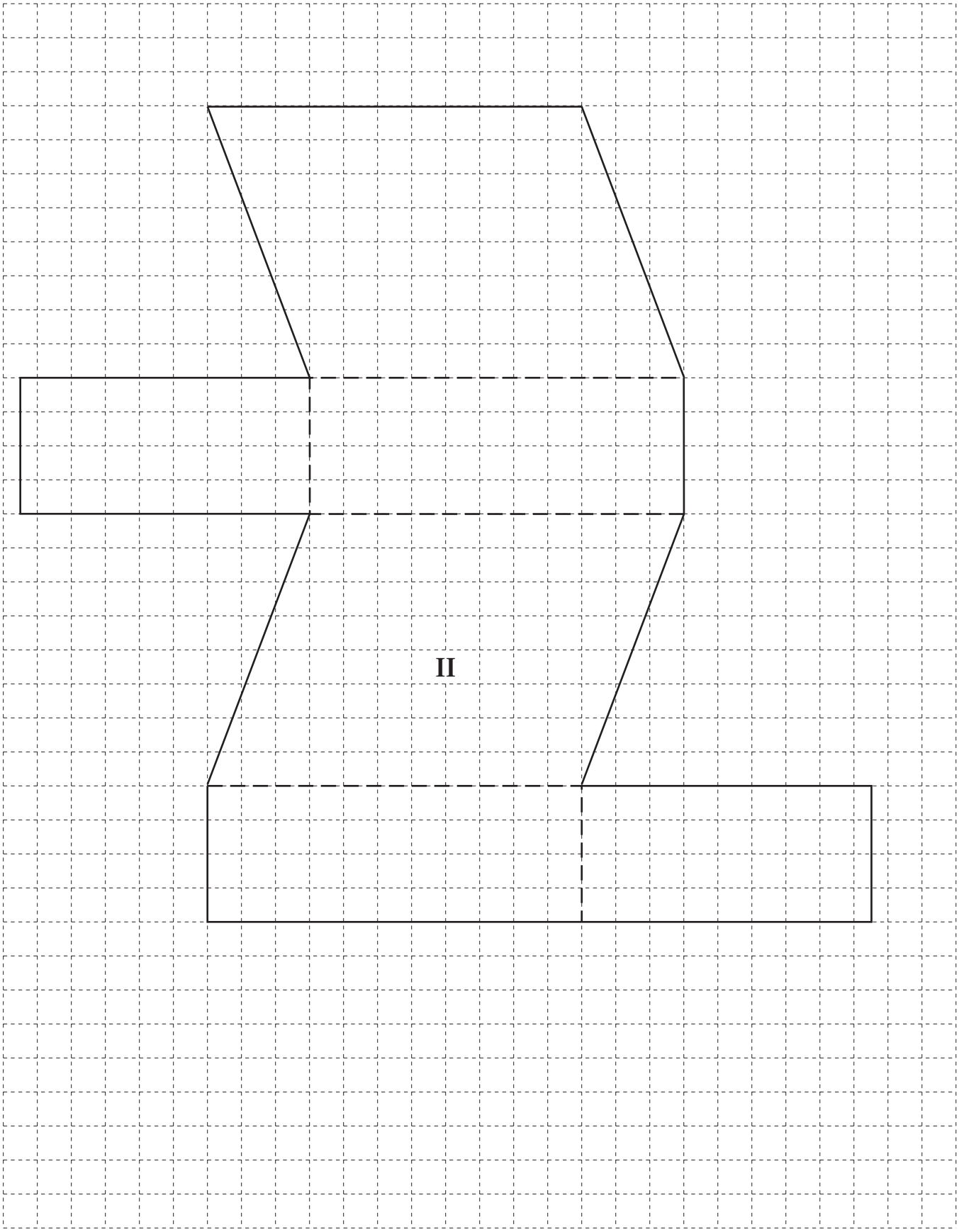
Focus Master B



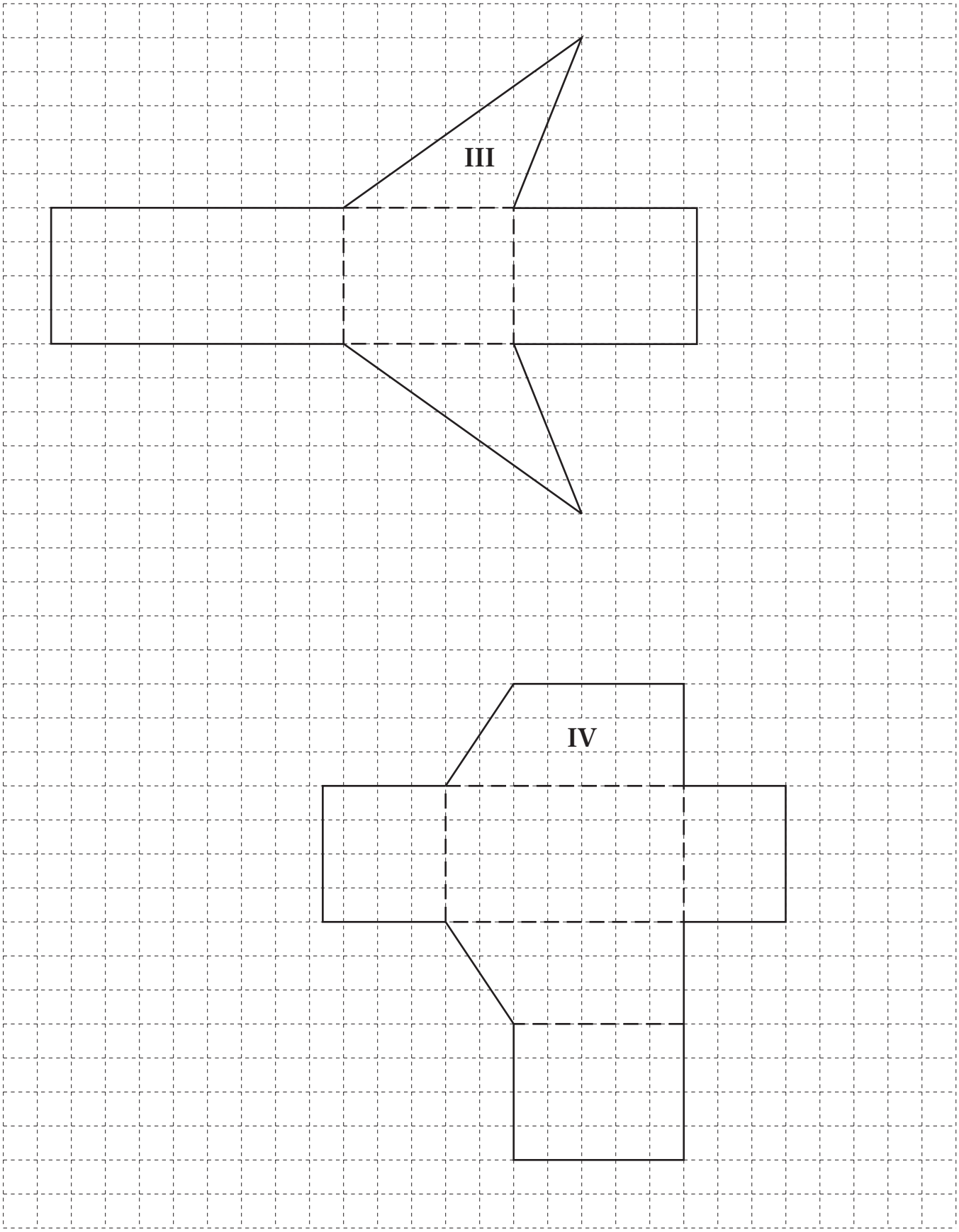
Focus Master C



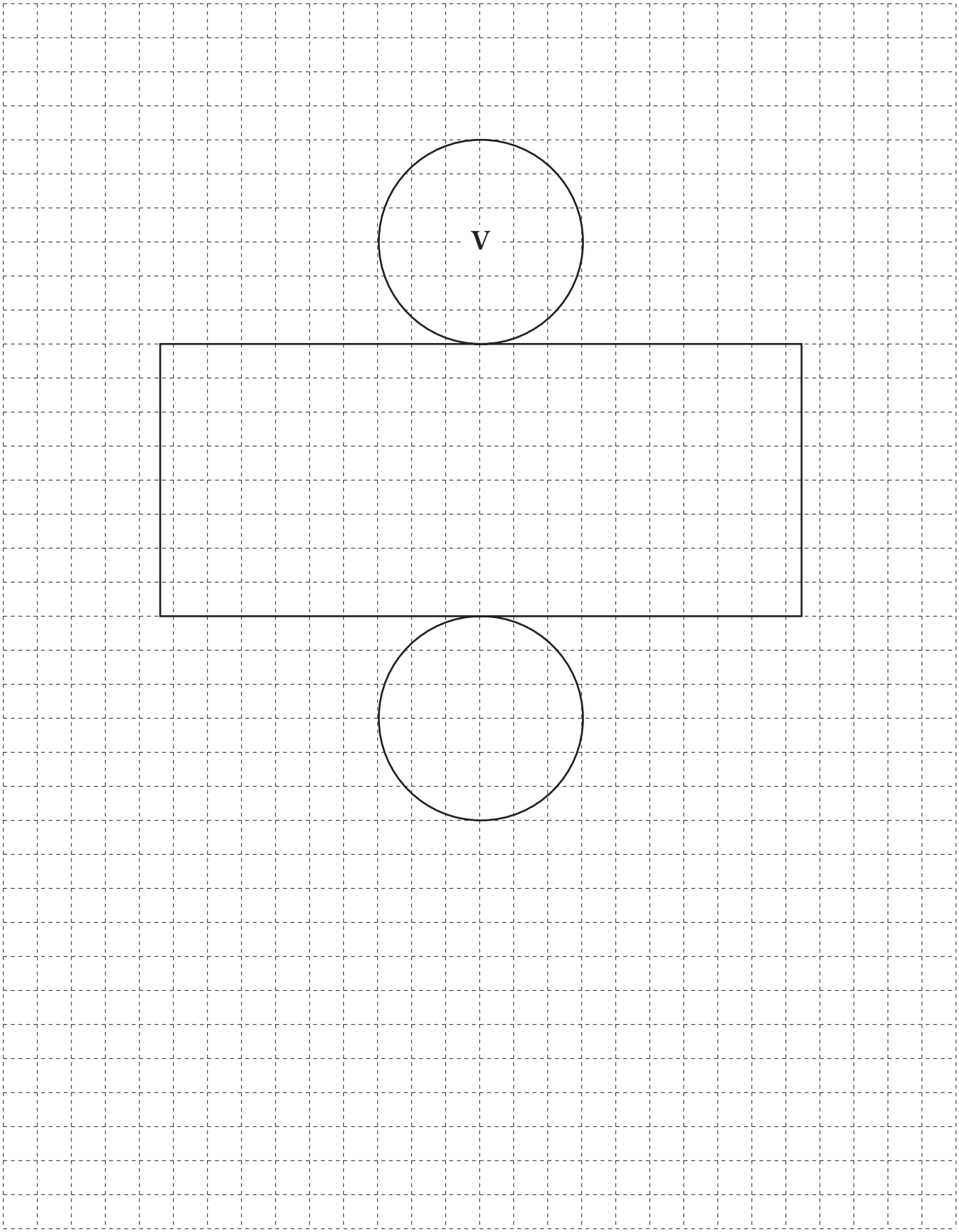
Focus Master C (page 2)



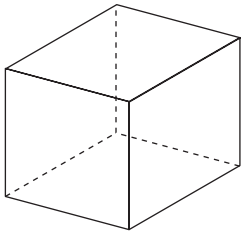
Focus Master C (page 3)



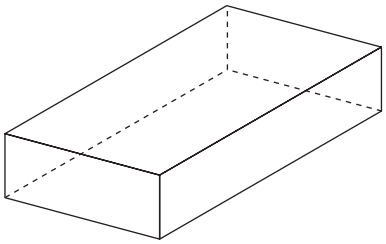
Focus Master C (page 4)



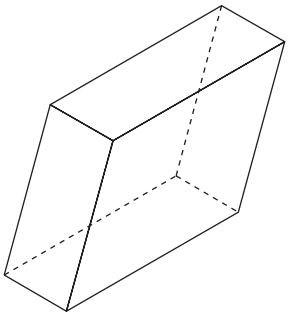
Focus Master D



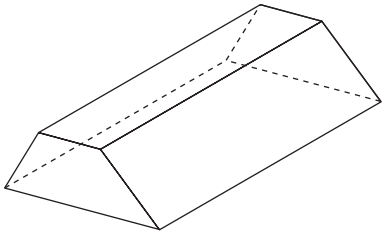
Cube



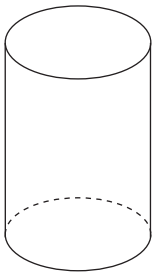
Noncubic Rectangular Prism



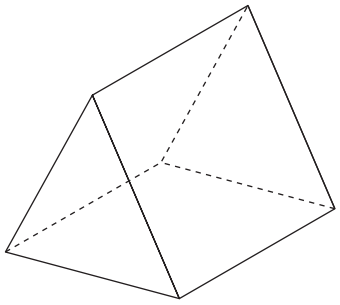
Nonrectangular Parallelepiped



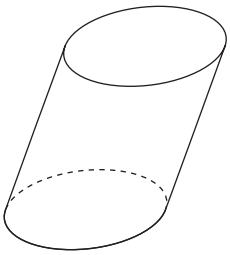
Trapezoidal Right Prism



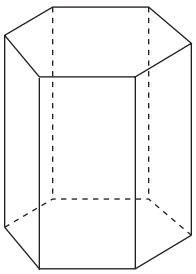
RIght Circular Cylinder



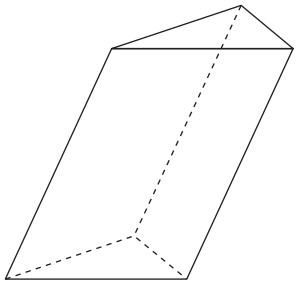
Triangular Prism



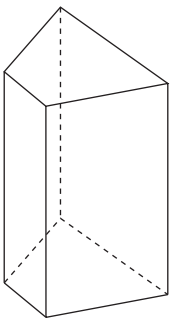
Oblique Circular Cylinder



Hexagonal Right Prism



Oblique Triangular Prism



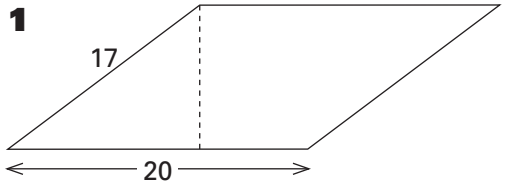
Quadrilateral Right Prism

Focus Student Activity 3.1

NAME _____ **DATE** _____

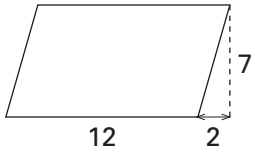
For each of the following polygons, use the given information to help you find the missing information. Mark each diagram or write equations or brief comments that communicate the steps of your thought processes. If it is not possible to find some of the missing information write NP and explain why. Note: diagrams are not drawn to scale; the dotted lines in the diagrams are altitudes; *a* represents area; *p,* perimeter; *h*, height (the length of the altitude); and *x*, *s*, and *b* are lengths that are marked on the diagrams.

1



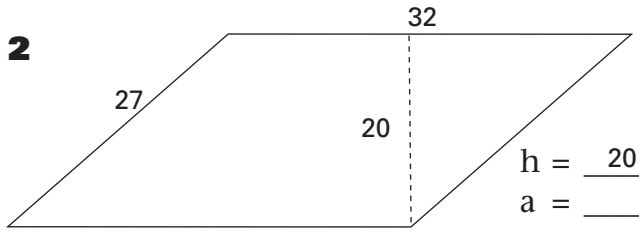
h = _____
a = 300
p = _____

4



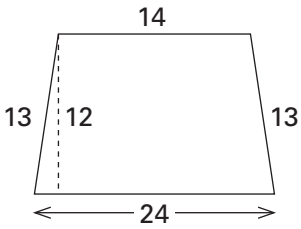
h = 7
a = _____
p = _____

2



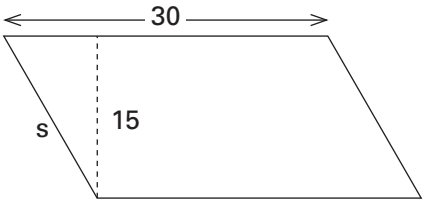
h = 20
a = _____
p = _____

5



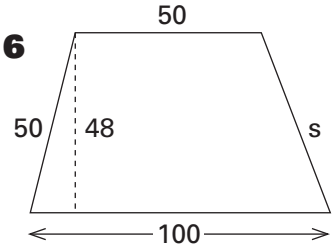
h = 12
a = _____
p = _____

3



h = _____
a = _____
p = 100
s = _____

6

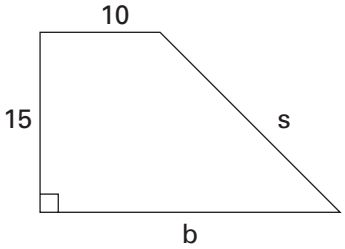


h = _____
a = _____
p = 260
s = _____

(Continued on back.)

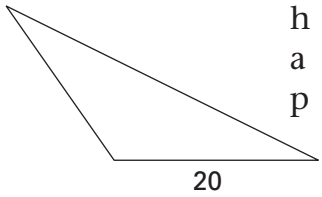
Focus Student Activity 3.1 (cont.)

7



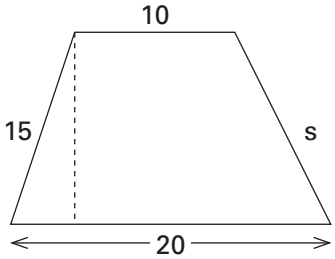
h = _____
a = 300
p = _____
s = _____
b = _____

10



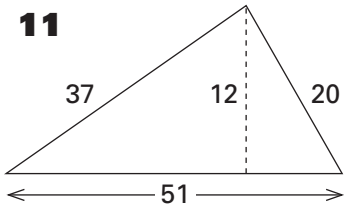
h = _____
a = 150
p = _____

8



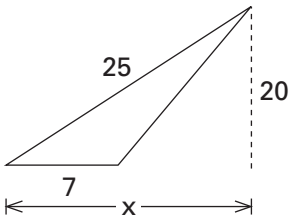
h = _____
a = 204
p = 62
s = _____

11



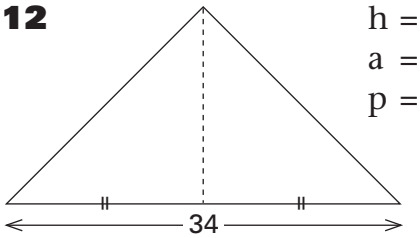
h = 12
a = _____
p = _____

9



h = _____
a = _____
p = _____
x = _____

12



h = _____
a = 216.75
p = _____

Focus Student Activity 3.2

NAME _____ **DATE** _____

Draw a diagram or sketch to solve each of the following problems. Use your calculator to carry out computations, and record all of the calculations that you do. Express each answer to the nearest tenth.

1 A circular lid has a diameter of 7 cm. What is its circumference, to the nearest tenth of a centimeter? Remember to show a sketch, the keys you press to compute the answer, and your answer.

2 A circular fish pond has a circumference of 5.8 meters. What is its diameter?

3 The diameter of one circle is twice as long as the diameter of another circle. How do the circumferences of the circles compare?

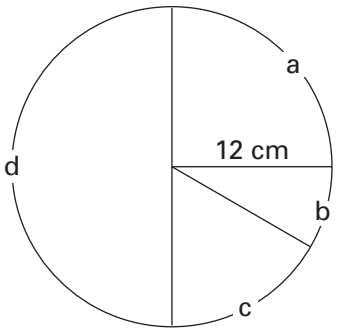
(Continued on back.)

Focus Student Activity 3.2 (cont.)

4 A circular patio has a circumference of 18.7 meters. What is its radius?

5 For a pony ride at the fair, Pete Pony is tied to the end of a 10 foot pole. The other end of the pole is connected to a "hub" that rotates 360°. One day at the fair, Pete completely circled the hub 220 times. How many miles did Pete travel on that day? 5,280 feet = 1 mile.

6 For the circle shown at the right, find a close approximation of the length of each of the indicated arcs (a, b, c, and d). Use math symbols to communicate your method of determining each arc length.



a = _____ Method:

b = _____ Method:

c = _____ Method:

d = _____ Method:

Focus Student Activity 3.3

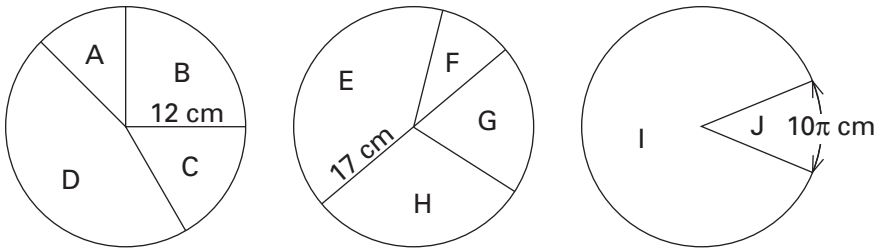
NAME _____ **DATE** _____

Write your responses to problems 1-7 on another sheet of paper. Be sure to put a copy of the problem next to your responses.

1 Write and illustrate two methods that you could use to show a younger student how to approximate the area of a circle with a radius of 7 cm.

2 Explain and sketch a diagram to show how a parallelogram can be used to find the area of a circle with a radius of 17 cm.

3 For each of sectors A-J shown below, without using a protractor, predict the area of the sector and the arc length along the edge of the sector. Record your predictions and reasoning. Then use a protractor to verify your predictions. Record your measurements and calculations.



4 Papa Paulo's Pizza Crust Company sells a crust with a 13 inch diameter for $2.29. They have decided to add a square crust to their product line. Assuming Papa Paulo wants to price all of his crusts at the same rate, what should he charge for a square crust with side length 13 inches?

5 On Carmen's clock, the second hand is 15 cm long and the hour hand is 10 cm long. Explain your methods of solving each of the following problems about her clock. Give answers to the nearest tenth of a centimeter.

a) How far does the tip of the second hand travel in 45 minutes?

b) How far does the tip of the hour hand travel in 45 minutes?

(Continued on back.)

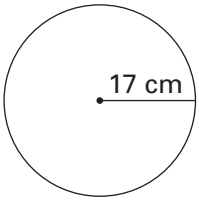
Focus Student Activity 3.3 (cont.)

6 In order to get a sign post that rises 8 feet above the ground to stand at right angles, Zane fastened one end of a wire to the top of the post and the other end to the ground 3 feet from the base of the post. If she needs 6 extra inches of wire on each end for fastening, how long (to the nearest inch) must she cut the wire? Make a diagram and show your reasoning.

7 What is the area (to the nearest square centimeter) of the largest circle that can be cut from a square piece of sheet metal 73 cm on a side? Explain your reasoning.

8 The following figures show the radius, diameter, circumference, or area of a circle. Use the information given to *mentally approximate* the requested measures. Use math symbols to communicate your *mental* calculations.

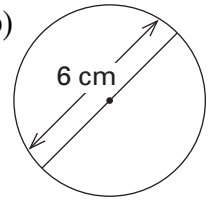
a)



Circumference ≈_____
Mental calculations:

_____

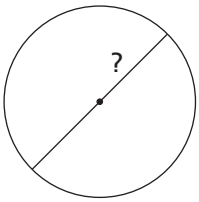
b)



Area ≈_____
Mental calculations:

_____

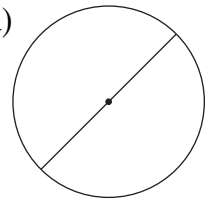
c)



Area = 75 cm^2
Radius ≈_____
Mental calculations:

_____

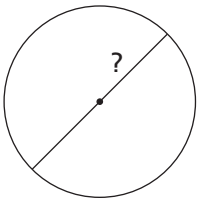
d)



Circumference = 21 cm
Area ≈_____
Mental calculations:

_____

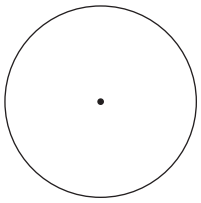
e)



Circumference = 15 cm
Radius ≈_____
Mental calculations:

_____

f)



Area = 48 cm^2
Circumference ≈_____
Mental calculations:

_____

Focus Student Activity 3.4

NAME _____ **DATE** _____

1 Suppose you need to fill ½ of a right cylindrical can with water. Assuming that you have no measuring tools and no other containers, how can you determine when the can is half full?

2 When an object is placed in a container of water it changes the level of the water. The amount of water that is *displaced* is equal to the volume of the object. This fact is useful for finding the volume of irregularly shaped objects. Suppose that:

a) You drop a rock into a 5" × 5" × 12" square prism that is filled with water to a height of 7". The rock displaces the water level by 4". What is the volume of the rock?

b) You drop some coins in a right circular cylinder with radius 7 cm and altitude 15 cm. The coins displace the water level from 6 cm to 13 cm. What is the volume of the coins to the nearest cubic centimeter?

c) Gilly Goldfish and his sisters all jump out of their "filled-to-the-brim" rectangular-prism-shaped fish tank. The water level in this 12" by 14" by 18" tank drops ½" below the 18" height. What is the total volume of Gilly and his sisters?

3 The perimeter of a face of Cube X is $4(\sqrt[3]{125})$ inches. Determine the surface area and volume of Cube X. Show your methods and reasoning.

4 The following questions relate to a rectangular parallelepiped with volume 36 cubic units:

a) What are all the possible whole number dimensions this parallelepiped could have?

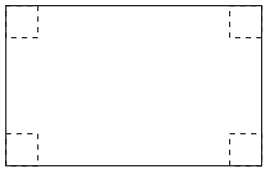
b) What is the minimum possible surface area for this parallelepiped if the dimensions are only whole numbers? if the dimensions are not whole numbers?

(Continued on back.)

Focus Student Activity 3.4 (cont.)

c) What is the maximum possible surface area for this parallelepiped? Explain.

5 A box (without a lid) can be formed by cutting squares out of corners of a rectangular sheet of paper, folding the "flaps" that remain after the cuts, and taping . Assume that you can cut out squares whose sides have only whole number lengths.



a) Determine the dimensions of the squares to cut from the corners of a 40" by 50" sheet in order to to maximize the volume of the box.

b) Repeat a) for a 36" by 36" square sheet.

c) What are the dimensions of the box with minimum surface area that can be formed from a 40" × 50" sheet? the box with maximum surface area?

6 Generalize about the dimensions, surface area, and volume of all pairs of trapezoidal prisms for which:

a) one prism is a reduction of the other by a scale factor of $\frac{1}{2}$.

b) one prism is a reduction of the other by a scale factor of $\frac{2}{3}$.

c) one prism is an enlargement of the other by a factor of $\frac{7}{3}$.

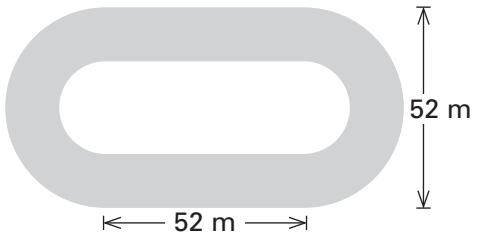
d) one prism is an enlargement of the other by a factor of n.

Follow-up Student Activity 3.5

NAME _____ **DATE** _____

For this activity use $\pi \approx 3.14$. Record your responses on separate paper, and include a statement of each problem next to your work.

1 The outer edge of the racetrack shown below encloses a square region and 2 half circles.



a) If the racetrack is 6 meters wide, what is the area, to the nearest tenth of a square meter, of *only* the racetrack (shown by the shaded part of the diagram)? Explain your reasoning.

b) How much farther (to the nearest tenth of a meter) would a person running along the *outside* edge of the racetrack run than a person running along the *inside* edge? Explain.

2 For a square, a nonsquare rectangle, and a circle:

a) Suppose each encloses a region with area 100 cm^2. Order these figures from least to greatest perimeter. Explain your reasoning.

b) Suppose each has perimeter 100 cm. Compare their areas. Explain.

3 Given: a nonsquare rectangle with dimensions l and w; a square with side length s; a circle with radius r; a rectangular prism with dimension l, w, and h; and a right circular cylinder with radius r and height h. Support your conclusions about the following:

a) Suppose the sides of the square, both dimensions of the rectangle, and the radius of the circle are all doubled; tripled; multiplied by 5; by ¼; by n. What is the effect of each factor on the perimeter and area of each figure?

(Continued on back.)

Follow-up Student Activity (cont.)

b) What is the effect on surface area and volume if the 3 dimensions of the prism, and the radius and height of the cylinder are doubled? all tripled? multiplied by 5? by ¼? by *n*?

4 A guard dog is chained to a clothesline that is 15 feet long and 5 feet high. The dog's chain is 10 feet long and can slide from one end of the clothesline to the other. Determine the area and perimeter of the dog's territory. Show your reasoning and conclusions.

5 The Big Wheeler ferris wheel has a radius of 25 feet, and the Little Wheeler has a radius of 20 feet. Both ferris wheels makes 4 complete revolutions in one ride. A ride on Big Wheeler costs $1.00. What is a fair price for a ride on Little Wheeler? Explain your reasoning.

6 Outline a presentation about a) and b) below. Then give your presentation to an adult. Turn in your outline, an explanation of anything you did differently from your outline, and 3 "I appreciate…" and 2 "I wish…" statements from the adult.

a) The relationships between the circumference, diameter, and radius of a circle.

b) Visual "proofs" of formulas for: the area of a square, nonsquare rectangle, parallelogram, trapezoid, right triangle, nonright triangle, and circle; and the volume and surface area of a rectangular prism and a right circular cylinder. For each formula that you prove, give a specific example.

7 Challenge. Suppose the earth is a smooth sphere and a piece of string is wrapped tightly around it at the equator. At the point where the string meets end-to-end with no overlap, 8 feet of string is added. Then, at the equator, the earth is rewrapped by this longer piece so that the distance from the equator to the string is equal all around the earth. Which do you think is more likely to fit between the earth and the lengthened piece of string (predict before problem solving): a) a piece of paper, b) your fist, c) you crawling, or d) you walking? Finally, determine which of a)-d) is actually most likely to fit, and support your conclusion with mathematical reasoning. (Hint: you do not need to know the measure of the earth's radius.)

Arithmetic Sequences

Lesson 4

THE BIG IDEA

Determining the number of handshakes if everyone in a room shakes hands provides a context for introducing the use of staircases to represent and add consecutive whole numbers and for illustrating the use of visual thinking in problem solving. Staircase models enable students to "see" formulas for computing the sum of consecutive whole numbers and to develop general formulas for the sum of the terms in any arithmetic sequence.

CONNECTOR

OVERVIEW

Students predict the number of handshakes required if all students in the class shake hands with each other. They design an organized procedure for physically carrying out the shaking of hands so that counting the number of handshakes is possible.

MATERIALS FOR TEACHER ACTIVITY

- ✔ One small slip of paper per student.
- ✔ Connector Student Activity 4.1, 1 copy per student.
- ✔ Straightedge for each student.

FOCUS

OVERVIEW

The students develop "staircase methods" of computing the sums of terms in arithmetic sequences. These methods lead to formulas and other generalizations about arithmetic sequences.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Cubes (1-cm or 2-cm linking cubes), 30 per student.
- ✔ Focus Masters A and B, 1 transparency of each.
- ✔ Focus Master C, 1 copy per pair of students and 1 transparency.
- ✔ Focus Student Activity 4.2 (optional), 1 copy per student.
- ✔ Butcher paper, 1 sheet per pair of students.
- ✔ Butcher paper strips (optional), 16 strips per pair of students, and 8 for the teacher.
- ✔ Marking pens for each pair of students.

FOLLOW-UP

OVERVIEW

Students use the staircase model to find the missing information in mystery sequences, and to illustrate various algebraic expressions for the terms and the sum of an arithmetic sequence.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 4.3, 1 copy per student.

LESSON IDEAS

FAITHFUL MODELS
Throughout *Math Alive!* emphasis is placed on the development of mathematically "faithful" models, i.e., models that are meaningful and useful as mathematical topics become more complex. It is not intended that other models be discounted or rejected, or that students are given the feeling the teacher's way is the best way. Rather, it is hoped that students will understand that emphasis is placed on a particular model because the model has far reaching mathematical implications. The staircase model for summing arithmetic sequences is an example.

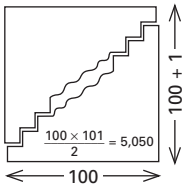
JOURNAL
At the completion of this lesson, you might ask students to do the following in their journals:

Write 3 problems for a "test" about arithmetic sequences. Create problems that you feel represent the 3 most important ideas from this lesson. Next to each problem, record the answer and your methods or reasoning.

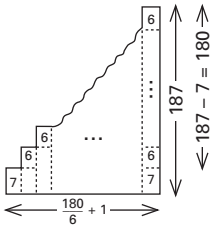
FOLLOW-UP
It is important that students have several days to work on this Follow-up, and that they have opportunities to discuss places they get "stuck." To help students get "unstuck," you might regularly provide opportunities for them to ask for clues (not solutions) from the class. Or, you might record the Follow-up problem numbers on the board and ask students to write their names next to specific problems for which they feel comfortable serving as a resource to their classmates. You could provide time for students to confer with their "peer resources" during class and/or suggest they do so outside of class.

SELECTED ANSWERS

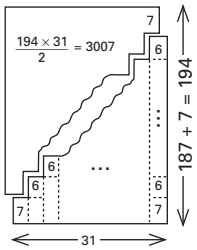
2.



3. b) ii) Since a 6 is added for every step *after* the first step, there are $^{(187-7)}/_{6} + 1 = 31$ steps in the staircase.

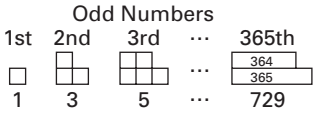


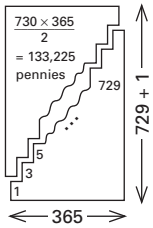
iii)



5. a) $S_n = 1 + 2 + 3 + ... + n$, where n is the number of students in your class, so the total cost is $\$1.50 \times S_n$.

b) The number of pennies Holly would earn on the 365th day is the 365th odd number:

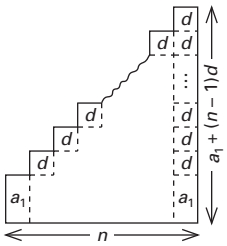




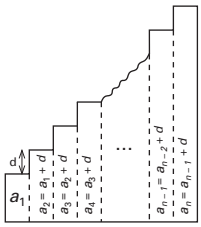
The total number of pennies she would earn is 1 + 3 + 5 + ... + 729. So, Holly would earn $1332.25 for the year.

7. a) 1, 5, 9, 13, 17,... 225, 229, 233, 237, 241
b) 29, 37, 45, 53, 61,... 125, 133, 141, 149, 157
c) 7, 8, 9, 10, 11,... 42, 43, 44, 45, 46

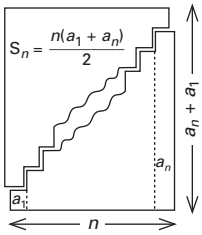
8. a) The height of the nth step, a_n, is equal to the height of the first step, a_1, plus $(n - 1)$ groups of d [i.e., the value of the last term, a_n, is found by adding $(n - 1)$ groups of the common difference, d, to the first term, a_1].



b) The height of the nth step of the staircase is d more than the height of the step that precedes [i.e., the value of the nth term of the series is d more than the value of the $(n - 1)$st term].



c) The sum of the heights of the steps in the staircase is ½ the area of the rectangle formed by joining 2 of the staircases (i.e., the sum of the first n terms of the series is equal to ½ the product of the number of terms and the sum of the first and last terms). Or, the sum of the first n terms is equal to the product of the number of terms times the average of the first and the last terms.



Connector Teacher Activity

OVERVIEW & PURPOSE

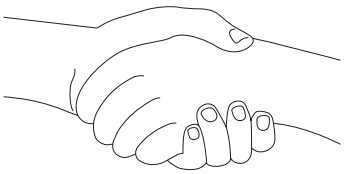
Students predict the number of handshakes required if all students in the class shake hands with each other. They design an organized procedure for physically carrying out the shaking of hands so that counting the number of handshakes is possible.

MATERIALS

✔ One small slip of paper per student.
✔ Connector Student Activity 4.1, 1 copy per student.
✔ Straightedge for each student.

ACTIONS

1 Distribute a small slip of paper to each student. Arrange the students in groups. Mention that when people get together, they often shake hands with one another. Ask the students to each privately guess the number of handshakes there would be if everyone in the room shook hands with everyone else, and to anonymously record their guess on the slip of paper. Make certain the students understand what constitutes a single handshake.



2 Collect the guesses and, without comment, record them on the chalkboard or overhead.

3 Pick a student, or ask for a volunteer, to assist you. Explain to your assistant that you want her or him to help check the guesses against the actual number of handshakes.

4 Ask the students to get up and shake hands with one another. Ask your assistant to count the handshakes. If, in a minute or so, your assistant does not inform you of the hopelessness of this task, suggest that you would like to ask the class for help in getting an accurate count.

5 Get the students' attention. Ask them to suggest procedures for shaking hands that will allow your assistant to count the number of handshakes.

COMMENTS

1 Asking for individual guesses will encourage the students to formulate their own thoughts about the problem. The idea at this point is to get their first intuitions, not to calculate or problem solve.

In this activity, 2 people shaking hands is counted as 1 handshake. You could illustrate this by shaking hands with a student while saying, "This is one handshake."

2 Avoid reactions that might be construed as value judgements about students' guesses.

3 The assistant will be asked to do an impossible job—this may influence who you pick.

4 The students may be hesitant to start shaking hands with each other. Encourage them by moving around the room, randomly shaking hands with students. The intent is to create a setting in which it is impossible for your assistant to count all the handshakes taking place. This provides a graphic picture of the need for a systematic procedure.

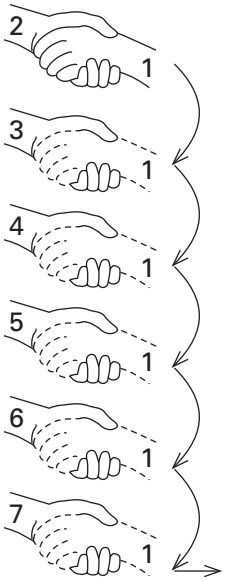
5 A number of procedures may be suggested. You may need to clarify some of the suggestions, but avoid judging one better than another.

Following are two procedures frequently suggested by students:

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



6 Select one of the procedures suggested by the students in Action 5, and ask for 6 students to demonstrate that method for determining the number of handshakes if everyone in a group of 6 students shakes hands with one another. Discuss.

Student		No. of Shakes
First Student	1	
Second Student	1 2	1
Third Student	1 3 2 3	+2
Fourth Student	1 4 2 4 3 4	+3
Fifth Student	1 5 2 5 3 5 4 5	+4
Sixth Student	1 6 2 6 3 6 4 6 5 6	+5
Total		+15

COMMENTS

5 (continued.)
Procedure A: Everyone lines up in a row. The first person walks down the row, shaking hands with each person (see the illustration at the left), then sits down. Then the second person does the same. This procedure continues with the third, fourth, etc., until only one person is left in the row.

Procedure B: One person goes to the center of the room. Then a second person goes to the center and shakes hands with the person there. Next, a third person goes to the center and shakes hands with the two persons already there. One at a time, people continue going to the center of the room and shaking hands with all the people there until everyone is in the center of the room.

In the above procedures, the total number of handshakes could be determined by adding the number of handshakes completed by each person as they enter the center of the room or pass down the row.

6 You may want to let the students pick the procedure to try. You might have the student who suggested the procedure carry it out with a group of students, while the rest of the class observes. The size of the group is not important, although it should be large enough that students can see how the procedure would be carried out if everyone in the room participated. Be sure the students see that the number of handshakes for a group of 6 is the sum of the counting numbers from 1 through 5.

Shown at the left is a diagram for 6 students carrying out Procedure B in Comment 5. The first student goes to the center of the room. A second student goes to the center of the room and shakes hands with the student there. Then a third student goes to the center and shakes hands with the 2 students who are already there. This process continues until the sixth student goes to the center and shakes hands with the 5 students who are there. As each student shakes hands, you can record the number of handshakes on the chalkboard or overhead. In this procedure, for 6 students, the number of handshakes is $1 + 2 + 3 + 4 + 5 = 15$.

If Procedure A in Comment 5 is carried out with a group of 6 students, the first student in the row will shake 5 hands before sitting down. The next person will shake hands with the 4 other students remaining in the row—he or she has already shaken hands with the person who sat down. The next person will shake 3 hands, the next 2. The next to last person will shake hands once (with

Connector Teacher Activity (cont.)

ACTIONS

7 Discuss with the students how the method used in Action 6 could be extended to determine the total number of handshakes if everyone in the room shook hands with one another. What calculation would be needed to find this number? Note: at this point, it is not important that students actually compute the answer; rather, they only need to identify which numbers must be added.

8 Ask the students to imagine carrying out the handshake procedure in a room of 50 people (and/or some other large number of people), and to describe the computation required to find the total number of handshakes in that group.

9 (Optional) Read the following aloud and ask the students to imagine the handshaking in their mind's eye.

> **When the 9 justices of the Supreme Court convene, they each shake hands with one another.**

Next read a) below and ask the students to discuss their ideas with their partners. Discuss as a large group. Then repeat for b) and c).

> **a) What computation is needed to determine the number of handshakes among the 9 justices?**
>
> **b) Suppose only 5 of the judges shake hands with each other; then the other 4 arrive. How many more handshakes will there be?**
>
> **c) The judges form two groups. Suppose the 6 in one group have shaken hands with each other; so have the 3 in the other group. How many more handshakes will there be if the 2 groups meet?**

COMMENTS

the last person). There is no one left for the last person to shake hands with. Thus, for 6 students, the number of handshakes is 5 + 4 + 3 + 2 + 1 = 15.

Notice, in both procedures, the number of handshakes for 6 students is the sum of the first 5 positive whole numbers.

7 You can ask the students to imagine carrying out the procedure you used in Action 6 with everyone in the room participating. If, for example, there are 32 people in the room, the number of handshakes will be 1 + 2 + 3 + … + 31. Note: in the Focus activity students will develop methods of computing this sum.

8 Again, it isn't expected that students compute the answer here; rather, they simply tell what calculations are required to determine the number of handshakes.

Recalling images of the class experience in Action 6 can help students to visualize the situation for any size group and see that, in general, the number of handshakes is the sum of the whole numbers beginning with 1 and ending with 1 less than the number of people shaking hands.

9 If students have a difficult time imagining the actions described in the problem, you could have 9 students act out the problem while the rest of the class observes.

a) 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36. Some students may compute this sum mentally.

b) The first of the 4 to arrive will shake 5 hands, the next 6, the next 7, and the last to arrive will shake 8 hands. So there will be 5 + 6 + 7 + 8, or 26, more handshakes.

Students may have other ways of arriving at the answer.

c) Each of the 6 will shake 3 hands. So there will be 6×3, or 18, more handshakes.

Connector Teacher Activity (cont.)

ACTIONS

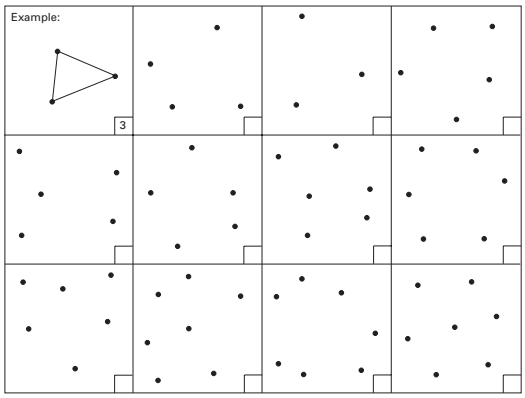
10 Give each student a copy of Connector Student Activity 4.1 and ask them to complete Problem 1. Discuss their results. Then have the students complete and discuss Problems 2-4.

Arithmetic Sequences Lesson **4**

Connector Student Activity 4.1

NAME _____ **DATE** _____

1 In each square below use a straightedge to draw straight line segments that connect each dot to *all* other dots in the square. Then, in the corner of each square, write how many segments you drew.



2 Tell about any patterns you notice or conjectures you have about counting segments between dots.

(Continued on back.)

COMMENTS

10 This could be used as homework. Students may find that using different colored pencils to sketch the lines simplifies counting.

Lesson **4** Arithmetic Sequences

Connector Student Activity (cont.)

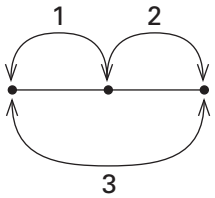
3 A student in another class said she thought that finding the number of segments was just like solving the "handshake problem." What do you think she meant?

4 Suppose you have to find the number of lines it takes to connect 125 dots. Describe a way to find this number without having to draw the segments and dots. (Just tell your method—you don't need to compute the answer!)

Following are answers to Problems 1-4 on Connector Student Activity 4.1.

1)	**3**	6	3	10
	10	10	15	15
	15	21	21	21

Notice that in the last square, there are 3 different sets of 3 collinear points. For each set of 3 collinear points there are 3 segments (representing 3 handshakes) formed when the 3 points are connected, as shown below. Note: points that lie on the same line are *collinear*.



Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

2) Here are examples of observations students have given:

Sets containing the same number of points have equal numbers of connecting segments, regardless the arrangements of the points.

The number of lines for increasing numbers of points form a pattern: their differences are the consecutive whole numbers, as illustrated here:

Number of Points:	2	3	4	5	6	7
Number of Lines:	1	3	6	10	15	21
Differences:	2	3	4	5	6	

If the number of segments connecting n points is p, then the number of segments connecting n + 1 points is n + p.

3) This activity is similar to the handshake problem (i.e., determining the number of handshakes completed when everyone in the class shook hands) because 2 connecting arms are like a line segment connecting 2 points.

4) Find the sum of the whole numbers from 1 through 124.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

The students develop "staircase methods" of computing the sums of terms in arithmetic sequences. These methods lead to formulas and other generalizations about arithmetic sequences.

MATERIALS

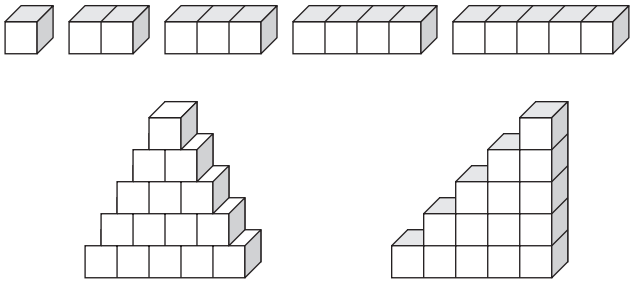
- ✔ Cubes (1-cm or 2-cm linking cubes), 30 per student.
- ✔ Focus Masters A and B, 1 transparency of each.
- ✔ Focus Master C, 1 copy per pair of students and 1 transparency.
- ✔ Focus Student Activity 4.2 (optional), 1 copy per student.
- ✔ Butcher paper, 1 sheet per pair of students.
- ✔ Butcher paper strips (optional), 16 strips per pair of students, and 8 for the teacher.
- ✔ Marking pens for each pair of students.

ACTIONS

1 Arrange the students in pairs and distribute 30 cubes to each student. Write the expression "1 + 2 + 3 + 4 + 5" on the chalkboard or overhead. Ask the students to think privately for a few moments regarding how they would arrange the cubes to model the meaning of this expression. Then ask them to make whatever arrangement came to mind. Emphasize that there are many ways to do this and you anticipate a variety of models.

2 Acknowledge, without judgement, the different models that students form. Discuss ways the students "see" the numbers 1 through 5 in their models and ways they "see" addition represented in the different models.

Models of 1 + 2 + 3 + 4 + 5



The Staircase Model

COMMENTS

1 Asking the students to think privately for a few moments before arranging the cubes will help them focus on the task and not wait to see what a classmate does.

Linking cubes, either 1-cm or 2-cm, are most convenient for this activity. However, nonlinking cubes will also work.

2 Students could demonstrate their methods by holding their models up for the class to see, or they could move about the room to see the different models displayed on the students' desks or tables.

Seeing the models will give you an idea about ways the students relate numbers and objects. For example, some may form numerals with the cubes. These students may associate school mathematics with symbols and their manipulation.

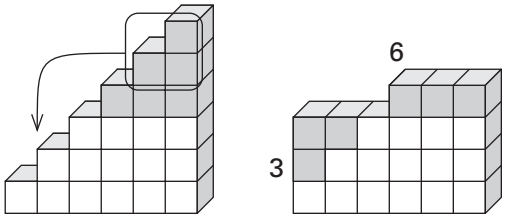
Some students may begin with rows or columns of cubes representing the numbers 1-5, and then push the cubes together so there are no gaps, thus showing addition as the process of "joining together" the numbers. If several students form staircases (see the staircase model at the left), although the models may look the same, the ways students view them as representations of 1 + 2 + 3 + 4 + 5 may differ. For example, some may "see" horizontal rows of 1, 2, 3, 4, and 5 cubes beginning with the top row and moving down the staircase, while others may see vertical columns of cubes. Still others may see diagonals.

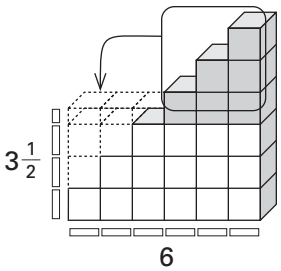
Focus Teacher Activity (cont.)

ACTIONS

3 Focus the students' attention on a "staircase" (see diagram on the previous page) from the models formed in Action 2. Ask each student to form a staircase model for the sum 1 + 2 + 3 + 4 + 5.

Ask each pair of students to determine and record different methods of "seeing" and counting (other than standard counting procedures) the total number of cubes in a staircase representing the sum 1 + 2 + 3 + 4 + 5. For each method that students invent, ask them to determine whether that method works for staircases representing 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 and 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10. Discuss the various methods. Encourage attempts to generalize.



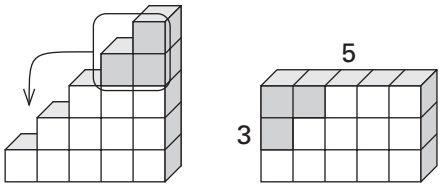


COMMENTS

3 If no students formed a staircase, pose one as another way of viewing the sum 1 + 2 + 3 + 4 + 5. It may be helpful to discuss the fact that, while there are many models possible, sometimes one model may reveal more mathematical relationships, and hence provide more insights, than another. For example, the staircase model is especially useful in finding the sum of consecutive whole numbers and, later, sums of other arithmetic sequences of numbers.

Note that students may invent formulas to represent their methods, or they may only give verbal explanations. Either is okay at this point. Notice that formulas are developed in Action 4. Following are several methods that students have suggested.

Method A. *We reshaped the staircase to form a 3 × 5 rectangle.*



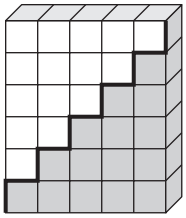
After studying other staircases, we conjecture that staircases with an odd number of steps can always be reshaped to form a rectangle with nothing left over. The length of the rectangle is equal to the number of steps in the staircase. We also conjecture that staircases with an even number of steps can be reshaped to form a rectangle whose length is equal to the number of steps in the staircase, but there is an extra half of one row of the rectangle (e.g., see the sum 1 + 2 + 3 + 4 + 5 + 6 at the left).

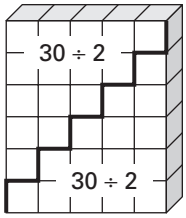
Some students may go on to generalize in more detail. For example, if one uses Method A to find the sum of the numbers from 1 to n, when n is odd, one dimension of the rectangle formed is n and the other dimension $(n + 1)/2$. If n is even, the shape formed contains a row of $n/2$ cubes plus an n by $n/2$ rectangle.

Method B. *We imagined leveling off the staircase to make a rectangle. We think of leveling off the staircase as finding the average height of the steps. Our conjecture is that the average, or leveled-off, height of any staircase with an odd number of steps is the height of the middle step. If there is an even number of steps, we think the average height of the staircase is the average height of the two middle steps (also the average height of the first and last steps). The number of cubes in the staircase is the number of steps times the average height of the staircase.*

Focus Teacher Activity (cont.)

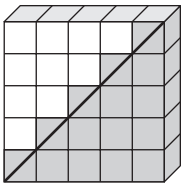
ACTIONS





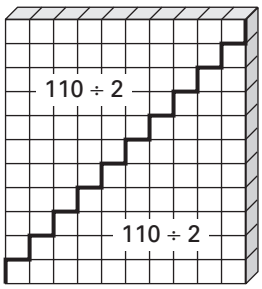
Two Staircases Form a Rectangle

1 + 2 + 3 + 4 + 5 = 30 ÷ 2 = 15



4 If the method of combining two staircases to compute the sum 1 + 2 + 3 + 4 + 5, as illustrated in Method C of Comment 3, was not suggested, have the students try it now. Discuss.

5 Place a transparency of Focus Master A (see next page) on the overhead, revealing the first sentence of Part a) only. Have students think privately about this situation, then slowly reveal Parts i)-iii) one at a time, having students think privately about the questions. Then have the students talk over their ideas with their partners. Invite volunteers to illustrate their reasoning at the overhead. Repeat for b)-e).



1 + 2 + 3 + … + 10 = 110 ÷ 2 = 55

COMMENTS

Method C: *We placed 2 copies of the staircase together to form a rectangle that contains 5 × 6 = 30, cubes* (see diagram at the left). *Since the rectangle is made from 2 identical staircases, each staircase contains 30 ÷ 2, or 15, cubes. Also, since each staircase was built to contain 1 + 2 + 3 + 4 + 5 cubes, then 1 + 2 + 3 + 4 + 5 = $^{(5 \times 6)}/_{2}$ = $^{30}/_{2}$ = 15.*

If students bring up formulas for generalizing the method of combining 2 staircases to form a rectangle, you may wish to refer to the discussion in Comment 5.

Method D: *We viewed the staircase for 1 + 2 + 3 + 4 + 5 as part of a 5 by 5 square* (see diagram at the left). *The number of cubes below the diagonal line is half of a 5 × 5 square. Since there are 5 half-squares above the diagonal, the total number of cubes in the staircase is (5^2 ÷ 2) + (5 × ½) = 12½ + 2½ = 15. We think that this method works for any staircase sum. For example, a staircase representing the sum 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 can be viewed as half of an 8 × 8 square plus 8 half squares, or $^{64}/_{2}$ + 4 = 36. So, a staircase representing the counting numbers from 1 to n contains a total of (n^2 ÷ 2) + (n ÷ 2) cubes.*

4 See Comment 3, Method C, for discussion ideas.

5 The intent here is for students to develop comfort with using staircase methods to compute sums of consecutive counting numbers. Emphasis is on the method of combining 2 staircases because it conveniently extends in later actions to staircases with steps that increase by numbers other than 1.

It is helpful to pause for a few moments before revealing the questions about each situation. This allows students to form images of the situation and "see" relationships before focusing on getting answers. Students may be excited by the strength of this model, and hence, may ask for other problems to solve that are similar to those on Focus Master A. Or, they may enjoy posing problems for one another to solve.

a) Two staircases form a 10 × 11 rectangle containing 110 cubes. Hence each staircase contains $^{110}/_{2}$ = 55 cubes. Thus 1 + 2 + 3 + … + 10 = 55.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Arithmetic Sequences Lesson **4**

Focus Master A

a) Imagine, build, or sketch a staircase of cubes for the sum of the first 10 positive integers, 1 + 2 + 3 + … + 10.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does this rectangle contain?
iii) How many cubes are in each of the 2 staircases?

b) Now imagine or sketch a staircase of cubes representing the sum 1 + 2 + 3 + … + 100.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does the rectangle contain?
iii) How many cubes are in each staircase?

c) Imagine or sketch a staircase of cubes representing the sum 1 + 2 + 3 + … + n, where n is a whole number.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does the rectangle contain?
iii) How many cubes are in each staircase?

d) Suppose a staircase contains a total of 465 cubes. The first step contains 1 cube, and each step is 1 cube higher than the preceding step.

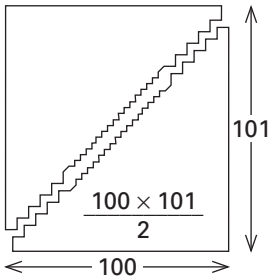
i) Reason from a sketch to determine how many steps it contains.

e) Challenge. Sketch a staircase of cubes to represent the sum 23 + 24 + 25 + … + 77.

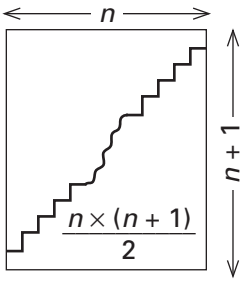
i) Reason from the sketch to find the number of steps in the staircase.
ii) Use your sketch to find the number of cubes in the staircase.

COMMENTS

5 (continued.)
b) You might remind the students that a "rough" sketch need not show every step. A staircase representing the sum 1 + 2 + 3 + … + 100 contains $(100 \times 101)/2$ cubes, as illustrated here:



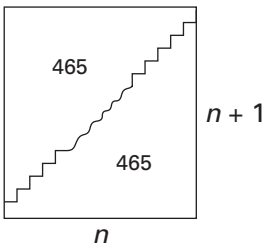
c) If the sum 1 + 2 + 3 + … + n is represented as a staircase, 2 copies of the staircase fit together to form an $n \times (n + 1)$ rectangle. Thus, 1 + 2 + 3 + … + $n = (n \times (n + 1))/2$, as illustrated at the right.



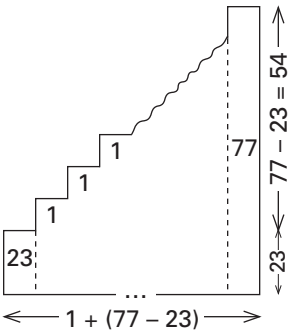
$= 1 + 2 + 3 + \ldots + n$
$= n \times (n + 1)/2$

d) Some students may begin by adding whole numbers 1 + 2 + 3 + … to obtain the sum 465. However, this will be tedious, even with a calculator, and is intended to motivate the use of other strategies.

Notice that 2 staircases can be placed together, as shown here, to form a rectangle with $2 \times 465 = 930$ cubes and with dimensions that are consecutive whole numbers. A calculator can be used to find that $30 \times 31 = 930$, so there are 30 steps.

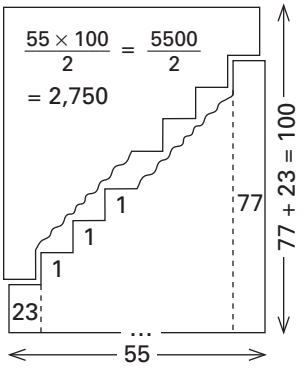


e) Allow plenty of time for the pairs to "puzzle" over this problem. One line of reasoning is that, since the difference between the first and last step is 77 – 23 = 54, there must be 54 + 1 = 55 steps. That is, each step is 1 cube higher than the previous step, so the difference between the first and last step indicates the number of steps *after* the first step. Thus, the total number of steps is 1 more than the difference.



Focus Teacher Activity (cont.)

ACTIONS



6 Slowly read the following statements in bold print aloud to the students, pausing frequently between sentences to allow students to form mental images. Ask the students to think *privately,* closing their eyes if they wish:

> **Create a picture in your mind's eye of the handshake procedure used earlier by our class. Recall…**

(Note to teacher: describe here the way your students lined up to shake hands, the way your students came to the center of the room to shake hands, or whatever method was used by your class in Action 6 of the Connector.)

> **Now imagine that all the students in our class today use this process to shake hands with each other. What are the numbers that must be added to determine the total number of handshakes made by the class?**
>
> **Next imagine a staircase representing these numbers. What does the staircase look like? How many steps does it have?**
>
> **Now imagine another staircase, identical to the one representing the total number of handshakes in the class. Imagine joining these 2 staircases together to form a rectangle. What are the dimensions of this rectangle?**
>
> **Finally, make a rough sketch showing your 2 staircases joined to form a rectangle; label the dimensions of this rectangle; and use that rectangle to determine the total number of handshakes if all the students in the class shake hands with each other.**

Discuss, comparing the students' results to the class' predictions that were made during the Connector.

COMMENTS

The total number of cubes in the staircase can be determined by combining matching staircases (see diagram at the left) to form a rectangle with dimensions 55 by (23 + 77); one staircase contains $(55 \times 100) \div 2 = 2{,}750$ cubes.

6 The questions that you read are intended to promote private reflection and are to be answered only in each student's mind.

If there are 32 people in the room, for example, the number of handshakes will be the sum of 1 + 2 + 3 + … + 31. A staircase representing this sum will have 31 steps. Two staircases, each representing this sum, will form a 31×32 rectangle. The number of handshakes will be half this amount: $(31 \times 32) \div 2 = 496$. This computation can be done with a calculator or mentally $(31 \times 32) \div 2 = 31 \times 16 = (30 \times 16) + (1 \times 16) = 480 + 16 = 496$.

Focus Teacher Activity (cont.)

ACTIONS

7 Ask the students to imagine a room of 50 people. Have the pairs of students determine the number of handshakes if everyone in that room shakes hands with every other person. Discuss their reasoning. Repeat, as needed, for a room of 100 people; 75 people; the class next door; *p* people.

8 Place a transparency of Focus Master B on the overhead, revealing only Part I, Problem a) for the pairs to investigate. Discuss their results and reasoning. Repeat for one or more of Part I, Problems b)-d). Encourage the use of rough sketches to represent sums involving many numbers and to represent large numbers.

Lesson **4** Arithmetic Sequences

Focus Master B

Part I

a) Use cubes to form and combine matching staircases to compute the sum 2 + 4 + 6 + 8 + 10.

b) Sketch a staircase representing the first 20 consecutive even numbers. Then determine S_{20}, where the symbol S_{20} represents the sum of the first 20 consecutive even numbers. Then find S_{100}, the sum of the first 100 even numbers. Finally, based on your diagram, invent a formula for S_n.

c) Reasoning from relationships you can "see" in a staircase and without counting, determine the number of even numbers from 28 through 104.

d) Find the sum of the even numbers from 28 through 104; from 8 through 818; from 152 through 346.

COMMENTS

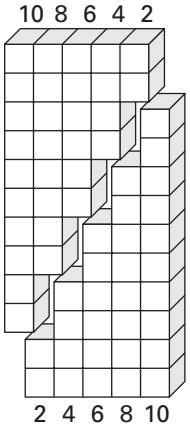
7 There are $(49 \times 50) \div 2 = 1225$ handshakes in a room of 50 people. This computation is easily done with a calculator.

You might ask the pairs of students to write instructions for computing the number of handshakes. They will have various ways of describing how to do this. Work with the students to arrive at descriptions which give correct answers and are unambiguous. Refrain from judging one correct method as better than another; allow the students to make their own judgements.

You can use this situation to remind the students how formulas evolve from written descriptions (see *Math Alive! Course II*, Lesson 3). For example, [the number of handshakes] = 1 + 2 + 3 + … + [one less than the number of people]. Representing the phrases in brackets by letters, so that *h* stands for the number of handshakes and *p* for the number of people in the room, one gets this formula: $h = 1 + 2 + 3 + \ldots + (p - 1)$, or $(p-1)((p-1)+1)/2 = (p-1)p/2$.

8 The intent here is to extend the staircase methods to sums involving steps of 2. Allow plenty of time for students to make and test conjectures.

a) Twin staircases can be used to determine this sum, as illustrated below:



$$2 + 4 + 6 + 8 + 10 = (5 \times 12)/2 = 30$$

Notice the 2 staircases form a 5 by 12 rectangle containing 60 cubes, so 1 staircase contains 30 cubes.

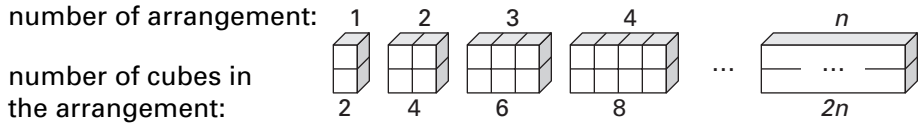
b) Note: the symbol S_{20} is read "S sub 20." The 20 is called a *subscript*. If students have difficulty determining the 20th, 100th, and *n*th even numbers, you might ask them to recall the sequence of cubes arrangements

Focus Teacher Activity (cont.)

ACTIONS

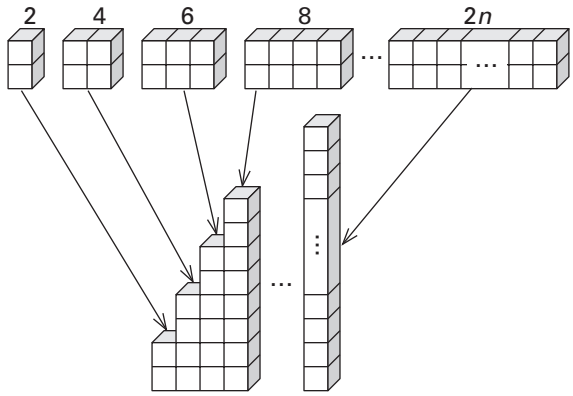
COMMENTS

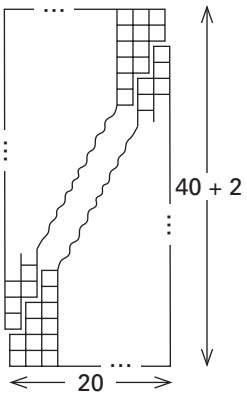
representing even numbers as explored in Lesson 4 of *Math Alive! Course II* and Lesson 6 of *Course I*. This sequence is illustrated here:



Notice that the number of cubes in each arrangement is dependent upon n, the number of the arrangement. Since a "2 by n" rectangle of $2n$ cubes represents the nth even number, a formula for a_n (read as "a sub n"), the nth term of the sequence of even numbers, is $a_n = 2n$. Hence, an arrangement representing the 20th even number contains $2 \times 20 = 40$ cubes.

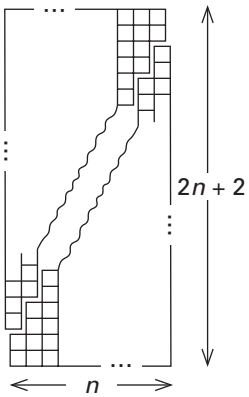
Each arrangement of cubes can be rearranged into single columns that differ by 2 cubes, as shown here:





$S_{20} = \frac{20(40+2)}{2} = 420$

Figure A



$S_n = \frac{n(2n+2)}{2}$

Figure B

Then staircase methods can be used to compute the sum, $S_{20} = 2 + 4 + 6 + \ldots + 40$, as in Figure A at the left. Similarly, S_n, the sum of the first n even numbers could be determined as shown in Figure B at the left.

Rather than rearranging the representations of the even numbers into single columns, some students may form or sketch "fat" stair steps that are 2 cubes wide, as illustrated at the left in Figure C.

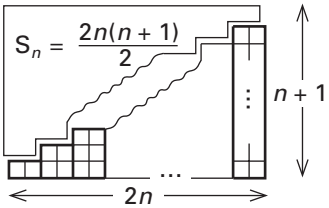


Figure C

Figures B and C illustrate that $n(2n+2)/2$ and $2n(n+1)/2$ are *equivalent* expressions (i.e., they are different representations of the same amount).

c) The statement "even numbers 28 through 104" implies that 28 and 104 are both included. One line of

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

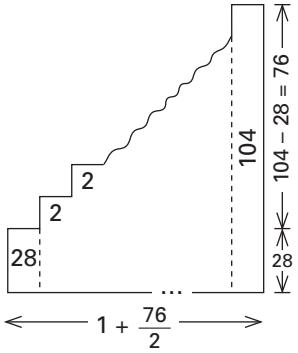


Figure D

Since each step is 2 cubes higher than the previous step and the difference between the first and last step is 104 – 28 = 76, there are $^{76}/_{2}$ = 38 steps *after* the first step. So there are 38 + 1 = 39 steps.

9 Write the expression 1 + 3 + 5 + 7 + 9 on the overhead. Reveal Part II a) from Focus Master B for the pairs to investigate. Discuss. Then repeat for one or more of Part II b)-d).

Lesson **4** Arithmetic Sequences

Focus Master B

Part I

a) Use cubes to form and combine matching staircases to compute the sum 2 + 4 + 6 + 8 + 10.

b) Sketch a staircase representing the first 20 consecutive even numbers. Then determine S_{20}, where the symbol S_{20} represents the sum of the first 20 consecutive even numbers. Then find S_{100}, the sum of the first 100 even numbers. Finally, based on your diagram, invent a formula for S_n.

c) Reasoning from relationships you can "see" in a staircase and without counting, determine the number of even numbers from 28 through 104.

d) Find the sum of the even numbers from 28 through 104; from 8 through 818; from 152 through 346.

Part II

a) Use cubes to form and combine matching staircases to compute the sum of 1 + 3 + 5 + 7 + 9.

b) Sketch and label a staircase representing S_n, the sum of the first *n* consecutive odd numbers. By reasoning from your sketch, invent a formula for S_n, the sum of the first *n* odd numbers.

c) Reasoning from relationships you can "see" in a staircase and without counting, determine the number of odd numbers from 79 through 245.

d) Use staircases to find the sum of the odd numbers from 79 to 245; from 31 through 331.

COMMENTS

8 (continued.)
reasoning about Problem c) is shown in Figure D at the left.

d) Figure E at the right shows one method of adding the even numbers from 28 to 104.

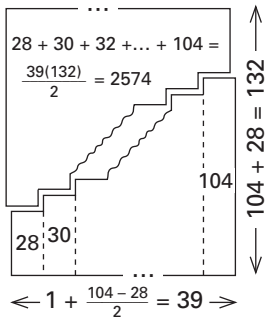
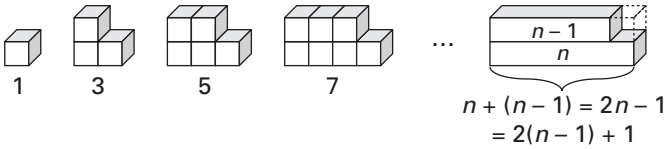


Figure E

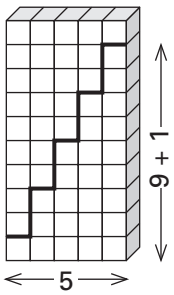
Another approach to adding the even numbers from 28 to 104 is to determine the difference between the sum of the first 14 even numbers (i.e., from 2 to 28) and the sum of the first 52 even numbers (from 2 to 104).

9 Students may benefit by revisiting the model for odd numbers explored in *Courses I* and *II*. Based on the following diagram, a_n, the *n*th term in the sequence of odd numbers, has value $2n - 1$ or, the *equivalent expressions,* $n + (n - 1)$ or $2(n - 1) + 1$.



Staircase methods of computing sums of consecutive odd numbers are similar to those for consecutive even numbers.

Part II a) $\frac{5(9+1)}{2}$ (see diagram at the right for computing the sum of odd numbers from 1 through 9.)



b) Students' formulas may vary. For example, some may say that $S_n = \frac{n(\text{first term} + \text{last term})}{2}$. Since the first term of the sequence 1, 3, 5, 7,... $2n - 1$ is 1 and the last term is $2n - 1$, a formula is $S_n = \frac{n(1 + (2n - 1))}{2}$.

Focus Teacher Activity (cont.)

ACTIONS

10 Give each pair of students a copy of Focus Master C, a sheet of butcher paper, and marking pens. When the requested work is completed, have the pairs post and discuss their results.

Arithmetic Sequences Lesson **4**

Focus Master C

Any sequence of numbers that can be represented by a staircase whose steps increase by a constant amount is called an *arithmetic sequence*. That is, in an arithmetic sequence there is a constant difference between any term and the term that follows it. The constant is called the *common difference*. (Note: each number in a sequence of numbers is called a *term* of the sequence.)

For example, the sequence of even numbers and the sequence of odd numbers are each an arithmetic sequence whose common difference is 2.

Explore the following problems and display your responses on a poster. Next to each response write the corresponding problem.

a) Create 2 different arithmetic sequences that each contain 7 terms and with common difference 3. Show a staircase model of each sequence. Note: you don't need to show every cube in your diagram, but label the size of each step.

b) Create an arithmetic sequence that contains 10 terms. The 1st term is 18 and the 2nd term is 23. Show how to use staircase methods to find the sum of the numbers in this sequence.

c) Do you think the staircase method sometimes/always/never works for finding the sum of the first *n* terms in *any* arithmetic sequence? Use diagrams and/or formulas to support your conclusions. Be sure to label your diagrams carefully and tell what any variables you use represent.

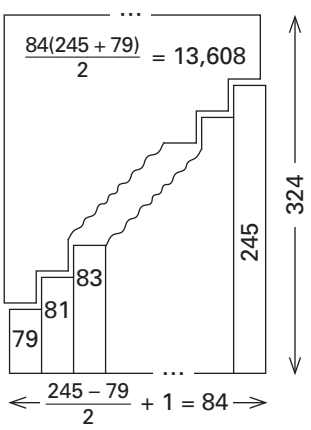
d) Following are descriptions of 3 arithmetic sequences. Sketch a staircase model (you don't have to show every step) of each sequence and label the values of the first and last term, the common difference, and the number of terms for each sequence.

(Continued on back.)

COMMENTS

c) Using reasoning similar to that for Part I, Problem c) in Comment 8, there are $1 + ((245 - 79) \div 2) = 84$ steps in a staircase of odd numbers from 79 through 245.

d) The sum of odd numbers from 79 through 245 is 13,608, as illustrated here. The sum of odd numbers from 31 through 331 is 27,331.



10 Depending on time available and your students' comfort with the mathematical ideas explored, you may want to assign only selected problems from Focus Master C (or have students choose problems). And, you may want to preview the review process they will use in Action 11 as well as your intentions for evaluation of the posters (see *Starting Points* for suggestions). If students are simultaneously working on the Follow-up as homework, this activity will lend insights to their work on the Follow-up and vice versa.

Prior to beginning work on their posters, it is helpful to first spend some time discussing students' ideas regarding what effective group work looks and sounds like in the classroom and your expectations regarding group participation.

Note that when the terms of a sequence are added, the indicated sum is called a *series*. For example, 1, 3, 5, 7, 9 is an arithmetic sequence, whereas 1 + 3 + 5 + 7 + 9 is an *arithmetic series*. A sequence or series that has a finite number of terms is *finite*; otherwise, it is *infinite*.

a) Sequences may vary, depending on the students' choices for a first term. For example: 8, 11, 14, 17, 20, 23, 26; or 15, 18, 21, 24, 27, 30, 33.

b) There is only one possibility here: 18, 23, 28, 33, 38, 43, 48, 53, 58, 63.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **4** Arithmetic Sequences

Focus Master C (cont.)

Next to each model explain how you determine *d*, the common difference.

i) The first term of this sequence is 28 and the last term is 208. There are 61 terms.
ii) There are 37 terms in this sequence, the first term is 8, and the last term is 332.
iii) Challenge. This sequence has *n* terms, the first term is a_1, and the last term is a_n.

e) Write a formula for a_n, the value of the *n*th term of the arithmetic sequence: 11, 18, 25, 32, 39, … (Examples: $a_1 = 11$ and $a_2 = 18$.)

f) Challenge. Write a formula (using only the variables *l*, *n*, *a*, and *d*) for *l*, the last term of any arithmetic sequence with first term *a*, common difference *d*, and *n* terms. Use a diagram and a brief explanation to show how and why your formula works. Label your diagram to show what each variable represents.

g) Repeat Problem d) for these 3 sequences, but this time explain how you determined *n*, the number of terms in each sequence.

i) The first term of this sequence is 32, the common difference is 2, and the last term is 160.
ii) The first term is 7, the common difference is 10, and the last term is 97.
iii) Challenge. The first term is a_1, the common difference is *d*, and the last term is a_n.

COMMENTS

10 (continued.)
c) Allow plenty of time for groups to investigate this. It may be helpful to prompt groups to explore whether/how their methods work for sequences with various common differences and various first terms.

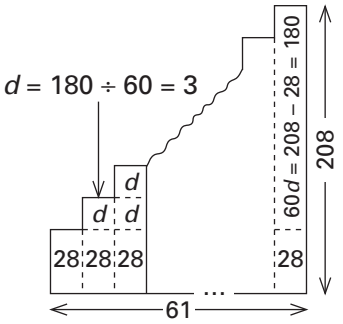
The method of combining 2 identical staircases to form a rectangle whose total area is double the area of one staircase can be used to find the sum of the terms in any arithmetic sequence. Here are examples of 3 formulas given by students:

Sum = $\dfrac{\left(\frac{L-F}{X}+1\right)(L+F)}{2}$, *where X = the common difference, L = the last term, and F = the first term.*

$\frac{n(l+f)}{2}$ = sum of the first *n* terms in any arithmetic sequence, where *l* is the last term, and *f* is the first term.

Sum = $\frac{n(H+V)}{2}$ and $H = V + (n-1)r$, where *n* = the number of terms, *H* = the number of cubes in the highest step, *V* = the number of cubes in the first step, and *r* = the common difference.

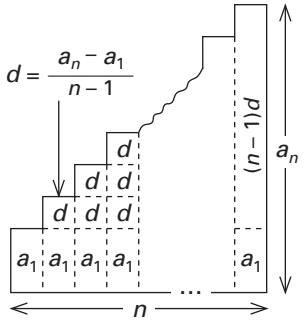
d) i) Reasoning from the following diagram, the height of the last step can be determined by adding the common difference 60 times to the height of the first step. Since the height of the last step is 208 and the height of the first step is 28, then the amount added is 208 – 28 = 180. Therefore, the common difference must be 180 ÷ 60 = 3.



d) ii) The height of the last step, 332, is found by adding 36 times the common difference to the first term, 8. Hence, the common difference is $^{(332-8)}/_{36} = 9$. Students may use various approaches to determine this, and it is not expected that they solve algebraic equations. Rather, emphasize reasoning from the diagram. Symbolic procedures will be developed later in the course, as ways of recording the students' reasoning processes.

Focus Teacher Activity (cont.)

ACTIONS



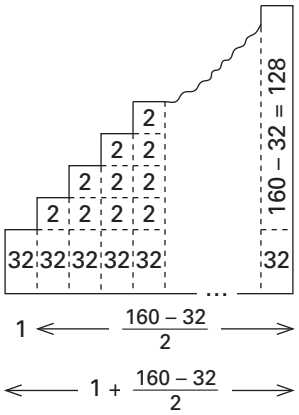
COMMENTS

d) iii) This problem may be frustrating for some students if they try to devise a formula. It is appropriate to keep explanations verbal here, and some students may be sufficiently challenged by describing specific situations such as i) and ii). The diagram at the left illustrates that, since the height of the last step is determined by adding $n - 1$ groups of d to the height of the first step, one can subtract the heights of the first and last steps (to find the amount added) and then divide by $n - 1$ to find d.

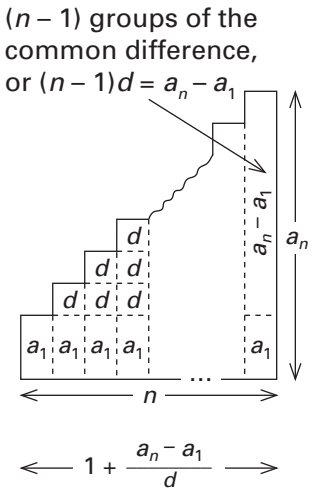
e) One possibility is $a_n = 11 + (n - 1)7$.

f) $l = a + (n - 1)d$

g) i) One line of reasoning is shown in the diagram below.



Since there are $^{128}/_{2}$ = 64 groups of 2 added to the height of the first step to make the height of the last step, there must be 64 + 1 = 65 steps.



g) ii) Reasoning similar to that used for i) leads to the solution, $n = 10$. Note that students may at first overlook counting the first term, and hence, incorrectly state there are 9 terms.

g) iii) Like d) iii) above, this problem may be frustrating for some students if they try to devise a formula. As appropriate, emphasize verbal explanations here, or limit the exploration to specific situations such as i) and ii).

Notice the number of groups of d that are contained in the difference between the first and last term is 1 less than n, the number of steps. Hence, $n = \frac{(a_n - a_1)}{d} + 1$.

Focus Teacher Activity (cont.)

ACTIONS

11 (Optional) Distribute 1 copy of Focus Student Activity 4.2 to each student. Have the pairs do the following:

a) Each pair of students exchanges posters with another pair, then reviews the poster and completes a Feedback Sheet about it, attaching the completed Feedback Sheet to the back of the poster.

b) Pairs exchange posters with yet another pair of students, reviewing that poster and completing a new Feedback Sheet (*without* reading Feedback Sheets attached previously). When completed, these new Feedback Sheets are attached to the back of the poster, and students compare their feedback to that given previously.

c) Repeat b) as time allows.

d) Return the posters to their authors, allowing time for the authors to read the feedback and make additions and adjustments, as they see fit.

Post the edited posters and Feedback Sheets in the classroom and discuss as needed.

12 (Optional) Discuss the students' ideas about their development in relation to the class goals (see Focus Master A from Lesson 1 of this course).

COMMENTS

11 If pairs of students review more than 2 posters, they will need additional Peer Feedback Sheets. See *Starting Points* for other ideas regarding peer feedback by students.

Arithmetic Sequences Lesson **4**

Focus Student Activity 4.2

Peer Feedback Sheet

REVIEWEES _____ **REVIEWERS** _____

We appreciate…	We question…
We wish…	We learned…

12 This provides a way to "check back in" on the goals discussed in Action 12 of the Focus of Lesson 1 of this course, and to help students see that they have been working on many goals during this activity.

Before discussing, you might have the students briefly describe in their journals examples of moments from the lesson that illustrate evidence of their work/progress in each goal area. For example, here are some examples of student comments:

I remember "seeing" how the staircase method worked on staircases with any sized steps when Dylan shared his method at the overhead. I felt ownership of the method after that, because I could use it to solve problems. This shows my growth in the goals visual thinking, concept understanding, reasoning and problem solving, and openness to new ideas. (It also shows that Dylan is growing in his mathematical communication.)

Focus Teacher Activity (cont.)

ACTIONS

Exploring Symmetry Lesson **1**

Focus Master A

Our Goals as Mathematicians

We are a community of mathematicians working together to develop our:

a) visual thinking,

b) concept understanding,

c) reasoning and problem solving,

d) ability to invent procedures and make generalizations,

e) mathematical communication,

f) openness to new ideas and varied approaches,

g) self-esteem and self-confidence,

h) joy in learning and doing mathematics.

COMMENTS

I used to only think of even numbers as 2, 4, 6,… , and I couldn't determine things like the 100th even number or the sum of the 50th and 70th even numbers without lots of tedious counting. Now, I just picture 2 rows of 100 tile or 2 rows of 50 tile added to 2 rows of 70 tile! This is an example of growth in my visual thinking, concept understanding, reasoning and problem solving, and ability to make generalizations. I am also more confident when I understand how something works.

I think the method we learned for adding staircase numbers is what visual thinking is all about. First we used the cubes to build staircases that represent arithmetic sequences. Next we joined two staircases to form rectangles so we could easily find the sum of the terms in the sequences. Then we imagined and sketched bigger staircases to find their sums and solve other problems about them. Finally, we shared our ideas and invented formulas that we could see in our pictures. We didn't have to memorize anything that someone else invented. I think this shows we are developing in every goal area!

TEACHER NOTES:

Follow-up Student Activity 4.3

NAME _____ **DATE** _____

Record all of your responses on separate paper. Include a statement of each problem next to your work about the problem.

1 Describe your understanding of the meaning of an *arithmetic sequence*. Then give an example of a sequence that is an arithmetic sequence and one that is not.

2 Show how to use staircases to find the sum of the first 100 counting numbers.

3 For each of series a) and b) below do the following:

i) Sketch and label a staircase that represents the series.
ii) Explain how to reason from your sketch to determine the number of steps in the staircase.
iii) Show how to use staircase methods to add the numbers in the series.

a) $2 + 4 + 6 + \ldots + 164$

b) $7 + 13 + 19 + 25 + \ldots + 181 + 187$

4 Draw a diagram to show how the staircase method can be used to find the sum of the first n terms of any arithmetic sequence. Label your diagram to show what any values or variables you use represent. Write a brief step-by-step explanation of the staircase method for adding the terms in an arithmetic sequence.

5 Draw diagrams to show how staircase methods can be used to solve each of the following problems. Label your diagrams in detail, and make sure that each step of your solution process is clearly communicated. Show all calculations that you do, and explain what they represent. Then tell how you verified your solution.

a) Suppose that your class takes a field trip to the fair, and every student goes on exactly one ride with each of the other students in the class. If only 2 students can ride together and each ride costs $1.50, how much will the class spend?

(Continued on back.)

Follow-up Student Activity (cont.)

b) Holly asked for the following allowance: 1¢ on the first day, 3¢ on the second day, 5¢ on the 3rd day, 7¢ on the 4th day, 9¢ on the 5th day, and so on, continuing for the 365 days in the year. How much would Holly earn on the 365th day? Altogether how much would she earn for the year?

6 For each of the following arithmetic series, create an interesting problem about a situation outside of school, so that the solution to the problem would involve finding the sum of the series. Then record the answer to your problem and show your solution methods.

a) 14 + 15 + 16 + … + 176

b) 49 + 51 + 53 + … + 97

7 Explain or draw diagrams to show your methods of determining each of the following Mystery Arithmetic Sequences. List the first 5 and last 5 terms in each arithmetic sequence.

a) The first term of Mystery Arithmetic Sequence A is 1 and the last term is 241. There are 61 terms in the sequence.

b) The common difference in Mystery Arithmetic Sequence B is 8. There are 17 terms and the last term is 157.

c) The sum of the 25th through the 29th terms of Mystery Arithmetic Sequence C is 165. The first term of the sequence is 7 and there are 40 terms in the sequence.

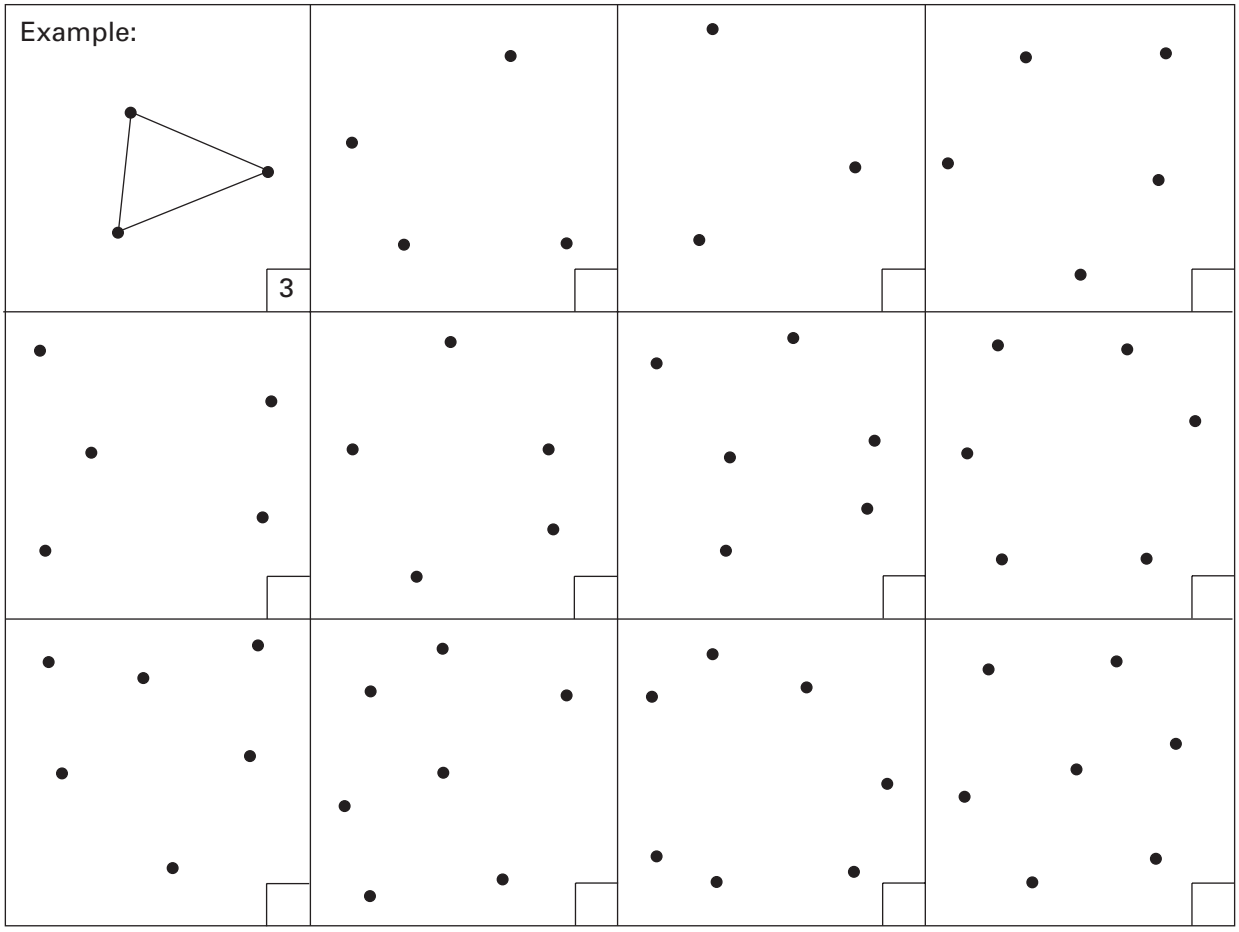
8 Challenge: Several high school mathematics texts list the following formulas regarding arithmetic series, where a_n = the *n*th term of the sequence, d = the common difference, and S_n = the sum of the first n terms of the series. For each of the following formulas: i) sketch and label a diagram of staircases to show what you think the formula means and what you think each part of the formula represents; ii) write a sentence or two to explain your thinking about what the formula means.

a) $a_n = a_1 + (n - 1)d$ b) $a_n = a_{n-1} + d$ c) $S_n = \frac{n(a_1 + a_n)}{2}$

Connector Student Activity 4.1

NAME _____ **DATE** _____

1 In each square below use a straightedge to draw straight line segments that connect each dot to *all* other dots in the square. Then, in the corner of each square, write how many segments you drew.



2 Tell about any patterns you notice or conjectures you have about counting segments between dots.

(Continued on back.)

Connector Student Activity (cont.)

3 A student in another class said she thought that finding the number of segments was just like solving the "handshake problem." What do you think she meant?

4 Suppose you have to find the number of lines it takes to connect 125 dots. Describe a way to find this number without having to draw the segments and dots. (Just tell your method—you don't need to compute the answer!)

Focus Master A

a) Imagine, build, or sketch a staircase of cubes for the sum of the first 10 positive integers, 1 + 2 + 3 + … + 10.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does this rectangle contain?
iii) How many cubes are in each of the 2 staircases?

b) Now imagine or sketch a staircase of cubes representing the sum 1 + 2 + 3 + … + 100.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does the rectangle contain?
iii) How many cubes are in each staircase?

c) Imagine or sketch a staircase of cubes representing the sum 1 + 2 + 3 + … + n, where n is a whole number.

i) What are the dimensions of a rectangle formed by combining 2 of these staircases?
ii) How many cubes does the rectangle contain?
iii) How many cubes are in each staircase?

d) Suppose a staircase contains a total of 465 cubes. The first step contains 1 cube, and each step is 1 cube higher than the preceding step.

i) Reason from a sketch to determine how many steps it contains.

e) Challenge. Sketch a staircase of cubes to represent the sum 23 + 24 + 25 + … + 77.

i) Reason from the sketch to find the number of steps in the staircase.
ii) Use your sketch to find the number of cubes in the staircase.

Focus Master B

Part I

a) Use cubes to form and combine matching staircases to compute the sum 2 + 4 + 6 + 8 + 10.

b) Sketch a staircase representing the first 20 consecutive even numbers. Then determine S_{20}, where the symbol S_{20} represents the sum of the first 20 consecutive even numbers. Then find S_{100}, the sum of the first 100 even numbers. Finally, based on your diagram, invent a formula for S_n.

c) Reasoning from relationships you can "see" in a staircase and without counting, determine the number of even numbers from 28 through 104.

d) Find the sum of the even numbers from 28 through 104; from 8 through 818; from 152 through 346.

Part II

a) Use cubes to form and combine matching staircases to compute the sum of 1 + 3 + 5 + 7 + 9.

b) Sketch and label a staircase representing S_n, the sum of the first *n* consecutive odd numbers. By reasoning from your sketch, invent a formula for S_n, the sum of the first *n* odd numbers.

c) Reasoning from relationships you can "see" in a staircase and without counting, determine the number of odd numbers from 79 through 245.

d) Use staircases to find the sum of the odd numbers from 79 to 245; from 31 through 331.

Focus Master C

Any sequence of numbers that can be represented by a staircase whose steps increase by a constant amount is called an *arithmetic sequence.* That is, in an arithmetic sequence there is a constant difference between any term and the term that follows it. The constant is called the *common difference*. (Note: each number in a sequence of numbers is called a *term* of the sequence.)

For example, the sequence of even numbers and the sequence of odd numbers are each an arithmetic sequence whose common difference is 2.

Explore the following problems and display your responses on a poster. Next to each response write the corresponding problem.

a) Create 2 different arithmetic sequences that each contain 7 terms and with common difference 3. Show a staircase model of each sequence. Note: you don't need to show every cube in your diagram, but label the size of each step.

b) Create an arithmetic sequence that contains 10 terms. The 1st term is 18 and the 2nd term is 23. Show how to use staircase methods to find the sum of the numbers in this sequence.

c) Do you think the staircase method sometimes/always/never works for finding the sum of the first *n* terms in *any* arithmetic sequence? Use diagrams and/or formulas to support your conclusions. Be sure to label your diagrams carefully and tell what any variables you use represent.

d) Following are descriptions of 3 arithmetic sequences. Sketch a staircase model (you don't have to show every step) of each sequence and label the values of the first and last term, the common difference, and the number of terms for each sequence.

(Continued on back.)

Focus Master C (cont.)

Next to each model explain how you determine *d*, the common difference.

i) The first term of this sequence is 28 and the last term is 208. There are 61 terms.
ii) There are 37 terms in this sequence, the first term is 8, and the last term is 332.
iii) Challenge. This sequence has *n* terms, the first term is a_1, and the last term is a_n.

e) Write a formula for a_n, the value of the *n*th term of the arithmetic sequence: 11, 18, 25, 32, 39, … (Examples: $a_1 = 11$ and $a_2 = 18$.)

f) Challenge. Write a formula (using only the variables *l*, *n*, *a*, and *d*) for *l*, the last term of any arithmetic sequence with first term *a*, common difference *d*, and *n* terms. Use a diagram and a brief explanation to show how and why your formula works. Label your diagram to show what each variable represents.

g) Repeat Problem d) for these 3 sequences, but this time explain how you determined *n*, the number of terms in each sequence.

i) The first term of this sequence is 32, the common difference is 2, and the last term is 160.
ii) The first term is 7, the common difference is 10, and the last term is 97.
iii) Challenge. The first term is a_1, the common difference is *d*, and the last term is a_n.

Focus Student Activity 4.2

Peer Feedback Sheet

REVIEWEES _____ **REVIEWERS** _____

We appreciate…

We question…

We wish…

We learned…

Follow-up Student Activity 4.3

NAME _____ **DATE** _____

Record all of your responses on separate paper. Include a statement of each problem next to your work about the problem.

1 Describe your understanding of the meaning of an *arithmetic sequence*. Then give an example of a sequence that is an arithmetic sequence and one that is not.

2 Show how to use staircases to find the sum of the first 100 counting numbers.

3 For each of series a) and b) below do the following:

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a) Suppose that your class takes a field trip to the fair, and every student goes on exactly one ride with each of the other students in the class. If only 2 students can ride together and each ride costs $1.50, how much will the class spend?

(Continued on back.)

Follow-up Student Activity (cont.)

b) Holly asked for the following allowance: 1¢ on the first day, 3¢ on the second day, 5¢ on the 3rd day, 7¢ on the 4th day, 9¢ on the 5th day, and so on, continuing for the 365 days in the year. How much would Holly earn on the 365th day? Altogether how much would she earn for the year?

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8 Challenge: Several high school mathematics texts list the following formulas regarding arithmetic series, where a_n = the nth term of the sequence, d = the common difference, and S_n = the sum of the first n terms of the series. For each of the following formulas: i) sketch and label a diagram of staircases to show what you think the formula means and what you think each part of the formula represents; ii) write a sentence or two to explain your thinking about what the formula means.

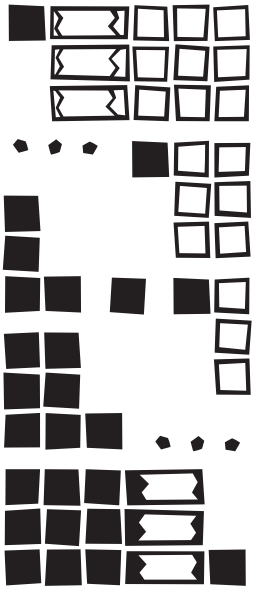
a) $a_n = a_1 + (n - 1)d$ b) $a_n = a_{n-1} + d$ c) $S_n = \frac{n(a_1 + a_n)}{2}$

Extended Counting Piece Patterns

Lesson **5**

THE BIG IDEA

Counting piece patterns whose arrangement numbers include both the negative and positive integers provide a meaningful context in which to develop strategies for representing equations whose solutions are integers. Using Algebra Pieces to represent and solve such equations builds intuitions about the meaning of a variable and promotes the invention of general strategies for solving equations and systems of equations.



CONNECTOR

OVERVIEW

Students review the use of Algebra Pieces to represent sequences of counting pieces and algebraic expressions and to solve equations. They also review relationships between a sequence of arrangements and its coordinate graph.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Algebra Pieces (not including the *n*-frames, see *Blackline Masters* and Comment 1), 1 set per student.
- ✔ Connector Master A (optional), 1 copy for teacher demonstration.
- ✔ Connector Master B, 1 copy per pair of students and 1 transparency.
- ✔ Connector Master C, 1 copy per student and 1 transparency.
- ✔ Connector Master D, 1 transparency.
- ✔ Blank counting pieces and *n*-strips (see *Blackline Masters* and Comment 1), 1 sheet per pair of students.
- ✔ Algebra Pieces for the overhead.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 2 sheets per pair of students.

FOCUS

OVERVIEW

Counting piece patterns are extended to include arrangements corresponding to negative, as well as positive, integers. Students devise symbolic methods of recording their Algebra Piece actions to solve equations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Algebra Pieces (including frames, see *Blackline Masters*), 1 set per student.
- ✔ Algebra Pieces for the overhead.
- ✔ Focus Masters A-C, 1 transparency of each.
- ✔ Focus Master D, I copy per student and 1 transparency.
- ✔ Focus Student Activity 5.1, 1 copy per student and 1 transparency.

FOLLOW-UP

OVERVIEW

Students sketch Algebra Piece representations of the *n*th arrangement of sequences and use those representations as a basis for solving linear and quadratic equations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 5.2, 1 copy per student.

LESSON IDEAS

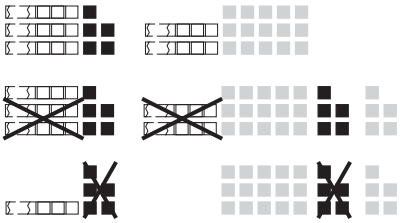
LOOKING BACK & AHEAD
Based on activities in Lessons 9, 10, 26, and 27 from *Math Alive! Course II*, this Connector activity reviews use of the Algebra Pieces to represent the *n*th arrangement of a sequence of counting piece arrangements to solve equations related to such sequences, and to represent algebraic expressions. These ideas are extended in the Focus activity and in Lessons 11-14 of this course.

LESSON PACING
The amount of time that classes spend on this lesson will vary depending on their prior experience with visual patterning and the use of Algebra Pieces. If students are not familiar with the use of bicolored counting pieces to represent integers and integer operations, it may be helpful to review Lessons 5-8 of *Math Alive! Course II* prior to starting this lesson.

QUOTE
It is essential that in grades 5-8 students explore algebraic concepts in an informal way to build a foundation for the subsequent study of algebra. Such informal explorations should emphasize physical models, data, graphs, and other mathematical representations rather than facility with formal algebraic manipulation.
NCTM Standards

SELECTED ANSWERS

1. A B



Remove $2n$ from A and B
Add 0 to B

Remove 5 from A and B

$3n + 5 = 2n - 15$
$n = -20$

2.

$3n + 4 = 190$
$3n = 186$
$n = {}^{186}/_{3} = 62$

3. a) $n = 6$
b) $n = {}^{-}14$
c) $n = {}^{-}5$
d) $n = 12$
e) no solutions
f) True for all values of n (i.e., for any number the 2 sides of the equation are equal).

5. Student example: (Note recording should reflect the order and nature of students' actions and thoughts).

$8n + 36 = 4(n + 1)$	
$8n + 36 = 4n + 4$	Form equivalent expressions.
$8n + 36 - 4n - 4 =$ $4n + 4 - 4n - 4$	Compare collections and subtract like amounts from each.
$4n + 32 = 0$	Simplify.
$4n + \cancel{32} = 0 + \cancel{32} - 32$	Add 0 to right collection, then subtract like amounts from each.
${}^{4n}/_{4} = {}^{-32}/_{4}$	Divide both collections by 4.
$n = {}^{-}8$	Simplify.

6. Student example:

$7(n + 3) = 5(n - 3) + 6$	
$7n + 21 = 5n - 15 + 6$	Form equivalent expressions.
$7n + 21 = 5n - 9$	Simplify.
$7n + 21 + {}^{-}5n + {}^{-}21 =$ $5n - 9 + {}^{-}5n + {}^{-}21$	Add 5 ${}^{-}n$-frames and 21 red units to both collections.
$2n = {}^{-}30$	Simplify.
${}^{2n}/_{2} = {}^{-30}/_{2}$	Divide both collections by 2.
$n = {}^{-}15$	Simplify.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students review the use of Algebra Pieces to represent sequences of counting pieces and algebraic expressions and to solve equations. They also review relationships between a sequence of arrangements and its coordinate graph.

MATERIALS

- ✔ Algebra Pieces (not including the *n*-frames, see *Blackline Masters* and Comment 1), 1 set per student.
- ✔ Connector Master A (optional), 1 copy for teacher demonstration.
- ✔ Connector Master B, 1 copy per pair of students and 1 transparency.
- ✔ Connector Master C, 1 copy per student and 1 transparency.
- ✔ Connector Master D, 1 transparency.
- ✔ Blank counting pieces and *n*-strips (see *Blackline Masters* and Comment 1), 1 sheet per pair of students.
- ✔ Algebra Pieces for the overhead.
- ✔ 1-cm grid paper (optional, see *Blackline Masters*), 2 sheets per pair of students.

ACTIONS

1 Arrange the students in pairs and distribute Algebra Pieces (*not* including the *n*-frames) to each student. Give each pair a sheet of blank counting pieces and strips. Briefly review with the students how the pieces are named and what they represent.

Area Pieces

counting piece unit — *n*-strip — n^2-mat

Linear/Edge Pieces

1 linear unit — *n* linear units

COMMENTS

1 Students were introduced to the use of counting pieces to represent integer operations in Lessons 5-8 of *Math Alive! Course II.* If the majority of students in your class have not experienced those lessons, it is recommended that you do so before proceeding with this lesson. Similarly, it is recommended that students be familiar with the use of Algebra Pieces from Lessons 26 and 27 of *Course II*.

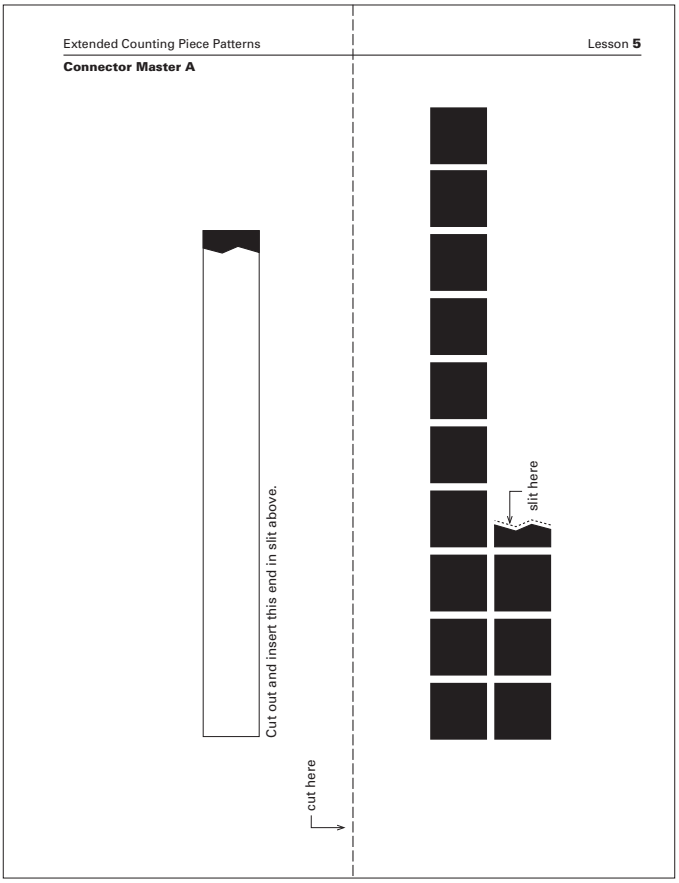
Algebra Pieces consist of 2 kinds of pieces: *area pieces*, which include a counting piece unit (a single counting piece), an *n*-strip (representing a strip of *n* counting pieces), an n^2-mat (representing an $n \times n$ array of counting pieces); and *linear pieces* (also called *edge pieces*) which include linear units (representing the length of the edge of 1 counting piece) and *n* linear units (representing the length of the edge of an *n*-strip). These pieces are black on one side and red on the other. Black pieces have a positive value and red pieces have a negative value.

The white spaces on the *n*-strips are intended to suggest that the strips can be mentally elongated or shortened to contain *an arbitrary number of* counting piece units, whatever *n* might be. To illustrate this, Connector Master A (see next page) contains a master for an *n*-strip that can be elongated or shortened.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



2 Give each pair of students a copy of Connector Master B and 2 sheets of 1-cm grid paper, and ask the pairs to complete Part I. Discuss and compare their results.

COMMENTS

1 (continued.)

Similarly, the n^2-mats are designed with white spaces to suggest that the mats can simultaneously stretch or shrink in 2 directions in order to form larger or smaller squares.

Cardstock or plastic Algebra Pieces are available from The Math Learning Center. A master for making the pieces yourself is also included in *Blackline Masters*. Each student needs at least one master's worth of Algebra Pieces (2 n^2-mats, 10 n-strips, 8 counting piece units) plus 17 or more additional counting piece units. To make the Algebra Pieces and counting pieces, if two-colored printing is not available, pieces can be printed on white cardstock or paper. In this case, red is represented by screened gray, as illustrated throughout this lesson. With reasonable care, two-sided copies can be made on standard copy machines. The two sides can be justified well enough to make usable pieces. Edge pieces can be made by cutting units and n-strips into 3rds or 4ths. Each student needs about 5 n-strip edge pieces and 7-10 unit edge pieces. Overhead Algebra Pieces can also be ordered from MLC (these are smaller than student pieces in order to better fit on the overhead screen), or they can be made using clear and red transparency film.

The blank counting pieces and n-strips (see *Blackline Masters*) can be used to indicate that pieces have been removed from a collection of Algebra Pieces. That is, cutting or folding Algebra Pieces can be avoided by laying the blank pieces over the top of collections to imply that counting pieces or n-strips have been removed. You might provide each group with a few of these blank pieces. Note that the blank pieces are different from the white n-frames, which are introduced in the Focus activity.

blank counting piece blank n-strip

2 All of the ideas in this Connector activity were explored in *Math Alive! Course II*. Depending on your students' backgrounds, you may prefer to explore the sequences one at a time, discussing the students' results and questions about each sequence before proceeding to the next.

Connector Teacher Activity (cont.)

ACTIONS

Lesson **5** Extended Counting Piece Patterns

Connector Master B

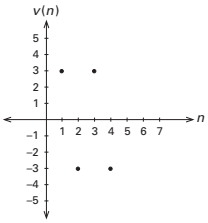
Following are representations of 5 different sequences.

1. 

2. $v(n) = -13 + 5(n - 1)$

3. 

4. 3, –3, –9, –15, –21, –27 …

5. 

Extended Counting Piece Patterns Lesson **5**

Connector Master B (cont.)

Part I

For each of Sequences 1-5 represented on the previous page, do the following (if the information is already given, write "given" and then skip that part):

a) Form and then sketch counting piece arrangements for the first 6 terms of the sequence.

b) Form and then sketch an Algebra Piece representation of the *n*th arrangement of the sequence.

c) Write a formula for $v(n)$, the value of the *n*th arrangement of the sequence. "Loop" a diagram to show how your formula works.

d) If a_n represents the *n*th term of the sequence, record the values of the following: a_1, a_2, a_3, a_4, a_5, a_6, a_{20}, a_{73}, and a_{200}.

e) Make a coordinate graph, plotting and labeling the first 5 or more terms in the sequence.

Part II

1. Which of Sequences 1-5 are arithmetic sequences? Explain your reasoning.

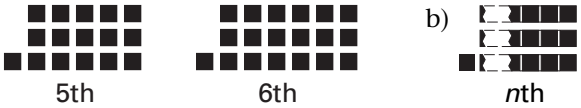
2. *Challenge*: For any of Sequences 1-5 that are arithmetic sequences, show how to use the method of combining two matching staircases to find S_{50}, the sum of the first 50 terms of those sequences.

COMMENTS

One way to conduct student sharing is to post 5 sheets of poster paper; at the top of each poster place an enlarged copy of a different *one* of the representations given on page 1 of Connector Master B. Invite volunteers to use marking pens to sketch the missing representations for Part I, a) through e) on the posters.

Here are some possible representations:

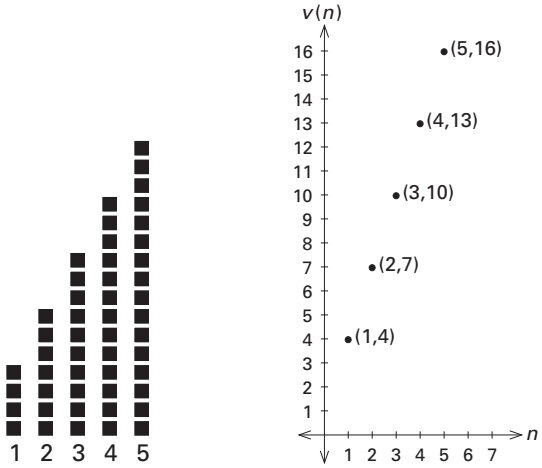
Sequence 1:

a) 

c) You may wish to discuss equivalent expressions that are based on different ways students "see" the *n*th arrangement. Two possibilities are $v(n) = 3n + 1$ and $v(n) = 3(n + 1) - 2$.

d) 4, 7, 10, 13, 16, 19, 61, 220, 601

e) If students have difficulty determining the coordinate graph, you might review the following methods, which were developed in the Connector and Focus activities of *Math Alive! Course II*, Lesson 9. Form minimal collections of counting pieces to represent each of the first 5 arrangements; then rearrange the pieces in each collection to form a column as in a bar graph; use the number of an arrangement and the height of its corresponding column to determine the coordinates of that arrangement on a coordinate graph. This process is illustrated below.

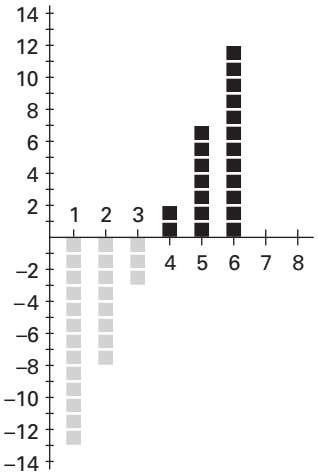


Bar Graph Coordinate Graph

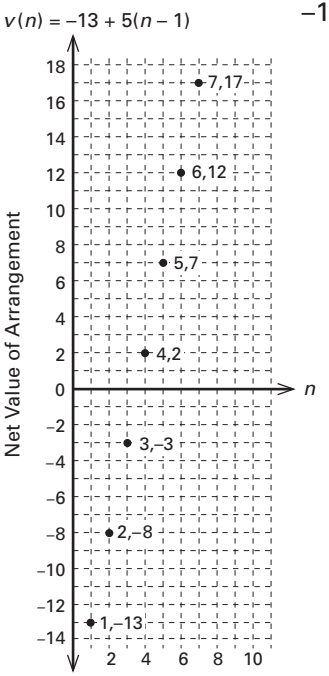
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



Sequence 2
Bar Graph



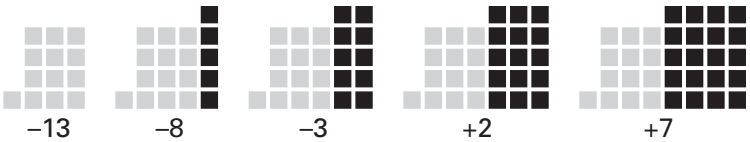
Sequence 2
Coordinate Graph

COMMENTS

2 (continued.)
It may be helpful to briefly review the use of *coordinates* to refer to points of a graph. The coordinates of a point on a graph are an *ordered pair* of numbers, the first of which tells how many units to count from zero along the *horizontal axis* (to the right if the first coordinate is positive, and to the left if the first coordinate is negative), and the second coordinate tells how many units to count from zero along the *vertical axis* (above zero if the second coordinate is positive, and below zero if the second coordinate is negative). It is customary to label the horizontal and vertical axes by the quantities they represent.

Sequence 2:

a) Here is one possibility:



e) The method of forming the minimal collections for the arrangements and then placing these in columns to determine the coordinate graph is illustrated at the left.

Sequence 3:

c) $v(n) = n^2 + n + 1$

Sequence 4:

a) Students may arrange the tile in a number of ways. Many will probably start with single columns and then look for patterns in ways to group the pieces. Some may note that each term is 6 less than its preceding term, and since the first term is 3, then the sequence could be formed as follows:

c) $v(n) = 3 - 6(n - 1)$, or $3 + {}^{-}6(n - 1)$, or $3 + 6[-(n - 1)]$, or $3 + {}^{-}(6n - 6)$

Again, you could use the equilvalent algebraic expressions that students record to prompt discussion of notation and its relationship to ways students "see" the pieces (see also Actions 4-6). For example, the expressions given for Sequence 4, part c), are equivalent, but each may represent a different way of seeing the pieces

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

or viewing the actions associated with forming each arrangement. For example, students may explain the above expressions, respectively, as: 3 units minus 6 groups of $(n - 1)$ units [or $(n - 1)$ groups of 6 units]; 3 units added to a rectangle with dimensions $^{-}6$ by $(n - 1)$; 3 units plus 6 groups of $-(n - 1)$; or 3 plus the opposite of $(6n - 6)$. Note that students may use extra parentheses to better communicate the order of their thought processes. When reading notation aloud, it is helpful to indicate an expression is contained in parentheses by using the phrase "the quantity" prior to or after the expression. For example, $^{-}6(n - 1)$ can be read, "negative 6 times the quantity n minus one." Or, in another instance, the expression $(n + 1)^2$ can be read "n plus one, the quantity squared."

Some questions may surface regarding the use and placement of negative signs. In standard practice, it is difficult to determine whether a – sign is being used as part of the symbol for a negative integer or to designate the opposite of a positive integer. Both the negative integer $^{-}3$ and the opposite of the positive integer $^{+}3$ are frequently denoted symbolically as –3. Since the opposite of the positive integer $^{+}3$ is the negative integer $^{-}3$, it doesn't matter which of these two interpretations is given to the symbol –3.

Notice that the – sign occurs in 3 different ways in arithmetical notation. Besides its use in denoting the opposite of a number and its use in designating a negative number, it is also used to denote the operation of subtraction. Generally, it is clear from the context what use is intended.

Sequence 5:

c) Here is one way a formula could be written:

$$v(n) = \begin{cases} 3 \text{ if } n \text{ is odd} \\ -3 \text{ if } n \text{ is even} \end{cases}$$

d) $a_{73} = 3$; $a_{200} = -3$

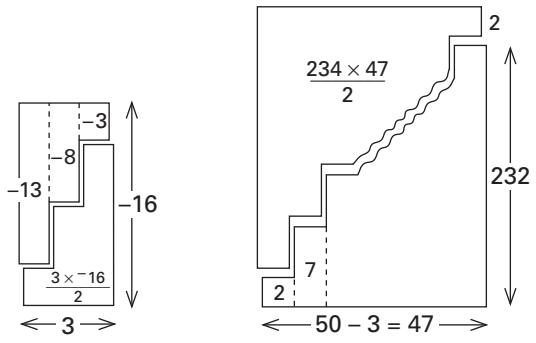
3 Have the pairs complete Problem 1 and Problem 2 (optional) from Part II of Connector Master B. Discuss.

3 Of those given on Part I of Connector Master B, Sequences 1, 2, and 4 are arithmetic sequences. When there are negative numbers involved, students may use staircases in innovative ways to find S_{50}. For example, in Sequence 2, $v(50) = {}^{-}13 + 5(49) = 232$, so determining S_{50} requires summing $^{-}13 + {}^{-}8 + {}^{-}3 + 2 + 7 + \ldots + 232$.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



$$\frac{3 \times {}^{-}16}{2} + \frac{234 \times 47}{2} = 5{,}475$$

4 Write the following expressions on the overhead and ask the pairs to form an Algebra Piece representation of each expression. Have volunteers share their results at the overhead, discussing questions and observations that surface regarding notation and order of operations. Encourage students to record equivalent expressions they see in each model. Discuss conjectures and generalizations that students pose.

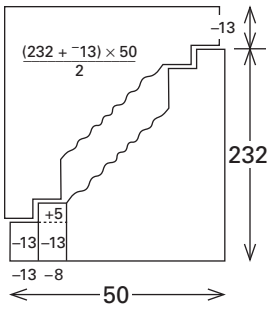
a) $3n + 2$ c) $3n - 2$ e) $-3(n + 2)$

b) $3(n + 2)$ d) $3(n - 2)$ f) $-3(n - 2)$

COMMENTS

3 (continued.)
Some students may separate the negative terms from the rest of the sequence, as shown in the example at the left.

Here is another method students have suggested:



$$S_{50} = \frac{219 \times 50}{2} = 5{,}475$$

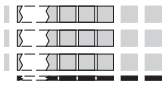
4 The use of parentheses and order of operations were first explored in Lesson 4 of *Math Alive! Course I*, and reviewed in Lesson 3 of *Course II*. See those lessons for additional discussion ideas.

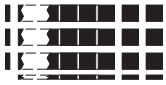
In general, rules for order of operations are as follows: computations within parentheses should be carried out first; then exponents are evaluated; next products and quotients should be computed in the order they occur from left to right; and finally sums and differences should be computed in the order they occur from left to right.

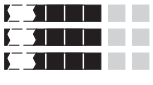
Following are examples of Algebra Piece representations of the given expressions. Students' representations will vary according to the way they interpret the expressions.

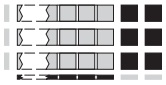
a) 
$3n + 2$

c) 
$3n - 2$

e) 
$-3(n + 2)$

b) 
$3(n + 2)$

d) 
$3(n - 2)$

f) 
$-3(n - 2)$

Some students may "see" expressions such as $3(n + 2)$ as 3 groups of $(n + 2)$, while others will "see" a rectangle with dimensions 3 by $(n + 2)$. Similarly, for e) some students may view $-3(n + 2)$ as "the opposite of a 3 by $(n + 2)$ rectangle" and others may see a rectangle with

Connector Teacher Activity (cont.)

ACTIONS

5 Give each student a copy of Connector Master C and have the pairs of students complete Problems 1 and 2. Discuss their results. Then have the pairs complete Problems 3 and 4. Select volunteers to read aloud their step-by-step instructions for Problem 3, while the other students try to carry out the instructions. Encourage students to help one another refine their instructions, clarifying ambiguities and comparing their algebraic expressions for Problem 4. Finally have the pairs complete one or more of Problems 5a)-5i), sharing their results as they did for Problem 3.

Lesson **5** Extended Counting Piece Patterns

Connector Master C

1. Use Algebra Pieces to do the following:

a) Form a collection that is ½ of a collection of 2 *n*-strips.

b) Now, add 4 units to the collection formed in a).

2. Write an algebraic expression that describes your actions in Problem 1.

3. One way to think about algebraic expressions is as representations of actions with Algebra Pieces. For the expression $\frac{4+2n}{2}$ write the steps of Algebra Piece actions, in order, that the expression could represent. At the end of each step, sketch the resulting collection of Algebra Pieces.

4. In the final collection that you formed for Problem 3, can you "see" an algebraic expression that is equivalent to the original expression? If so, record that expression.

5. Repeat Problems 3 and 4 for the following expressions:

a) $\frac{4n-2n}{2}$

b) $\frac{4-2n}{2}$

c) $4^2 - 2n^2$

d) $4^2 - (2n)^2$

e) $(4 - 2n)^2$

f) $4 + 4n - 8 \div 4$

g) $\frac{4+4n-8}{4}$

h) $6 - 12n \div 3 + 3$

i) $6 - 12n \div (3 + 3)$

6. Record any general observations, AHA!s, conjectures, or important ideas you noticed.

COMMENTS

dimensions –3 by (n + 2). Still others may interpret the expression –3(n + 2) as "form 3 groups of (n + 2), then flip the pieces to form the opposite collection." Each of these interpretations is reasonable. You might encourage students to use edge pieces when they see rectangles, as it helps you see their thinking and it reduces the likelihood of errors when working with products involving negatives.

5 Students could work on Problems 3-5 as homework. Note that for Problems 3 and 5, there may be some disagreement over the actions implied by the symbols. The point here is to reinforce the idea that algebraic expressions are representations of ways of "seeing" or "maneuvering" the Algebra Pieces. There are multiple meanings for multiplication, division, and subtraction, and there are different actions associated with the different meanings. (See the Lesson 3 Connector activity of *Math Alive! Course II*, for discussion ideas related to the meanings of the 4 basic operations, order of operations, and several number properties.)

2) Most students will probably record one of the following: ½(2n) + 4; ^{2n}⁄₂ + 4; or 2n ÷ 2 + 4.

To assure understanding of the directions it may be helpful to have pairs complete 5a) and discuss as a large group before proceeding. The following illustrate the resulting collections associated with each expression in Problem 5.

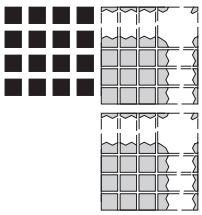
5a)

$$\frac{4n-2n}{2} = n$$

5b)

$$\frac{4-2n}{2} = 2 - n$$

5c)



$$4^2 - 2n^2 = 16 - 2n^2$$

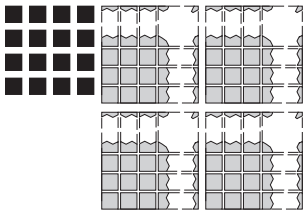
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

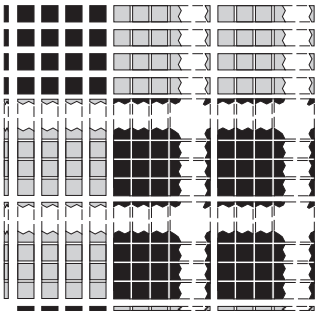
5 (continued.)

5d)



$4^2 - (2n)^2 = 16 - 4n^2$

5e) Notice in this representation, edge pieces are used to represent $4 - 2n$ as the length of the edge of a square. The equivalent expression, $16 - 16n + 4n^2$ represents the value of the square.



$(4 - 2n)^2 = 16 - 16n + 4n^2$

5f)

$4 + 4n - 8 \div 4 = 2 + 4n$

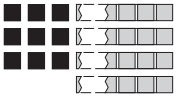
6 Place a transparency of Connector Master D on the overhead, revealing Sequences A and B only. Ask the pairs of students to use their Algebra Pieces to form a representation of the *n*th arrangement of Sequence A and a representation of the *n*th arrangement of Sequence B. Then ask them to find a way to *use their Algebra Pieces to determine* for what *n* the *n*th arrangement of Sequence A has the same net value as the *n*th arrangement of Sequence B. Have volunteers demonstrate their Algebra Piece methods, and verify that their results are correct.

COMMENTS

5g) Some students may view this as a collection of 4 black units, 4 *n*-strips, and 8 red units shared equally in 4 groups that are equivalent to 1 *n*-strip and 1 red unit. Others may view it as a rectangle with 1 edge of 4 linear units and value 4 *n*-strips and 4 red units. The other dimension is $(n - 1)$ linear units.

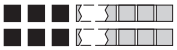
$\frac{4 + 4n - 8}{4} = n - 1$

5h)



$6 - 12n \div 3 + 3 = 9 - 4n$

5i)



$6 - 12n \div (3 + 3) = 6 - 2n$

If needed, here are additional problems students could explore using the format of Problems 3 and 4 on Connector Master C:

i) $4(2n - 1)$

ii) $4(2n) - 1$

iii) $2n^2 - 1$

iv) $(2n - 1)^2$

v) $(2n)^2 - 1$

vi) $-(2n - 1)$

vii) $-2n - 1$

viii) $3(-n) + {}^{-}2(n + 1) + 4 \div 2$

ix) $3 - 5(n + 4) + 6n \div 2 + 3$

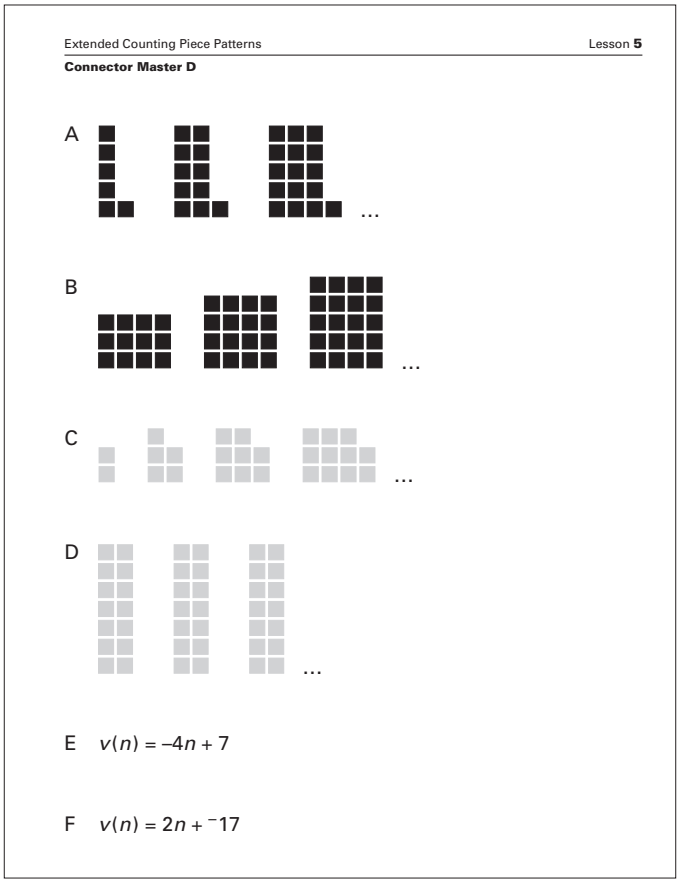
x) Challenge: $[-4({}^{-}n - 1) - (6n - 12) \div 3] \div 2 + 3$

6 While some students may be able to solve this problem using guess and check methods, the point here is to review and develop strategies of solving the problems with the Algebra Pieces. The Algebra Piece methods discussed here were explored in Lessons 26 and 27 of *Math Alive! Course II*, and will be extended in the Focus activity. Ultimately students will develop methods of using symbols to record their actions with the Algebra Pieces. However, the emphasis here is on developing conceptual understanding through experiences with the pieces.

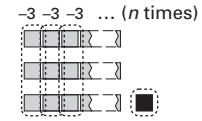
It is important that students interpret the task properly. Note they must determine when corresponding arrangements have the same value. This is called *solving equations simultaneously*. Below are possible Algebra Piece representations of the *n*th arrangements for sequences A and B. Comparing these 2 arrangements, one sees they have 4 *n*-strips and 1 unit in common. Hence, they will

Connector Teacher Activity (cont.)

ACTIONS



7 Repeat Action 6 for Sequences C and D on Connector Master D. Then, if it has not already come up, point out that, for Sequences C and D whose nth arrangements have values $-3n + 1$ and -14, respectively, determining the value of n for which the nth arrangements of the 2 sequences have the same net value is referred to as *solving the equation* ${}^{-}3n + 1 = -14$.

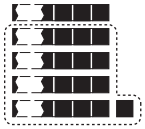


$v(n) = n({}^{-}3) + 1$

COMMENTS

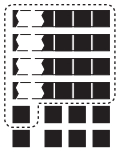
have the same value if the remaining n strip in A has the same value as the remaining 7 units in B. Thus, each n strip has value 7, and the two arrangements have the same value when n is 7.

A



$v(n) = 5n + 1$

B



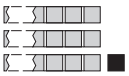
$v(n) = 4n + 8$

Notice we have determined that 7 is the solution of $5n + 1 = 4n + 8$. The notation $v(n)$ is read "the value of the nth arrangement" (note this is *not* the same as the value of n). See Lesson 10 of *Math Alive! Course II*, for additional discussion of notation. Other representations and equivalent expressions for the value of the nth arrangements are possible, and will lead to other formulations of the above equation. However, regardless of the representation, the solution will be $n = 7$.

The solution, $n = 7$, can be verified by forming (or sketching or imagining) the 7th arrangement in each sequence, and noting that each 7th arrangement has value 36.

7 Following is one Algebra Piece representation of the nth arrangements of Sequence C. The 3 red n-strips can be thought of as the opposite of a collection of 3 black n-strips. Since 3 black n-strips have a value of $3n$, the opposite collection has value $-(3n)$ or, dropping the parentheses, $-3n$. Alternatively, since each red n-strip has value $-n$, the value of 3 of them can be written as $3(-n) = -3n$. Since adding a black unit is equivalent to removing a red unit, and since there is 1 unit removed from one $-n$-strip in each arrangement, 1 black unit is added to the nth arrangement.

C



$v(n) = -3n + 1$

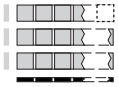
The collection of 3 red n-strips can also be viewed as n collections of 3 red tile, that is, n collections each of value -3, for a total value of $n(-3)$. This view is illustrated at the left.

(Continued next page.)

Connector Teacher Activity (cont.)

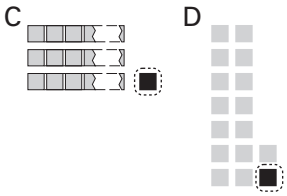
ACTIONS

$v(n) = {}^{-}3n + 1$



$v(n) = {}^{-}3n - {}^{-}1$

$v(n) = -14$



COMMENTS

7 (continued.)
Another way of viewing the *n*th arrangement is as a $^{-}3$ by *n* rectangle with one black unit added to the collection, as shown at the left. Hence, $v(n) = {}^{-}3n + 1$.

Yet another representation is as a $^{-}3$ by *n* rectangle with a red unit removed from the rectangle (e.g., see the rectangle at the left, with one red square missing from the upper right corner). Hence, $v(n) = {}^{-}3n - {}^{-}1$. Recall that removing red pieces from a collection has the same effect on its value as adding a like number of black pieces. Thus, ${}^{-}3n - {}^{-}1$ and ${}^{-}3n + 1$ are equivalent expressions.

Sequence D has *constant* value because the value of the arrangements never change, regardless the value of *n*. Hence $v(n) = -14$, for all *n* (see diagram at the left).

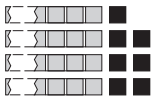
Since there are no pieces in common to the *n*th arrangements for Sequences C and D, and because a red unit and a black unit have total net value zero, adding a red unit and black unit to the *n*th arrangement of Sequence D does not change the net value of the arrangement. However, in so doing, the 2 arrangements now have 1 black unit in common, as illustrated at the left. Hence, 3 red *n*-strips have the same value as 15 red units, and thus 1 red *n*-strip has value –5. The number of the arrangement for which both sequences have the same value is equal to the number of red unit squares in the red *n*-strip, or 5. (Alternatively, one could determine the value of *n* by noting that *n* is the opposite of the value of the red-strip, or $-({}^{-}5) = 5$.)

Thus, $n = 5$ is the solution of the equation ${}^{-}3n + 1 = -14$.

8 Repeat Action 6 for Sequences E and F on Connector Master D. Then have the students record the equation they have solved.

8 The *n*th arrangements of these sequences could be represented as follows:

E



$v(n) = {}^{-}4n + 7$

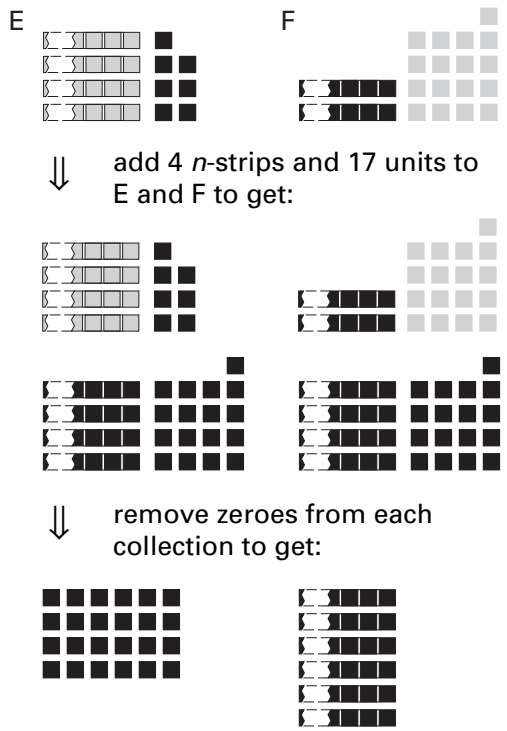
F

$v(n) = 2n + {}^{-}17$

Notice there are no pieces common to both collections of Algebra Pieces. However, since adding zeroes to the collections does not affect their net values, one can add 4 red *n*-strips and 4 black *n*-strips to the *n*th arrangement of Sequence F and 17 red units and 17 black units to the *n*th arrangement of Sequence E. Thus, since the circled portions shown below have the same value, Sequences E and F have the same value when $6n = 24$, so $n = 4$.

Connector Teacher Activity (cont.)

ACTIONS



9 Pose one or more of the following equations and ask the students to develop strategies of *using the Algebra Pieces* to find solutions.

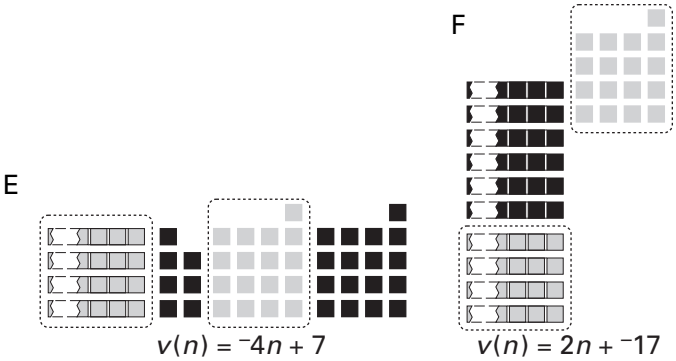
a) $2n + 6 = 5n - 15$

b) $-6n - 25 = -115$

c) $4n - 1 = 4n + 1$

d) $2(n - 2) = 2n - 4$

COMMENTS



Another strategy students may use is to keep the 2 arrangements equal by adding collections of equal value to both arrangements. For example, they may add 4 black *n*-strips and 17 black units to each arrangement, as illustrated at the left. Then, removing combinations of pieces that have net value zero leaves 6 *n*-strips with value 24. Hence, *n* has value 4.

Or, they may do the following: add 2 red *n*-strips and 7 red units to each collection; then remove combinations of pieces that have net value 0, leaving 6 red *n*-strips with net value –24. Thus, 1 red *n*-strip has net value –4 and *n* equals the opposite of –4 (i.e., $n = -(-4) = 4$).

It is important here to keep emphasis on Algebra Piece procedures. You may have some students who prefer the method of adding zeroes to solve an equation, and others may prefer the method of keeping the equations balanced. In the Focus activity, students will develop symbolic methods of recording their Algebra Piece procedures.

9 It may be helpful to remind the students that solving these equations is equivalent to finding the value of *n* for which the *n*th arrangements of the sequences represented by the expressions on each side of the equal sign have the same net value. To encourage flexibility in their thinking, encourage students to try out methods other students share.

Choose from these equations according to the needs and interest of your students. If students have considerable difficulty solving these equations, you might review Lessons 26 and 27 (and perhaps Lessons 9 and 10) from *Math Alive! Course II*, for other discussion ideas. Note that equations whose solutions are nonpositive

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
	9 (continued.) integers are introduced in the Focus activity of this lesson, and nonintegral solutions are explored in Lesson 12 of this course. a) $n = 7$ b) $n = 15$ c) This equation has no solutions (i.e, it is impossible for 1 more than 4 times a number ever to be equal in value to 1 less than 4 times that same number). Or, another way to verify there are no solutions to the equation $4n - 1 = 4n + 1$ follows: If one assumes that a collection of 4 n-strips and 1 red unit has the same value as a collection of 4 n-strips and 1 black unit, adding 4 red n-strips to both collections results in 1 red unit equal in value to 1 black unit, an impossibility. d) Notice these two expressions are equivalent. The same collection of Algebra Pieces can be used to represent both sequences, so for every value of n, the nth arrangements of these sequences are equal in value. Hence, for all values of n the equation is a true statement.

Focus Teacher Activity

OVERVIEW & PURPOSE

Counting piece patterns are extended to include arrangements corresponding to negative, as well as positive, integers. Students devise symbolic methods of recording their Algebra Piece actions to solve equations.

MATERIALS

- ✔ Algebra Pieces (including frames, see *Blackline Masters*), 1 set per student.
- ✔ Algebra Pieces for the overhead.
- ✔ Focus Masters A-C, 1 transparency of each.
- ✔ Focus Master D, 1 copy per student and 1 transparency.
- ✔ Focus Student Activity 5.1, 1 copy per student and 1 transparency.

ACTIONS

1 Arrange the students in groups and distribute Algebra Pieces (without the *n*-frames) to each student. Form or sketch the following collection of counting piece arrangements on the overhead, white board, or black board. Ask the groups to form the same arrangements. Then ask them to form additional arrangements which maintain the pattern of the collection. Discuss.

2 If it hasn't come up, discuss the students' ideas about ways the collection might be extended indefinitely in two directions.

3 Place a transparency of Focus Master A on the overhead, revealing Diagram A only. Discuss the students' ideas about ways the arrangements in the extended sequence shown in Diagram A might be numbered.

COMMENTS

1 The students may add arrangements that extend the pattern in one direction only (left or right). If so, ask them to extend the pattern in the other direction also.

2 One way to extend the collection is shown in Diagram A on Focus Master A (see below). Going to the right, a column of 3 black tile is added to an arrangement to get the next arrangement. Going to the left, a column of 3 red tile is added.

3 One way of numbering the arrangements is to select one of them and number it 0. Arrangements to the right of this arrangement are successively numbered 1, 2, 3, etc. Those to the left are successively numbered –1, –2, –3, etc.

A collection of arrangements which extends indefinitely in two directions and is numbered so there is an arrangement which corresponds to every integer, positive, negative, and zero, will be called an *extended sequence*.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

3 (continued.)
Mathematically speaking, a set of arrangement numbers is called an *index set* and an individual arrangement number is called an *index*. Thus, an arrangement whose number is –3 could be referred to as "the arrangement whose index is –3." Instead of using this language, we shall refer to this arrangement as "arrangement number –3" or "the –3rd arrangement." In the language of index sets, a *sequence* is a collection of arrangements whose index set is the set of positive integers and an *extended sequence*—a phrase coined for our purposes—is a set of arrangements whose index set is the set of all integers. On occasion, once a set of arrangements has been determined to be an extended sequence, it will be referred to simply as a sequence, the word "extended" being understood.

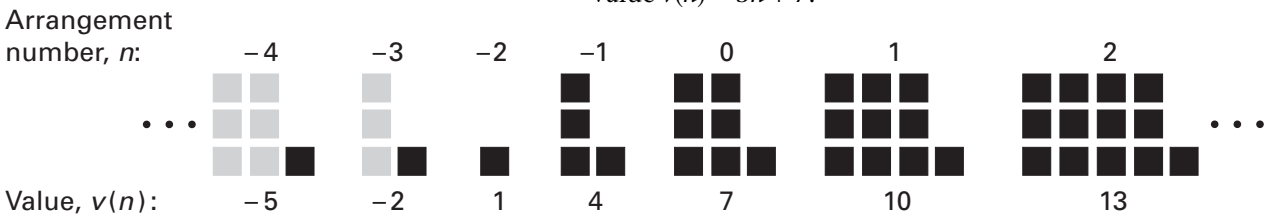
Using the language of functions (see *Math Alive! Course II*, Lesson 10), the set of arrangement numbers is the *domain* of the function, while the set of values of the arrangements is the *range* of the function.

4 Ask the students to number the arrangements in Diagram A from Focus Master A as suggested in Comment 3 and then write an expression for $v(n)$, the value of the arrangement numbered n. Repeat for several different numberings.

4 The expression $v(n)$ will depend on the choice of numbering. If the arrangement consisting of a single black tile is numbered 0, then $v(n) = 3n + 1$. Notice this formula holds for all integers n, positive, negative or zero.

$v(n) = 3n + 1$

A different numbering will result in a different expression for $v(n)$. For example, if the arrangement which consists of 7 black tile is numbered 0, one could view the nth arrangement as 1 black unit added to 3 strips with value $(n + 2)$. Hence, the nth arrangement has value $v(n) = 3n + 7$.



$v(n) = 3(n + 2) + 1$ or $v(n) = 3n + 7$

Focus Teacher Activity (cont.)

ACTIONS

5 Ask the students to assume the arrangements in Diagram A on Focus Master A are numbered so the 0th arrangement contains the single black counting piece (and no other pieces). Ask the students to describe a) the 50th arrangement and b) the –100th arrangement.

6 Assuming the numbering used in Action 5, ask the students to build a representation of the *n*th arrangement, for *n* positive, of the extended sequence in Diagram A of Focus Master A. Repeat for *n* negative. Discuss.

7 Form the following collection of counting piece arrangements and ask the groups to do the same.

Repeat Actions 1-6 for the above collection, by having the students do the following:

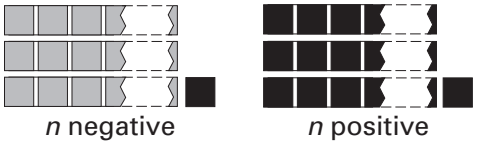
a) Form further arrangements that maintain the pattern and extend the collection in both directions.

COMMENTS

5 The 50th arrangement has 50 columns with 3 black tile per column and an adjoining black tile. Alternatively, it can be described as 3 rows of 50 black tile with an adjoining black tile. Other descriptions are possible.

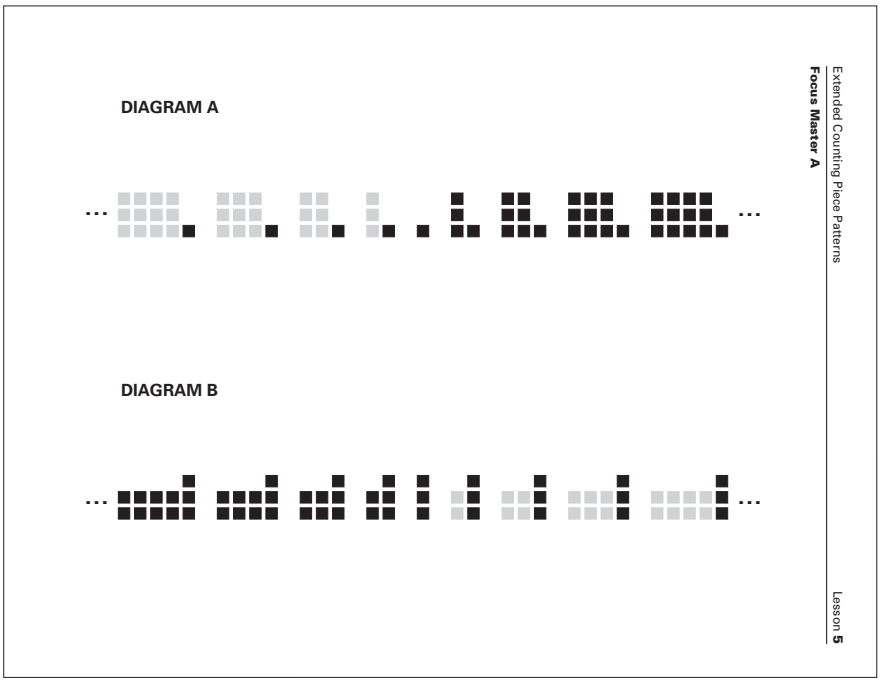
The –100th arrangement (that is, the arrangement numbered –100) has 3 rows of 100 red tile each and an adjoining black tile. Note that the number of red tile in each row is the opposite of the number of the arrangement.

6 One way of forming the arrangements is shown below.



$v(n) = 3n + 1$

7 a) The collection can be extended to the right by adding a column of 2 red tile to an arrangement to get the next arrangement. It can be extended to the left by adding a column of 2 black tile to successive arrangements. This is illustrated in Diagram B on Focus Master A, shown below.

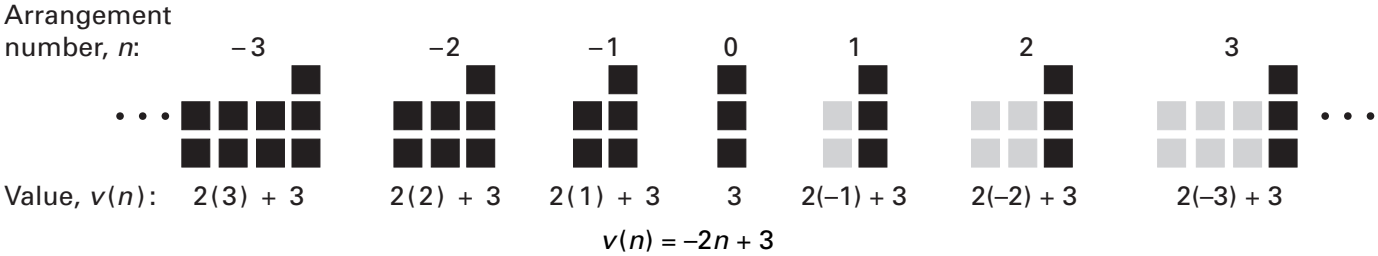


(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

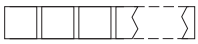
7 (continued.)
b) Select a numbering for the extended sequence, and write an expression for $v(n)$, the value of the nth arrangement.



c) Assuming the single column of 3 black counting pieces is designated the 0th arrangement, describe the 50th arrangement and the –100th arrangement.

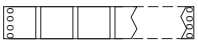
d) Assuming the numbering of c) above, form a representation of the nth arrangement for n positive, and the nth arrangement for n negative.

8 Distribute and discuss the role of *n-frames* and *–n-frames*. Then place a transparency of Focus Master A on the overhead, and ask the students to use the n-frames and $-n$-frames to build representations of the nth arrangement of the extended sequences shown in Diagrams A and B, based on the numberings given in Actions 5 and 7c). Repeat, as appropriate, for other numberings.



n-frame

Has value n for all n, positive, negative, or zero.



$-n$-frame

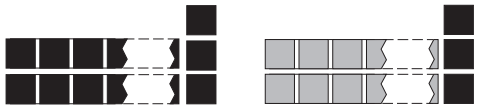
Has value $-n$ for all n, positive, negative, or zero.

COMMENTS

b) If the single column of 3 black tile is designated the 0th arrangement, then $v(n) = 2(-n) + 3 = -2n + 3$. Other numberings will result in different expressions.

c) Using the given numbering, the 50th arrangement has 2 rows of 50 red tile each and a column of 3 adjoining black tile. The –100th arrangement has 2 rows of 100 black tile each and an adjoining column of 3 black tile. Other descriptions are possible.

d) One way of forming the arrangements is shown here:



n negative n positive

$v(n) = -2n + 3$

8 In Actions 6 and 7d), the strips used to form the arrangements for positive n are a different color than those used for negative n, e. g., in the sequence for which $v(n) = -2n + 3$, red strips are used for positive n and black strips for negative n.

Frames are introduced to provide pieces that are sometimes red and sometimes black (i.e., for modeling situations in which n could be positive, negative or zero). An *n-frame* contains black tile if n is positive, red tile if n is negative, and no tile if n is 0. In all cases, the total value of the tile it contains is n. Thus, if n is positive, it contains n black tile and if n is negative, it contains $-n$ red tile. For example, if $n = -50$, an n-frame contains $-(-50)$, or 50, red tile. The value of 50 red tile is –50.

A *–n-frame* is the opposite of an n-frame. It contains red tile if n is positive, black tile if n is negative, and no tile if n is 0. In all cases, the total value of the tile it contains is $-n$. Thus, if n is positive, it contains n red tile and if n is negative, it contains $-n$ (or $|n|$) black tile. A $-n$-frame is distinguished from an n-frame by the small o's on each end.

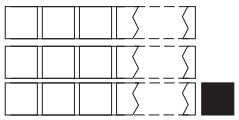
Focus Teacher Activity (cont.)

ACTIONS

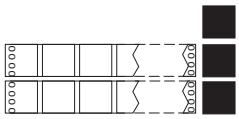
COMMENTS

Masters for *n*-frames and –*n*-frames are included in *Blackline Masters* (*n*-frames are included in each set of plastic or cardstock Algebra Pieces available from The Math Learning Center). They are intended to be printed back-to-back so that turning over an *n*-frame results in a –*n*-frame, and conversely. Each student needs approximately 12 *n*-frames.

Shown below are *n*th arrangements for each of the two extended sequences shown on Focus Master A.



$v(n) = 3n + 1$



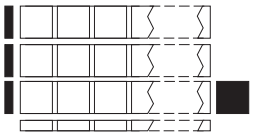
$v(n) = -2n + 3$

Some students may view $3n + 1$ as one unit added to a rectangle. Hence, it is useful to have edge pieces whose color, like that of frames, differs for positive and negative *n*. Such pieces—referred to as edge frames—are obtained by cutting frames into thirds:

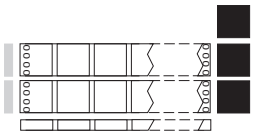
Edge Frames

Has value *n* for *n*, positive, negative, or zero.

Has value –*n* for *n*, positive, negative, or zero.

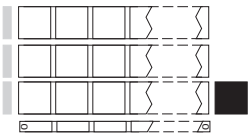


$v(n) = 3n + 1$

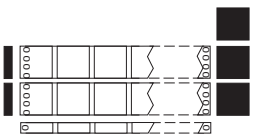


$v(n) = -2n + 3$

Thus, $v(n) = 3n + 1$ and $v(n) = -2n + 3$ could both be represented in two ways as shown at the left. The rectangle with value $3n$ can have edges with values 3 and *n*, or –3 and –*n*. Similarly, a rectangle with value $-2n$ can have edges of value –2 and *n*, or 2 and –*n*.



$v(n) = 3n + 1$



$v(n) = -2n + 3$

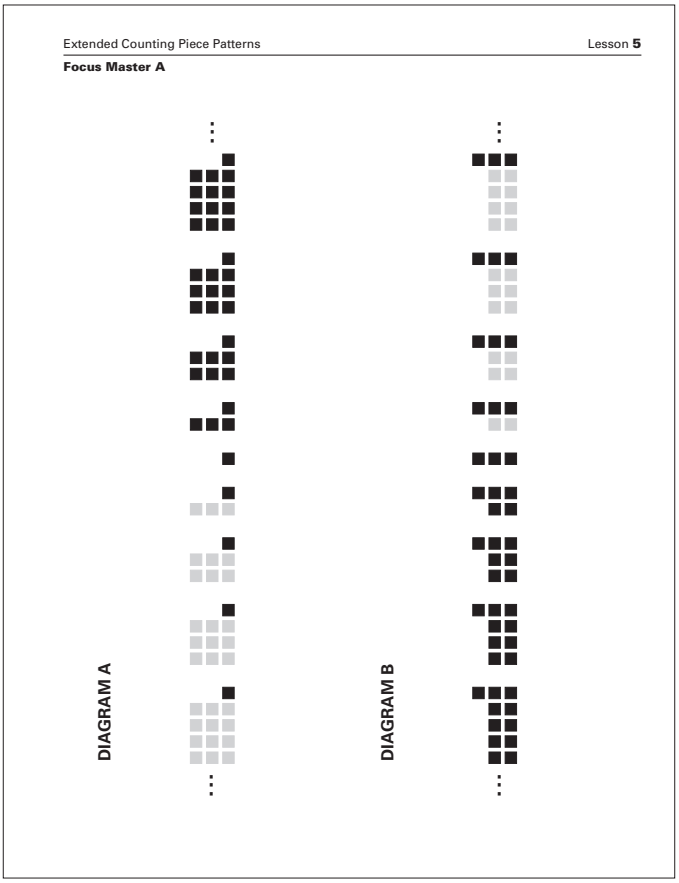
Students may question whether there are frames for n^2-mats. Note that an n^2-mat is black when *n* is positive and black when *n* is negative. Similarly, a $-n^2$-mat is red regardless the sign of *n*. Hence, frames for n^2-mats are not needed. Rather than pointing this out to students, you might have them investigate the idea in their groups.

Focus Teacher Activity (cont.)

ACTIONS

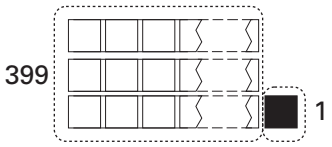
9 Assuming the numbering given in Action 5, ask the students to reason from the *n*th arrangement to determine, for the extended sequence shown in Diagram A on Focus Master A, the number of the arrangement which has value a) 400, b) –200.

Then, assuming the numbering given in Action 7c) for the extended sequence shown in Diagram B, ask the students to reason from the *n*th arrangement to determine the number of the arrangement in Diagram B which has value a) 165, b) –75.



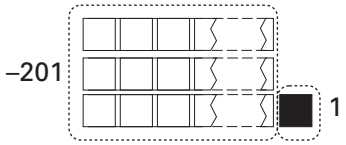
COMMENTS

9 If an arrangement in the extended sequence shown in Diagram A has value 400, the 3 *n*-frames in the *n*th arrangement shown below have a total value of 399. Hence, each has value 133, Thus, $n = 133$ and it is the 133rd arrangement which has value 400.



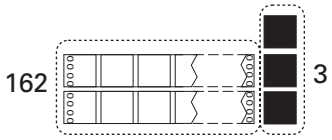
$v(n) = 3n + 1 = 400$

If an *n*th arrangement has value –200, one red tile can be added, changing the value to –201. The red and black tile cancel each other, so the 3 *n*-frames in the figure have a total value of –201, as illustrated in the diagram below. Hence, each has value –67 and thus $n = -67$.



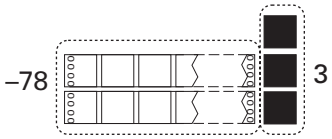
$v(n) = 3n + 1 = -200$

If the value of the *n*th arrangement of the sequence in Diagram B of Focus Master is 165, as shown below, then each of the 2 –*n*-frames in the figure has a value of 81. Hence, $-n = 81$ in which case, $n = -81$. So, it is the –81st arrangement which has value 165.



$v(n) = -2n + 3 = 165$

If the value of the *n*th arrangement shown below is –75, 3 red tile can be added, changing the value to –78. The 3 red and 3 black cancel each other, so each –*n*-frame in the figure has value $^{-78}/_{2} = -39$. Hence, $-n = -39$. Thus, $n = 39$ and it is the 39th arrangement which has value –75.



$v(n) = -2n + 3 = -75$

Focus Teacher Activity (cont.)

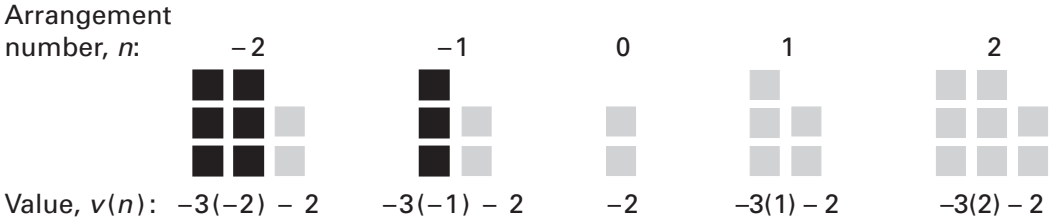
ACTIONS

10 Write the equation $v(n) = -3n - 2$ on the overhead. Ask the students to build the –2nd, –1st, 0th, 1st and 2nd arrangements of an extended sequence for which $v(n) = -3n - 2$. Have volunteers demonstrate their results. Then have the students form a single arrangement of Algebra Pieces that represents the nth arrangement of this extended sequence.

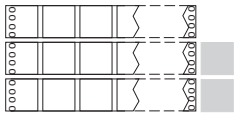
11 Ask the students to reason from the nth arrangement formed in Action 10 to determine which arrangement, if any, of the extended sequence has value a) 100, b) 200, c) –200. Ask the students to record the equations that have been solved.

COMMENTS

10 Here is one possibility for the requested arrangements:

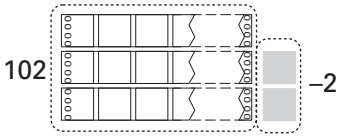


Since $v(n) = -3n - 2 = 3(-n) - 2$, one possible representation of the nth arrangement is the following:



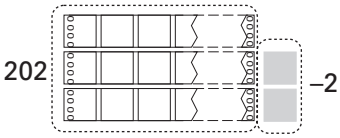
$v(n) = -3n - 2$

11 a) If an nth arrangement has value 100, each $-n$-frame is $102 \div 3$ or 34. Thus $n = -34$.



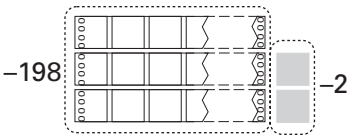
$v(n) = -3n - 2 = 100$

b) This situation is not possible, since 202 is not a multiple of 3.



$v(n) = -3n - 2 = 200$
This is not possible.

c) In this case, $-n = -198 \div 3 = -66$. Hence $n = 66$.

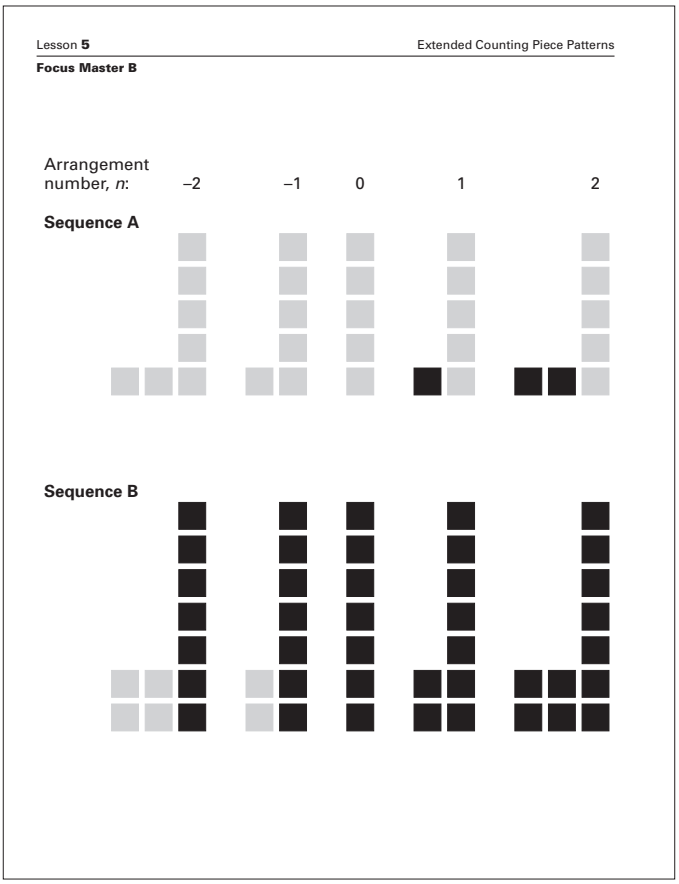


$v(n) = -3n - 2 = -200$

Focus Teacher Activity (cont.)

ACTIONS

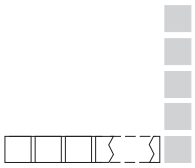
12 Place a transparency of Focus Master B on the overhead and ask the students to build an Algebra Piece representation of the *n*th arrangement of each sequence. Then ask them to determine for which *n* these two arrangements have the same value. Discuss the equation that has been solved.



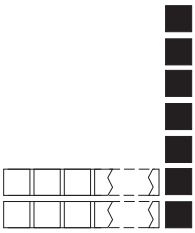
13 Repeat Action 12 for Sequences C and D on Focus Master C.

COMMENTS

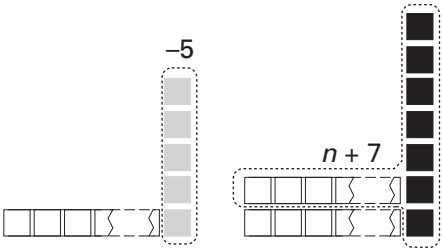
12 A representation for the *n*th arrangement of A:



A representation for the *n*th arrangement of B:

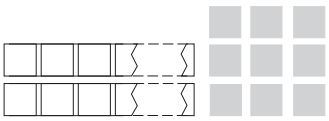


The two arrangements have the same value if the circled portions shown below have the same value. The portion on the left has value –5, and the portion on the right has value $n + 7$. The portion on the right will have value –5 if the enclosed *n*-frame has value –12, i.e., if $n = -12$. Thus, the solution to the equation $n - 5 = 2n + 7$ is $n = -12$.

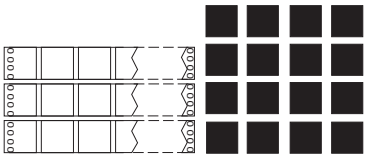


13 The *n*th arrangements are shown below.

Sequence C



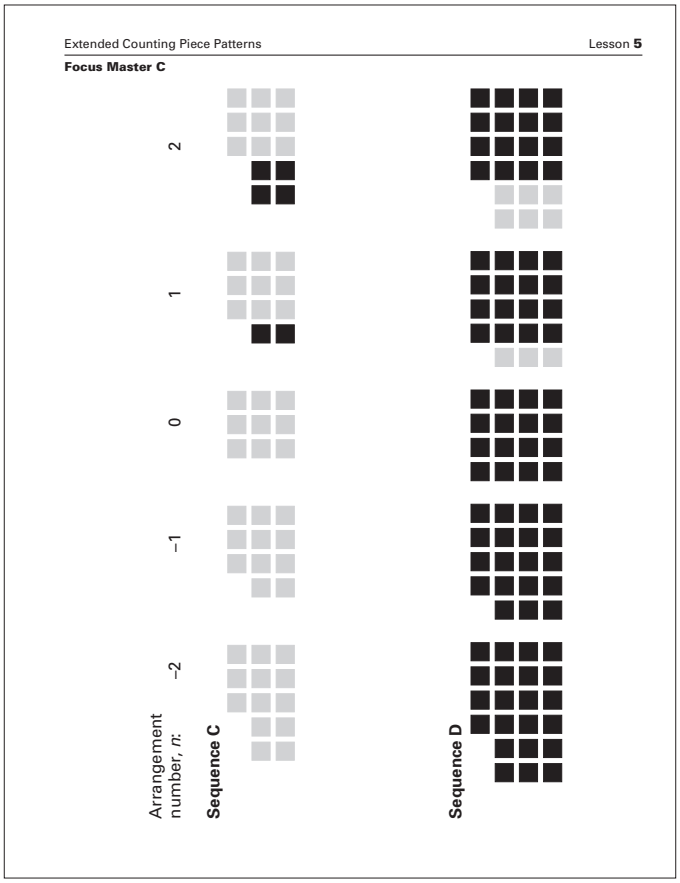
Sequence D



The values of the arrangements, as they appear above, are difficult to compare. However, adding 3 *n*-frames and 3 –*n*-frames to the *n*th arrangement for Sequence C,

Focus Teacher Activity (cont.)

ACTIONS



14 Ask the students to use Algebra Pieces or sketches to help them solve the following equations and to relate each equation to sequences of counting piece arrangements. Then ask them to verify their solutions. Discuss the students' observations and encourage conjectures and generalizations.

a) $4n + 7 = -133$

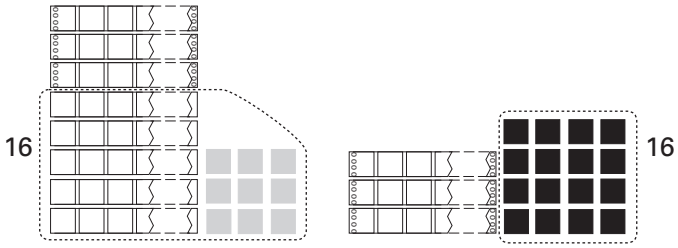
b) $8 - 5n = -142$

c) $4n + 5 = 3n - 8$

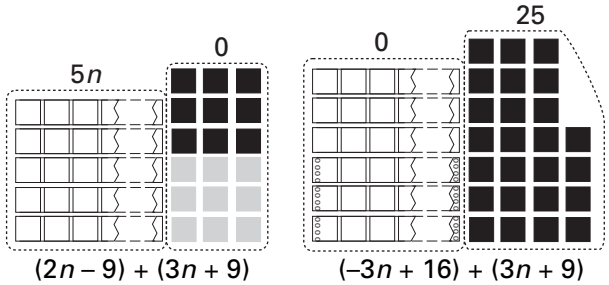
d) $8n - 4 = 6n + 10$

COMMENTS

as shown below, does not change its value. The two arrangements have the same value if the circled portions have the same value, i.e., if $5n + {}^{-}9 = 16$. Hence, $5n = 25$ and $n = 5$.



An alternative solution is based on the observation that if the same value is added to two arrangements that are equal in value, the resulting arrangements will have equal values. Shown here, values $3n + 9$ have been added to arrangements with values $2n - 9$ and $-3n + 16$. The resulting arrangements have equal values provided $5n = 25$ or, simply, $n = 5$.



The equation $2n - 9 = -3n + 16$ has been solved.

14 The intent here is to use Algebra Pieces to solve equations. In terms of sequences of counting piece arrangements, solving each equation is equivalent to determining the value of n for which the nth arrangements of two sequences have the same value. The expression on the left side of the equation is the value of the nth arrangement of one of the two sequences, and the expression on the right side of the equation is the value of the nth arrangement of the other sequence. Although some students may use symbols to represent their Algebra Piece actions, it is not expected that students use symbolic procedures here. Rather, the insights and intuitions that are prompted by these hands-on experiences will be useful later when students invent ways of recording the procedures they use with the pieces, and ultimately, to use the symbols as representations of their actions with the pieces.

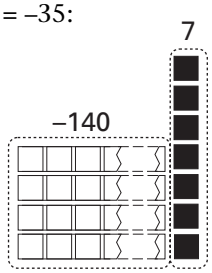
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

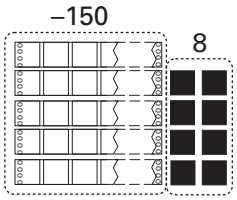
COMMENTS

14 (continued.)
a) Solving the equation $4n + 7 = -133$ is equivalent to determining the value n for which the nth arrangement of the sequence $v(n) = 4n + 7$ has the same value as the nth arrangement of the sequence $v(n) = -133$. An arrangement whose value is $4n + 7$ has value –133 if the 4 circled n-frames shown below have total value –140. This is the case if $n = -35$:



If, $4n + 7 = -133$, then $4n = -140$,
and $n = (-140) \div 4 = -35$

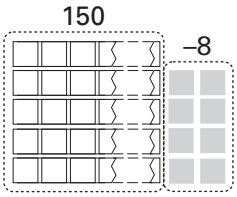
A solution to an equation can be verified by substituting the solution in the original equation to see if the result is a true statement. For example, substituting $n = {}^{-}35$ in the equation $4n + 7 = -133$ produces the true statement, $4(-35) + 7 = -133$. Substituting any other value for n produces a statement that is not true. Students may suggest other verification methods, such as solving the problem using a different method to see if that method produces the same solution. Note that while going back over the steps of one's methods is a good idea, when doing so, one often overlooks or repeats errors that were made.



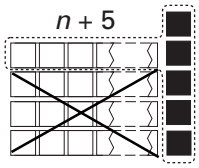
$8 - 5n = -142$,
so, $-5n = -150$
and $-n = {}^{-150}/_{5} = -30$
Thus, $n = 30$

b) An arrangement with value $8 - 5n$ has value –142 when each $-n$-frame has value –30, that is, when $n = 30$ (illustrated at the left):

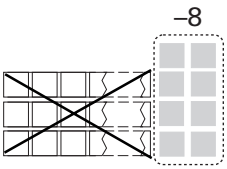
Alternatively, an arrangement with value $8 - 5n$ has value –142 when the opposite arrangement has value 142, as shown here:



$5n - 8 = 142$,
so, $5n = 150$
and $n = {}^{150}/_{5} = 30$



$4n + 5$

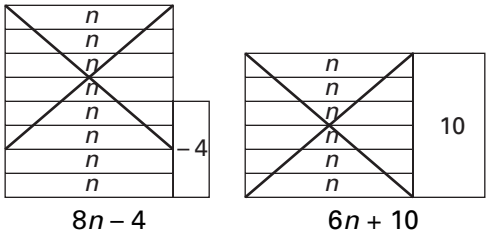


$3n - 8$

c) Arrangements with values $4n + 5$ and $3n - 8$ have the same value when the two circled portions in the diagram at the left each have value –8, i.e., when $n + 5 = -8$, in which case $n = -13$.

Focus Teacher Activity (cont.)

ACTIONS



15 Give each student a copy of Focus Student Activity 5.1 to complete. Discuss, encouraging students to make observations about similarities, differences, and general strategies they notice. Ask them to discuss the strategies they prefer and why.

Lesson **5** Extended Counting Piece Patterns

Focus Student Activity 5.1

NAME _____ DATE _____

Luise, Bob, and Patty used Algebra Pieces to solve the equation $3(n - 4) = 5(n + 4)$. Then they wrote the following to represent each step of their thoughts and actions with the Algebra Pieces. Next to each line of the three methods, write a brief explanation of what thoughts or actions you think are represented by the algebra statement on that line.

Luise's Method

$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 - 3n = 5n + 20 - 3n$
$-12 = 2n + 20$
$-12 + {}^{-}20 = 2n + 20 + {}^{-}20$
$-32 = 2n$
$-16 = n$

Bob's Method

$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 + 5n + {}^{-}5n = 5n + 20$
$3n - 12 - 5n = 20$
${}^{-}2n - 12 = 20$
${}^{-}2n - 12 = 20 + 12 + {}^{-}12$
${}^{-}2n = 32$
${}^{-}2n \div 2 = 32 \div 2$
${}^{-}n = 16$
$-({}^{-}n) = -16$
$n = -16$

Patty's Method

$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 + {}^{-}3n + {}^{-}20 = 5n + 20 + {}^{-}3n + {}^{-}20$
$-32 = 2n$
$\frac{1}{2}(-32) = \frac{1}{2}(2n)$
$-16 = n$

16 Write equation a) below on the overhead or board. Ask the students to use their Algebra Pieces, or sketches of pieces, to solve the equation. Then have them use numbers and algebraic symbols only (no words or pictures) to communicate each step of their Algebra Piece methods. Discuss and repeat for b)-d).

a) $n + 10 = 1 - 2n$

b) $6n - 64 = 2n$

c) $3n - 81 = 6n + 84$

d) $5n - 170 = 190 - 4n$

COMMENTS

d) As the sketches at the left show, $8n - 4$ and $6n + 10$ have the same value when $2n - 4$ has value 10. This is so if $2n$ has value 14, in which case $n = 7$.

15 The intent here is to reinforce the notion that algebraic symbols are representations of thoughts and procedures with Algebra Pieces, and to expose students to a few conventions for using algebraic symbols. Here are examples of observations made by students:

Each symbolic solution begins by stating the equation to solve.

Actions are recorded on both sides of the equation simultaneously, just as actions are carried out with the pieces.

Sometimes an expression is restated in simpler terms. Note: this is called *combining like terms* or *simplifying* the expression.

The final line in the solution tells the value of n.

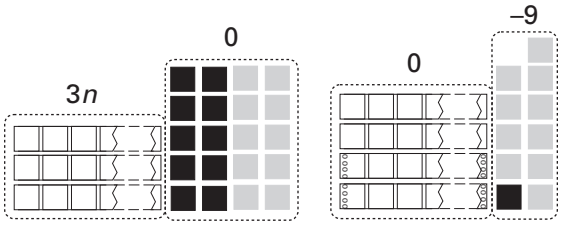
Two expressions that are equal remain equal if the same action is carried out on both expressions, or if zero is added to either expression.

16 Note that students' recordings will vary and some students will include more detail in their recordings than others. It is important not to be rigid or suggest "rules" for recording students' thoughts. It is hoped that students develop a view that the purpose of symbols is to provide a "shorthand" way of recording thought processes and carrying out actions with Algebra Pieces mentally (e.g., the variable n is associated with a mental image of an n-frame).

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

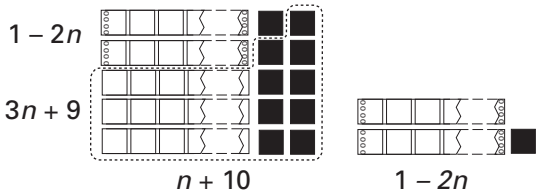


$(n + 10) + (2n - 10)$ $(-2n + 1) + (2n - 10)$

COMMENTS

16 (continued.)
a) Note that solving $n + 10 = 1 - 2n$ is equivalent to finding the value of n for which the arrangements of sequences with net values $v_1(n) = n + 10$ and $v_2(n) = 1 - 2n$ are equal in value.

Notice (see diagram below) that adding 2 $-n$-frames and 2 n-frames to an arrangement with value $n + 10$ doesn't change the value of the arrangement, but doing so results in a collection that has pieces in common with the arrangement with value $1 - 2n$. Comparing the arrangements with net values $v_1(n) = n + 10$ and $v_2(n) = 1 - 2n$ shows the two arrangements have the same value if $3n + 9 = 0$ or $n = -3$.



Here is one student's recording, representing her thoughts when solving $n + 10 = 1 - 2n$:

$n + 10 = 1 - 2n$
$n + 10 + 2n + {}^{-}2n = 1 - 2n$, add zero to $n + 10$
$3n + 10 = 1$ (remove 2 $-n$-frames from each nth arrangement)
$3n + 10 - 1 = 1 - 1$, add -1 to both nth arrangements
$3n + 9 = 0$
$3n + 9 = 0 + 9 - 9$, add zero to zero
$3n = {}^{-}9$
${}^{3n}/_{3} = {}^{-9}/_{3}$
$n = {}^{-}3$

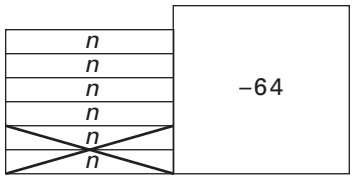
An alternative solution, illustrated at the left, is based on the last method of Comment 13. Starting with collections with values $n + 10$ and $1 - 2n$, respectively, and then adding 2 n-frames and 10 red tile to each collection results in two collections with values $3n$ and -9, respectively. Hence $n + 10 = 1 - 2n$ provided $3n = -9$. This is so if $n = -3$. A recording of the these procedures might look like the following:

$n + 10 = 1 - 2n$
$n + 10 + 2n + {}^{-}10 = 1 - 2n + 2n + {}^{-}10$
$3n = -9$
$n = -3$

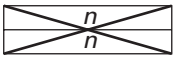
Note that some students may include the statement ${}^{3n}/_{3} = {}^{-9}/_{3}$ prior to stating $n = -3$ in the sequence of steps

Focus Teacher Activity (cont.)

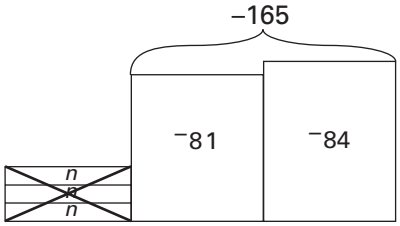
ACTIONS



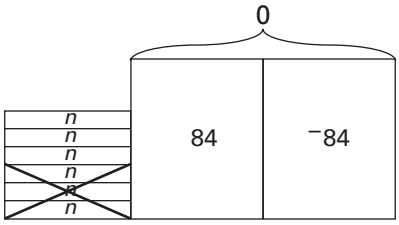
$6n - 64$



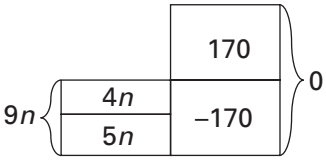
$2n$



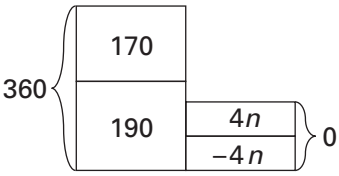
$(3n - 81) + {}^{-}84$



$(6n + 84) + {}^{-}84$



$(5n - 170) + (4n + 170)$



$(190 - 4n) + (4n + 170)$

17 (Optional) Give each student a copy of Focus Master D (see next page), pointing out that the problems on Master D were written by students in another *Math Alive! Course III,* classroom. Ask the groups to each create an "answer key" for the problems. Suggest that each answer should include the important information that a teacher would expect from a student who understands the ideas in the problem. When completed, have the groups exchange their "keys," making note of ideas that have been overlooked and ideas that are especially important.

COMMENTS

shown above. This represents the process of separating the 3 n-strips and 9 red units into 3 groups in order to show that 1 n-strip has value –3. Either representation is correct; adding the step $^{3n}/_{3}$ simply shows the student's thought processes more completely.

b) As illustrated at the left, one sees that sketches for $6n - 64$ and $2n$ have the same value if $4n - 64 = 0$, which happens when $n = 16$. Symbolically, students may record such thought processes as follows:

$$\begin{aligned} 6n - 64 &= 2n \\ 4n - 64 &= 0 \\ 4n &= 64 \\ n &= 16 \end{aligned}$$

c) In the 2 diagrams at the left a section representing ⁻84 has been added to sketches for $3n - 81$ and $6n + 84$. The sketches have equal values if $3n = -165$ or $n = (-165) \div 3 = -55$. A symbolic representation of this line of thinking might be:

$$\begin{aligned} 3n - 81 &= 6n + 84 \\ 3n - 81 + {}^{-}84 &= 6n + 84 + {}^{-}84 \\ 3n - 165 &= 6n \\ -165 &= 3n \\ {}^{-165}/_{3} &= n \end{aligned}$$

d) As shown in the sketch at the left, if $4n + 170$ is added to $5n - 170$ and $190 - 4n$, the results have equal values provided $n = 40$. Symbolically, such reasoning could be represented as follows:

$$\begin{aligned} 5n - 170 &= 190 - 4n \\ 5n - 170 + (4n + 170) &= 190 - 4n + (4n + 170) \\ 9n &= 360 \\ n &= {}^{360}/_{9} = 40 \end{aligned}$$

17 These are the actual wordings of the students' problems. You might suggest that students refine any questions they feel should be worded differently.

Rather than having the students answer all of these problems, you might select from the problems according to the needs of your students. Or, you might have the class identify the problems they feel would be most challenging and then have the groups write answers for those.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **5** Extended Counting Piece Patterns

Focus Master D

1. When you see the "–" symbol, what do you think of?

2. When you are done solving an equation, how do you test your answer to make sure that it is correct?

3. What are common things that you do when solving an equation?

4. Define n and $-n$.

5. Two important ideas in solving an equation are addition and subtraction of negative and positive pieces. Generalize and explain these processes and give an example of each.

6. The n-frame is an important tool to understand. Explain what they are used for, how they are used, and why.

7. What is an algebra equation you can solve in your mind, by simply imagining the pieces?

8. Explain when, if ever, $-n$ is less than, more than, and/or equal to zero. Give examples and go into detail.

9. Why are there no n^2-frames? Build a convincing argument and explain your thinking on this question.

10. Tell what is meant by an "extended" sequence of counting piece arrangements.

Extended Counting Piece Patterns Lesson **5**

Focus Master D (cont.)

11. Figure out what is wrong with the way Jane solved the following equation. Then solve it the way you think would use the correct procedures and least amount of steps.

Jane's Steps:

1. $2n + 3 + 3(n - 4) = 8n + 3$
2. $5n + {}^{-}9 = 8n + 3$
3. $5n + {}^{-}9 = 8n + 3 - 5n$
4. $5n + {}^{-}9 - 3n = 3n + 3 - 3n$
5. $2n + {}^{-}9 = 3$
6. $2n + {}^{-}9 + 9 = 3 + 9$
7. ${}^{2}n/2 = {}^{12}/2$
8. $n = 6$

12. Explain how the equation $-3n + 2 = 4n - 12$ relates to sequences of counting piece arrangements.

13. Explain how edge pieces and/or edge frames are used to represent products of whole numbers, integers, and algebraic expressions.

14. Can you think of other thoughtful questions about the ideas in this lesson? Try to think of ones that require *understanding* of important ideas.

COMMENTS

17 (continued.)
Note that answers for Problem 14 could list several examples of additional questions.

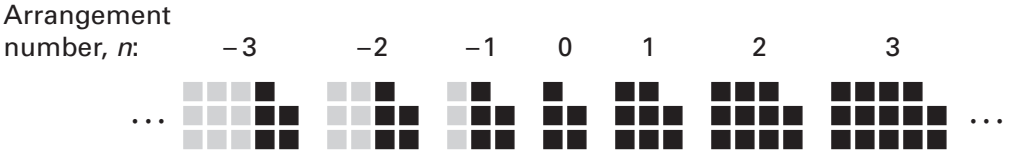
Groups could use the Peer Feedback Sheets from Action 11 of Lesson 4 (see Focus Student Activity 4.2) for giving feedback to each other.

Follow-up Student Activity 5.2

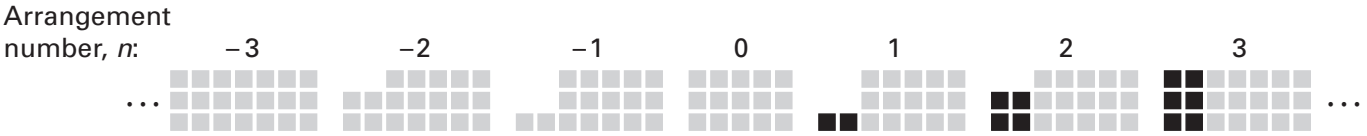
NAME _____ **DATE** _____

Complete the problems on this Follow-up on separate paper.

Sequence A



Sequence B



1 a) Sketch the Algebra Piece representations of the nth arrangement of Sequence A and the nth arrangement of Sequence B.

b) Draw diagrams to show each step of Algebra Piece procedures for finding the value of n for which Sequences A and B have the same net value. Write brief comments, as needed, to help communicate your methods.

c) Tell what equation you solved in b).

2 Sketch the –3rd through 3rd arrangements of a sequence of counting piece arrangements with net value $v(n) = 3n + 4$. Then show how to use Algebra Pieces to determine the value of n for which $v(n) = 190$.

3 Draw diagrams to show how to use Algebra Pieces to solve, if possible, the following equations. Write brief comments to explain what you do in each step. If there is no solution, explain why. If there is more than one solution, explain how many and why.

a) $7n + 2 = 8n - 4$

b) $4n^2 + 3n - 5 = (2n + 1)^2 + 8$

c) $3(2n - 3) = 9n + 6$

d) $-16 + 24n = 272$

e) $3 + n = -3 + n$

f) $7(n + 2) = 7n + 14$

(Continued on back.)

Follow-up Student Activity (cont.)

4 For each of the following conditions, write an equation (not already on this Follow-up) which meets the given conditions. Then make a diagram or write a brief explanation to show why your equation satisfies the conditions.

a) This equation has exactly one solution and that solution is negative.

b) This equation has no solutions.

c) This equation has an infinite number of solutions.

5 Use Algebra Pieces to solve the equation $8n + 36 = 4(n + 1)$. Then:

a) using algebraic symbols only, record each step of your thought processes and Algebra Piece methods;

b) write a brief explanation of the thoughts and actions represented by each step you wrote in a).

6 Solve the equation $7(n + 3) = 5(n - 3) + 6$ using whatever methods you choose. Explain or illustrate each step of your thought processes and actions. Then tell how you can be sure that your solution is correct.

7 Give one or more different equations (not already on this Follow-up) for each of the following. Show or explain your methods of solving each equation.

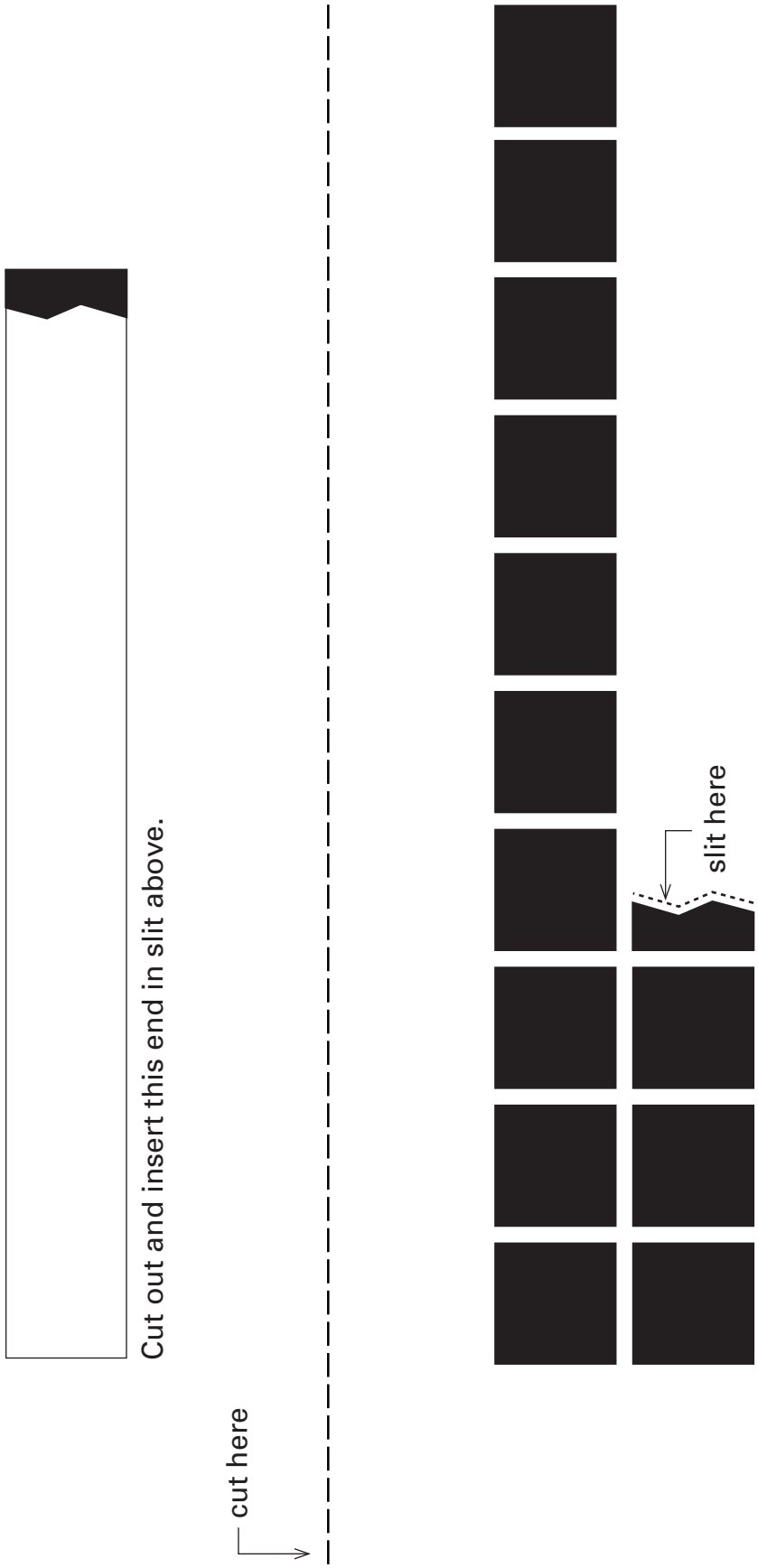
a) an algebra equation you think is most convenient to solve by simply imagining the Algebra Pieces in your mind's eye (i.e., without building or sketching models or writing equations);

b) an algebra equation you think is most convenient to solve by using algebraic symbols to represent the Algebra Pieces;

c) an algebraic equation that you can solve and you think is difficult.

8 Write several "tips" you recommend that others keep in mind when solving equations or representing expressions with Algebra Pieces.

Connector Master A

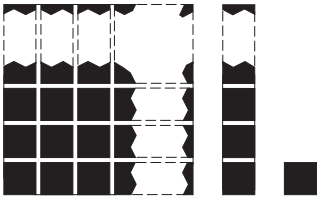


Connector Master B

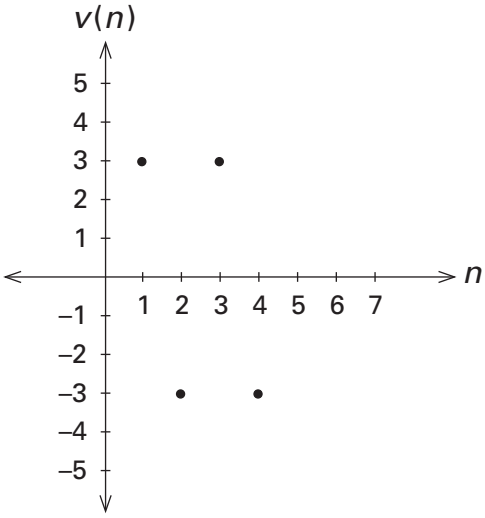
Following are representations of 5 different sequences.

1. 

2. $v(n) = -13 + 5(n - 1)$

3. 

4. 3, –3, –9, –15, –21, –27 …

5. 

Connector Master B (cont.)

Part I

For each of Sequences 1-5 represented on the previous page, do the following (if the information is already given, write "given" and then skip that part):

a) Form and then sketch counting piece arrangements for the first 6 terms of the sequence.

b) Form and then sketch an Algebra Piece representation of the *n*th arrangement of the sequence.

c) Write a formula for $v(n)$, the value of the *n*th arrangement of the sequence. "Loop" a diagram to show how your formula works.

d) If a_n represents the *n*th term of the sequence, record the values of the following: a_1, a_2, a_3, a_4, a_5, a_6, a_{20}, a_{73}, and a_{200}.

e) Make a coordinate graph, plotting and labeling the first 5 or more terms in the sequence.

Part II

1. Which of Sequences 1-5 are arithmetic sequences? Explain your reasoning.

2. *Challenge*: For any of Sequences 1-5 that are arithmetic sequences, show how to use the method of combining two matching staircases to find S_{50}, the sum of the first 50 terms of those sequences.

Connector Master C

1. Use Algebra Pieces to do the following:

a) Form a collection that is ½ of a collection of 2 *n*-strips.

b) Now, add 4 units to the collection formed in a).

2. Write an algebraic expression that describes your actions in Problem 1.

3. One way to think about algebraic expressions is as representations of actions with Algebra Pieces. For the expression $\frac{4+2n}{2}$ write the steps of Algebra Piece actions, in order, that the expression could represent. At the end of each step, sketch the resulting collection of Algebra Pieces.

4. In the final collection that you formed for Problem 3, can you "see" an algebraic expression that is equivalent to the original expression? If so, record that expression.

5. Repeat Problems 3 and 4 for the following expressions:

a) $\frac{4n-2n}{2}$

b) $\frac{4-2n}{2}$

c) $4^2 - 2n^2$

d) $4^2 - (2n)^2$

e) $(4 - 2n)^2$

f) $4 + 4n - 8 \div 4$

g) $\frac{4+4n-8}{4}$

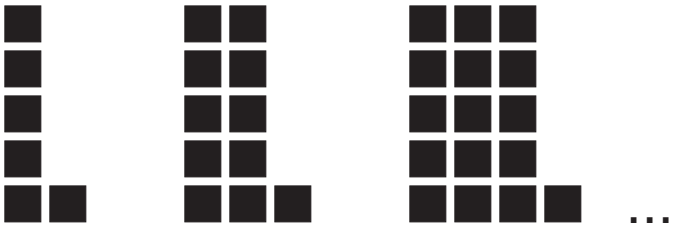
h) $6 - 12n \div 3 + 3$

i) $6 - 12n \div (3 + 3)$

6. Record any general observations, AHA!s, conjectures, or important ideas you noticed.

Connector Master D

A



B

C

D

E $v(n) = -4n + 7$

F $v(n) = 2n + {}^{-}17$

Focus Master A

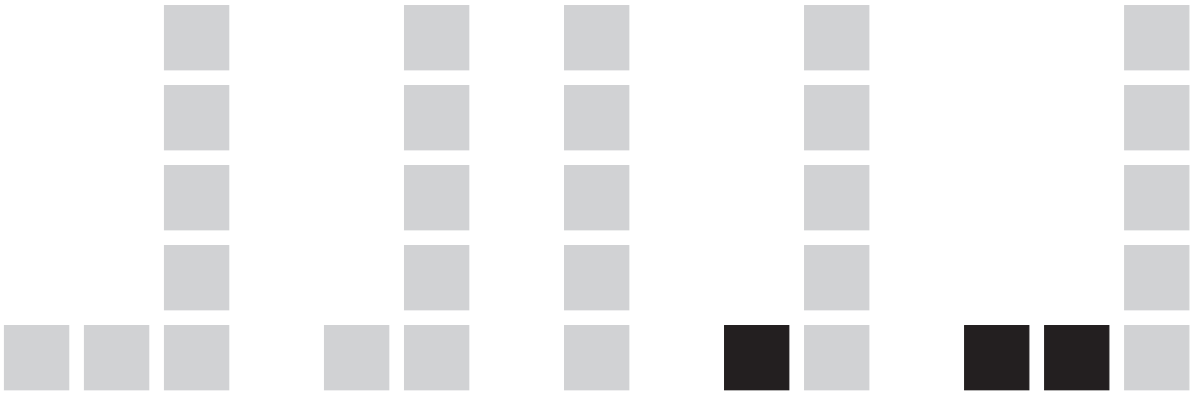
DIAGRAM A

DIAGRAM B

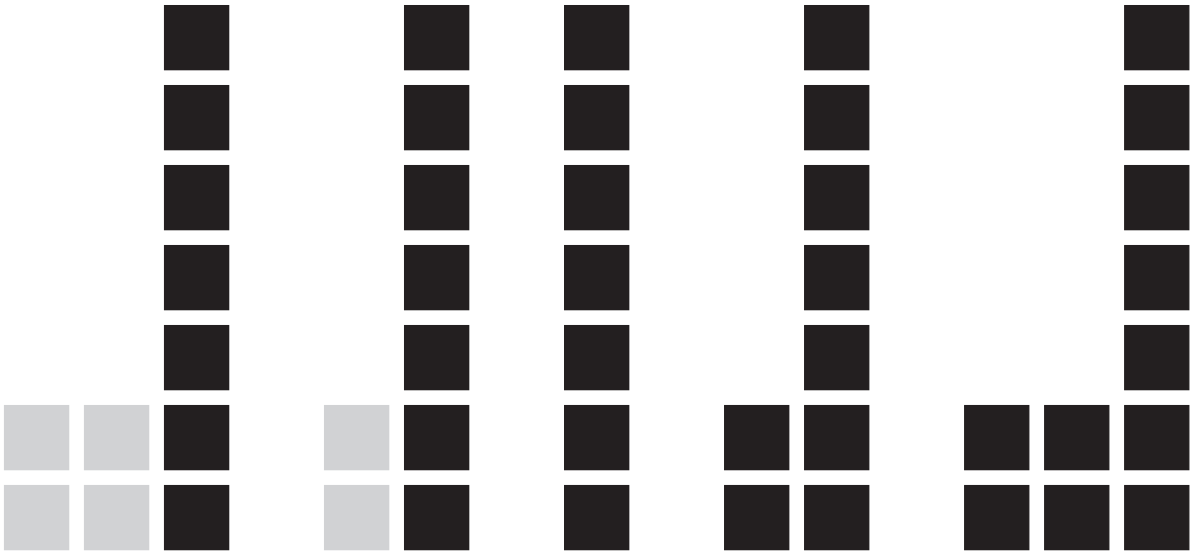
Focus Master B

Arrangement number, *n*: –2 –1 0 1 2

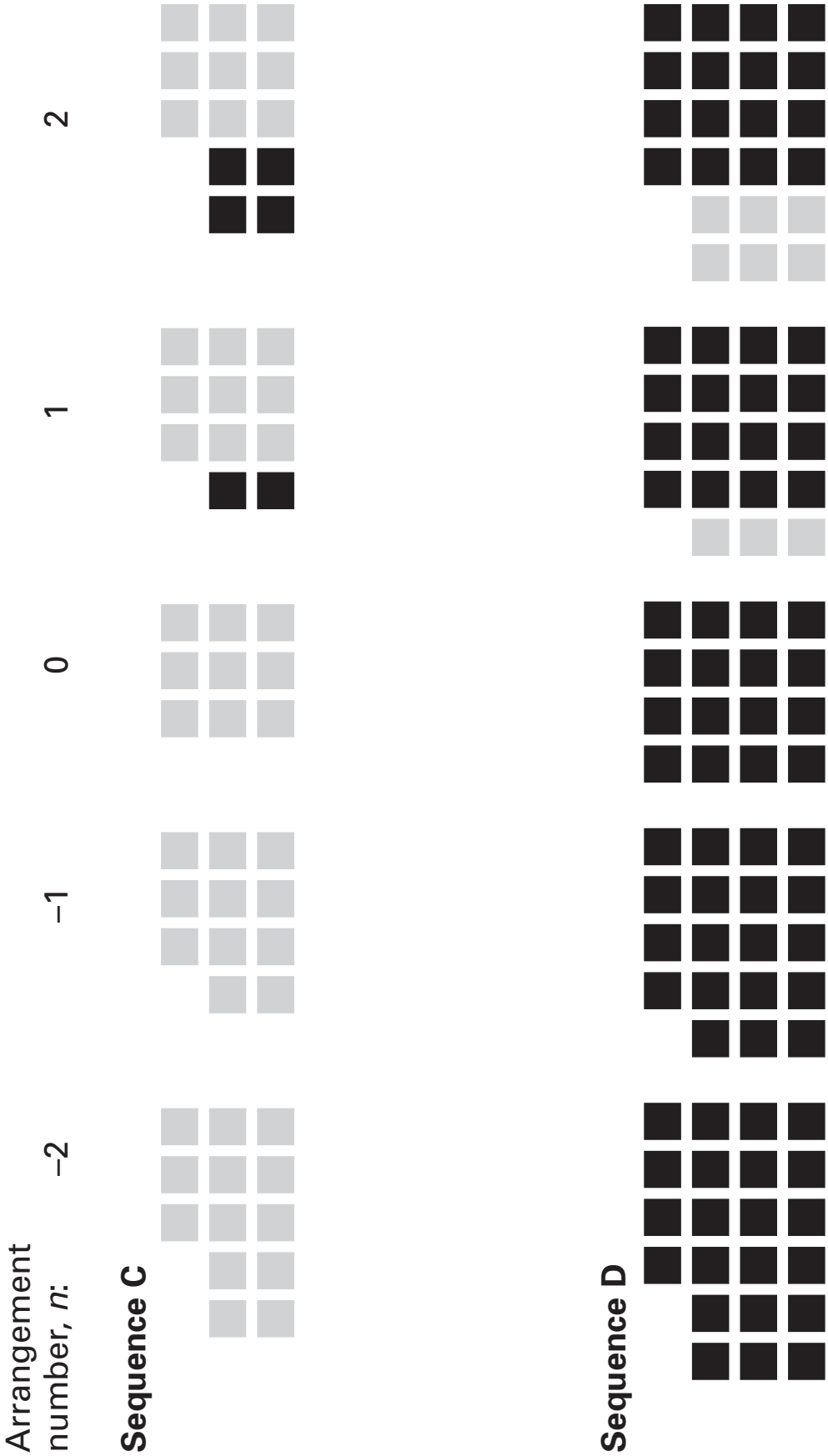
Sequence A



Sequence B



Focus Master C



Focus Master D

1. When you see the "–" symbol, what do you think of?

2. When you are done solving an equation, how do you test your answer to make sure that it is correct?

3. What are common things that you do when solving an equation?

4. Define n and $-n$.

5. Two important ideas in solving an equation are addition and subtraction of negative and positive pieces. Generalize and explain these processes and give an example of each.

6. The n-frame is an important tool to understand. Explain what they are used for, how they are used, and why.

7. What is an algebra equation you can solve in your mind, by simply imagining the pieces?

8. Explain when, if ever, $-n$ is less than, more than, and/or equal to zero. Give examples and go into detail.

9. Why are there no n^2-frames? Build a convincing argument and explain your thinking on this question.

10. Tell what is meant by an "extended" sequence of counting piece arrangements.

(Continued on back.)

Focus Master D (cont.)

11. Figure out what is wrong with the way Jane solved the following equation. Then solve it the way you think would use the correct procedures and least amount of steps.

Jane's Steps:

1. $2n + 3 + 3\,(n - 4) = 8n + 3$
2. $5n + {}^{-}9 = 8n + 3$
3. $5n + {}^{-}9 = 8n + 3 - 5n$
4. $5n + {}^{-}9 - 3n = 3n + 3 - 3n$
5. $2n + {}^{-}9 = 3$
6. $2n + {}^{-}9 + 9 = 3 + 9$
7. ${}^{2n}/_{2} = {}^{12}/_{2}$
8. $n = 6$

12. Explain how the equation $-3n + 2 = 4n - 12$ relates to sequences of counting piece arrangements.

13. Explain how edge pieces and/or edge frames are used to represent products of whole numbers, integers, and algebraic expressions.

14. Can you think of other thoughtful questions about the ideas in this lesson? Try to think of ones that require *understanding* of important ideas.

Focus Student Activity 5.1

NAME _____ **DATE** _____

Luise, Bob, and Patty used Algebra Pieces to solve the equation $3(n - 4) = 5(n + 4)$. Then they wrote the following to represent each step of their thoughts and actions with the Algebra Pieces. Next to each line of the three methods, write a brief explanation of what thoughts or actions you think are represented by the algebra statement on that line.

Luise's Method

$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 - 3n = 5n + 20 - 3n$
$-12 = 2n + 20$
$-12 + {}^{-}20 = 2n + 20 + {}^{-}20$
$-32 = 2n$
$-16 = n$

Bob's Method

$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 + 5n + {}^{-}5n = 5n + 20$
$3n - 12 - 5n = 20$
${}^{-}2n - 12 = 20$
${}^{-}2n - 12 = 20 + 12 + {}^{-}12$
${}^{-}2n = 32$
${}^{-}2n \div 2 = 32 \div 2$
${}^{-}n = 16$
$-({}^{-}n) = -16$
$n = -16$

Patty's Method

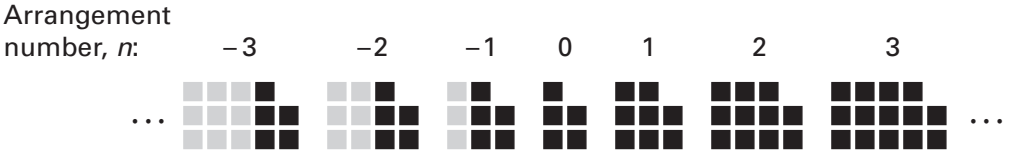
$3(n - 4) = 5(n + 4)$
$3n - 12 = 5n + 20$
$3n - 12 + {}^{-}3n + {}^{-}20 = 5n + 20 + {}^{-}3n + {}^{-}20$
$-32 = 2n$
$\frac{1}{2}(-32) = \frac{1}{2}(2n)$
$-16 = n$

Follow-up Student Activity 5.2

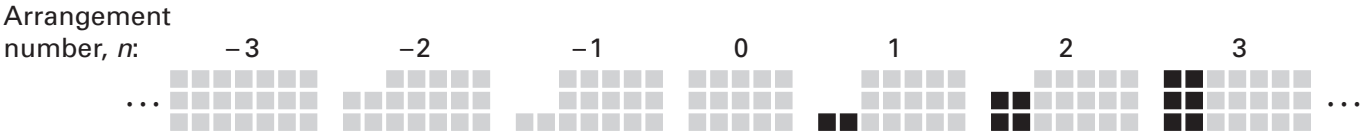
NAME _____ **DATE** _____

Complete the problems on this Follow-up on separate paper.

Sequence A



Sequence B



1 a) Sketch the Algebra Piece representations of the nth arrangement of Sequence A and the nth arrangement of Sequence B.

b) Draw diagrams to show each step of Algebra Piece procedures for finding the value of n for which Sequences A and B have the same net value. Write brief comments, as needed, to help communicate your methods.

c) Tell what equation you solved in b).

2 Sketch the –3rd through 3rd arrangements of a sequence of counting piece arrangements with net value $v(n) = 3n + 4$. Then show how to use Algebra Pieces to determine the value of n for which $v(n) = 190$.

3 Draw diagrams to show how to use Algebra Pieces to solve, if possible, the following equations. Write brief comments to explain what you do in each step. If there is no solution, explain why. If there is more than one solution, explain how many and why.

a) $7n + 2 = 8n - 4$

b) $4n^2 + 3n - 5 = (2n + 1)^2 + 8$

c) $3(2n - 3) = 9n + 6$

d) $-16 + 24n = 272$

e) $3 + n = -3 + n$

f) $7(n + 2) = 7n + 14$

(Continued on back.)

Follow-up Student Activity (cont.)

4 For each of the following conditions, write an equation (not already on this Follow-up) which meets the given conditions. Then make a diagram or write a brief explanation to show why your equation satisfies the conditions.

a) This equation has exactly one solution and that solution is negative.

b) This equation has no solutions.

c) This equation has an infinite number of solutions.

5 Use Algebra Pieces to solve the equation $8n + 36 = 4(n + 1)$. Then:

a) using algebraic symbols only, record each step of your thought processes and Algebra Piece methods;

b) write a brief explanation of the thoughts and actions represented by each step you wrote in a).

6 Solve the equation $7(n + 3) = 5(n - 3) + 6$ using whatever methods you choose. Explain or illustrate each step of your thought processes and actions. Then tell how you can be sure that your solution is correct.

7 Give one or more different equations (not already on this Follow-up) for each of the following. Show or explain your methods of solving each equation.

a) an algebra equation you think is most convenient to solve by simply imagining the Algebra Pieces in your mind's eye (i.e., without building or sketching models or writing equations);

b) an algebra equation you think is most convenient to solve by using algebraic symbols to represent the Algebra Pieces;

c) an algebraic equation that you can solve and you think is difficult.

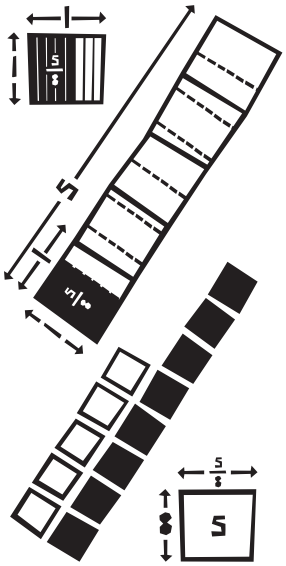
8 Write several "tips" you recommend that others keep in mind when solving equations or representing expressions with Algebra Pieces.

Fraction Concepts

Lesson 6

THE BIG IDEA

Modeling and comparing the four concepts of a fraction—part-to-whole, division, area, and ratio—foster insights and facilitate problem solving. Understanding these concepts of a fraction, leads to the discovery of important properties of numbers and equivalent fractions and lays groundwork for the invention of strategies for manipulating algebraic expressions.



CONNECTOR

OVERVIEW

Students use diagrams and write situations to model and explain the 4 different concepts of a fraction and to illustrate equivalent fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 copy per group and 1 transparency.

FOCUS

OVERVIEW

Students use line-dividing techniques to explore the division concept of a fraction and relate it to the part-to-whole concept. The area concept of a fraction is used to establish several number properties and procedures for obtaining equivalent fractions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activities 6.1-6.5, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Straightedges, 1 per student.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Marking pens for each group.

FOLLOW-UP

OVERVIEW

Students explain the meanings of the part-to-whole, ratio, division, and area concepts of a fraction and subdivide regions to illustrate equivalent fractions and properties of fractions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 6.6, 1 copy per student.
- ✔ 1-cm grid paper, 1 per student.
- ✔ Focus Master A, 1 per student.

LESSON IDEAS

QUOTE

In grades 5-8, computation and estimation should be integrated with the study of the concepts underlying fractions, decimals, integers, and rational numbers, as well as with the continuing study of whole numbers.

...As they begin to understand the meaning of operations and develop a concrete basis for validating symbolic processes and situations, students should design their own algorithms and discuss, compare, and evaluate these with their peers and teacher. Students should analyze the way the various algorithms work and how they relate to the meaning of the operation and to the numbers involved.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

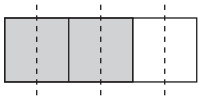
2. Methods of obtaining pairs of equivalent fractions are shown for each concept of a fraction.

Using part-to-whole:

Divide whole into 3 equal parts

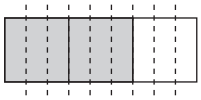
$\frac{2}{3}$

Divide each part into 2 parts



$\frac{2}{3} = \frac{4}{6}$

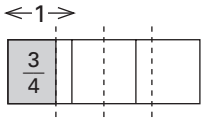
Divide each part into 3 parts



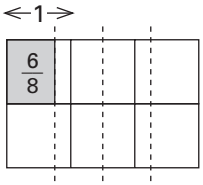
$\frac{2}{3} = \frac{6}{9}$

Using division concept:

Divide 3 area units into 4 equal parts.



Combine 2 copies of the 1st figure to produce 6 area units with 8 equal parts. One of these parts is $6 \div 8 = 6/8$.

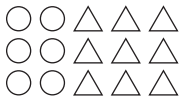


$\frac{3}{4} = \frac{6}{8}$

Using ratio concept:

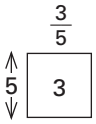
Ratio of circles to triangles is $2/3$

Ratio of circles to triangles is $4/6 = 2/3$

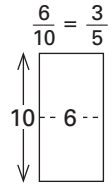


Ratio of circles to triangles is $6/9 = 2/3$

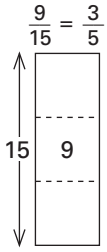
Using area concept:



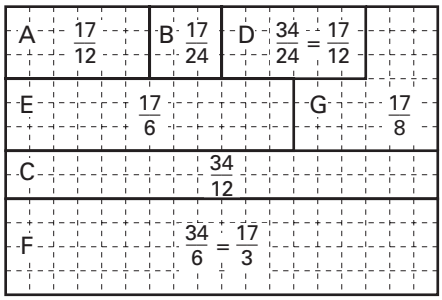
Doube area and length of rectangle



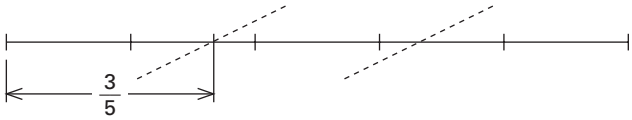
Triple area and length of rectangle



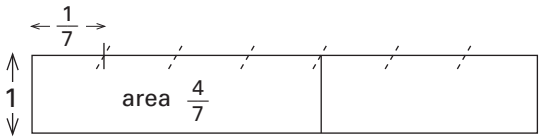
6. One possible set of subdivisions.



7. a) Divide 5 linear units into 3 equal parts.



b) Divide the length of the rectangle, which is 4, into 7 equal parts.



Connector Teacher Activity

OVERVIEW & PURPOSE

Students use diagrams and write situations to model and explain the 4 different concepts of a fraction and to illustrate equivalent fractions.

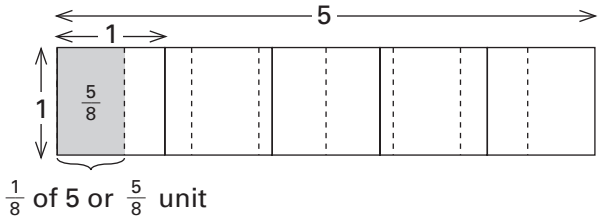
MATERIALS

✔ Connector Master A, 1 copy per group and 1 transparency.

ACTIONS

1 Arrange the students in groups. Write "5/8" on the overhead and ask the groups to discuss different ways of viewing the meaning of this symbol. Invite volunteers to share their group's ideas, illustrating with diagrams at the overhead. If students don't bring up the *part-to-whole*, *division*, *ratio*, and *area* concepts of a fraction, initiate such discussion.

Division Concept

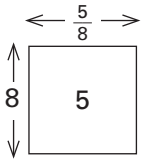


Ratio Concept

5 white for every 8 black

The ratio of white to black is 5 to 8, or 5/8.

Area Concept

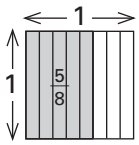


COMMENTS

1 The *part-to-whole*, *division*, *ratio*, and *area concepts of a fraction* are illustrated below. Students were introduced to these meanings of fractions in *Math Alive! Courses I* and *II.*

According to the *part-to-whole* concept the symbol 5/8 suggests "divide a unit into 8 equal parts, 5/8 unit is 5 of those parts," i.e., 5/8 is 1/8 + 1/8 + 1/8 + 1/8 + 1/8, or 5 times 1/8 unit, or 5/8 of 1 unit. For example, one might sketch a region with area 1 unit and subdivide it into 8 equal parts; 5/8 is the area of 5 of those parts, as shown below.

Part-to-whole Concept



According to the *division* concept the symbol 5/8 suggests "divide 5 units, collectively, into 8 equal parts, 5/8 is one of those parts," i.e., 5/8 represents 5 ÷ 8 or 1/8 of 5. For example, one could sketch a rectangle with area 5, then divide that rectangle into 8 equal parts; each part has area 5 ÷ 8 = 5/8 area unit.

According to the *ratio* concept, a fraction is used to compare one amount to another. For example, if the number of objects in one set is 5/8 the number of objects in another set, one can say the ratio of the numbers of objects in the sets is 5 to 8, or 5/8.

According to the *area* concept, the fraction 5/8 is one dimension of a rectangle with area 5 and other dimension 8.

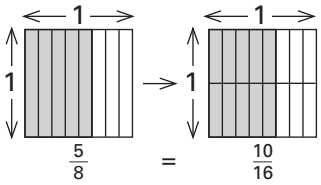
Note that "five-eighths" frequently evokes a part-to-whole image, particularly for students whose school experiences with fractions have been limited primarily to discussions and models involving the part-to-whole concept. The Focus activity of this lesson emphasizes understanding of the division and area concepts of a fraction.

Connector Teacher Activity (cont.)

ACTIONS

2 Ask the groups to discuss their ideas about the meaning of *equivalent fractions*, and to demonstrate visually how fractions equivalent to $\frac{5}{8}$ can be determined using each of the following concepts of a fraction: part-to-whole, area, division, and ratio. Invite volunteers to illustrate their group's ideas at the overhead.

Part-to-whole Concept

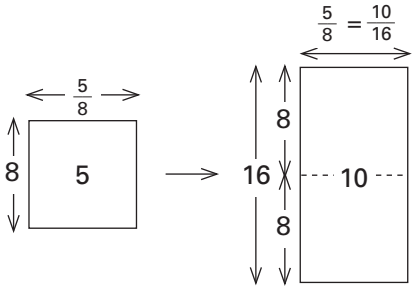


Doubling the number of equal parts in the first figure also doubles the number of shaded parts, but does not change the amount shaded. Hence, $\frac{5}{8} = \frac{10}{16}$.

COMMENTS

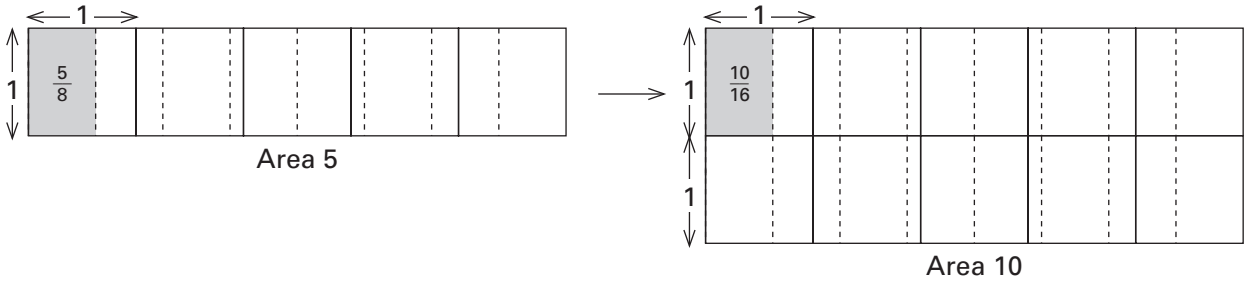
2 Two fractions are *equivalent* if they represent equal portions of the same unit. The process of forming equivalent fractions is illustrated below, for the part-to-whole, area, division, and ratio concepts of a fraction.

Area Concept



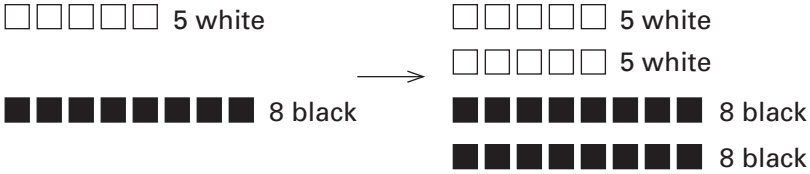
Doubling the length of the first figure also doubles its area, but the top dimension is not changed. Therefore, $5 \div 8 = \frac{5}{8} = 10 \div 16 = \frac{10}{16}$.

Division Concept



Combining 2 copies of the first figure produces a figure with area 10 and 16 equal parts. Each equal part has area $10 \div 16 = \frac{10}{16} = \frac{5}{8}$.

Ratio Concept



Doubling the number of white and black squares produces a second set with a ratio of 10 white to 16 black. But for every 5 white in the 2nd set, there are 8 black, so the ratio $\frac{5}{8} = \frac{10}{16}$.

Connector Teacher Activity (cont.)

ACTIONS

3 Ask the groups to each write a situation in which one might "think about" $\frac{3}{4}$ using the part-to-whole concept. Repeat for the ratio concept, the division concept, and the area concept. Invite volunteers to share their situations orally with their groupmates, who in turn, identify the fraction concept illustrated. Discuss.

4 Give each group a copy of Connector Master A and ask them to indicate the fraction concept/concepts that they use to think about each situation. Discuss.

Fraction Concepts Lesson **6**

Connector Master A

1. The county plans to evenly distribute 7 signs along the 5 miles of highway construction.

2. Tia and 4 friends are going to the movies. They have 3 candy bars to share equally.

3. The 4 members of the Wilkinson family plan to share a pizza equally.

4. The 32 members of Ms. Callahan's class plan to share 8 pizzas equally.

5. On a 10 kilometer relay, the 3 members of the Run-abouts each ran an equal distance.

6. The 7 members of the Greenspace Garden Project each were allocated an equal portion of the garden.

7. The Bergmans own a rectangular strip of land that runs 7 miles along the freeway. The area of the strip is 5 square miles. The Bergmans need to replace the fencing around the strip.

COMMENTS

3 Situations that students pose will vary, and students may find that some situations can be viewed according to more than one interpretation of the meaning of a fraction. Some possibilities for situations follow.

Part to whole concept: in Rick's Rug Store, $\frac{3}{4}$ of the rugs are machine made.

Ratio concept: the ratio of camcorders to still cameras in Clyde's Camera Shop is $\frac{3}{4}$.

Division concept: the 3-hour broadcast period was divided into 4 equal parts so that each DJ broadcast for $\frac{3}{4}$ hour.

Area concept: a rectangular brass sign with area 3 square feet and length 4 feet has width $\frac{3}{4}$ feet.

4 Groups may feel that more than one concept applies to some situations. For example, for Situation 2, some students may suggest the candy bars could be distributed by breaking the candy according to the part-to-whole concept, i.e., by breaking each candy bar into 5 equal parts and distributing 1 part from each candy bar to each girl; each girl receives a total of 3 of the $\frac{1}{5}$-bars. Others may suggest using the division concept by lining up the bars end-to-end and dividing that length into 5 equal parts; each girl receives 1 part, or $\frac{1}{5}$ of 3. Still others may view the situation as the ratio 3 candy bars for every 5 girls or $\frac{3}{5}$. In this case, they might use either of the preceding methods as the basis for determining each girl's share.

One possible concept of a fraction follows for each of the remaining situations from Connector Master A:

1) By the division concept, the 5 miles are divided into 7 equal parts of length $\frac{5}{7}$ mile.

3) By the part-to-whole concept, the whole pizza is divided into 4 equal parts, so that each family member receives $\frac{1}{4}$ pizza.

4) Using equivalent ratios, if there are 8 pizza for 32 people, then there is a ratio of 1 pizza for 4 people, and $\frac{1}{4}$ pizza for 1 person.

5) Using the division concept, each member runs $\frac{10}{3}$ kilometers.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
	4 (continued.) 6) By using the part-to-whole concept and dividing the garden into 7 equal parts, each person receives $1/7$ of the garden. 7) The area concept can be used to determine that the missing dimension of the rectangle is $5/7$ mile.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use line-dividing techniques to explore the division concept of a fraction and relate it to the part-to-whole concept. The area concept of a fraction is used to establish several number properties and procedures for obtaining equivalent fractions.

MATERIALS

- ✔ Focus Student Activities 6.1-6.5, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Straightedges, 1 per student.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Marking pens for each group.

ACTIONS

1 Arrange the students in groups. Draw a segment with endpoints A and B on the overhead and, using a transparency of Focus Master A, demonstrate the parallel line method of dividing $\overline{AB}$ into equal parts.

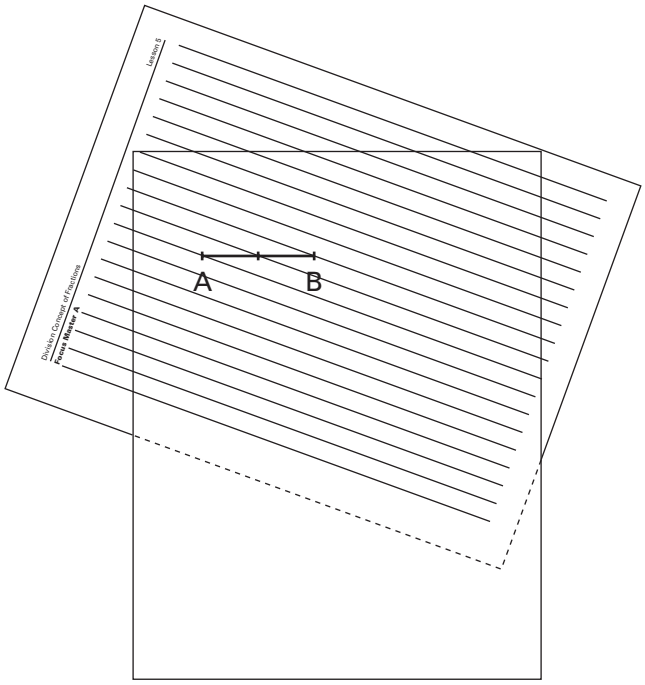


Figure 1

2 Distribute a copy of Focus Student Activity 6.1 (see next page) to each student. Have the students subdivide the segments in Problem 1 as indicated. Discuss their methods and observations.

COMMENTS

1 To carry out the parallel line method of dividing the segment, place a transparency of Focus Master A beneath the transparency which has the segment on it. For example, Figure 1 below shows $\overline{AB}$ divided into 2 equal parts. Figure 2 shows $\overline{AB}$ divided into 3 equal parts.

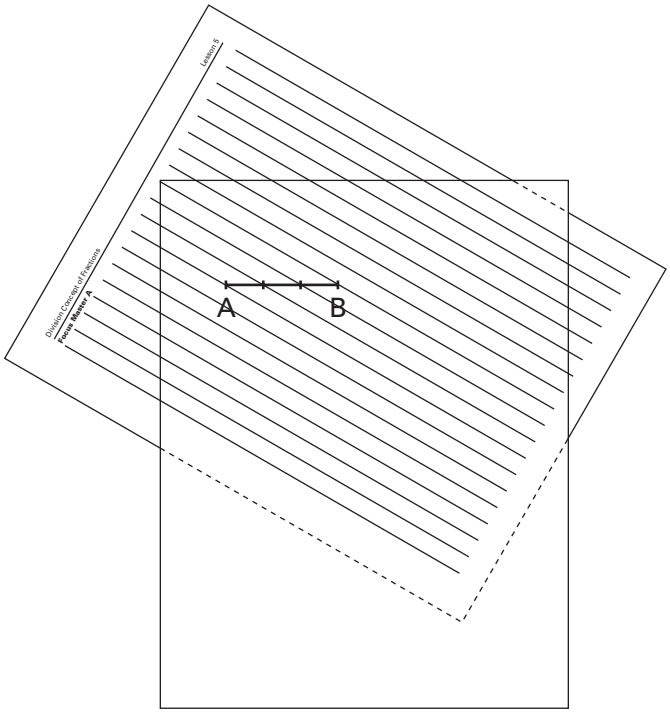


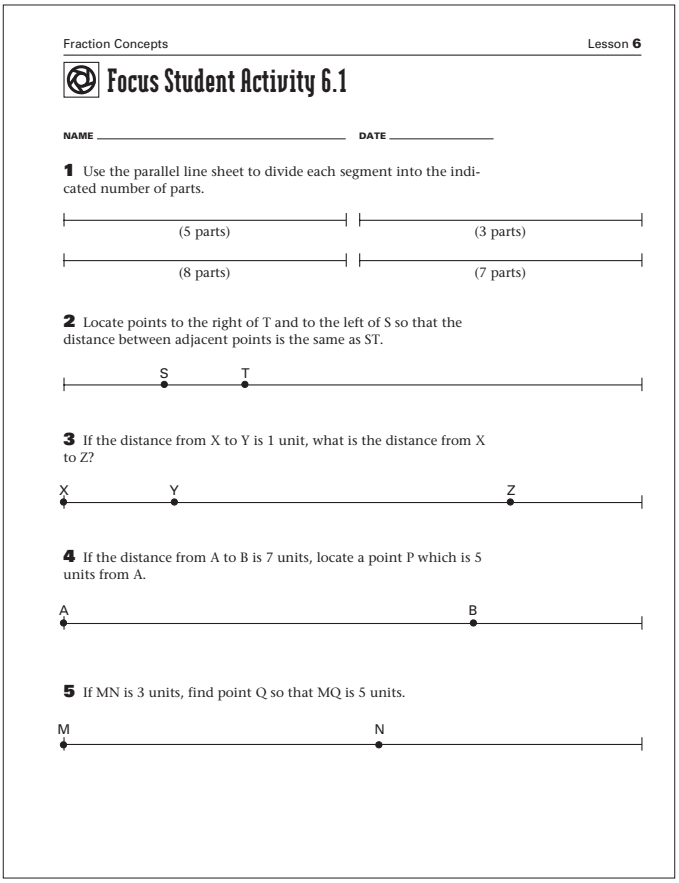
Figure 2

2 This can be done by placing the parallel line sheet under the activity sheet and proceeding as shown in Comment 1.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



Fraction Concepts Lesson **6**

Focus Student Activity 6.1

NAME _____ DATE _____

1 Use the parallel line sheet to divide each segment into the indicated number of parts.

(5 parts) (3 parts)

(8 parts) (7 parts)

2 Locate points to the right of T and to the left of S so that the distance between adjacent points is the same as ST.

S T

3 If the distance from X to Y is 1 unit, what is the distance from X to Z?

X Y Z

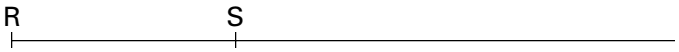
4 If the distance from A to B is 7 units, locate a point P which is 5 units from A.

A B

5 If MN is 3 units, find point Q so that MQ is 5 units.

M N

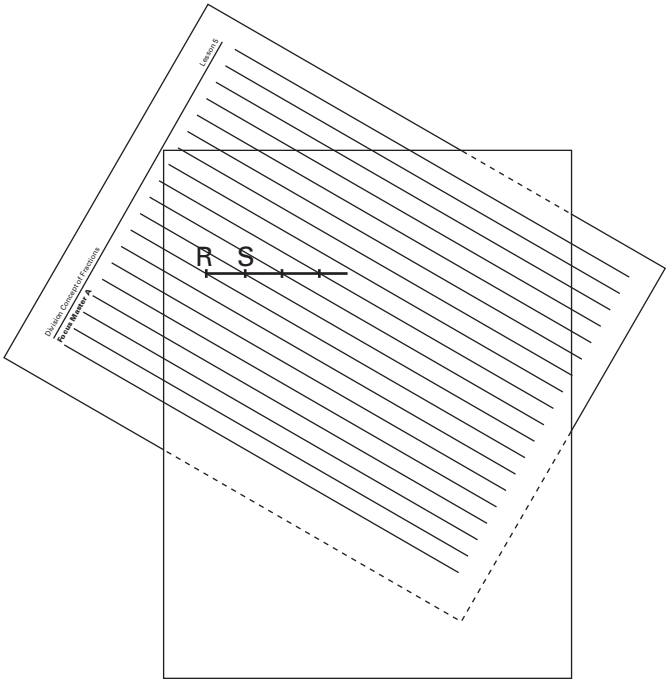
3 Give each student a straight edge. Draw the following diagram on the overhead and ask the students to each use a straightedge to draw a similar diagram. Tell the class that you want to mark points to the right of S so that the distance between each pair of consecutive points is equal to the length of $\overline{RS}$. Ask them to determine how this can be done using equally spaced parallel lines and without using their ruler to measure. Discuss their ideas.



COMMENTS

2 (continued.)
Students' observations may vary. For example, some may notice that congruent corresponding angles are formed at the points where $\overline{AB}$ intersects the parallel lines. Others may notice the pairs of vertical angles formed at each point of intersection. Properties of parallel lines are investigated in Lesson 10; explorations here may prompt insights later.

3 Students may enjoy demonstrating their methods at the overhead. Some may use 1 space for each interval.

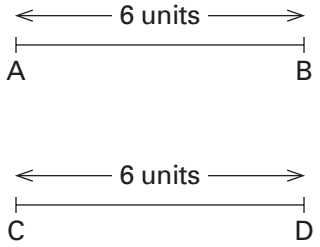


Focus Teacher Activity (cont.)

ACTIONS

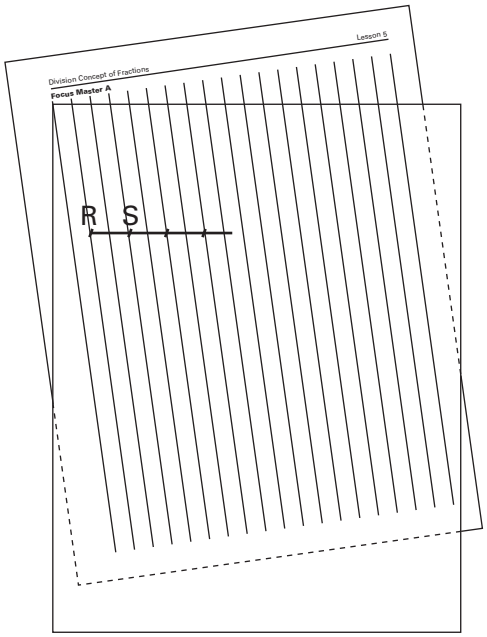
4 Ask the students to use their parallel lines (and no rulers) to solve the remaining problems on Focus Student Activity 6.1. Discuss their results and methods.

5 Ask each student to draw 2 identical line segments that are about half the width of a sheet of paper, to suppose that each segment has length 6 units, and to label the endpoints of one segment A and B and the endpoints of the other C and D (see below). Ask the students to use parallel lines to divide $\overline{AB}$ into 3 equal parts and $\overline{CD}$ into 5 equal parts. Then have them determine the length of 1 part of $\overline{AB}$ and 1 part of $\overline{CD}$. Discuss. If the division concept is not suggested by a student, bring it up for discussion.



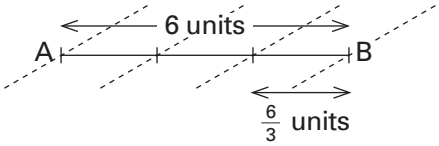
COMMENTS

Others may use 2 spaces for each interval.

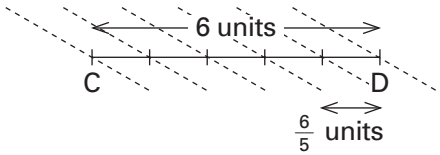


4 Students may find more than one way to solve these problems.

5 Actions 5-12 emphasize the division concept of a fraction which, in addition to being useful to solve numerical problems, is used extensively throughout the study of algebra. Using the division concept of a fraction, the length of 1 part of $\overline{AB}$ is viewed as $6 \div 3$, or $\frac{1}{3}$ of 6, or $\frac{6}{3}$.



Some students may report that the length of one part of $\overline{CD}$ is $1\frac{1}{5}$ units, and others may suggest the equivalent, $\frac{6}{5}$, based on the division concept. Notice that, when interpreted as $\frac{1}{5}$ of 6, or $6 \div 5$, the symbol $\frac{6}{5}$ conveys the action of dividing 6 units into 5 equal parts.



You might point out that, in algebra, mixed numbers are typically expressed as improper fractions.

Focus Teacher Activity (cont.)

ACTIONS

6 Distribute one copy of Focus Student Activity 6.2 to each student. Have the students complete Problem 1a)-1d). Discuss. Then have them complete and discuss Problem 2.

Lesson **6** Fraction Concepts

Focus Student Activity 6.2

NAME _____ DATE _____

1 Use the parallel line sheet to divide each segment into the indicated number of parts. Then write a fraction name for each part.

		Length of One Part
3 units	(4 parts)	a.
4 units	(3 parts)	b.
5 units	(2 parts)	c.
6 units	(7 parts)	d.

2 Use the parallel lines to locate the indicated fraction on the given number line.

3 units — Locate $\frac{3}{5}$.

5 units — Locate $\frac{5}{2}$.

7 units — Locate $\frac{7}{10}$.

3 HI is $\frac{1}{4}$ of a unit. Find point J so that HJ is 1 unit.

H I

4 UV is $\frac{3}{5}$ of a unit. Find point M so that UM is 3 units.

U V

5 UV is $\frac{3}{5}$ units. Find point W so that UW is 1 unit.

U V

7 Sketch the diagram shown in a) below on the overhead. Ask the groups to determine the length of $\overline{AC}$. Discuss their results and reasoning. Then repeat for b) and c).

a) A B C

$\frac{2}{3}$

b) A B C

$\frac{7}{3}$

c) A B C

$\frac{4}{5}$

COMMENTS

6 You might encourage students to compare their results with their groupmates as they work. If a student gets stumped by a problem, you might suggest they ask the class for "clues."

1a) $\frac{3}{4}$ (i.e., $3 \div 4$, or $\frac{1}{4}$ of 3)
1b) $\frac{4}{3}$ (i.e., $4 \div 3$, or $\frac{1}{3}$ of 4)
1c) $\frac{5}{2}$ (i.e., $5 \div 2$, or $\frac{1}{2}$ of 5)
1d) $6 \div 7$ (i.e., $6 \div 7$, or $\frac{1}{7}$ of 6)

2) To locate $\frac{3}{5}$ on a segment with length 3 units, the segment can be divided into 5 equal parts, using the parallel lines.

Note that the intent in this problem is to illustrate the division concept rather than the part-to-whole concept. Hence, the length of the unit has been chosen sufficiently small so that the parallel lines and the part-to-whole concept cannot be used to obtain $\frac{1}{5}$ unit, which could then be tripled to obtain $\frac{3}{5}$ unit.

7 Here are examples of reasoning students have used:

a) *Based on the division concept of a fraction, $\frac{2}{3}$ represents $2 \div 3$ or one-third of two. So, $\overline{AC}$ is three-thirds of 2 or, simply, 2.*

Since $\frac{2}{3}$ is the length obtained when 2 units are divided into 3 equal parts, then 3 lengths of $\frac{2}{3}$ must be 2 units.

Three groups of $\frac{2}{3}$ are $\frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{6}{3} = 2$ units.

b) *Since $\frac{7}{3}$ is the length of 1 part when 7 units are divided into 3 equal parts, then 3 lengths of $\frac{7}{3}$ must be 7 units.*

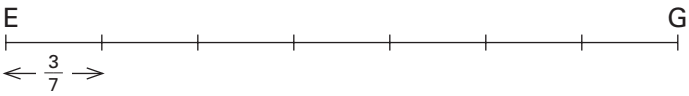
Since $\frac{7}{3}$ is $\frac{1}{3}$ of 7, then 3 of $\frac{7}{3}$ is $\frac{3}{3}$ of 7, or 7.

c) *5 lengths of $\frac{4}{5}$ units is 4 units.*

Focus Teacher Activity (cont.)

ACTIONS

8 Sketch the following line segment on the overhead and ask the groups to copy the sketch. Ask the groups to determine the length of $\overline{EG}$ and locate a point F so that $\overline{EF}$ is 1 unit long. Invite volunteers to demonstrate their methods and reasoning.



9 Have the students determine methods of using the parallel line sheet (Focus Master A) to complete Problems 3-5 on Focus Student Activity 6.2. Discuss.

COMMENTS

8 Notice that $\overline{EG}$ is formed by 7 lengths of $\frac{3}{7}$. Since $\frac{3}{7}$ results from dividing 3 into 7 equal parts, 7 lengths of $\frac{3}{7}$ is 3 units.

To locate point F, $\overline{EG}$ must be divided into 3 equal parts, which can be done by using the parallel lines on Focus Master A.

9 Some students may reason according to the part-to-whole concept, and hence, try to subdivide $\overline{UV}$ into 3 congruent parts to locate $\frac{1}{5}$ unit, planning to locate $\frac{5}{5}$ by using 5 groups of $\frac{1}{5}$. However, this is impossible because of the distance between the parallel lines on Focus Master A. If this comes up, you might assure the students that it is possible to locate 3 units using the given set of parallel lines (i.e., rather than using parallel lines that are closer together to locate $\frac{1}{5}$). To do this: locate 3 units by marking off 5 lengths of $\frac{3}{5}$. Once the students have found 3 units in Problem 4, this distance can be divided into 3 equal parts to obtain 1 unit for Problem 5.

Focus Teacher Activity (cont.)

ACTIONS

10 Give each student a copy of Focus Student Activity 6.3 to complete. Ask for volunteers to demonstrate their methods and results for selected problems.

Fraction Concepts Lesson **6**

Focus Student Activity 6.3

NAME _____ DATE _____

1 If the length of $\overline{AB}$ is $\frac{1}{7}$ of a unit, find and label point C so that the length of $\overline{AC}$ is 1 unit.

A B

2 If the length of $\overline{RS}$ is $\frac{3}{5}$ of a unit, find and label point T so that the length of $\overline{RT}$ is 3 units.

R S

3 Find and label point U on the line in Problem 2 so that $\overline{RU}$ is 1 unit in length.

4 If the length of $\overline{WX}$ is 1 unit, find and label point Y so that the length of $\overline{WY}$ is $\frac{5}{3}$ units.

W X

5 If $\overline{DH}$ is 9 units in length, how long is $\overline{DG}$? $\overline{EG}$? $\overline{FG}$?

D E F G H

6 The length of $\overline{PQ}$ is $\frac{3}{8}$ of a unit. How long is $\overline{LU}$? $\overline{LQ}$? $\overline{PR}$?

L M N P Q R S T U

7 The length of $\overline{AH}$ is 3 units. How long is $\overline{AB}$? $\overline{DE}$? $\overline{AC}$? $\overline{DF}$?

A B C D E F G H

(Continued on back.)

COMMENTS

10 This activity could be completed as homework. Prior to a class discussion of results, you might have the students compare results with their groupmates and identify problems for discussion by the class.

Lesson **6** Fraction Concepts

Focus Student Activity 6.3 (cont.)

8 If the length of $\overline{AB}$ is 3 units, find and label point D so that the length of $\overline{AD}$ is 5 units.

A B

9 Locate point C on the line above so that the length of $\overline{AC}$ is $\frac{5}{2}$ units.

10 The length of $\overline{EF}$ is $\frac{1}{4}$ of a unit. Find and label point G so that the length of $\overline{EG}$ is 1 unit.

E F

11 If the length of $\overline{AD}$ is $\frac{4}{3}$ units, locate and label point E so that $\overline{AE}$ is 4 units in length.

A D

12 If the length of $\overline{NP}$ is $\frac{5}{4}$ units, find and label point Q so that $\overline{NQ}$ is 1 unit in length.

N P

13 Locate and label point R on the line above so that the length of $\overline{NR}$ is $\frac{4}{3}$ units.

14 Sketch an equilateral triangle, a rectangle, a regular hexagon, and a rhombus. Label their areas 13, 23, 37, and 91 square units. Subdivide each into 4 congruent subregions and label the area of each subregion.

11 Give each group a copy of Focus Master B and ask them to complete thought starters a) and b). Discuss and compare the groups' responses. Repeat for one or more of thought starters c)-e).

11 These thought starters could also be completed independently by students in their journals and then discussed. Note that some students may be challenged by the general cases in c)-e); encouraging them to provide oral explanations based on several examples may be helpful. Here are examples of visual procedures for a)-e):

a) Given: $\frac{3}{5}$

$\frac{1}{5}$ $\frac{1}{5}$ $\frac{1}{5}$

Step 1: Divide $\frac{3}{5}$ into 3 equal groups of $\frac{1}{5}$ unit.

$5 \times \frac{1}{5} = 1$ (5 groups of $\frac{1}{5}$)

Step 2: Form a length of $5 \times \frac{1}{5} = 1$ unit.

3 groups of 1 unit

Step 3: Form a length of $3 \times 1 = 3$ units.

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS

Lesson **6** Fraction Concepts

Focus Master B

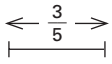
a) Given a length of $\frac{3}{5}$, here is a method of forming a length of 3, and this method relies on thinking about $\frac{3}{5}$ as $\frac{3}{5}$ of 1 unit (i.e., using the part-to-whole concept of $\frac{3}{5}$):

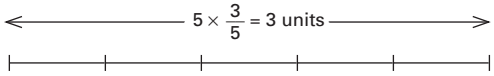
b) Given a length of $\frac{3}{5}$, here is a method of forming a length of 3, and this method relies on thinking about $\frac{3}{5}$ as $\frac{1}{5}$ of 3 or $3 \div 5$ (i.e., according to the division concept of $\frac{3}{5}$):

c) Given a length $\frac{a}{b}$, where *a* and *b* are positive integers, following is a set of instructions for forming the length *a*, and this method relies on the part-to-whole concept of a fraction:

d) Given a length $\frac{a}{b}$, where *a* and *b* are positive integers, following is a set of instructions for forming the length *a*, and this method relies on the division concept of a fraction:

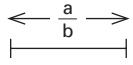
e) Given a length $\frac{a}{b}$, where *a* and *b* are positive integers, here is a set of instructions for constructing the length 1 unit, and this method does not require subdividing the length $\frac{a}{b}$ into *a* equal parts.

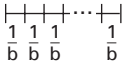
b) Given: 



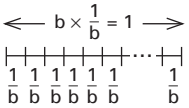
(5 groups of $\frac{3}{5}$)

Form a length of $5 \times \frac{3}{5} = 3$ units.

c) Given: 

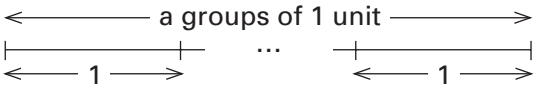


Step 1: Divide $\frac{a}{b}$ into a groups of $\frac{1}{b}$.



(b groups of $\frac{1}{b}$)

Step 2: Form the length $b \times \frac{1}{b} = 1$ unit.

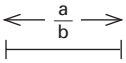


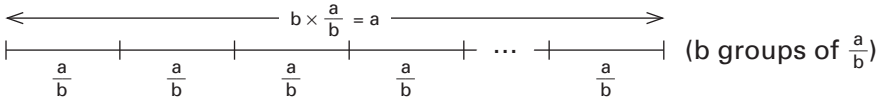
Step 3: Form the length $a \times 1 = a$ units.

(Continued next page.)

Focus Teacher Activity (cont.)

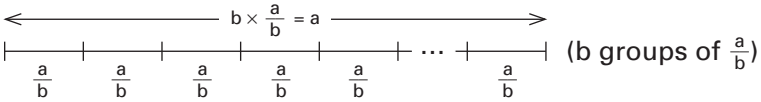
ACTIONS

d) Given: 

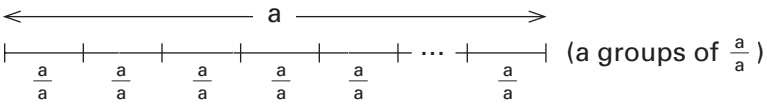


Form the length $b \times \frac{a}{b} = a$ units.

e) Given: 



Step 1: Form the length $b \times \frac{a}{b} = a$ units.



Step 2: Divide the length a into a equal parts of length $\frac{a}{a} = 1$.

12 Give each student a copy of Focus Student Activity 6.4 to complete. Invite volunteers to show their results and reasoning at the overhead on a transparency of the activity.

COMMENTS

11 (continued.)

12 These problems extend the division concept of a fraction from problems involving length to problems involving area. These ideas will be useful in Lesson 7 when students compute with fractions and algebraic fractions.

Focus Teacher Activity (cont.)

ACTIONS

Fraction Concepts Lesson **6**

Focus Student Activity 6.4

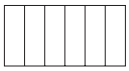
NAME _____ DATE _____

Complete the following. Use a straightedge, but no ruler. Express all fractions in improper form.

1 Assuming the area of the rectangle shown below is $\frac{3}{7}$ square units, sketch a rectangle with area 3 units.

2 The area of the shaded region is $\frac{x}{4}$ square unit. What is the area of the whole circle?

3 The area of this field is 13 acres. What is the area of each part?



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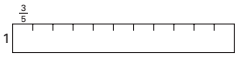
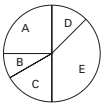


a) Shade a region with area $\frac{3}{2}$ square units.

b) Draw a bold line around a rectangle with area 6 square units.

5 The area of this circle is X square units. Find the area of each region.



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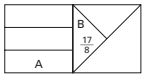
COMMENTS

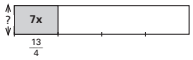
Lesson **6** Fraction Concepts

Focus Student Activity 6.4 (cont.)

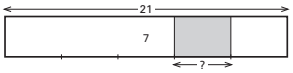
6 The area of region B is $\frac{17}{8}$. Find the area of region A.



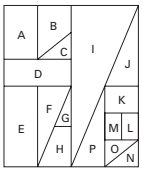
7 The shaded region has area $7x$ and length $\frac{13}{4}$. Find the area, length, and height of the large rectangle.



8 The area of the large outer rectangle is 7 and its length is 21. Find the area and length of the shaded region.



9 The area of the large outer rectangle below is 29. Find the area of each subregion.



10 The rectangle below has area X and length $\frac{2}{3}$. Extend the right edge of the rectangle to form a rectangle with length 2. Record its area and mark the rectangle to show your methods.

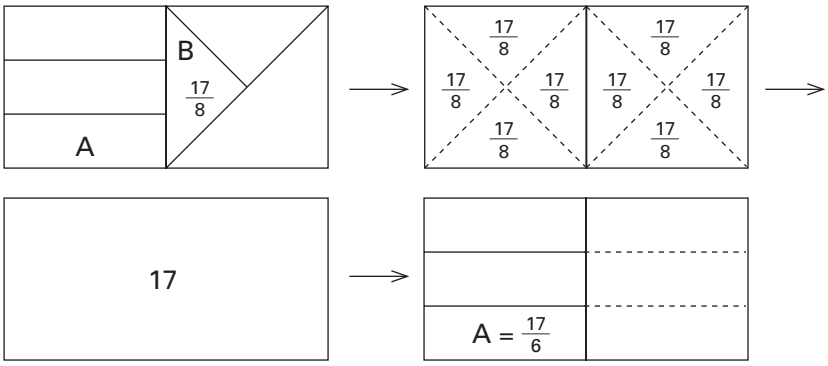
Answers are as follows:

2) x square units

3) $\frac{13}{6}$ acres

5) A) $\frac{x}{4}$
B) $\frac{x}{12}$
C) $\frac{x}{6}$
D) $\frac{1}{8}$ of x, or $\frac{x}{8}$
E) $\frac{3}{8}$ of x, or ($\frac{1}{8}$ of x) $\times$ 3, or $\frac{x}{8} \times 3$, or $\frac{3x}{8}$ units

6) One line of reasoning is illustrated here:



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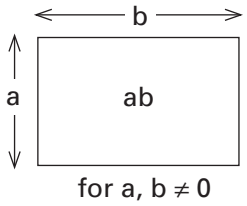
Focus Teacher Activity (cont.)

ACTIONS

13 Sketch the following diagram on the overhead and ask the groups to each write several mathematical equations that represent relationships in the diagram, including some equations involving fractions. Discuss.



COMMENTS

12 (continued.)
7) Area, $28x$ square units; length, 13 linear units; height, $^{28x}/_{13}$ linear units.

8) Area, $^{7}/_{5}$ square units; length, $^{21}/_{5}$ linear units

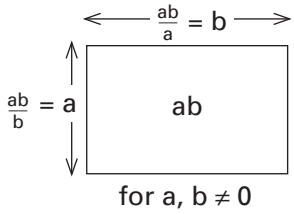
9) For example, the large outer rectangle can be subdivided into 12 regions congruent to A; hence, based on the division concept of a fraction, the area of region A is $^{29}/_{12}$. Students may use other strategies, and hence, give fractions that are equivalent to those listed below. You might suggest that students not "simplify" fractions but, rather, express them according to how they "see" the area of each region.

B, $^{29}/_{16}$; C, $^{29}/_{48}$; D, $^{29}/_{12}$; E, $^{29}/_{8}$; F, $^{29}/_{16}$; G, $^{29}/_{64}$; H, $^{87}/_{64}$ (3 of $^{29}/_{64}$) ; I, $^{29}/_{4}$; J, $^{29}/_{16}$; K, $^{29}/_{24}$; L, $^{29}/_{48}$; M, $^{29}/_{48}$; N, $^{29}/_{48}$; O, $^{29}/_{48}$; P, $^{29}/_{16}$.

10) Area, 3X.

13 Based on the area model for multiplication, the product of the dimensions of a rectangle equals the area of the rectangle. This shows that $a \times b = b \times a$, an illustration of the *commutative property* for multiplication.

Based on the area model for division, and the area concept of a fraction, since the area of the rectangle is ab and one dimension is a then the other dimension is $ab \div a$, or $\frac{ab}{a}$. But since the other dimension is b, $\frac{ab}{a} = b$. Similarly, $\frac{ab}{b} = a$.



Some students may ask why a and b are restricted to nonzero values. Notice that Action 14 addresses this issue.

Focus Teacher Activity (cont.)

ACTIONS

14 Write i)-iii), as shown below, on the overhead and ask the groups to determine, if possible, the value of x in each equation, assuming neither b nor c are equal to 0. Ask them to show visually why their answers are correct. Invite volunteers to share their group's results. Then have the class work together to write summary statements regarding *multiplication and division properties of zero*.

i) $0/b = x$

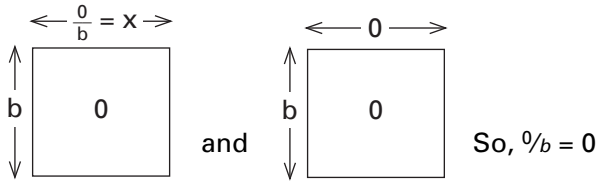
ii) $0/0 = x$

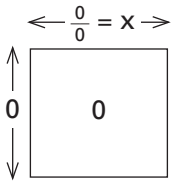
iii) $c/0 = x$

COMMENTS

14 Note that b and c are restricted to nonzero values to avoid redundancy in the three problems. The following solutions for i)-iii) are based on the area models for multiplication and division, and the area concept of a fraction. Students may offer other visual arguments that are based on the grouping and sharing methods of division.

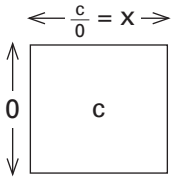
i) Based on the area model for division and the division concept of a fraction, $0/b = x$ can be represented using a rectangle with area 0 and dimensions b by $0/b = x$, as shown below. Since $0 \times b = 0$, then it must be that $x = 0/b = 0$.





ii) The fraction $0/0$ represents one dimension of a rectangle with area 0 and other dimension 0, as shown at the left.

Since $x \times 0 = 0$ for all possible values of x (i.e., for all real numbers), there are an infinite number of possibilities for $x = 0/0$, and so $0/0$ is *not defined*.



iii) The fraction $c/0$ represents one dimension of a rectangle with area c and other dimension 0, as shown at the left.

There are no possible values of x for which $x \times 0 = c$, where c is a nonzero number. Hence, division by 0 is not possible, and no fraction exists whose denominator is zero. Therefore, it is necessary to place restrictions on a fraction to assure that the denominator is never 0. For example, for a fraction a/b, a can be any value, and b can be any nonzero value.

Having the students verbalize summary statements helps them to clarify and strengthen their understanding of the properties of zero.

Focus Teacher Activity (cont.)

ACTIONS

15 Give each group a copy of Focus Master C. Ask the groups to complete i) and ii) for diagrams a) and b) only. Discuss. Have the class reach agreement on summary statements involving the *multiplication and division properties of 1*, the *multiplicative identity*, and *multiplicative inverses*. Use this as a context for discussing the term *reciprocal*.

Fraction Concepts Lesson **6**

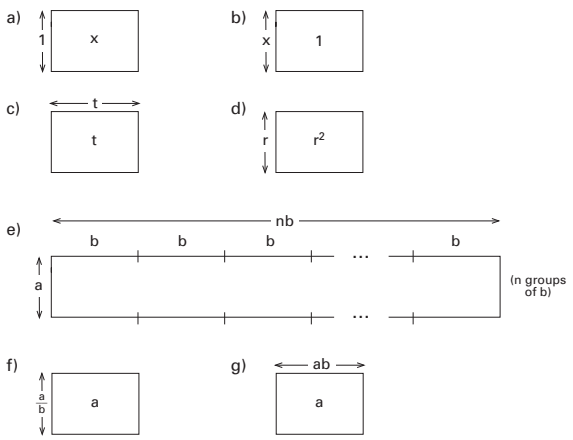
Focus Master C

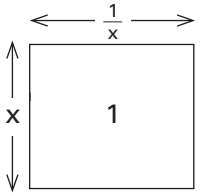
For each of the following diagrams:

i) Label dimensions and areas not shown in the diagrams. Then write several equations that represent mathematical relationships in the diagram. Be sure to include some equations that involve fractions.

ii) For each equation in i), give 2 different examples involving specific numbers in place of the variables. Relate each example to the diagram. If there are specific values that are not possible for some variables, list those values and tell why they are not possible.

Diagrams are not necessarily to scale.





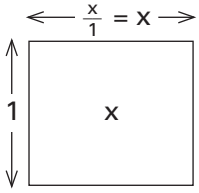
$$1 \div x = \frac{1}{x}$$
$$\frac{1}{x} \cdot x = 1$$
$$x \cdot \frac{1}{x} = 1$$
$$1 \cdot x = x$$
$$1 \div \frac{1}{x} = \frac{1}{1/x} = x$$

COMMENTS

15 Keep emphasis here on relationships students can "see" or glean from the diagrams, based on the area models for division and multiplication and the division concept of a fraction. To reinforce the division concept of a fraction, you might encourage students to write all division statements as fractions.

As students share their examples, to prompt generalizations, you might ask them to determine whether there are any values of x for which their equations are *not* true.

a) Here are several equations that students may suggest for this diagram:



$$x \div 1 = \frac{x}{1} = x$$
$$\frac{x}{1} \cdot 1 = x$$
$$x \cdot 1 = x$$
$$1 \cdot x = x$$
$$x \div x = \frac{x}{x} = 1$$

Since for all numbers x, $x \times 1 = 1 \times x = x$, 1 is called the *multiplicative identity*.

Asking students to create examples that illustrate their statements involving variables builds understanding for the meaning of a variable and may prompt additional generalizations.

b) Several possible equations associated with diagram b) from Focus Master C are shown at the left. In this case, x cannot be equal to 0 (see Comment 14 iii)). The values x and $1/x$ are called *multiplicative inverses* of each other since, for all nonzero values of x, $x \times 1/x = 1$. For example, since a rectangle with dimensions 2 by ½ has area $2 \times ½ = 1$, 2 and ½ are multiplicative inverses of each other. The numbers x and $1/x$, for all nonzero values of x, are also called *reciprocals* of each other. The product of a number and its reciprocal is always 1.

Questions regarding reciprocals of numbers such as ¾ may arise. If so, note that multiplication and division of fractions are explored in Lesson 7. You may want to delay discussion until then, or read that lesson for discussion ideas now.

Focus Teacher Activity (cont.)

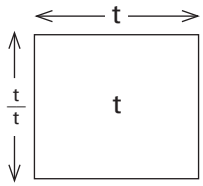
ACTIONS

16 Have the students complete i) and ii) for diagrams c)-g) on Focus Master C. Discuss their results. Encourage all attempts to generalize.

COMMENTS

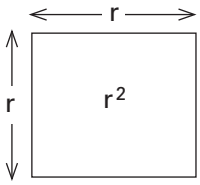
16 The intent here is for students to apply the division concept and make conjectures and generalizations based on the results. Remind students that some conjectures may not be resolved until later in this lesson or until explorations in Lesson 7. It is helpful to resist affirming or correcting conjectures for them.

c) To avoid a situation such as described in Comment 14 ii), *t* cannot equal 0. Several equations based on diagram c) are shown here:



$$\frac{t}{t} = 1$$
$$t \bullet \frac{t}{t} = t$$
$$t \bullet 1 = t$$
$$t \div \frac{t}{t} = \frac{t}{1/t} = t$$

d) Since the area is r^2 and one dimension is *r*, then the other dimension is r and the diagram must be a square.

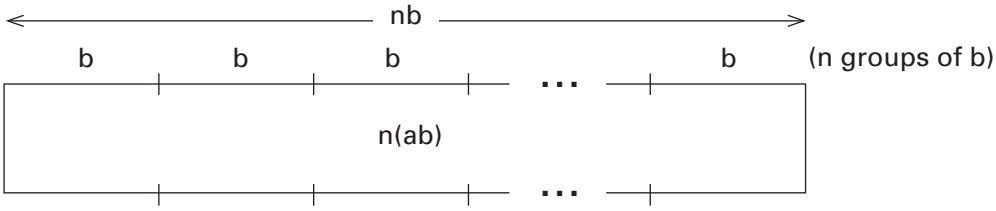


$$r^2 \div r = \frac{r^2}{r} = r$$
$$r \bullet r = r^2$$

Some students may bring up the fact that the length of the edge of a square of area r^2 is $\sqrt{r^2}$; hence, another equation for diagram d) is $\sqrt{r^2} = r$. See Lesson 28 of *Math Alive! Course II* and Lesson 9 of this course for additional discussion ideas.

e) Here are some equations that diagram e) may prompt:

$$a(nb) = n(ab)$$
$$\frac{n(ab)}{a} = nb$$
$$\frac{n(ab)}{nb} = a$$



$$n(ab) = \underbrace{a(b + b + b + \ldots + b)}_{n \text{ times}} = \underbrace{ab + ab + ab + \ldots + ab}_{n \text{ times}}$$

In Lesson 20 of *Math Alive! Course II*, students explored "maneuvers" on rectangles that lead to strategies for computing sums and differences of fractions. In Lesson 7 of this course, such maneuvers are also used to find products and quotients of fractions. One useful maneuver is illustrated by diagram e) on Focus Master C—expanding one dimension of a rectangle by a factor of *n* increases the area by a factor of *n*, and vice versa. Note that this idea is utilized in f) below, and in Action 17 of this activity.

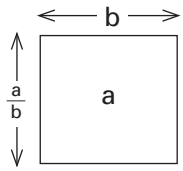
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

16 (continued.)
f) Shown below are some equations that represent relationships in diagram f) from Focus Master C.



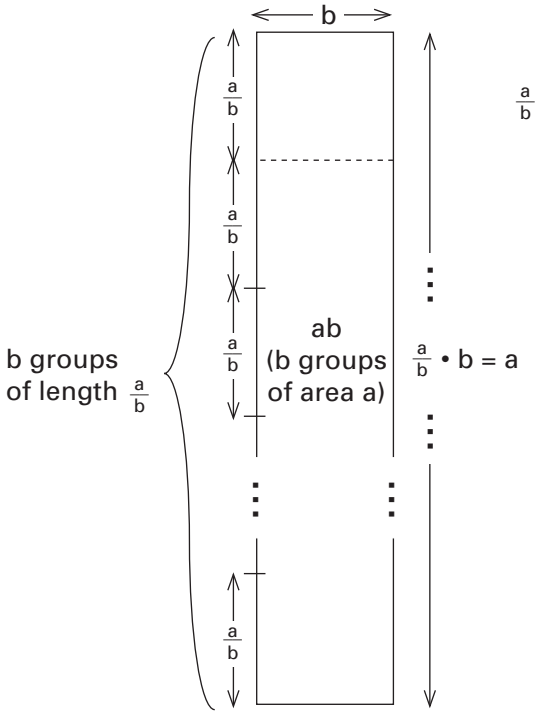
$$b \cdot \frac{a}{b} = a$$

$$\frac{a}{b} \cdot b = a$$

$$a \div \frac{a}{b} = \frac{a}{a/b} = b$$

$$a \div b = \frac{a}{b}$$

Note that some students may apply the "rectangle maneuver" that was described in e) above to this rectangle. Since b groups of length a/b form the total length a, as established in Action 11, and since b groups of area a produce area ba, then $\frac{a}{a/b} = \frac{ab}{a} = b$, as illustrated below.



$$\frac{ab}{a} = b$$

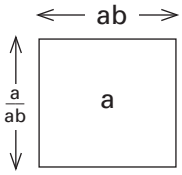
$$\frac{a}{b} \cdot b = a$$

$$\frac{a}{a/b} = b$$

$$\frac{a}{a/b} = \frac{ab}{a}$$

$$\frac{a}{b} \cdot b = a$$

g) Following are some equations that diagram g) from Focus Master C may prompt. Note that students may give other equations, based on properties established on previous diagrams.



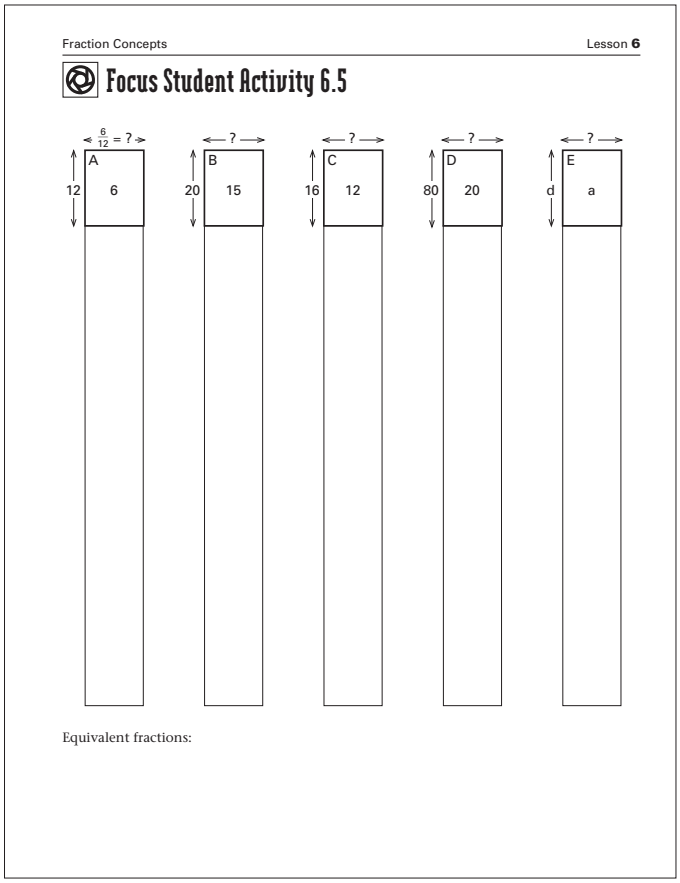
$$ab \cdot \frac{a}{ab} = a$$

$$\frac{a}{a/ab} = ab$$

Focus Teacher Activity (cont.)

ACTIONS

17 Place a transparency of Focus Student Activity 6.5 on the overhead. Point out that the bold Rectangle A has area 6 and side dimension 12. Hence, its top dimension is $^6/_{12}$. Ask for volunteers to subdivide the large rectangle below Rectangle A to form several rectangles that are not congruent to Rectangle A but with top dimension equivalent to $^6/_{12}$. Ask them to label the area and dimensions of these rectangles. Discuss the *equivalent fractions* illustrated.



COMMENTS

17 The methods utilized here are based on those explored in Lesson 20 of *Math Alive! Course II* for creating *equal quotients*.

In Figure 1 below, the side dimension and area of Rectangle A have been doubled to form a rectangle with area 12, side dimension 24, and top dimension $^6/_{12} = {}^{12}/_{24}$.

In Figure 2, the area and side dimension of Rectangle A have been divided by 2, forming a new rectangle with area 3, side dimension 6, and top dimension $^3/_6$.

In Figure 3, a new rectangle has been formed with area and side dimension that are $^1/_3$ of the area and side dimension of Rectangle A. The top dimension does not change.

Since the top dimension of all the rectangles formed are the same, the fractions that represent the top dimensions are *equivalent*. Thus, Figures 1, 2, and 3 below show 3 fractions that are equivalent to $^6/_{12}$: $^{12}/_{24}$, $^3/_6$, and $^2/_4$.

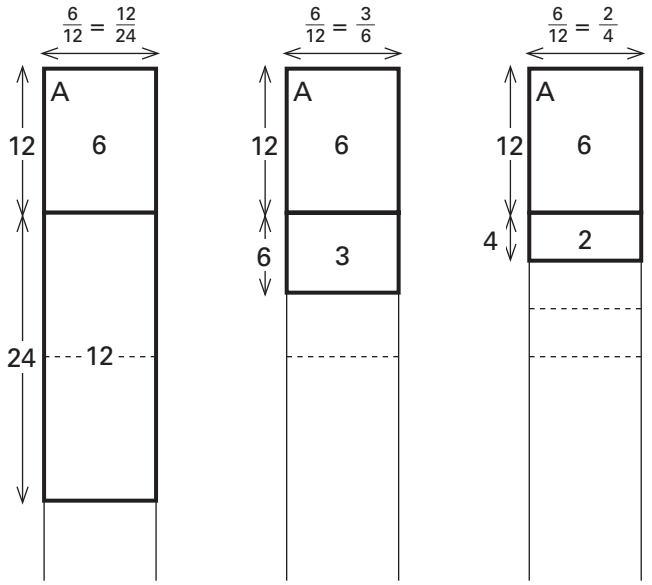


Figure 1 Figure 2 Figure 3

Students may also notice that combinations of rectangles can be used. For example, combining the two rectangles of areas 6 and 12 in Figure 1 produces a rectangle of area 18 with side dimension 36 and top dimension $^{18}/_{36} = {}^6/_{12} = {}^{12}/_{24}$.

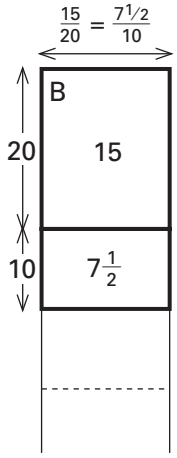
Focus Teacher Activity (cont.)

ACTIONS

18 Distribute one copy of Focus Student Activity 6.5 to each student and ask them to repeat Action 17 for Rectangle B to determine fractions that are equivalent to $^{15}/_{20}$. Discuss their results, observations, and conjectures. Then repeat for rectangles C and D. Use this as a context for discussing the meaning of *simplifying fractions*.

COMMENTS

18 The rectangles that students form may vary, producing a variety of equivalent fractions. Some students may form *complex fractions*, i.e., fractions whose numerator and/or denominator contain fractions. For example, cutting the area and side dimension of Rectangle B into 2 congruent parts produces a rectangle with area $7\frac{1}{2}$, side dimension 10, and top dimension $\frac{7\frac{1}{2}}{10} = {}^{15}/_{20}$.



A fraction representing the top dimension of a rectangle is in *simplest form* when there is no rectangle possible with a smaller whole number area and whole number side dimension. That is, the numerator and denominator of the fraction cannot *both* be divided evenly by a number other than 1 (the numerator and denominator are relatively prime).

19 Repeat Action 17 for Rectangle E. Discuss the students' conjectures and generalizations.

19 One important observation is that multiplying or dividing the numerator (the area) and the denominator (the side dimension) by the same number produces an equivalent fraction, since the top dimension is not altered. For example, Figures 4 and 5 below illustrate why, for $d \neq 0$, $^{a}/_{d} = {}^{2a}/_{2d} = \frac{a/3}{d/3}$.

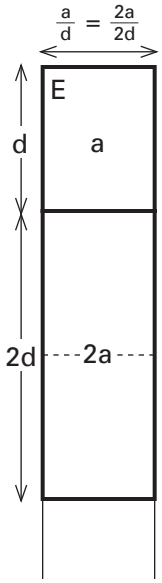


Figure 4

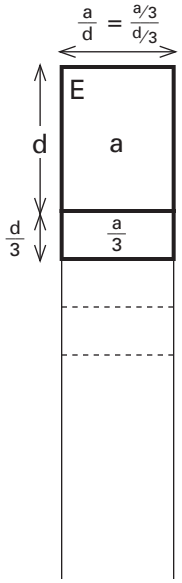


Figure 5

Focus Teacher Activity (cont.)

ACTIONS

20 Mention to the class that several important mathematical properties have been established during this lesson. Ask the class to reach consensus on a plan for writing and displaying a summary of some of these properties, including diagrams supported by equations and verbal statements. This summary will be used for ongoing reference by the class. Carry out the class plan.

COMMENTS

20 The intent here is to provide another opportunity for students to identify their questions, clarify their thinking, and build comfort and facility with "thinking" according to the area and division concepts of a fraction and the area methods of multiplication and division. It is not expected that students memorize lists of properties; rather these properties will become familiar through use during subsequent lessons. Students frequently "reinvent" and "regeneralize" some relationships several times before those relationships become integrated into the students' natural way of thinking.

There are several ways this action could be carried out. For example, each group could make a set of mini-posters illustrating several important properties and ideas. The class could vote on those to leave on display, based on their clarity and depth. Since the posters will be used for ongoing reference, completed posters should be reviewed by the class and adapted as needed. If you teach several classes of students, each class's posters could be concealed from other classes until all have completed the activity. Comparing results from class to class may prompt new discussions. The classes could choose a single set of posters to leave on display.

If students are keeping journals, you might ask them to make an extended entry that emphasizes generalizations based on the area model for multiplication and division, and the area and division concepts of a fraction.

TEACHER NOTES:

Follow-up Student Activity 6.6

NAME _____ **DATE** _____

Write all of your responses on other sheets of paper. Be sure to write the problem next to your response.

1 For each of these concepts of a fraction—*part-to-whole*, *ratio*, *division*, and *area*—do the following:

a) explain the meaning of a fraction a/b when it is viewed according to that concept;

b) write a situation in which you "think" according to that meaning, and make a sketch that illustrates your thinking.

2 Explain the meaning of *equivalent fractions*. Show how to use each of the 4 concepts of a fraction to determine 3 pairs of equivalent fractions.

3 For each of the 4 concepts of a fraction, show how to determine an infinite set of fractions that are equivalent to a/b, for a and b not equal to zero.

4 Suppose $a \neq 0$. Demonstrate *why*:

a) $0/a = 0$

b) $0/0$ is undefined

c) $a/0$ is not possible

d) 1 is the multiplicative identity; 1 is the division identity; 0 is the additive identity; and 0 is the subtractive identity.

5 Suppose a and b are whole numbers not equal to zero. Create at least 6 "visual proofs" of mathematical relationships that you can show on rectangles whose dimensions and areas are a, b, 1, or fractions whose numerators and/or denominators are a, b, or 1.

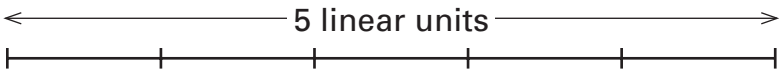
(Continued on back.)

Follow-up Student Activity (cont.)

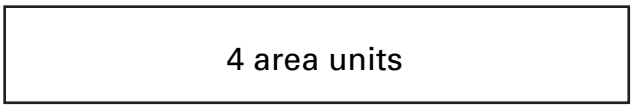
6 On the attached sheet of 1-cm grid paper, outline a rectangle that is 18 cm × 12 cm. Suppose it has area 17 square units for a certain area unit. Subdivide the rectangle to form rectangular subregions with the following areas and with no overlaps. Label the area of each subregion.

Region A, $^{17}/_{12}$ square units.
Region B, $^{17}/_{24}$ square units
Region C, $^{34}/_{12}$ square units
Region D, $^{34}/_{24}$ square units
Region E, $^{17}/_{6}$ square units
Region F, $^{34}/_{6}$ square units
Region G, $^{17}/_{8}$ square units

7 Use the parallel line sheet to mark off lengths or areas for the given fractions. Explain your reasoning.

a) 

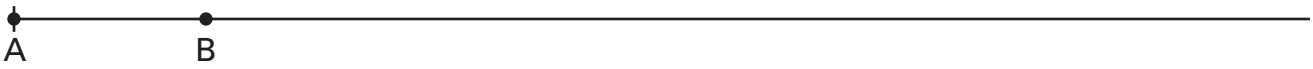
Show length $^{3}/_{5}$.

b) 

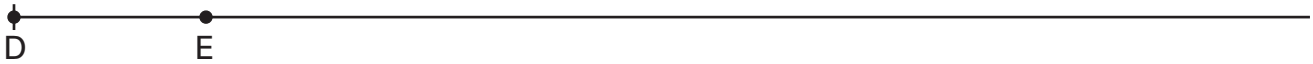
Show area $^{4}/_{7}$.

8 Use the parallel line sheet to locate and label points satisfying the given conditions. Mark the diagram to show your methods.

a) AB is $^{2}/_{3}$ units. Locate point C so that AC is 4 units.



b) DE is $^{4}/_{5}$ units. Locate point F so that DF is 3 units.



c) GH is 3 units. Locate point K so that GK is $2^{3}/_{5}$ units.

Connector Master A

1. The county plans to evenly distribute 7 signs along the 5 miles of highway construction.

2. Tia and 4 friends are going to the movies. They have 3 candy bars to share equally.

3. The 4 members of the Wilkinson family plan to share a pizza equally.

4. The 32 members of Ms. Callahan's class plan to share 8 pizzas equally.

5. On a 10 kilometer relay, the 3 members of the Run-abouts each ran an equal distance.

6. The 7 members of the Greenspace Garden Project each were allocated an equal portion of the garden.

7. The Bergmans own a rectangular strip of land that runs 7 miles along the freeway. The area of the strip is 5 square miles. The Bergmans need to replace the fencing around the strip.

Focus Master A

Focus Master B

a) Given a length of $3/5$, here is a method of forming a length of 3, and this method relies on thinking about $3/5$ as $3/5$ of 1 unit (i.e., using the part-to-whole concept of $3/5$):

b) Given a length of $3/5$, here is a method of forming a length of 3, and this method relies on thinking about $3/5$ as $1/5$ of 3 or $3 \div 5$ (i.e., according to the division concept of $3/5$):

c) Given a length a/b, where a and b are positive integers, following is a set of instructions for forming the length a, and this method relies on the part-to-whole concept of a fraction:

d) Given a length a/b, where a and b are positive integers, following is a set of instructions for forming the length a, and this method relies on the division concept of a fraction:

e) Given a length a/b, where a and b are positive integers, here is a set of instructions for constructing the length 1 unit, and this method does not require subdividing the length a/b into a equal parts.

Focus Master C

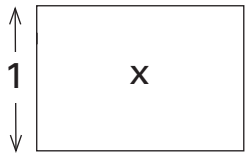
For each of the following diagrams:

i) Label dimensions and areas not shown in the diagrams. Then write several equations that represent mathematical relationships in the diagram. Be sure to include some equations that involve fractions.

ii) For each equation in i), give 2 different examples involving specific numbers in place of the variables. Relate each example to the diagram. If there are specific values that are not possible for some variables, list those values and tell why they are not possible.

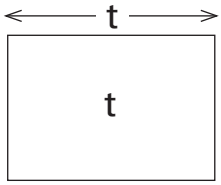
Diagrams are not necessarily to scale.

a)

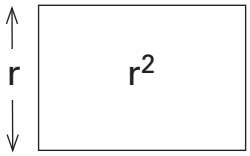


b)

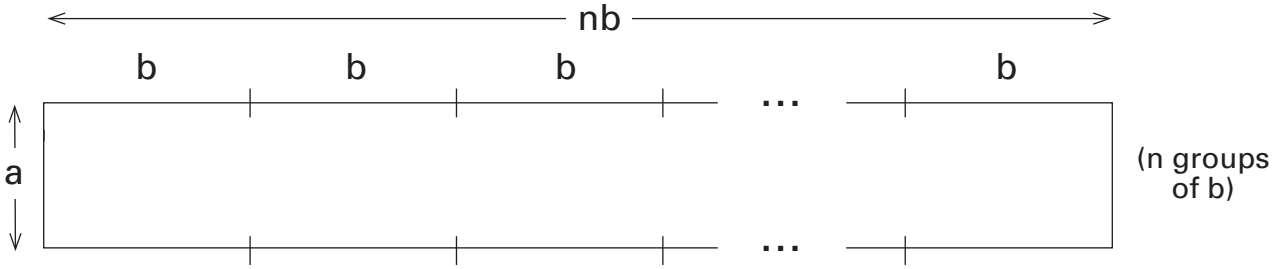
c)



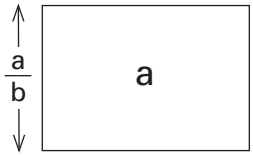
d)



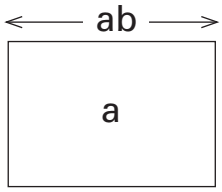
e)



f)



g)



Focus Student Activity 6.1

NAME _____ **DATE** _____

1 Use the parallel line sheet to divide each segment into the indicated number of parts.

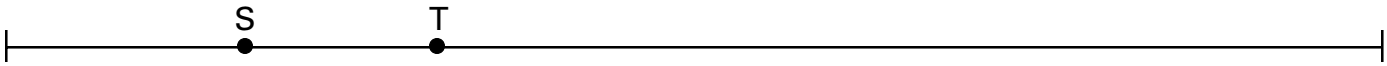
(5 parts)

(3 parts)

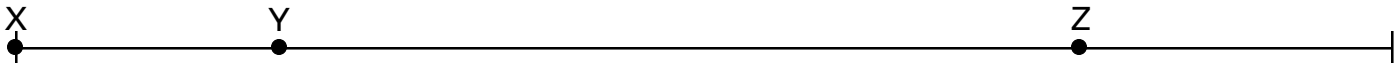
(8 parts)

(7 parts)

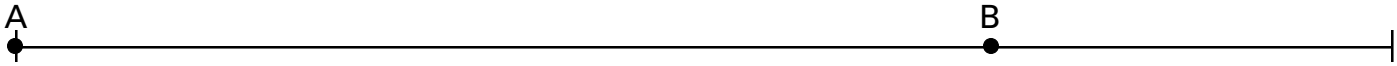
2 Locate points to the right of T and to the left of S so that the distance between adjacent points is the same as ST.



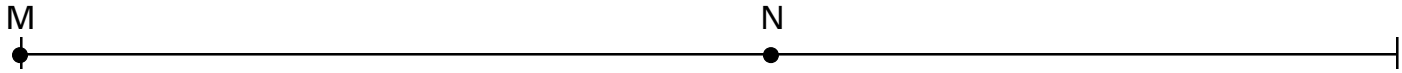
3 If the distance from X to Y is 1 unit, what is the distance from X to Z?



4 If the distance from A to B is 7 units, locate a point P which is 5 units from A.



5 If MN is 3 units, find point Q so that MQ is 5 units.



Focus Student Activity 6.2

NAME _____ **DATE** _____

1 Use the parallel line sheet to divide each segment into the indicated number of parts. Then write a fraction name for each part.

Length of One Part

3 units (4 parts) a. ☐

4 units (3 parts) b. ☐

5 units (2 parts) c. ☐

6 units (7 parts) d. ☐

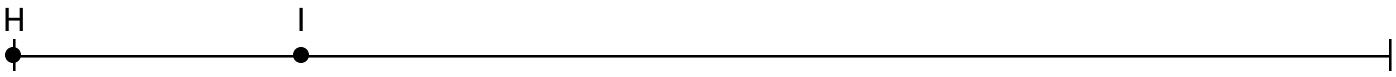
2 Use the parallel lines to locate the indicated fraction on the given number line.

3 units Locate $\frac{3}{5}$.

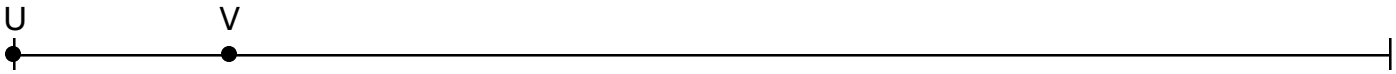
5 units Locate $\frac{5}{2}$.

7 units Locate $\frac{7}{10}$.

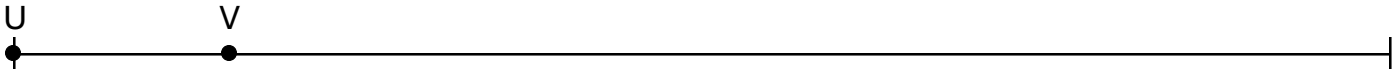
3 HI is $\frac{1}{4}$ of a unit. Find point J so that HJ is 1 unit.



4 UV is $\frac{3}{5}$ of a unit. Find point M so that UM is 3 units.



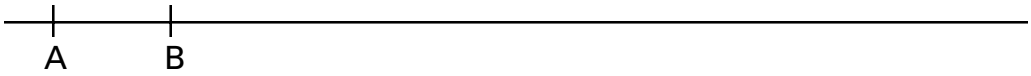
5 UV is $\frac{3}{5}$ units. Find point W so that UW is 1 unit.



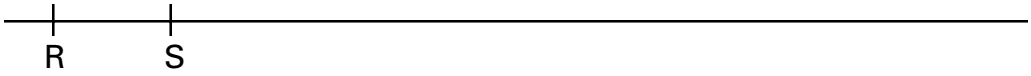
Focus Student Activity 6.3

NAME _____ **DATE** _____

1 If the length of $\overline{AB}$ is $1/7$ of a unit, find and label point C so that the length of $\overline{AC}$ is 1 unit.

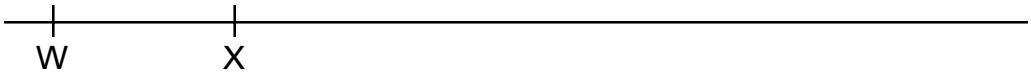


2 If the length of $\overline{RS}$ is $3/5$ of a unit, find and label point T so that the length of $\overline{RT}$ is 3 units.

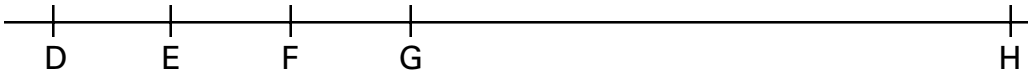


3 Find and label point U on the line in Problem 2 so that $\overline{RU}$ is 1 unit in length.

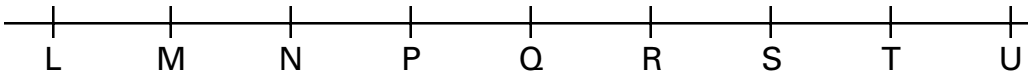
4 If the length of $\overline{WX}$ is 1 unit, find and label point Y so that the length of $\overline{WY}$ is $5/3$ units.



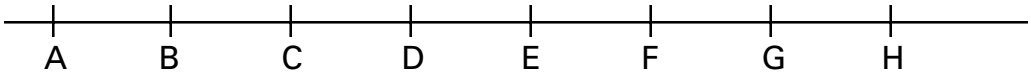
5 If $\overline{DH}$ is 9 units in length, how long is $\overline{DG}$? $\overline{EG}$? $\overline{FG}$?



6 The length of $\overline{PQ}$ is $3/8$ of a unit. How long is $\overline{LU}$? $\overline{LQ}$? $\overline{PR}$?



7 The length of $\overline{AH}$ is 3 units. How long is $\overline{AB}$? $\overline{DE}$? $\overline{AC}$? $\overline{DF}$?



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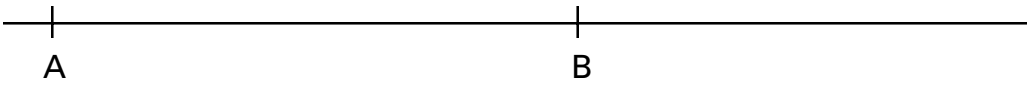
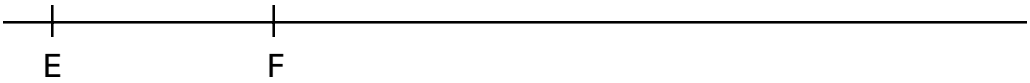
Focus Student Activity 6.3 (cont.)

8 If the length of $\overline{AB}$ is 3 units, find and label point D so that the length of $\overline{AD}$ is 5 units.

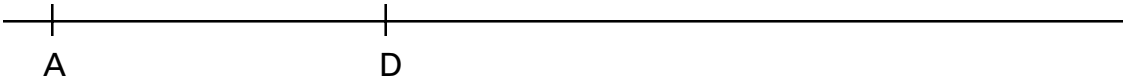


9 Locate point C on the line above so that the length of $\overline{AC}$ is $5/2$ units.

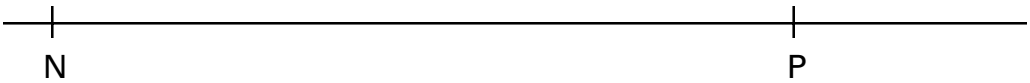
10 The length of $\overline{EF}$ is $1/4$ of a unit. Find and label point G so that the length of $\overline{EG}$ is 1 unit.



11 If the length of $\overline{AD}$ is $4/3$ units, locate and label point E so that $\overline{AE}$ is 4 units in length.



12 If the length of $\overline{NP}$ is $5/4$ units, find and label point Q so that $\overline{NQ}$ is 1 unit in length.



13 Locate and label point R on the line above so that the length of $\overline{NR}$ is $4/3$ units.

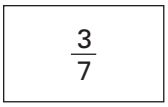
14 Sketch an equilateral triangle, a rectangle, a regular hexagon, and a rhombus. Label their areas 13, 23, 37, and 91 square units. Subdivide each into 4 congruent subregions and label the area of each subregion.

Focus Student Activity 6.4

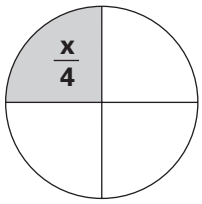
NAME _____ **DATE** _____

Complete the following. Use a straightedge, but no ruler. Express all fractions in improper form.

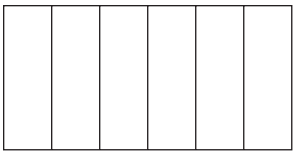
1 Assuming the area of the rectangle shown below is 3/7 square units, sketch a rectangle with area 3 units.



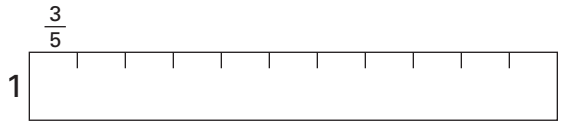
2 The area of the shaded region is x/4 square unit. What is the area of the whole circle?



3 The area of this field is 13 acres. What is the area of each part?



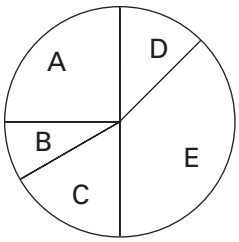
4



a) Shade a region with area 3/2 square units.

b) Draw a bold line around a rectangle with area 6 square units.

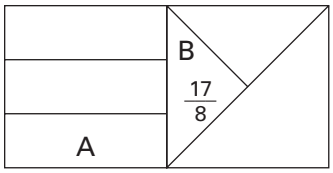
5 The area of this circle is X square units. Find the area of each region.



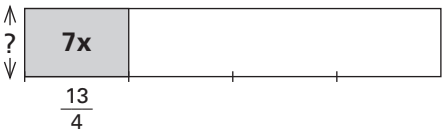
(Continued on back.)

Focus Student Activity 6.4 (cont.)

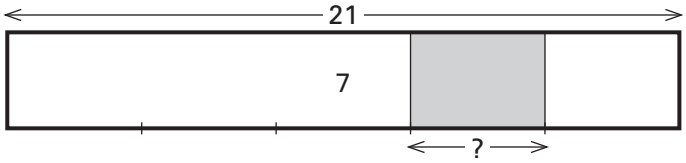
6 The area of region B is $17/8$. Find the area of region A.



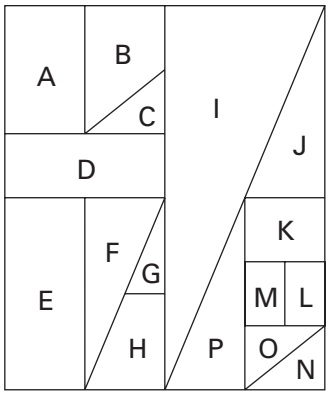
7 The shaded region has area $7x$ and length $13/4$. Find the area, length, and height of the large rectangle.



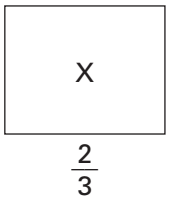
8 The area of the large outer rectangle is 7 and its length is 21. Find the area and length of the shaded region.



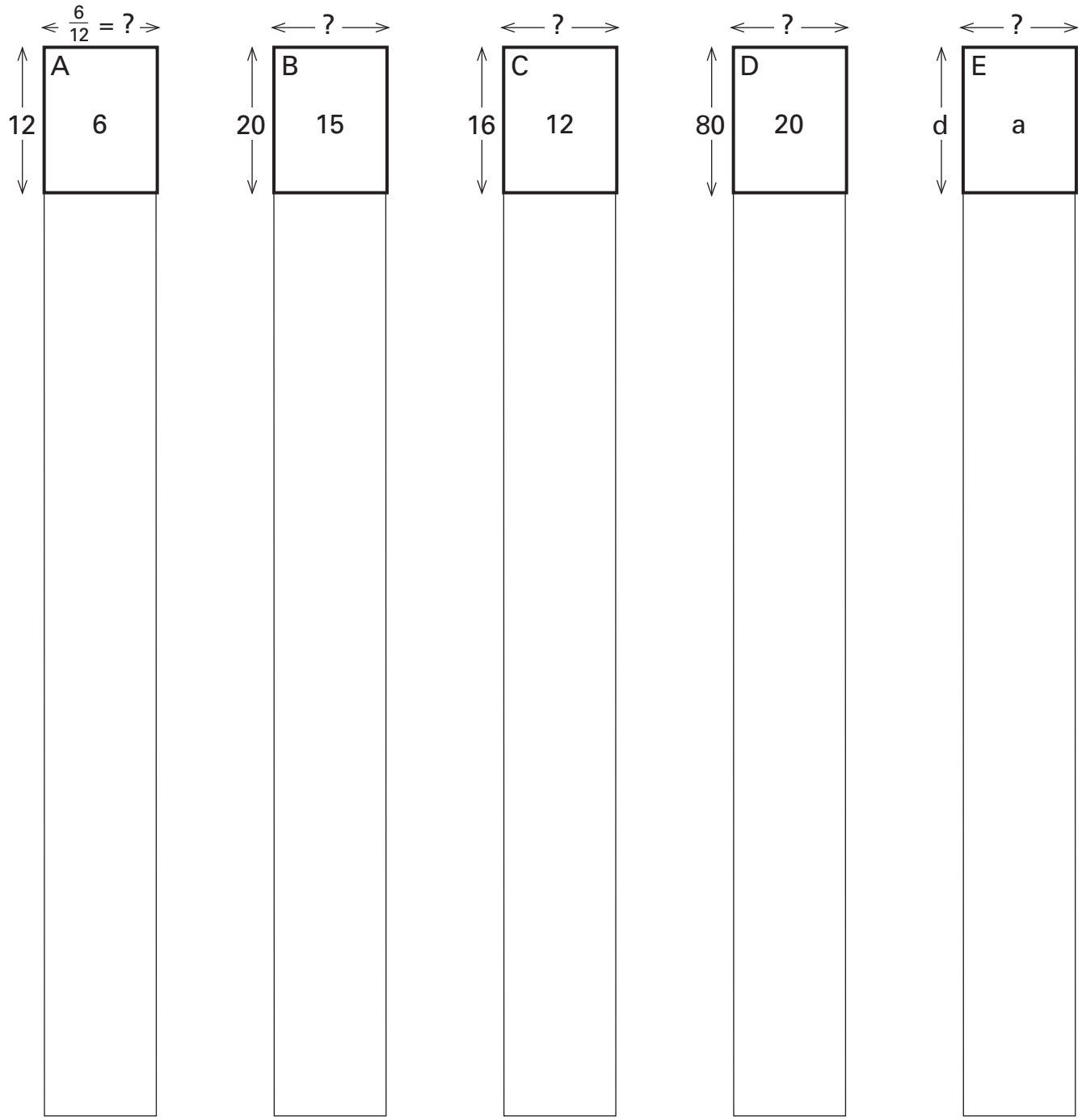
9 The area of the large outer rectangle below is 29. Find the area of each subregion.



10 The rectangle below has area X and length $2/3$. Extend the right edge of the rectangle to form a rectangle with length 2. Record its area and mark the rectangle to show your methods.



Focus Student Activity 6.5



Equivalent fractions:

Follow-up Student Activity 6.6

NAME _____ **DATE** _____

Write all of your responses on other sheets of paper. Be sure to write the problem next to your response.

1 For each of these concepts of a fraction—*part-to-whole*, *ratio*, *division*, and *area*—do the following:

a) explain the meaning of a fraction a/b when it is viewed according to that concept;

b) write a situation in which you "think" according to that meaning, and make a sketch that illustrates your thinking.

2 Explain the meaning of *equivalent fractions*. Show how to use each of the 4 concepts of a fraction to determine 3 pairs of equivalent fractions.

3 For each of the 4 concepts of a fraction, show how to determine an infinite set of fractions that are equivalent to a/b, for a and b not equal to zero.

4 Suppose $a \neq 0$. Demonstrate *why*:

a) $0/a = 0$

b) $0/0$ is undefined

c) $a/0$ is not possible

d) 1 is the multiplicative identity; 1 is the division identity; 0 is the additive identity; and 0 is the subtractive identity.

5 Suppose a and b are whole numbers not equal to zero. Create at least 6 "visual proofs" of mathematical relationships that you can show on rectangles whose dimensions and areas are a, b, 1, or fractions whose numerators and/or denominators are a, b, or 1.

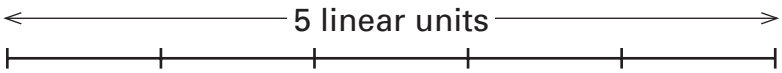
(Continued on back.)

Follow-up Student Activity (cont.)

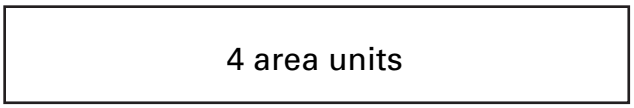
6 On the attached sheet of 1-cm grid paper, outline a rectangle that is 18 cm × 12 cm. Suppose it has area 17 square units for a certain area unit. Subdivide the rectangle to form rectangular subregions with the following areas and with no overlaps. Label the area of each subregion.

Region A, $\frac{17}{12}$ square units.
Region B, $\frac{17}{24}$ square units
Region C, $\frac{34}{12}$ square units
Region D, $\frac{34}{24}$ square units
Region E, $\frac{17}{6}$ square units
Region F, $\frac{34}{6}$ square units
Region G, $\frac{17}{8}$ square units

7 Use the parallel line sheet to mark off lengths or areas for the given fractions. Explain your reasoning.

a) 

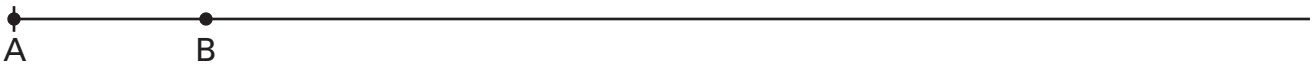
Show length $\frac{3}{5}$.

b) 

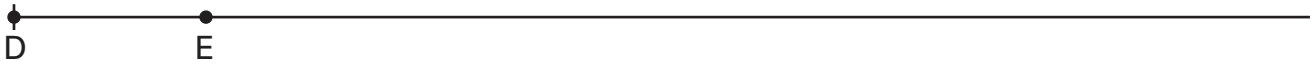
Show area $\frac{4}{7}$.

8 Use the parallel line sheet to locate and label points satisfying the given conditions. Mark the diagram to show your methods.

a) AB is $\frac{2}{3}$ units. Locate point C so that AC is 4 units.



b) DE is $\frac{4}{5}$ units. Locate point F so that DF is 3 units.



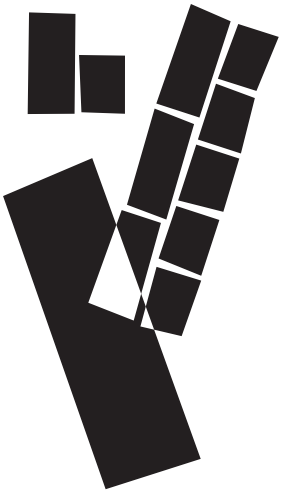
c) GH is 3 units. Locate point K so that GK is $2\frac{3}{5}$ units.

Properties, Operations & Algorithms

Lesson 7

THE BIG IDEA

Opportunities to invent algorithms for calculating with whole numbers, fractions, decimals, percentages, and algebraic expressions strengthen students' number sense and computational facility while building conceptual understanding. Models that represent the meanings of the operations provide a basis for "seeing" and proving important mathematical properties.



CONNECTOR

OVERVIEW

Students sketch diagrams to illustrate the meanings of the 4 basic operations, whole numbers, decimals, fractions, integers, and percents. They discuss the meaning of the term algorithm.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 transparency.
- ✔ Counting pieces and ¼" grid paper (optional), as needed by groups.

FOCUS

OVERVIEW

Students create visual proofs for several mathematical properties. They invent algorithms and examine algorithms invented by others for computing with fractions, decimals, whole numbers, integers, percents, and algebraic expressions. Given various conditions for numbers and operations, students create word problems involving everyday situations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Masters D and E, 1 transparency of each.
- ✔ Focus Student Activity 7.1, 1 copy per student and 1 transparency.
- ✔ Percent grids (see *Blackline Masters*), 1 copy per student and 1 transparency.
- ✔ Blank note cards, 12 per group.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Marking pens, scissors, and tape for each group.
- ✔ Calculators, 1 per student.

FOLLOW-UP

OVERVIEW

Students create visual proofs for several mathematical properties, and they show visual and symbolic strategies for solving several computations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 7.2, 1 copy per student.

LESSON IDEAS

***MATH ALIVE!* GOALS**
Throughout this course, an important goal is to have students work back and forth between visual and symbolic representations. Hence, it is intended that seeing symbols will trigger images that enable student sensemaking and problem solving, and that seeing and manipulating models will prompt meaningful symbolic recordings.

ASSESSMENT
To prompt extended responses for Problems 7 and 8 on the Follow-up, you might create, or have the students create, specific criteria regarding their work. Students' work on Problems 7 and 8 could be used as a portfolio entry (see *Starting Points* for information on using portfolios).

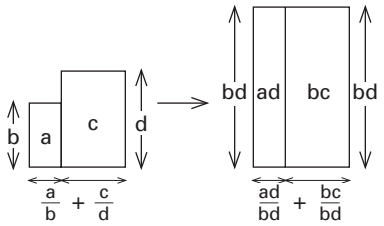
QUOTE
Understanding the fundamental operations of addition, subtraction, multiplication, and division is central to knowing mathematics. One essential component of what it means to understand an operation is recognizing conditions in real-world situations that indicate that the operation would be useful in those situations. Other components include building an awareness of models and the properties of an operation, seeing relationships among operations, and acquiring insight into the effects of an operation on a pair of numbers. These four components are aspects of operation sense.
NCTM Standards

SELECTED ANSWERS

1. a) True
   b) False
   c) True
   d) False
   e) False
   f) True
   g) False
   h) True
   i) True
   j) False

2. a) $6/35$
   b) $7/24$
   c) $3\frac{3}{8}$
   d) $-11/24$
   e) .63
   f) 22.4
   g) $5\frac{1}{3}$
   h) .452
   i) $-x/15$
   j) $6\frac{5}{28}$
   k) $\frac{(9x+13)}{10}$
   l) $19/3x$
   m) 2.43
   n) $7x/6$
   o) $1\frac{31}{56}$

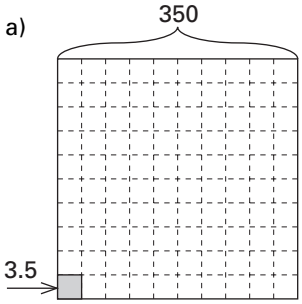
3. a) $a/b + c/d$



Since the total area of the 2nd figure is $ad + bc$, and since the length of the bases of the 2 figures are equal, the figures show that $a/b + c/d = \frac{(ad + bc)}{bd}$.

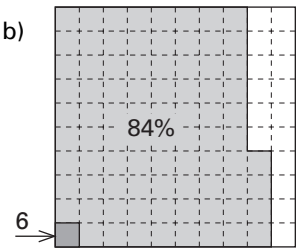
4. a) $17\frac{8}{9} + 6\frac{4}{9} = 18 + 6\frac{3}{9} = 24\frac{3}{9}$
   c) $36.98 - 4.28 = 37 - 4.3 = 32.7$
   e) $12 \times 1155 = 4 \times 3465 = 2 \times 6930 = 13860$
   g) $288 \div 60 = 144 \div 30 = 72 \div 15 = 24 \div 5 = 4\frac{4}{5}$

5. a)



Since the whole square has value 350, 1 small square has value 3.5. So, 60 ÷ 3.5 = 18, show that 18% of 350 = 63.

b)



Since 84 small squares have value 504, 1 small square has value 6. So, the whole square has value 600 and 84% of 600 = 504.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students sketch diagrams to illustrate the meanings of the 4 basic operations, whole numbers, decimals, fractions, integers, and percents. They discuss the meaning of the term algorithm.

MATERIALS

✔ Connector Master A, 1 transparency.

✔ Counting pieces and ¼" grid paper (optional), as needed by groups.

ACTIONS

1 Arrange the students in groups. Place a transparency of Connector Master A on the overhead and ask the students to carry out the instructions. Then have the groups each conduct a round-robin discussion of their understanding of each term. Provide tile (counting pieces) and/or ¼" grid paper as needed for group discussions. Invite volunteers to share their group's ideas with the class.

Properties, Operations, and Algorithms Lesson **7**

Connector Master A

Complete the following thought starter. If you think about a term from the list in more than one way, be sure to include all of your ideas.

Here is how I think about the meaning of each term below, together with an example or examples to illustrate my understanding of the term and ways it is related to other terms in the list…

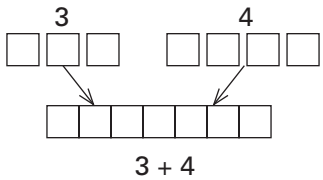
a) addition
b) subtraction
c) multiplication
d) division
e) whole number
f) fraction
g) decimal
h) integer
i) percent

COMMENTS

1 If students are keeping journals, you might suggest they write their responses to this thought starter in their journals. Since students could write extensively, you might set a time limit and suggest that students focus on what they think are key points.

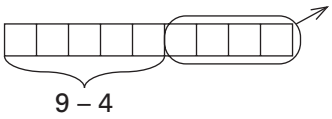
"Listening in" while students share their ideas with one another provides an opportunity for you to assess your students' understanding of number and operation concepts, and to determine whether there are ideas that need more extended review. Throughout *Math Alive! Courses I* and *II* students investigated the meanings of the operations and relationships among fractions, decimals, and percents; see those courses for other discussion ideas.

Most students will probably refer to addition as the process of *joining together* 2 sets of objects. For example:



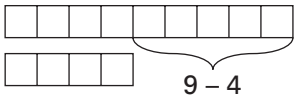
Joining sets together

In some contexts subtraction is most appropriately viewed as a *take-away* process. For example, 9 – 4 is the amount remaining after 4 objects are removed from a set of 9 objects:



Take-away

Or, in other contexts, the *comparison*, or *difference*, method of subtraction is most appropriate. For example, 9 – 4 is the difference between the numbers of objects in a set of 9 objects and a set of 4 objects:



Comparison or Difference

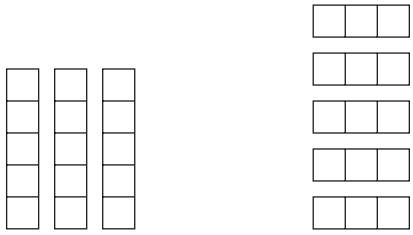
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)
One can view multiplication as *repeated addition*. For example, 3×5 can be viewed as adding 3 groups of 5, or 5 groups of 3:

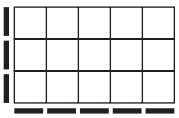


$3 \times 5 = 5 + 5 + 5$
3 groups of 5

$5 \times 3 = 3 + 3 + 3 + 3 + 3$
5 groups of 3

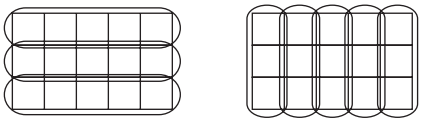
Repeated addition

Another meaning of multiplication is based on the *area* method. For example, 3×5 can be represented as a rectangle with dimensions 3 linear units by 5 linear units and area $3 \times 5 = 15$ square units:

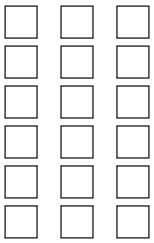


3 linear units $\times$ 5 linear units = 15 square units
Area

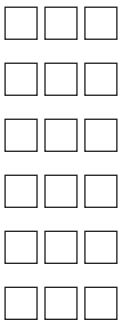
Notice that it is possible to "see" repeated addition in an area model, e.g., one can see 3 groups of 5, and 5 groups of 3, in an area model of 3×5:



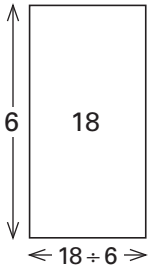
Division can be viewed according to the *grouping*, *sharing*, and *area* methods, as illustrated by the following diagrams representing $18 \div 6$:



Grouping Arrange 18 objects in groups of 6 objects. There are $18 \div 6$ groups.



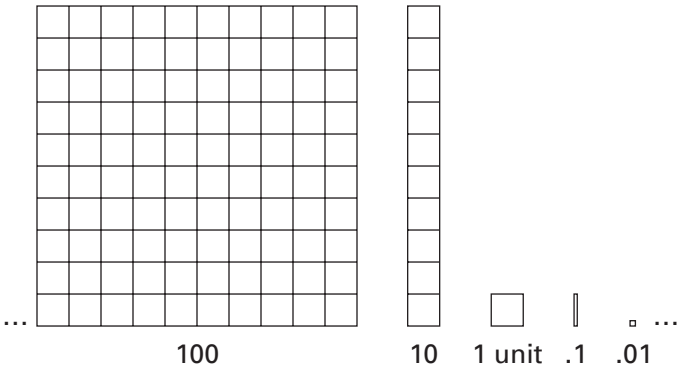
Sharing Distribute 18 objects in 6 equal shares. There are $18 \div 6$ in each share.

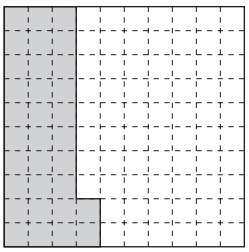


Area Arrange 18 square units to form a rectangle with dimensions 6 and $18 \div 6$.

Connector Teacher Activity (cont.)

ACTIONS





32% = .32 = $^{32}/_{100}$

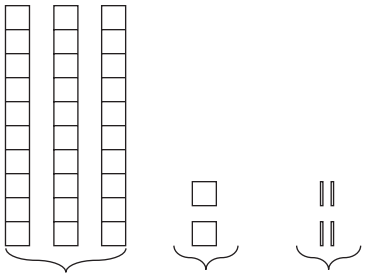
COMMENTS

The *whole numbers* are the numbers 0, 1, 2, 3, 4, … . The *place values* of the digits in a whole number, beginning with the digit in the far right position and moving to the left, are: units, tens, hundreds, thousands, ten-thousands, hundred thousands, and so on.

A *fraction* can be viewed according to the *part-to-whole, ratio,* and *division* concepts. See Lesson 6 for an in-depth discussion of these ideas. A *mixed number* is the sum of a whole number and a fraction.

A *decimal* number is a *base ten* number written using decimal notation. A whole number can be written as a decimal by placing a *decimal point* to the right of the units position, and a zero to the right of the decimal. A *decimal fraction* is a base ten number with no digits other than zeroes to the left of the decimal point. For any fraction $^{a}/_{b}$, its decimal equivalent is determined by computing $a \div b$. The place values of the digits to the right of a decimal point are, from left to right, tenths, hundredths, thousandths, ten-thousandths, hundred-thousands, and so forth.

In previous *Math Alive!* courses, students used base ten pieces such as those shown at the left to represent whole numbers and decimals. For example, using the unit given at the left, the number 32.4 is represented as follows:



$(3 \times 10) + (2 \times 1) + (4 \times .1) = 30 + 2 + .4$
$= 32.4$

The *integers* are the positive and negative whole numbers, together with zero. Students were introduced to the use of red and black counting pieces to represent and compute with integers in Lessons 5-8 of *Math Alive! Course II*. You might read those lessons for discussion ideas.

Percent means per hundred, i.e., when something (regardless its size or shape) is divided into 100 equal parts, one part is called one percent, or 1%. For example, 32% of the grid shown at the left is shaded. Since percent means per hundred, 32% can be written as a decimal, .32, or as a fraction, $^{32}/_{100}$.

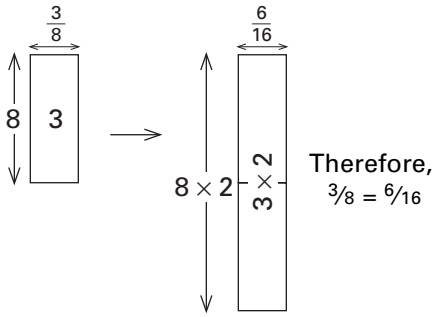
Connector Teacher Activity (cont.)

ACTIONS

2 Ask the groups to discuss their ideas about the meaning of the term *algorithm*. Discuss as a large group.

3 Ask the students to observe you without questioning or commenting as you silently make the following series of sketches. Ask the groups to write, in words only, a set of procedures that describe your actions. Invite volunteers to read aloud their group's wordings. Discuss.

Forming a fraction equivilant to $\frac{3}{8}$:



4 If the students have not suggested it, point out that in Action 3 you used an algorithm for constructing a fraction equivalent to $\frac{3}{8}$. Ask for their ideas regarding how your algorithm works for constructing other fractions equivalent to $\frac{3}{8}$. Discuss. Then ask the groups to adapt your algorithm to create an algorithm for forming fractions equivalent to the fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Invite volunteers to demonstrate their algorithms and receive feedback from the class.

COMMENTS

2 An algorithm is a special procedure for solving a certain type of problem, often involving the repeated use of the same basic process. Students may suggest a variety of examples.

3 This action and Action 4 are based on and reinforce ideas from Actions 17-19 of Lesson 6. So that the students' can follow the thought process involved, draw and label the rectangles in sequence, and do not speak as you draw. For example: write the heading, "Forming a fraction equivalent to $\frac{3}{8}$"; then draw the first rectangle; label its area 3; label its side dimension 8; label the top dimension $\frac{3}{8}$; then draw a rectangle which is double the length of the first rectangle; mark its area 3×2; mark its side dimension 8×2; mark its top dimension $\frac{6}{16}$; and then write the statement, "Therefore, $\frac{3}{8} = \frac{6}{16}$." If students have questions, you might just repeat the process rather than answering them orally.

The intent here is to get students to speculate about the steps of your thought processes; they document what your thoughts appear to be. Students can critique one another's wordings. Here are the procedures given by one group:

Draw a rectangle; label its area 3, its side dimension 8, and its top dimension $\frac{3}{8}$; then draw a rectangle whose side dimension and area are double the side dimension and area of the first rectangle; label its top dimension $\frac{6}{16}$; thus, $\frac{3}{8} = \frac{6}{16}$ because the top dimension of the rectangle has not changed.

4 In the following example, students have adapted the procedure given in Action 3 to form other fractions equivalent to $\frac{3}{8}$: *Draw a rectangle; label its area 3, its side dimension 8, and its top dimension $\frac{3}{8}$; then draw a rectangle whose side dimension and area are multiplied or divided by k (for k an integer); label its top dimension $\frac{3k}{8k}$ or $(\frac{3}{k})/(\frac{8}{k})$; thus, $\frac{3}{8} = \frac{3k}{8k} = (\frac{3}{k})/(\frac{8}{k})$.*

Here is how a group adapted the above procedure for any fraction $\frac{a}{b}$: *Draw a rectangle; label its area a, its side dimension b, and its top dimension $\frac{a}{b}$; then draw a rectangle whose side dimension and area are multiplied or divided by k (for k an integer not equal to zero); label its top dimension $\frac{ak}{bk}$ or $(\frac{a}{k})/(\frac{b}{k})$; thus, $\frac{a}{b} = \frac{ak}{bk} = (\frac{a}{k})/(\frac{b}{k})$.*

Students' recordings may look similar to symbolic algorithms traditionally memorized; however, the intent is for students' algorithms to be anchored in understanding and for the student to be an inventor rather than an imitator.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students create visual proofs for several mathematical properties. They invent algorithms and examine algorithms invented by others for computing with fractions, decimals, whole numbers, integers, percents, and algebraic expressions. Given various conditions for numbers and operations, students create word problems involving everyday situations.

MATERIALS

- ✔ Focus Master A, 1 copy per student and 1 transparency.
- ✔ Focus Masters B and C, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Masters D and E, 1 transparency of each.
- ✔ Focus Student Activity 7.1, 1 copy per student and 1 transparency.
- ✔ Percent grids (see *Blackline Masters*), 1 copy per student and 1 transparency.
- ✔ Blank note cards, 12 per group.
- ✔ Butcher paper, 1 sheet per group.
- ✔ Marking pens, scissors, and tape for each group.
- ✔ Calculators, 1 per student.

ACTIONS

1 Arrange the students in groups. Write the following statement on the overhead:

$$\frac{bc}{b} = c, \text{ for } b \text{ and } c \text{ integers and } b \neq 0.$$

Ask the groups to create a visual proof that the above statement is true, supported by explanations that are based on knowing the meanings of the basic operations and the meanings of a fraction. Then ask them to provide 2 examples by replacing the variables with numbers.

Invite volunteers to demonstrate their reasoning and examples at the overhead. If no students offer arguments based on the area concept of $^{bc}/_{b}$, introduce such an argument as one possibility.

COMMENTS

1 As shown in Figure 1 below, for integers b and c ($b \neq 0$), $^{bc}/_{b}$ can be viewed as the value of one edge of a rectangle with value bc and the other edge of value b. Further, as shown in Figure 2 below, if a rectangle has value bc and one edge b, the other edge has value c. Hence, it must be that $^{bc}/_{b} = c$. The restriction $b \neq 0$ is required since division by zero is undefined (see Lesson 6).

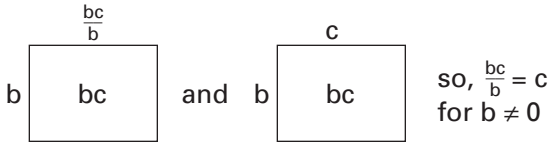


Figure 1 Figure 2

Two examples are: $\frac{12}{3} = \frac{(3)(4)}{3} = 4$, and $\frac{-14}{-2} = \frac{(-2)(7)}{(-2)} = 7$. You might remind the students that showing that a property holds for specific numbers does not constitute proof that the property holds for all numbers. However, just one example using numbers is sufficient to prove that a property does *not* hold for all numbers. Such an example is called a *counter example*.

Notice that properties established in Lesson 6 were based on areas and lengths, i.e., on positive numbers. In this action, by referring to the value of a rectangle and its edges, we have extended the area concept of a fraction and the area methods of multiplication and division to prove $^{bc}/_{b} = c$ for all integers except $b = 0$. See Lessons 5-8 of *Math Alive! Course II* for discussion ideas regarding models for operations involving integers.

Focus Teacher Activity (cont.)

ACTIONS

2 Give each student a copy of Focus Master A and ask the groups to create a visual proof that equation a) is true for all integers that meet the given conditions, and to give 2 examples using numbers to replace the variables. Discuss. Repeat for equations b)-u). Explain that once a property is proven, then it is okay to use that property to prove other properties. Encourage students to verbalize generalizations.

Properties, Operations, and Algorithms — Lesson **7**

Focus Master A

Assume that *a*, *b*, and *c* are integers such that denominators are not zero.

a) $a\left(\frac{c}{a}\right) = c$

b) $\left(\frac{1}{b}\right)b = 1$

c) $ab = ba$

d) $\frac{ab}{a} = b$

e) $\frac{b}{b} = 1$

f) $b\left(\frac{a}{b}\right) = a$

g) $\frac{a}{a/b} = b$

h) $\frac{b}{b/a} = a$

i) $b\left(\frac{a}{c}\right) = \frac{ba}{c}$

j) $a(b + c) = ab + ac$

k) $\frac{a+b}{a} = 1 + \frac{b}{a}$

l) $a(bc) = ab(c)$

m) $\frac{1}{1/a} = a$

n) $\frac{a^2+ab}{a} = a + b$

o) $\frac{a}{b} = \frac{ac}{bc}$

p) $\frac{abc}{b} = ac$

q) $\frac{abc}{c} = ab$

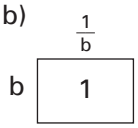
r) $\frac{abc}{bc} = a$

s) $\frac{1}{b/c} = \frac{c}{b}$

t) $\frac{a+b}{a+b} = 1$

u) $\frac{-ab}{b} = -a$

b)



$b \times \frac{1}{b} = 1$
for $b \neq 0$

If the value of a rectangle is 1 and the value of an edge of the rectangle is *b*, then the value of the other edge is $^1/_b$. So, since the product of the values of the edges equals the value of the rectangle, it must be that $b \times {}^1/_b = 1$. Note: *b* and $^1/_b$ are called *reciprocals* since their product is 1. The expression $^1/_b$ is also written b^{-1}. The *inverse for multiplication property* states that for every nonzero number, *b*, there is a reciprocal, $^1/_b$.

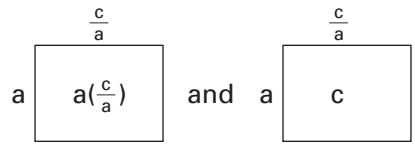
COMMENTS

2 Some of the properties represented by these equations were investigated in Lesson 6, Action 6, where students used their understanding of the operations and the meanings of fractions to solve equations; here students use their understandings of operations and fractions to prove that equations are true.

Notice that students are establishing these properties for integers; you may need to prompt students to consider examples involving negative integers. Later in this lesson, students have opportunities to investigate applications of these properties to fractions and decimals.

To keep track of students' questions and ideas and to provide "fuel" for later discussions, you might post a "We wonder… We conjecture…" poster and have students record properties, conjectures, and questions.

a) As shown in Figure 1 below, the product $a(^c/_a)$ represents the value of a rectangle with edges of value *a* and $^c/_a$. Further, as shown in Figure 2, a rectangle with value *c* and one edge of value *a*, has other edge of value $^c/_a$. Hence, $a(^c/_a) = c$. Note: students may create other valid visual proofs that $a(^c/_a) = c$.



so, $a(\frac{c}{a}) = c$
for $a \neq 0$

Figure 1 Figure 2

It is helpful to discuss equations b)-u) one or two at a time. Note that several of the equations represent the same properties but are stated with different variables; you might encourage students to point these out.

You may need to let the students know that it is okay to assume they know the *identity for multiplication property*, i.e., that for all real numbers, n, $n \times 1 = 1 \times n = n$.

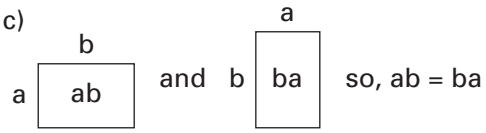
Shown at the left and on the next 2 pages are diagrams that can be used as the basis for proving each equation. A sample explanation accompanies the diagram for b). While all of the examples shown are based on the area methods of multiplication and division and the area concept of a fraction, students may use other visual models as the basis for proofs.

Before a volunteer gives a verbal explanation of their reasoning, you might have them sketch their proof and invite the other students to speculate about the reason-

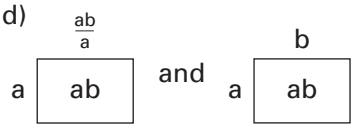
Focus Teacher Activity (cont.)

COMMENTS

c)



d)



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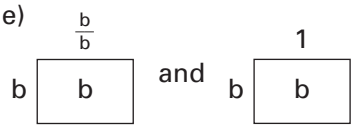
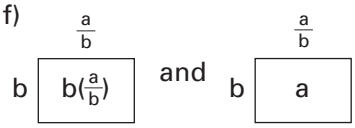
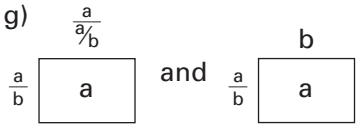


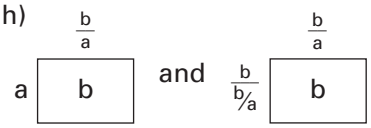
f)



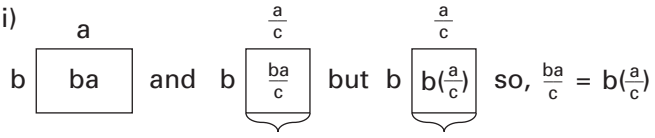
g)



h)



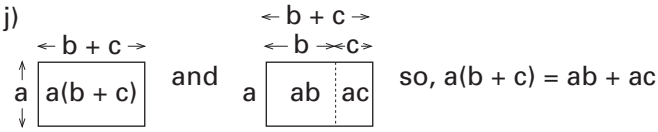
i)



Dividing the value of an edge by *c* also divides the value of the rectangle by *c*.

The value of a rectangle is the product of the value of its edges.

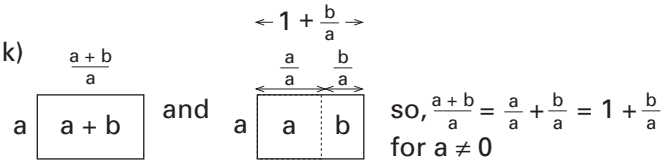
j)



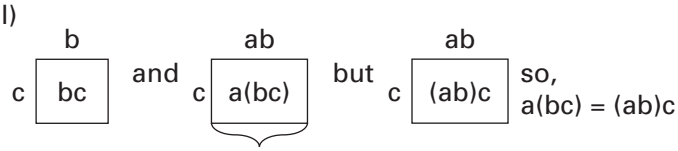
Note: this illustrates the *distributive property for multiplication over addition.*

ing behind the proof. Encourage students to give one another critical feedback regarding these proofs, but keep in mind that it is not intended that students perfect proofs, or that they memorize these properties or proofs. Rather, the purposes here are: to engage students in the process of building, refining, and debating conceptually based arguments; to provide a rich mathematical context for reviewing operations with integers; and to promote growth in students' number and operation sense.

k)



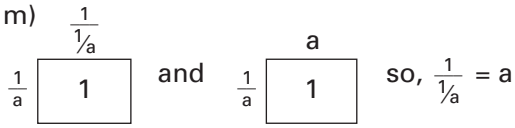
l)



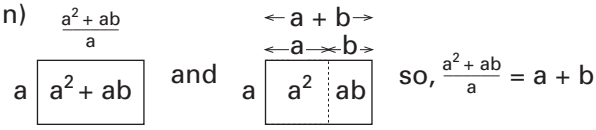
Multiply edge *b* by *a*.
This multiplies the value of the rectangle by *a*.

Note: the above diagrams illustrate the *associative property for multiplication*.

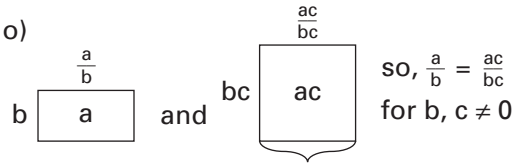
m)



n)



o)



Multiplying the value of the rectangle by *c* multiplies the value of one edge by *c*, but doesn't change the value of the other edge.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

2 (continued.)

p) $\frac{abc}{b}$: b [abc] and $\frac{ac}{b}$: b [bac = abc] so, $\frac{abc}{b} = ac$ for $b \neq 0$

q) $\frac{abc}{c}$: c [abc] and $\frac{ab}{c}$: c [c(ab) = abc] so, $\frac{abc}{c} = ab$ for $c \neq 0$

r) $\frac{abc}{bc}$: bc [abc] and $\frac{a}{bc}$: bc [abc] so, $\frac{abc}{bc} = a$ for $b,c \neq 0$

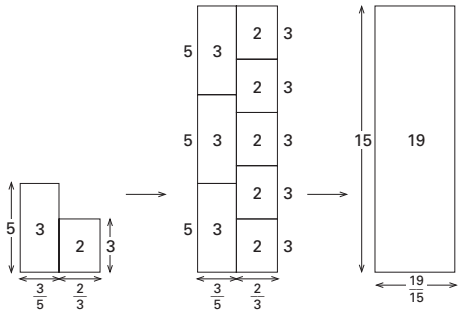
3 Write $\frac{3}{5} + \frac{2}{3}$ on the overhead and ask each student to think privately about strategies for computing this sum. Then give each group a copy of Focus Master B. Tell the students that these diagrams were made by a student to illustrate the sequence of her thought processes to solve $\frac{3}{5} + \frac{2}{3}$. Ask the groups to discuss their ideas about Julie's reasoning. Clarify as needed.

Lesson **7** Properties, Operations & Algorithms

Focus Master B

Julie's Method

$\frac{3}{5} + \frac{2}{3}$



COMMENTS

s) $\frac{1}{b/c}$: $\frac{b}{c}$ [1] and $\frac{c}{b}$: $\frac{b}{c} \times c = b$ [$1 \times c = c$] so, since the top edge did not change, $\frac{1}{b/c} = \frac{c}{b}$

Note: $\frac{b}{c} \times \frac{1}{b/c} = 1$ and $\frac{1}{b/c} = \frac{c}{b}$; so, $\frac{b}{c} \times \frac{c}{b} = 1$. This shows that $\frac{b}{c}$ and $\frac{c}{b}$ are reciprocals.

t) $\frac{a+b}{a+b}$: $a + b$ [$a + b$] and 1: a [a], b [b] so, $\frac{a+b}{a+b} = 1$ for $a + b \neq 0$

u) $\frac{-(ab)}{b}$: b [$-(ab) = -ab$] and $-a$: b [$(-a)b = -ab$] so, $\frac{-ab}{b} = -a$ for $b \neq 0$

3 Julie's approach is based on a series of "maneuvers" on adjacent rectangles. This approach was introduced in Lesson 20 of *Math Alive! Course II* for computing fraction sums and differences. You might reread that lesson for discussion ideas.

Following is Julie's description of her process:

I think of $\frac{3}{5}$ as the value of 1 edge of a rectangle with value 3 and other edge value 5, and I think of $\frac{2}{3}$ as the value of 1 edge of a rectangle with value 2 and other edge value 3. In my first diagram I positioned the 2 rectangles so they are adjacent and the bottom edges represent the sum $\frac{3}{5} + \frac{2}{3}$. Then I multiplied the value of a vertical edge of the 1st rectangle by 3 and the value of a vertical edge of the 2nd rectangle by 5. This forms a new rectangle with edge values 15 and ($\frac{3}{5} + \frac{2}{3}$). The value of this rectangle is 3 + 3 + 3 + 2 + 2 + 2 + 2 + 2 = 19. So, the bottom edge has value 19 ÷ 15 = $\frac{19}{15}$. Therefore, $\frac{3}{5} + \frac{2}{3} = \frac{19}{15}$. I can also see that $\frac{3}{5} + \frac{2}{3} = \frac{9}{15} + \frac{10}{15}$.

After students speculate about Julie's reasoning, you might read aloud her statement and encourage comparisons to the students' speculations.

Focus Teacher Activity (cont.)

ACTIONS

4 Ask the groups to use Julie's method as the basis for solving 3 different fraction addition problems, using fractions whose denominators are different from those on Focus Master B. Invite several volunteers to pose their addition problems to the class for solution using Julie's method. Discuss the students' reasoning and results, and their observations and generalizations about Julie's method.

5 Ask the groups to determine whether/how Julie's method could be used to determine sums of decimal fractions, for example .03 + .2. Ask for volunteers to use examples to support their conclusions. Discuss other algorithms that students know for computing fraction and decimal sums; encourage discussions of reasons why these methods work and ways they compare to Julie's methods.

COMMENTS

4 The intent here is to encourage students to identify features of Julie's process that generalize. Notice that, in Action 8, students create symbolic generalizations. Here are some examples of student responses:

We always start with 2 small adjacent rectangles. If the denominators are the same, their heights are the same, and they combine to form a larger rectangle. If the 2 small rectangles have different heights we expand the height and area of one or both of them to form a single larger rectangle.

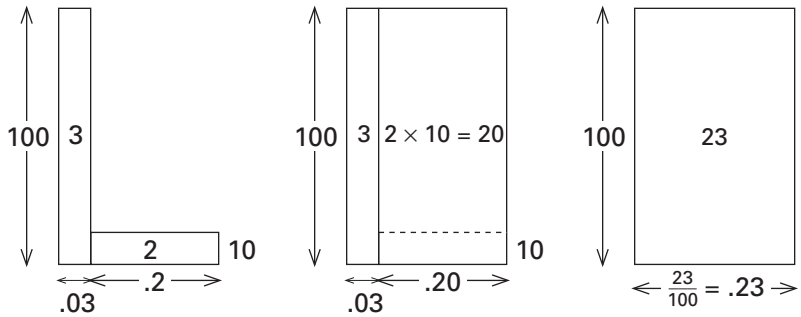
There are an infinite number of larger rectangles we can form by expanding the heights and areas of both of the rectangles.

Expanding the height of a rectangle by a factor automatically expands the area by the same factor, and vice versa.

To form a single large rectangle from 2 smaller adjacent rectangles, it always works to expand the height (and therefore the area) of each rectangle by the number that is the height of the other rectangle. Therefore, the height of the large rectangle is the product of the heights of the 2 smaller rectangles. If one denominator is a multiple of the other, then we only have to expand 1 rectangle.

If we only expand the height of the rectangles, the area changes by the same factor, but the other dimension of the rectangle doesn't change. We don't want to change the edge that represents a fraction we are adding, since that would change the problem.

5 Julie's method can be adapted for use with decimal sums. For example, to add .03 + .2, since .03 = $^3/_{100}$ and .2 = $^2/_{10}$:



Notice that this process creates an equivalent sum whose addends have the same number of decimal places.

Focus Teacher Activity (cont.)

ACTIONS

6 Ask the students to determine whether/how Julie's method works for the following sums of *algebraic fractions*:

a) $x/3 + 2x/5$

b) $2x/7 + 3x/2$

c) $\frac{(3x+2)}{3} + \frac{(2x+1)}{2}$

d) $\frac{(x+1)}{2} + \frac{x}{5}$

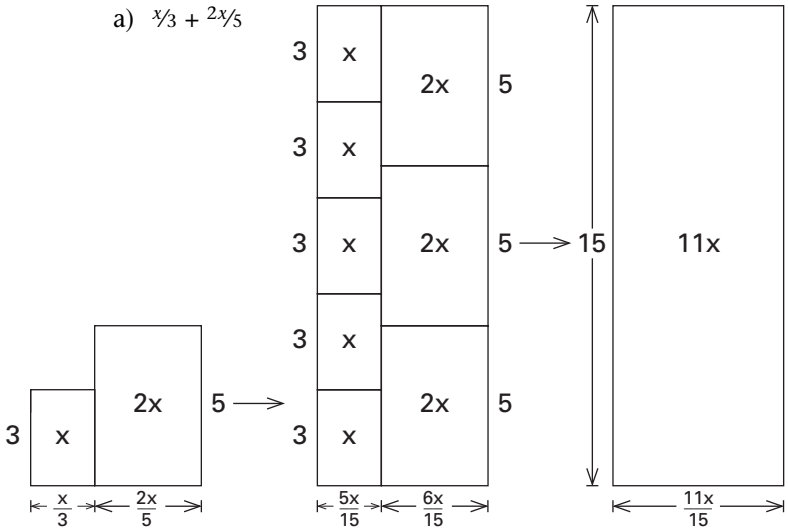
e) $4/x + 5/3$

f) $x/7 + 3/x$

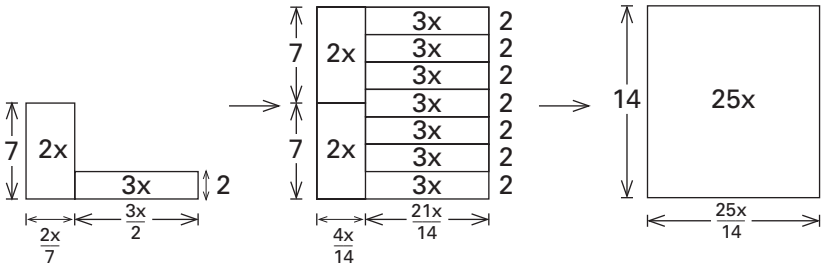
COMMENTS

6 *Algebraic fractions* are *rational expressions* with variables in the numerator and/or denominator. Julie's methods can be applied as follows:

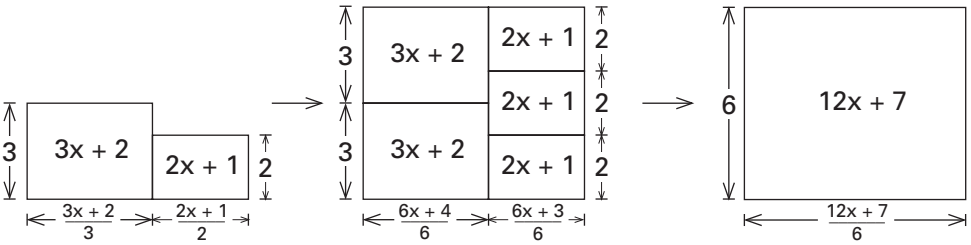
a) $x/3 + 2x/5$



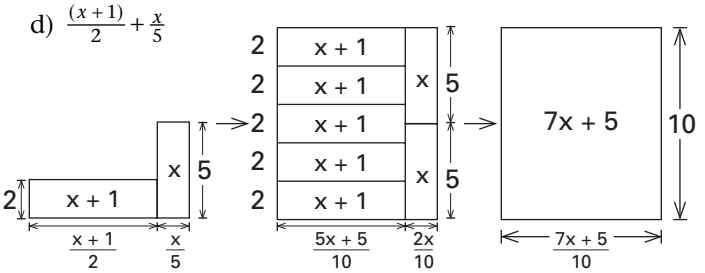
b) $2x/7 + 3x/2$



c) $\frac{(3x+2)}{3} + \frac{(2x+1)}{2}$



d) $\frac{(x+1)}{2} + \frac{x}{5}$

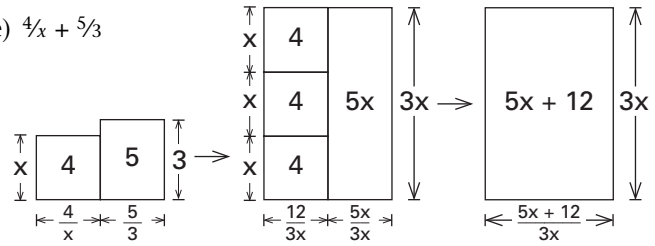


Focus Teacher Activity (cont.)

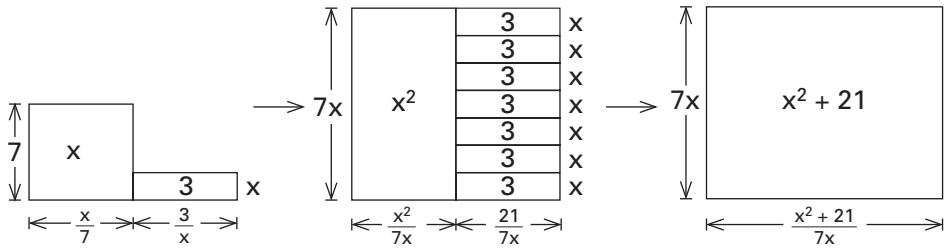
ACTIONS

COMMENTS

e) $^4/_x + {}^5/_3$



f) $^x/_7 + {}^3/_x$



7 Place a transparency of Focus Master B (see Action 3) on the overhead. Tell the students that mathematicians often use equations to represent the thought processes, actions, sketches, and mental images they use to solve computations. Ask the groups to record equations that could represent each step of Julie's method of computing $^3/_5 + {}^2/_3$, i.e., to use numbers and math symbols only to communicate as much detail as possible about Julie's thinking. Invite volunteers to show their equations to the class for discussion.

7 Here is one student's series of equations to represent Julie's methods:

Step 1: $\frac{3}{5} + \frac{2}{3} = ?$

Step 2: $\frac{3 \times 3}{3 \times 5} + \frac{5 \times 2}{5 \times 3} = \frac{9}{15} + \frac{10}{15}$

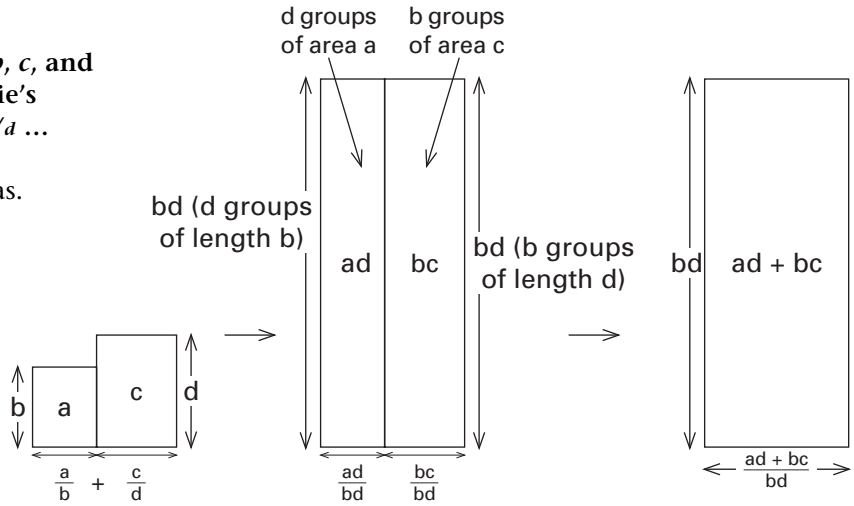
Step 3: $\frac{3+3+3+2+2+2+2+2}{3 \times 5} = \frac{19}{15}$

8 If it hasn't already come up, pose the following for completion by the groups.

> **For any fractions, $^a/_b$ and $^c/_d$, where *a*, *b*, *c*, and *d* are positive integers, according to Julie's method, here is an algorithm for $^a/_b + {}^c/_d$ …**

Ask for volunteers to show their group's ideas.

8 Here is a generalization of Julie's methods:



Some students may refer to the above general visual procedure as an algorithm; others may suggest the end result, $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$, as an algorithm.

Focus Teacher Activity (cont.)

ACTIONS

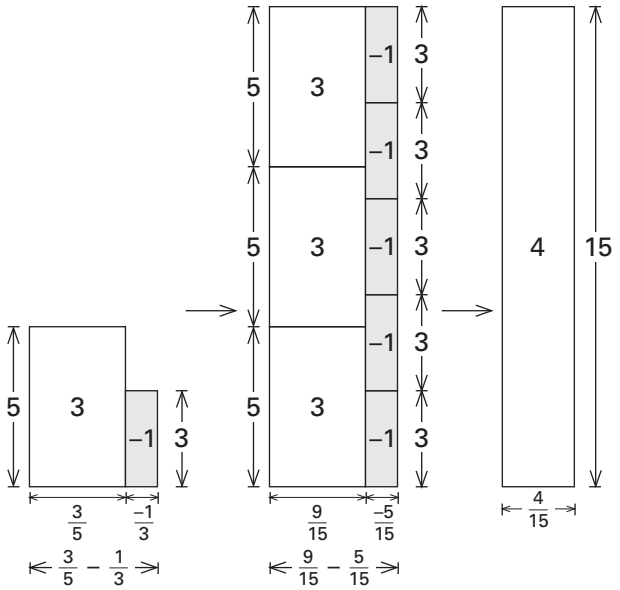
9 Give each group a sheet of butcher paper and marking pens. Ask each group to invent one or more algorithms for subtracting fractions, and to prepare a visual demonstration, verbal explanation, and symbolic representation of each algorithm. Ask them to include several specific examples to show how their methods work for common fractions, decimal fractions, and algebraic fractions. Have the groups post their results; then provide time for groups to peruse one another's algorithms. Discuss.

10 Select one or more of the students' subtraction algorithms from Action 9 for the class to use to solve the following:

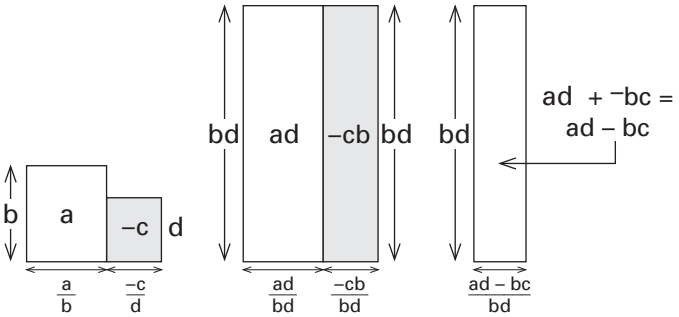
a) $\frac{4}{5} - \frac{2}{3}$ b) $\frac{7}{8} - \frac{3}{4}$ c) $\frac{5x}{7} - \frac{x}{2}$ d) $3\frac{1}{2} - 2\frac{2}{3}$

COMMENTS

9 Students may use a variety of procedures and, hence, invent a variety of algorithms. The following diagram illustrates how, if one thinks of subtraction as addition of the opposite, Julie's method can be applied to subtraction. Note: this approach also relies on the area method of multiplying and dividing integers developed in Lessons 7 and 8 of *Math Alive! Course II*.



In general, $\frac{a}{b} - \frac{c}{d}$ can be solved using Julie's method as shown below, to get $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.



Therefore, $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

10 a) $\frac{2}{15}$ b) $\frac{1}{8}$ c) $\frac{3x}{14}$ d) $\frac{5}{6}$

Focus Teacher Activity (cont.)

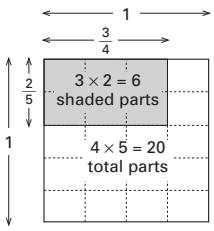
ACTIONS

11 Write $\frac{3}{4} \times \frac{2}{5}$ on the overhead and ask the students to think privately about strategies for computing this product. Then give each group a copy of Focus Master C and ask them to speculate about Linden's and Erica's thinking. Discuss. Next ask the groups to create 3 other multiplication problems involving fractions with unlike denominators, and to solve those problems using both Linden's and Erica's methods. Discuss. Then repeat for 3 multiplication problems involving decimal fractions, and 3 involving algebraic fractions. Discuss other methods students know.

Properties, Operations, and Algorithms Lesson **7**

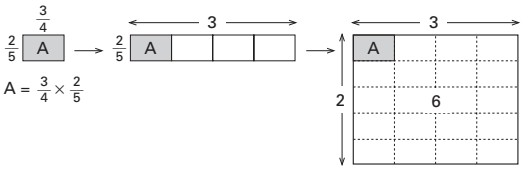
Focus Master C

Linden's Method: $\frac{3}{4} \times \frac{2}{5}$



Therefore, $\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.

Erica's Method: $\frac{3}{4} \times \frac{2}{5}$



So, $A = \frac{3}{4} \times \frac{2}{5} = (3 \times 2) \div (4 \times 5) = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.

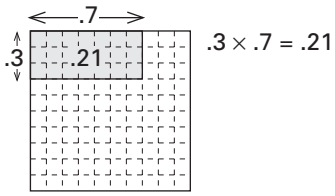
COMMENTS

11 Linden explained her thinking as follows: *The value of the large square is $1 \times 1 = 1$. The value of the shaded part is $\frac{3}{4} \times \frac{2}{5}$, and since there are $3 \times 2 = 6$ small squares shaded out of $4 \times 5 = 20$ total small squares, the value of the shaded part is $(3 \times 2) \div (4 \times 5) = \frac{6}{20}$ of the value of the large square, or $\frac{6}{20} \times 1 = \frac{6}{20}$. Therefore, $\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.*

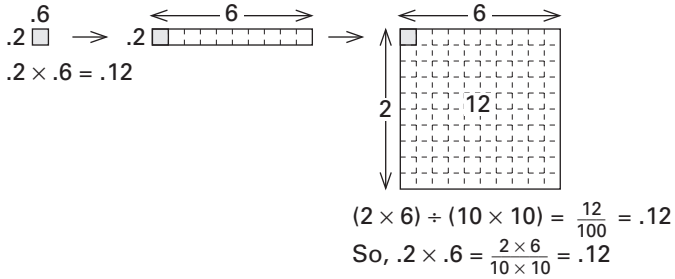
Erica gave the following explanation: *After expanding the horizontal edges by 4 and the vertical edges by 5, there are $4 \times 5 = 20$ congruent parts with total value $3 \times 2 = 6$. The value of 1 part is: $\frac{1}{20}$ of 6, or $\frac{6}{20}$. So, $A = \frac{3}{4} \times \frac{2}{5} = (3 \times 2) \div (4 \times 5) = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.*

Notice that Linden's method is based on the part-to-whole concept of a fraction, while Erica's is based on the division concept. Some students may point out that Linden's and Erica's methods show that $\frac{6}{20}$ of 1 equals $\frac{1}{20}$ of 6. They may also generalize from other problems that $\frac{a}{b} \times 1$ equals $\frac{1}{b} \times a$. Encourage such discussion.

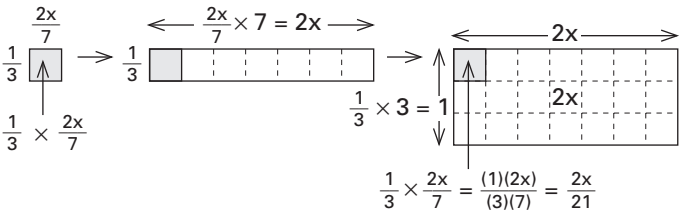
Example based on part-to-whole concept of a fraction:



Example based on division concept of a fraction:



Example based on division concept of a fraction:



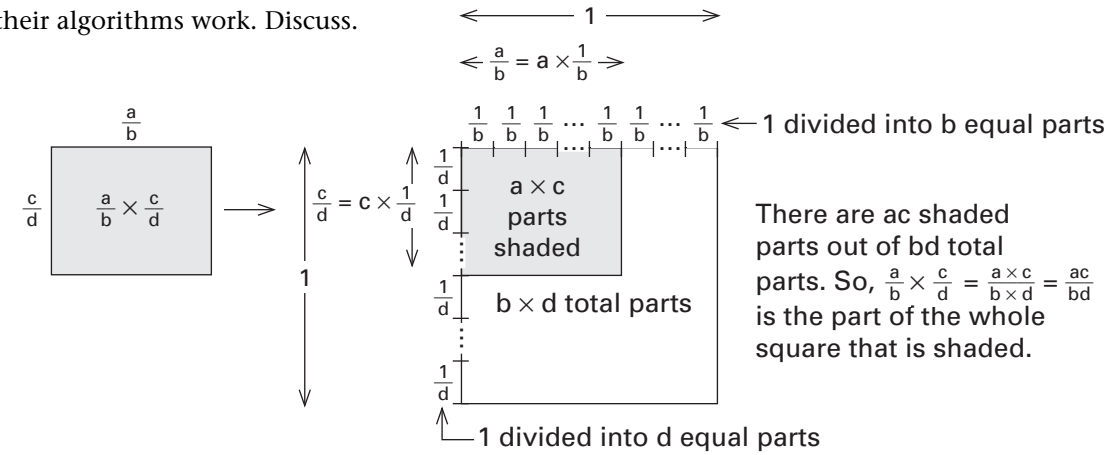
Focus Teacher Activity (cont.)

ACTIONS

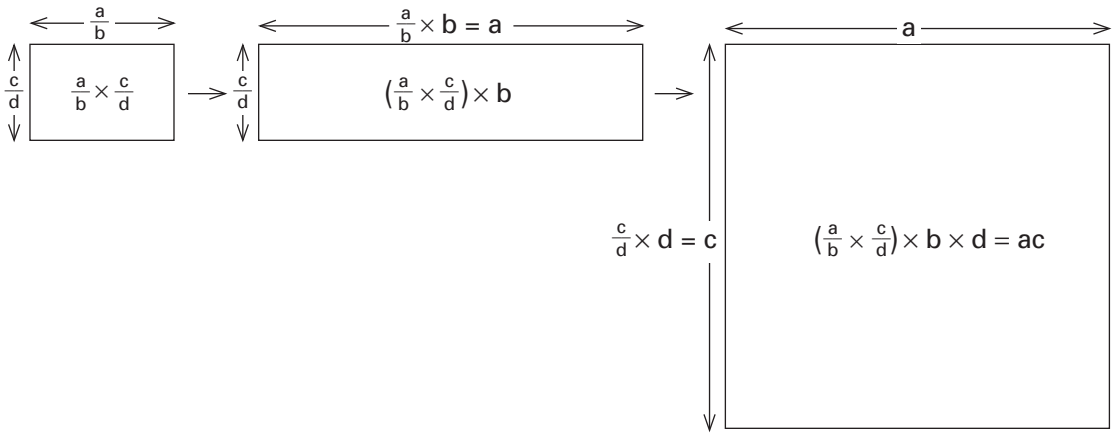
12 Ask the groups to write one or more algorithms for computing the product $\frac{a}{b} \times \frac{c}{d}$, where a, b, c, and d are integers and b and d are not equal to zero. Invite groups to show their algorithms to the class and to use visual models to show why their algorithms work. Discuss.

COMMENTS

12 Students' generalizations may be oral and based on examples. Here is a more formal generalization of Linden's method:

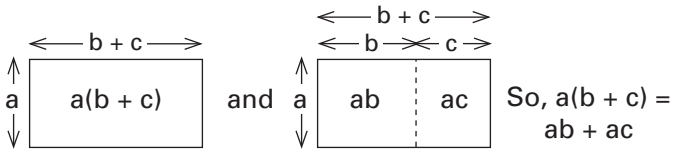


Here is a generalization of Erica's method:



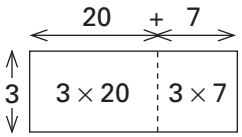
Multiplying one edge of the original rectangle by b and the other edge by d creates bd copies of the original rectangle. This new rectangle has value ac. So, the value of the original rectangle is $\frac{ac}{bd} = \frac{a}{b} \times \frac{c}{d}$.

13 Point out that according to the Distributive Property $a(b + c) = ab + ac$, as established during Action 2:



Ask the students to complete a) below. Discuss. Repeat for b) and c).

13 a)



So, $3 \times 27 = 3(20 + 7)$
$= 3(20) + 3(7)$
$= 60 + 21$
$= 81$

The same model can be used for each of the multiplications in b) and c). The linear and area units change; in the case of -32×-21 the signs of the edges also change, while a variable is involved in $(3x + 2)(2x + 1)$.

Focus Teacher Activity (cont.)

ACTIONS

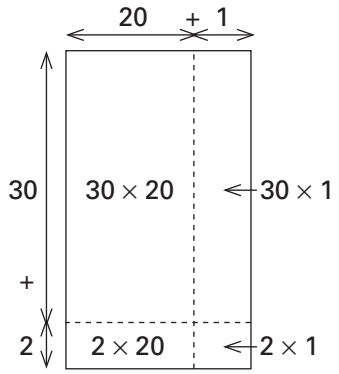
a) Show how to use a visual model for the distributive property to compute 3×27.

b) Show visually how to apply the distributive property twice to compute 32×21.

c) Determine whether/how the distributive property works for $(-32) \times (-21)$; for $3\frac{2}{3} \times 2\frac{1}{3}$; 3.2×2.1; for 320×210; for $(3x + 2)(2x + 1)$; and for 43×52. What stays the same and what changes for these problems?

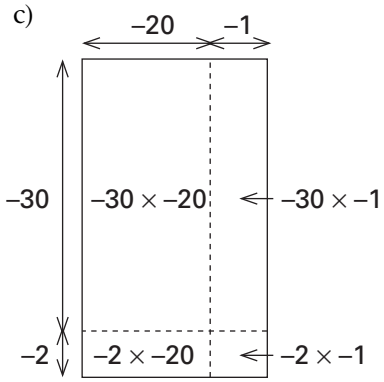
COMMENTS

b)

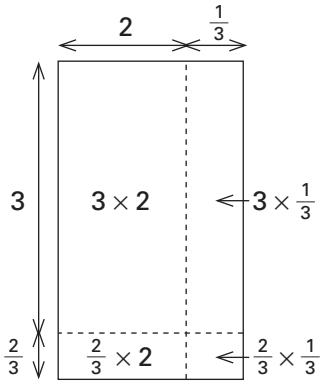


$$\begin{aligned} \mathbf{32 \times 21} &= (30 \times 21) + (2 \times 21) \\ &= 30\,(20 + 1) + 2(20 + 1) \\ &= [(30 \times 20) + (30 \times 1)] + [(2 \times 20) + (2 \times 1)] \\ &= 600 + 30 + 40 + 2 \\ &= 600 + 70 + 2 \\ &= 672 \end{aligned}$$

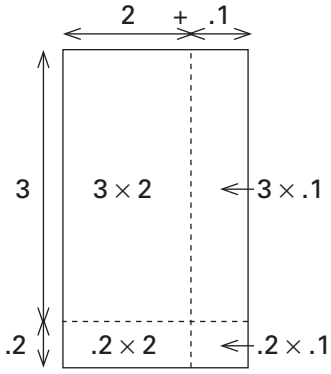
c)



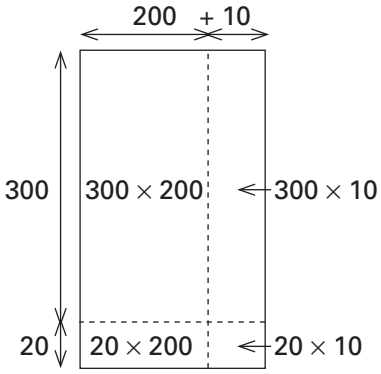
$$\begin{aligned} \mathbf{-32 \times -21} &= 600 + 30 + 40 + 2 \\ &= 600 + 70 + 2 \\ &= 672 \end{aligned}$$



$$\begin{aligned} \mathbf{3\tfrac{2}{3} \times 2\tfrac{1}{3}} &= 6 + \tfrac{3}{3} + \tfrac{4}{3} + \tfrac{2}{9} \\ &= 6 + \tfrac{7}{3} + \tfrac{2}{9} \\ &= 8\tfrac{5}{9} \end{aligned}$$



$$\begin{aligned} \mathbf{3.2 \times 2.1} &= 6 + .3 + .4 + .02 \\ &= 6 + .7 + .02 \\ &= 6.72 \end{aligned}$$



$$\begin{aligned} \mathbf{320 \times 210} &= 60{,}000 + 3{,}000 + 4{,}000 + 200 \\ &= 60{,}000 + 7{,}000 + 200 \\ &= 67{,}200 \end{aligned}$$

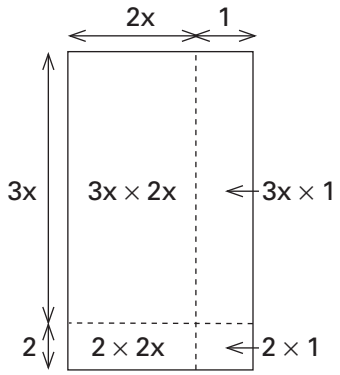
(Continued next page.)

Focus Teacher Activity (cont.)

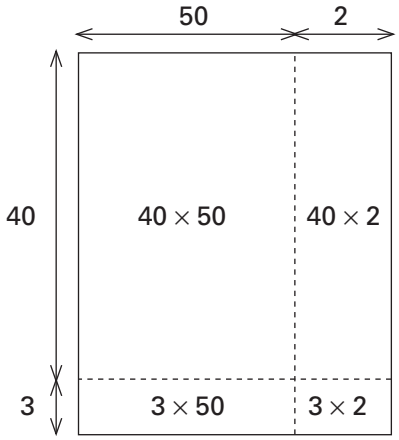
ACTIONS

COMMENTS

13 (continued.)

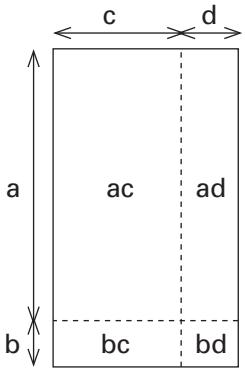


(3x + 2)(2x + 1) $= 6x^2 + 3x + 4x + 2$
$= 6x^2 + 7x + 2$



43 × **52** = 2,000 + 80 + 150 + 6
= 2236

In general,



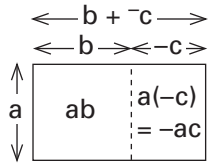
(a + b)(c + d) = ac + ad + bc + bd

14 Ask the groups to discuss the following:

Determine whether the distributive property holds for multiplication over subtraction. Show your reasoning.

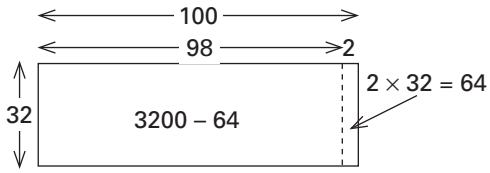
Discuss the groups' findings, and ask them to identify examples of computations for which this property is convenient.

14 If one views subtraction as "adding the opposite," then the product $a(b - c)$ can be viewed as $a(b + {}^{-}c)$ and the distributive property for multiplication over addition can be applied. For example:



$a(b - c) = a(b + {}^{-}c)$
$= ab + ({}^{-}ac)$
$= ab - ac$

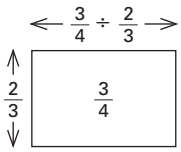
This property can be useful for mentally calculating products such as 32 × 98. Using the distributive property 32 × 98 = 32 (100 – 2) = 3200 – 64, as shown at the left. Note: students may be interested in discussing ways of mentally computing differences such as 3200 – 64 = 3200 – 60 – 4 = 3140 – 4 = 3136.



Focus Teacher Activity (cont.)

ACTIONS

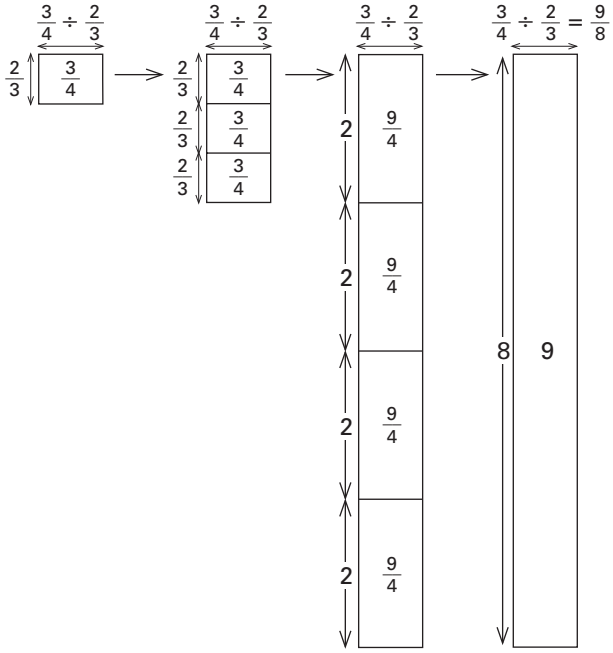
15 Draw the following diagram on the overhead:



Ask the groups to investigate visual strategies that show why ¾ ÷ ⅔ = ⅝. Discuss. Then ask the groups to adapt and/or apply their methods to division problems involving other common fractions. Have them determine which methods also work for decimal fractions, mixed numbers, and algebraic fractions. Invite volunteers to pose problems for the class to solve.

COMMENTS

15 It may be helpful to recall the various meanings of division discussed during the Connector activity. If not suggested by students, you might ask the groups to investigate ways to use "rectangle maneuvers" to compute ¾ ÷ ⅔. In the example shown below, notice the top dimension, ¾ ÷ ⅔ is not changed.



16 Tell the students that there is an old saying about division of fractions:

> **Yours is not to reason why, just invert and multiply!**

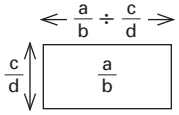
Suggest that you would like to rephrase this adage to read as follows,

> **Ours is to give reasons why we just invert and multiply!**

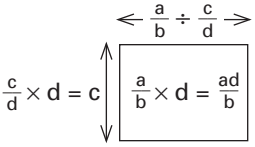
Have the groups report their conclusions and reasoning.

16 Students may be surprised to hear that this rhyme is how many teachers and parents learned division of fractions. Depending on lesson timing, you might pose this as a homework assignment, adding the requirement that they demonstrate "the reason why" to an adult and have the adult write "I appreciate…" and "I wish…" statements regarding the demonstration.

Here is one student's explanation using rectangle maneuvers such as those shown in Comment 15 and using several properties established in Action 2.

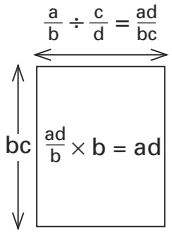


→



Multiply the side dimension $\frac{c}{d}$ by d. This multiplies the area $\frac{a}{b}$ by d.

→



Multiply the side dimension c and the area $\frac{ad}{b}$ by b.

Since the area is now ad and the side dimension is bc, the top dimension (which hasn't changed) is $\frac{ad}{bc}$. So, an algorithm for dividing fractions is: $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$. This is equivalent to computing $\frac{a}{b} \times \frac{d}{c}$. So, to divide $\frac{a}{b}$ by $\frac{c}{d}$, multiply $\frac{a}{b}$ by $\frac{d}{c}$. Notice that $\frac{d}{c}$ is $\frac{c}{d}$ inverted!

Focus Teacher Activity (cont.)

ACTIONS

17 If students haven't suggested a general algorithm, ask the groups to invent algorithms for solving $a/b \div c/d$ (b, c, $d \neq 0$), reflecting their strategies from Actions 15 and 16. Discuss.

18 Distribute calculators if the students do not have them. Ask the groups to discuss possibilities for using their calculators to compute fraction sums, differences, products, and quotients. Then discuss as a large group. Finally, place a transparency of Focus Master A (see Action 2) on the overhead and ask the groups to use their calculators to test examples of each property for fractions and decimals.

19 Write 397 + 268 on the overhead for students to compute mentally. Discuss their strategies. Use this as a context for discussing the *equal sums* strategy.

20 Place a transparency of Focus Master D on the overhead revealing only the equal sums strategy. Demonstrate the use of notecards for a "proof without words" that shows how and why the computation strategy *equal sums* works. Discuss the students' ideas regarding your thinking and reasoning. Then ask each group to list 5 computations for which they think the equal sums strategy would be particularly convenient, including at least one example involving whole numbers, one involving integers, one with fractions, one with decimals, and one with algebraic expressions.

Lesson **7** Properties, Operations, and Algorithms

Focus Master D

For real numbers a, b, and c.

Equal sums: $a + b = (a - c) + (b + c)$

COMMENTS

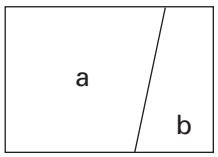
17 This activity provides a setting for reinforcing the meaning of division and for discussing the roles and uses of algorithms and the invention of algorithms. See Comments 15 and 16 for ideas that may come up here.

18 For example, to use the TI-83 to compute $3/4 + 2/3$, enter $3 \div 4 + 2 \div 3$, then activate the "frac" function from the MATH menu, and the calculator provides the answer in fraction form. Another method is to change fractions to their decimal form and then compute.

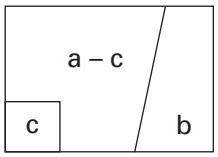
19 Several students may use the equal sums strategy to compute 397 + 268. Students explored this strategy in *Math Alive! Courses I* and *II*. One *equal sums* strategy is to compute 397 + 268 = (397 + 3) + (268 – 3) = 400 + 265 = 665. In general, this strategy involves forming a sum that is equal to a given sum by subtracting an amount from one addend and adding the amount subtracted to the other addend. This is a particularly useful strategy for computing sums mentally.

20 Following is an example of a "proof without words" that shows how and why the equal sums strategy works. The comments to the left of the diagrams provide a series of actions that you can carry out silently while the students observe.

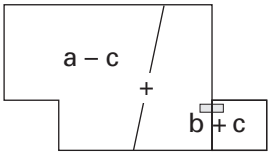
Step 1: Draw a line to subdivide a card into 2 different regions. Label the regions to show one region has area a, the other has area b, and the total area of the card is a + b.



Step 2: Draw and label a third region with area c so the 3 regions formed now have areas c, a – c, and b.



Step 3: Cut out region c. Translate and tape region c to region b. Since the total area of the card has not changed the original area, a + b = (a – c) + (b + c).



Focus Teacher Activity (cont.)

ACTIONS

21 Give each group of students 12 notecards, scissors, and tape. Reveal the remainder of Focus Master D at the overhead. Ask the groups to use the notecards to invent "proofs without words" that demonstrate how and why the *equal differences*, *equal products*, and *equal quotients* strategies work. For each strategy, ask the groups to provide 5 computations for which that strategy would be particularly convenient, with one example involving whole numbers, one for integers, one for fractions, one for decimals, and one for algebraic expressions. Invite volunteers to share their proofs and examples with the class. Discuss.

Lesson **7** Properties, Operations, and Algorithms

Focus Master D

For real numbers a, b, and c.

Equal sums: $a + b = (a - c) + (b + c)$

Equal differences: $a - b = (a + c) - (b + c)$, and
$a - b = (a - c) - (b - c)$

Equal products: $a \times b = (a \times n) \times (b \div n)$
That is, $ab = (an)(\frac{b}{n})$

Equal quotients: $a \div b = (a \times n) \div (b \times n)$, and
$a \div b = (a \div n) \div (b \div n)$
That is, $\frac{a}{b} = \frac{an}{bn}$, and $\frac{a}{b} = \frac{a/n}{b/n}$

COMMENTS

Here are several addition problems for which the equal sums strategy is useful:

$37 + 53 = (37 + 3) + (53 - 3) = 40 + 50$;

$38\frac{1}{4} + 29\frac{3}{4} = (38\frac{1}{4} - \frac{1}{4}) + (29\frac{3}{4} + \frac{1}{4}) = 38 + 30$;

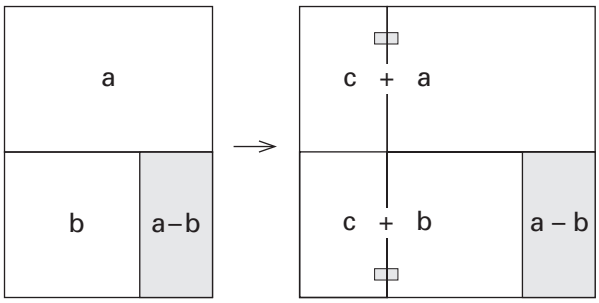
$151.2 + 23.8 = (151.2 + .8) + (23.8 - .8) = 152 + 23$;

$-39 + 41 = (-39 - 1) + (41 + 1) = -40 + 42$;

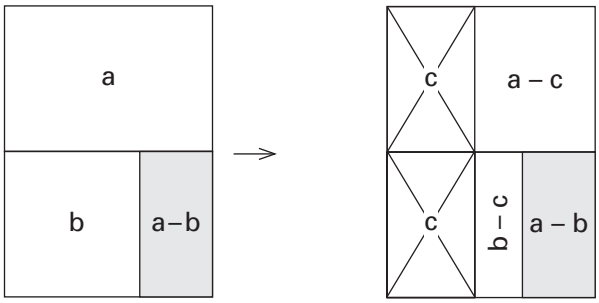
$72x + 13 + 38x + 57 = (72x + 8x) + (13 - 3) + (38x - 8x) + (57 + 3) = 80x + 10 + 30x + 60 = 110x + 70$.

21 Groups may need additional notecards, so it is helpful to have extras on hand. The following series of diagrams illustrates *equal differences*, $a - b = (a + c) - (b + c)$ and $a - b = (a - c) - (b - c)$:

This shows that a – b = (a + c) – (b + c):



This shows that a – b = (a – c) – (b – c):



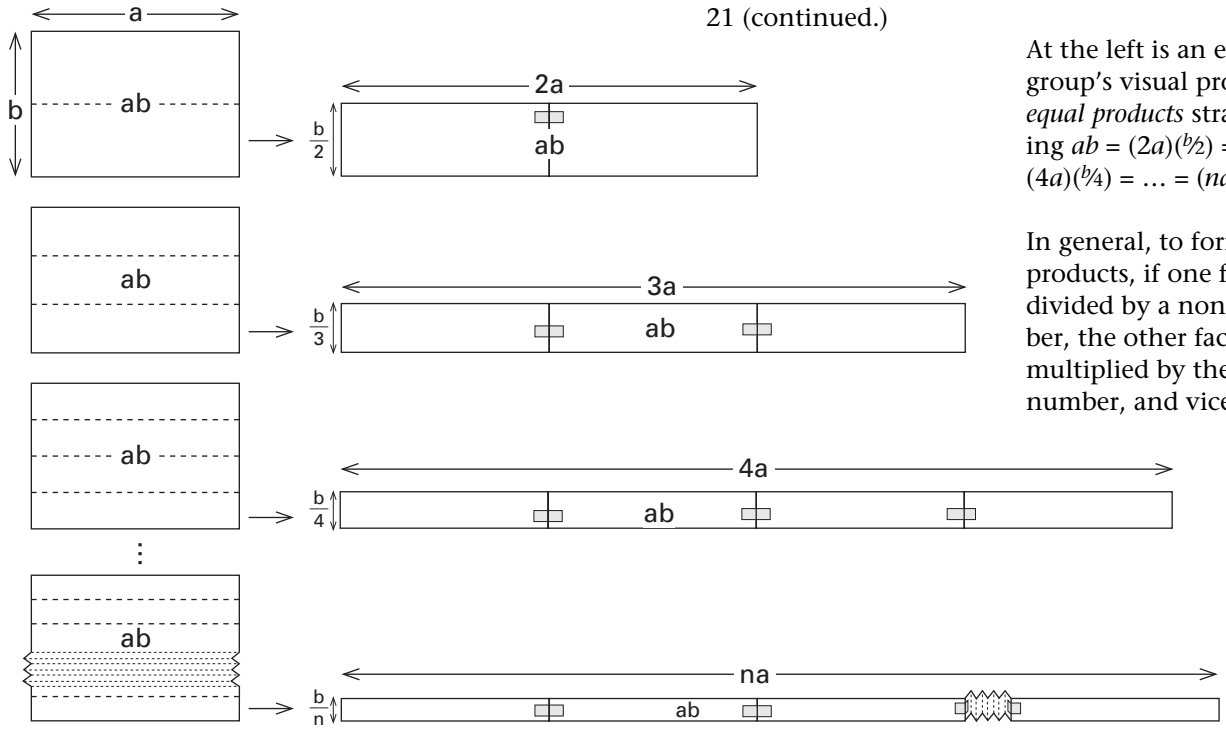
Notice, as long as equal amounts are added to, or subtracted from, the original numbers, the difference is not affected.

You may wish to have the students write descriptions of their thinking and reasoning for each step of their proofs, as shown by the captions for the steps given in Comment 20.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

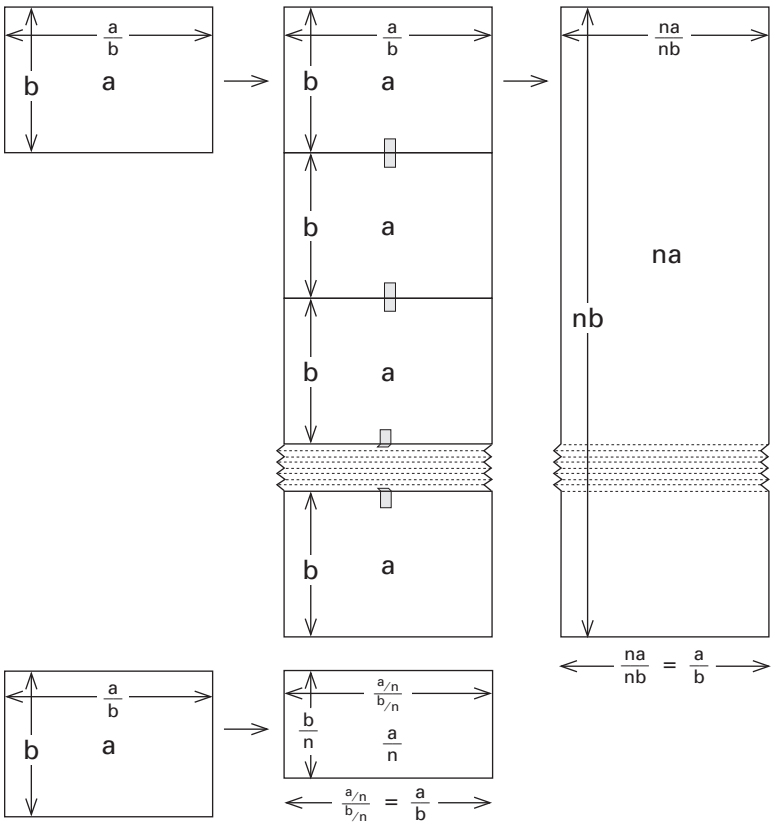


COMMENTS

21 (continued.)

At the left is an example of a group's visual proof of the *equal products* strategy, showing $ab = (2a)(b/2) = (3a)(b/3) = (4a)(b/4) = \ldots = (na)(b/n)$.

In general, to form equal products, if one factor is divided by a non-zero number, the other factor must be multiplied by the same number, and vice versa.



As established in previous actions, multiplying or dividing one dimension of a rectangle by a factor multiplies or divides the area by the same factor, and the other dimension remains unchanged. Hence, *equal quotients* can be formed by multiplying or dividing the divisor and the dividend by the same number. The diagram at the left shows one group's proof of the equal quotients strategy, $\frac{a}{b} = \frac{na}{nb} = \frac{a/n}{b/n}$.

The equal quotients strategy is useful for mentally calculating quotients. For example, one way to compute 312 ÷ 28 is to note that 312 ÷ 28 = 156 ÷ 14 = 78 ÷ 7 ≈ 11. The equal quotients strategy is also useful for approximating quotients and for determining equivalent fractions (see the Connector, Actions 3 and 4).

Note: The equal products and equal quotients strategies shown above illustrate inverse and direct variation, respectively (see Lesson 13 of this course).

Focus Teacher Activity (cont.)

ACTIONS

22 Write statement a) on the overhead and ask the groups to use a diagram and/or a logical argument to illustrate why the statement is *always*, *sometimes*, or *never* true. Ask them to provide examples to support their arguments. Repeat for statements b)-f).

a) To multiply by $1/b$ is equivalent to dividing by b.

b) Given a rectangle with edges a and b, to multiply one edge by c divides the area by c.

c) Given a rectangle with edges a and b, to add c to side a adds bc to the area.

d) If $a = b$ and $b = c$, then $a = c$.

e) If $a > 1$ then $1/a > 1$.

f) If $a/b = c/d$, then $ad = bc$ (for $b, d \neq 0$).

23 Tell the students that, in a *proportion* $a/b = c/d$, b and c are called the *means* of the proportion, and a and d are called the *extremes*, and the property that is illustrated by statement f) from Action 22 is sometimes called the *means-extremes property*. This property, i.e., if $a/b = c/d$, then $ad = bc$, can be used as an algorithm for solving proportions. Ask the students to determine how the means-extremes property can be used to help find the value of x in each of the following proportions.

a) $3/5 = x/25$

b) $4/7 = 32/x$

c) $x/2 = 8/x$

d) $3/x = 12/8$

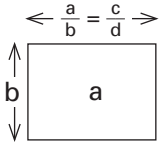
e) $x/8 = 9/12$

f) $a/x = b/c$

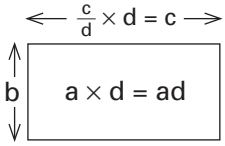
COMMENTS

22 Based on the needs of your students you might add other statements to this list. For example, if there is a mathematical relationship students have had difficulty grasping, you might write a statement based on that relationship.

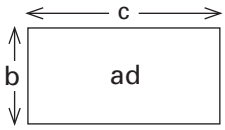
a) True for all real numbers b except 0.
b) True only for $c = 1$.
c) True for all real numbers a, b, and c.
d) True for all real numbers. This is called the *transitive property of equality*.
e) False. If $a > 1$, then $1/a < 1$.
f) True. Here is one group's visual proof and explanation that, if $a/b = c/d$, then $ad = bc$.



Start with a rectangle with area a and 1 dimension b. We are given that $a/b = c/d$, so the top dimension is $a/b = c/d$.



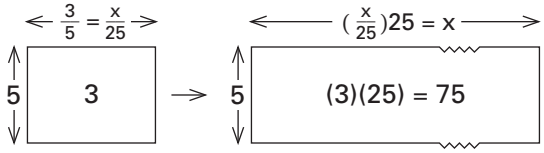
Multiply the top dimension and the area by d. The top dimension is now $c/d \times d = c$ and the area is ad.



The area of the rectangle is the product of its dimensions, or $b \times c = bc$. But the area is also ad. So, $ad = bc$.

23 Some students may use a visual procedure such as that shown in Comment 22. Others may use the results. Either is appropriate, as long as students understand why the property is true.

a) If $3/5 = x/25$, then:



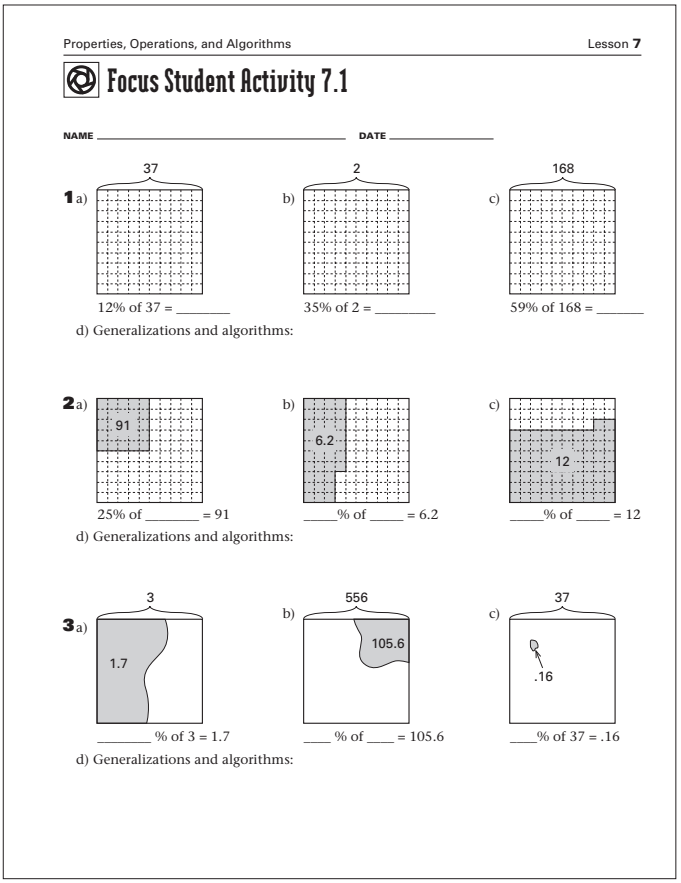
So, since the area is $75 = 5x$, then $15 = x$.

b) $4x = 224$, so $x = 56$
c) $x^2 = 16$, so $x = \pm 4$
d) $12x = 24$, so $x = 2$
e) $12x = 72$, so $x = 6$
f) $bx = ac$, so $x = ac/b$

Focus Teacher Activity (cont.)

ACTIONS

24 Give each student a copy of Focus Student Activity 7.1. Point out that the 37 at the top of diagram 1a) means the value of the whole square is 37. Ask the students to shade 12% of the square and to *reason from the diagram* to determine the value of the shaded part. Invite volunteers to demonstrate their methods on a transparency of the activity. Repeat by having them shade and determine the value of 35% of b) and 59% of c). Then ask the groups to discuss generalizations and algorithms regarding percent problems like those in 1a)-1c). Discuss.


Properties, Operations, and Algorithms Lesson **7**

Focus Student Activity 7.1

NAME _____ DATE _____

1 a) 37 — 12% of 37 = _____
b) 2 — 35% of 2 = _____
c) 168 — 59% of 168 = _____
d) Generalizations and algorithms:

2 a) 91 — 25% of _____ = 91
b) 6.2 — _____% of _____ = 6.2
c) 12 — _____% of _____ = 12
d) Generalizations and algorithms:

3 a) 3, 1.7 — _____ % of 3 = 1.7
b) 556, 105.6 — ____ % of ____ = 105.6
c) 37, .16 — ____% of 37 = .16
d) Generalizations and algorithms:

25 Repeat Action 24 for Problems 2 and 3 on Focus Student Activity 7.1.

COMMENTS

24 Calculators are appropriate here; the intent is for students to use relationships they can "see" in the diagrams to determine what to compute.

You might keep a supply of Percent Grids (see *Blackline Masters*) and a transparency of the grids on hand for students to use in testing their generalizations and algorithms.

Students may approach these percentage problems in a variety of ways. Although students are asked to generalize and invent algorithms, it is important not to suggest that the "bottom line" or best solution is to use symbolic procedures. What is important is that students have access to a number of strategies and choose the one that makes the most sense for a given context. Algorithms may be visual or symbolic.

1a) One way to solve this problem is to determine the value of 1 small square from the grid, which is 37 ÷ 100 = .37, and then multiply by 12 to determine the value of 12 small squares, 12 × .37 = 4.44. Other students may note that determining the value of 12% of 37 is equivalent to solving the proportion $^{12}/_{100} = {}^{x}/_{37}$. Hence, one way to solve this problem is to apply the means extremes property and determine the value of x for which $12 \times 37 = 100x$. Thus, $x = \frac{(12 \times 37)}{100}$.
1b) .7
1c) 99.12

25 If students have difficulties with these problems or need more conceptual work in percents, background on percents is in *Math Alive! Course I*, Lesson 39, and *Math Alive! Course II*, Lessons 21-23.

2a) One way to solve this problem is to determine the value of 1 small square from the grid, which is 91 ÷ 25 = 3.64, and then multiply this value by 100 to obtain 364, the value of the whole square. Another approach is to use proportions, $^{25}/_{100} = {}^{91}/_{x}$, and using the means and extremes property, $25x = 9100$, so $x = 364$.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

2b) ≈ 16.76
2c) $16\frac{2}{3}$

3a) One approach is to divide 3 by 100 to obtain .03 as the value of 1 small square, and then divide 1.7 by .03 to determine the number of small squares (the percent) corresponding to 1.7. Since 1.7 ÷ .03 = $56\frac{2}{3}$, $56\frac{2}{3}$% of 3 is 1.7.
3b) ≈ 18.99%
3c) .43%

26 Place a transparency of Focus Master E on the overhead. Give the groups their specific assignments. When problems and solution keys have been created, ask the groups to exchange and solve each other's problems.

26 Rather than having each group create and solve a problem for every condition listed, you might assign selected conditions to each group. You could have all groups solve all other groups' problems, or you could have each group exchange with only 2 or 3 other groups. Or, you could photocopy the groups' problems and distribute a full set to each student for completion as homework and discussion in their groups.

Properties, Operations, and Algorithms Lesson **7**

Focus Master E

For each of the following conditions, create an interesting and challenging math problem based on a situation from everyday life outside of school. You could write a separate problem for each condition listed, or you could combine more than one condition in a problem. On a separate sheet, show how to solve your problems.

1. Fractions
   a) addition of 3 mixed numbers with unlike denominators
   b) subtraction of 2 mixed numbers with unlike denominators
   c) multiplication of 2 mixed numbers with unlike denominators
   d) division of 2 fractions with unlike denominators

2. Decimals
   a) addition, the sum is greater than 150.713
   b) subtraction, the difference is less than .001
   c) multiplication, the product is a mixed decimal
   d) division, 14.452 is a multiple of the quotient

3. Integers
   a) addition, the sum is negative
   b) subtraction, the difference is greater than $^{-}50$
   c) multiplication, the product is a negative square number
   d) division, the quotient is less than $^{-}250$

4. Percents, *r*% of *s* equals *t*
   a) *r* is unknown
   b) *s* is unknown
   c) *t* is unknown

(Continued on back.)

Lesson **7** Properties, Operations, and Algorithms

Focus Master E (cont.)

5. Proportions, $m/n = p/q$
   a) *m* is unknown; *n*, *p*, and *q* are positive integers
   b) *n* is unknown; *m*, *p*, and *q* are positive integers

6. Algebraic expressions
   a) addition
   b) subtraction
   c) multiplication
   d) division

TEACHER NOTES:

Follow-up Student Activity 7.2

NAME _____ **DATE** _____

1 Determine if each equation below is true, assuming a, b, and c are integers such that denominators are not equal to zero. If an equation is true create a visual proof, and if it is false show a counter example.

a) $\frac{a-b}{a-b} = 1$

b) $\frac{a+b}{a} = 1 + b$

c) $\frac{a}{1/2} = 2a$

d) $a + (b \times c) = (a + b) \times (a + c)$

e) $\frac{a}{a+b} = \frac{1}{b}$

f) $\frac{a^2 - ab}{a} = a - b$

g) $a \div (b + c) = (a \div b) + (a \div c)$

h) $\frac{a}{b} = \frac{ac}{bc}$

i) $\frac{-abc}{c} = -ab$

j) $\frac{a}{1/b} = \frac{a}{b}$

2 Show how to solve each of the following computations, using whatever visual or symbolic method you wish. Be sure to show the steps of your thinking.

a) $\frac{3}{7} \times \frac{2}{5}$

b) $\frac{-1}{3} + \frac{5}{8}$

c) $\frac{3}{4} \div \frac{2}{9}$

d) $\frac{3}{8} - \frac{5}{6}$

e) $.2 + .43$

f) $.7 \times 32$

g) $.8 \div .15$

h) $.46 - .008$

i) $\frac{3x}{5} - \frac{2x}{3}$

j) $14\frac{5}{12} \div 2\frac{1}{3}$

k) $\frac{x+3}{2} + \frac{2x-1}{5}$

l) $\frac{6}{x} + \frac{1}{3x}$

m) $7.23 - 4.8$

n) $\frac{x}{3} \div \frac{2}{7}$

o) $3\frac{1}{8} - 1\frac{4}{7}$

3 Show one or more visual methods for computing each of the following, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. For each visual method, write a symbolic algorithm (using only numbers and math symbols) that is based on the method.

a) $\frac{a}{b} + \frac{c}{d}$ b) $\frac{a}{b} - \frac{c}{d}$ c) $\frac{a}{b} \times \frac{c}{d}$ d) $\frac{a}{b} \div \frac{c}{d}$

(Continued on back.)

Follow-up Student Activity (cont.)

4 Show how to solve each of the following computations using the equal sums, equal differences, equal products, or equal quotients strategy. Use numbers and math symbols to record your thought processes.

a) $17\frac{8}{9} + 6\frac{4}{9}$ c) $36.98 - 4.28$ e) 12×1155 g) $288 \div 60$

b) $14\frac{1}{9} - 8\frac{5}{9}$ d) $9.85 + 14.9$ f) $16 \div \frac{7}{3}$ h) $2\frac{2}{3} \times 27$

5 Show how to reason from diagrams to determine the missing value for each of the following percent problems. Next to each diagram show the steps in your reasoning and all of your calculations.

a) _____% of 350 = 63 b) 84% of _____ = 504 c) 28% of 92.5 = _____

6 An excerpt from Linden's journal is shown below. Write what you think she said in place of "…" and give examples to illustrate each statement.

I notice that the following aspects of each operation never change, regardless the type of number or variable that I use:

addition… subtraction… multiplication… division…

7 Suppose that a and b are integers. Tell whether each of the following is *always*, *sometimes*, or *never* true. If a statement is always true or never true, explain why. If a statement is sometimes true, explain the conditions necessary for it to be true. Give examples to support your reasoning.

$a + b$ is greater than both a and b

$a - b$ is greater than both a and b

$a \times b$ is greater than both a and b

$a \div b$ is greater than both a and b

8 Repeat Problem 7, but suppose that a and b are fractions between 0 and 1.

Connector Master A

Complete the following thought starter. If you think about a term from the list in more than one way, be sure to include all of your ideas.

Here is how I think about the meaning of each term below, together with an example or examples to illustrate my understanding of the term and ways it is related to other terms in the list…

a) addition
b) subtraction
c) multiplication
d) division
e) whole number
f) fraction
g) decimal
h) integer
i) percent

Focus Master A

Assume that a, b, and c are integers such that denominators are not zero.

a) $a\left(\frac{c}{a}\right) = c$

b) $\left(\frac{1}{b}\right)b = 1$

c) $ab = ba$

d) $\frac{ab}{a} = b$

e) $\frac{b}{b} = 1$

f) $b\left(\frac{a}{b}\right) = a$

g) $\frac{a}{a/b} = b$

h) $\frac{b}{b/a} = a$

i) $b\left(\frac{a}{c}\right) = \frac{ba}{c}$

j) $a(b + c) = ab + ac$

k) $\frac{a+b}{a} = 1 + \frac{b}{a}$

l) $a(bc) = ab(c)$

m) $\frac{1}{1/a} = a$

n) $\frac{a^2 + ab}{a} = a + b$

o) $\frac{a}{b} = \frac{ac}{bc}$

p) $\frac{abc}{b} = ac$

q) $\frac{abc}{c} = ab$

r) $\frac{abc}{bc} = a$

s) $\frac{1}{b/c} = \frac{c}{b}$

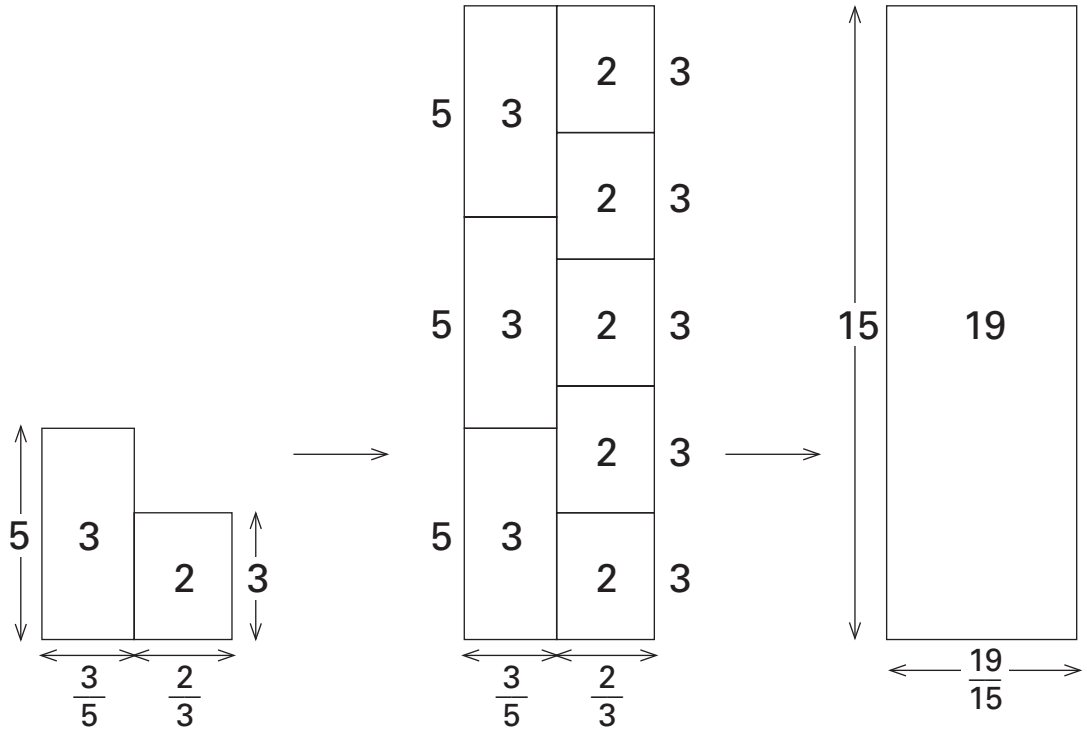
t) $\frac{a+b}{a+b} = 1$

u) $\frac{-ab}{b} = -a$

Focus Master B

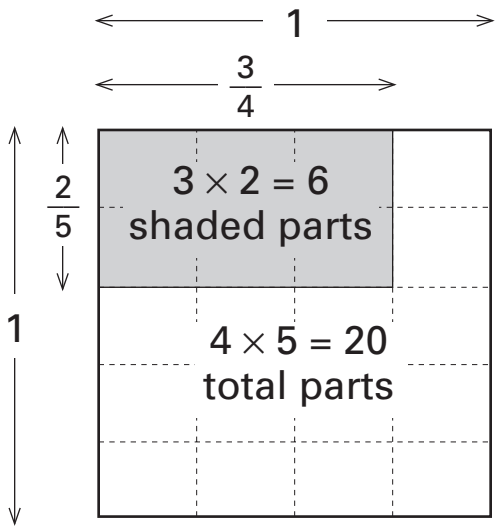
Julie's Method

$$\frac{3}{5} + \frac{2}{3}$$



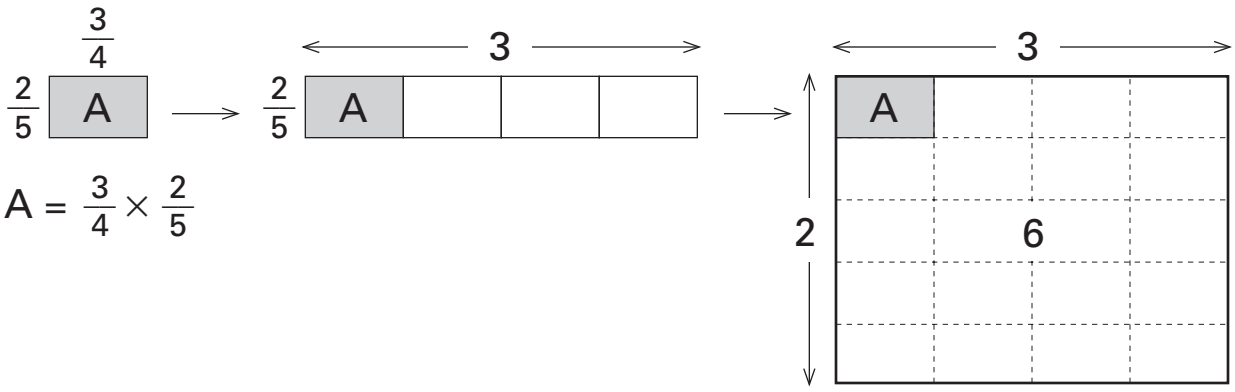
Focus Master C

Linden's Method: $\frac{3}{4} \times \frac{2}{5}$



Therefore, $\frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.

Erica's Method: $\frac{3}{4} \times \frac{2}{5}$



$A = \frac{3}{4} \times \frac{2}{5}$

So, $A = \frac{3}{4} \times \frac{2}{5} = (3 \times 2) \div (4 \times 5) = \frac{3 \times 2}{4 \times 5} = \frac{6}{20}$.

Focus Master D

For real numbers a, b, and c.

Equal sums: $a + b = (a - c) + (b + c)$

Equal differences: $a - b = (a + c) - (b + c)$, and
$a - b = (a - c) - (b - c)$

Equal products: $a \times b = (a \times n) \times (b \div n)$
That is, $ab = (an)(\frac{b}{n})$

Equal quotients: $a \div b = (a \times n) \div (b \times n)$, and
$a \div b = (a \div n) \div (b \div n)$
That is, $\frac{a}{b} = \frac{an}{bn}$, and $\frac{a}{b} = \frac{a/n}{b/n}$

Focus Master E

For each of the following conditions, create an interesting and challenging math problem based on a situation from everyday life outside of school. You could write a separate problem for each condition listed, or you could combine more than one condition in a problem. On a separate sheet, show how to solve your problems.

1. Fractions
   a) addition of 3 mixed numbers with unlike denominators
   b) subtraction of 2 mixed numbers with unlike denominators
   c) multiplication of 2 mixed numbers with unlike denominators
   d) division of 2 fractions with unlike denominators

2. Decimals
   a) addition, the sum is greater than 150.713
   b) subtraction, the difference is less than .001
   c) multiplication, the product is a mixed decimal
   d) division, 14.452 is a multiple of the quotient

3. Integers
   a) addition, the sum is negative
   b) subtraction, the difference is greater than $^{-}50$
   c) multiplication, the product is a negative square number
   d) division, the quotient is less than $^{-}250$

4. Percents, r% of s equals t
   a) r is unknown
   b) s is unknown
   c) t is unknown

(Continued on back.)

Focus Master E (cont.)

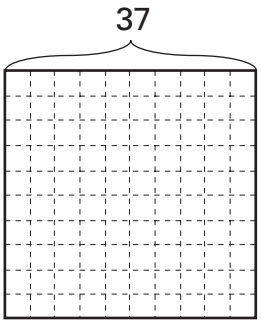
5. Proportions, $m/n = p/q$
   a) m is unknown; n, p, and q are positive integers
   b) n is unknown; m, p, and q are positive integers

6. Algebraic expressions
   a) addition
   b) subtraction
   c) multiplication
   d) division

Focus Student Activity 7.1

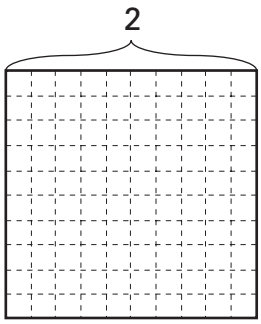
NAME _____ **DATE** _____

1 a)



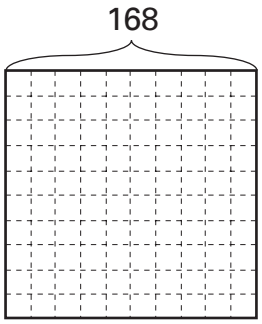
12% of 37 = _____

b)



35% of 2 = _____

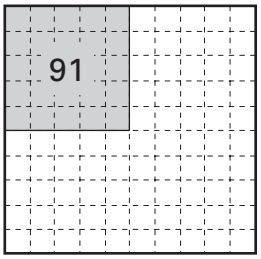
c)



59% of 168 = _____

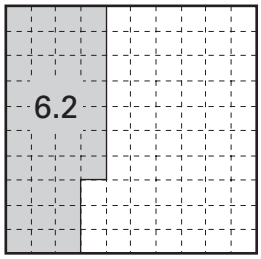
d) Generalizations and algorithms:

2 a)



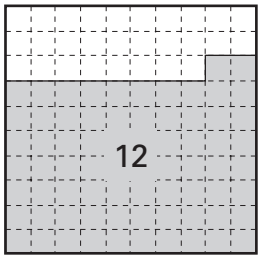
25% of _____ = 91

b)



_____% of _____ = 6.2

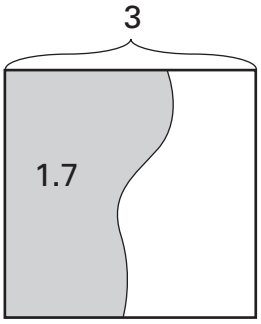
c)



_____% of _____ = 12

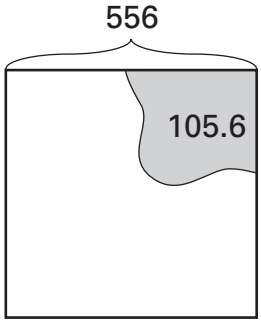
d) Generalizations and algorithms:

3 a)



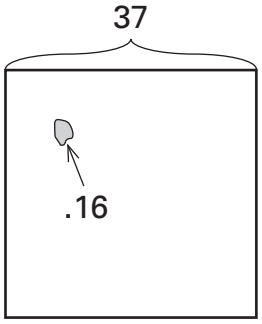
_____ % of 3 = 1.7

b)



____ % of ____ = 105.6

c)



____% of 37 = .16

d) Generalizations and algorithms:

Follow-up Student Activity 7.2

NAME _____ **DATE** _____

1 Determine if each equation below is true, assuming a, b, and c are integers such that denominators are not equal to zero. If an equation is true create a visual proof, and if it is false show a counter example.

a) $\frac{a-b}{a-b} = 1$

b) $\frac{a+b}{a} = 1 + b$

c) $\frac{a}{1/2} = 2a$

d) $a + (b \times c) = (a + b) \times (a + c)$

e) $\frac{a}{a+b} = \frac{1}{b}$

f) $\frac{a^2 - ab}{a} = a - b$

g) $a \div (b + c) = (a \div b) + (a \div c)$

h) $\frac{a}{b} = \frac{ac}{bc}$

i) $\frac{-abc}{c} = -ab$

j) $\frac{a}{1/b} = \frac{a}{b}$

2 Show how to solve each of the following computations, using whatever visual or symbolic method you wish. Be sure to show the steps of your thinking.

a) $\frac{3}{7} \times \frac{2}{5}$

b) $\frac{-1}{3} + \frac{5}{8}$

c) $\frac{3}{4} \div \frac{2}{9}$

d) $\frac{3}{8} - \frac{5}{6}$

e) $.2 + .43$

f) $.7 \times 32$

g) $.8 \div .15$

h) $.46 - .008$

i) $\frac{3x}{5} - \frac{2x}{3}$

j) $14\frac{5}{12} \div 2\frac{1}{3}$

k) $\frac{x+3}{2} + \frac{2x-1}{5}$

l) $\frac{6}{x} + \frac{1}{3x}$

m) $7.23 - 4.8$

n) $\frac{x}{3} \div \frac{2}{7}$

o) $3\frac{1}{8} - 1\frac{4}{7}$

3 Show one or more visual methods for computing each of the following, where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$. For each visual method, write a symbolic algorithm (using only numbers and math symbols) that is based on the method.

a) $\frac{a}{b} + \frac{c}{d}$ b) $\frac{a}{b} - \frac{c}{d}$ c) $\frac{a}{b} \times \frac{c}{d}$ d) $\frac{a}{b} \div \frac{c}{d}$

(Continued on back.)

Follow-up Student Activity (cont.)

4 Show how to solve each of the following computations using the equal sums, equal differences, equal products, or equal quotients strategy. Use numbers and math symbols to record your thought processes.

a) $17\frac{8}{9} + 6\frac{4}{9}$ c) $36.98 - 4.28$ e) 12×1155 g) $288 \div 60$

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5 Show how to reason from diagrams to determine the missing value for each of the following percent problems. Next to each diagram show the steps in your reasoning and all of your calculations.

a) _____% of 350 = 63 b) 84% of _____ = 504 c) 28% of 92.5 = _____

6 An excerpt from Linden's journal is shown below. Write what you think she said in place of "…" and give examples to illustrate each statement.

I notice that the following aspects of each operation never change, regardless the type of number or variable that I use:

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$a + b$ is greater than both a and b

$a - b$ is greater than both a and b

$a \times b$ is greater than both a and b

$a \div b$ is greater than both a and b

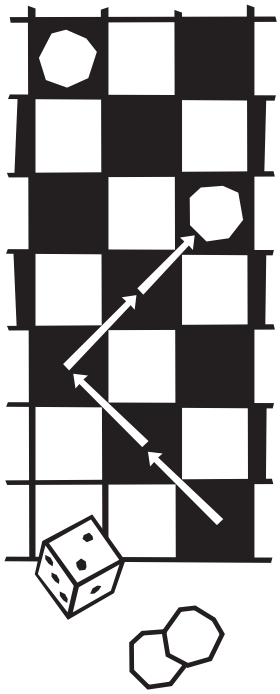
8 Repeat Problem 7, but suppose that a and b are fractions between 0 and 1.

Experimental & Theoretical Probability

Lesson **8**

THE BIG IDEA

Probability is useful for making decisions about a situation whose outcomes are not immediately obvious. Carefully organized experimentation with the situation, or a simulation of the situation, can provide powerful evidence regarding possible outcomes and can reveal insights about the theoretical probabilities of the situation.



CONNECTOR

OVERVIEW

Students conduct binomial experiments with equally likely outcomes and with unequally likely outcomes, and compare the experimental and theoretical probabilities of such outcomes.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Student Activity 8.1 (2 pages), 1 copy per group and 1 transparency.
- ✔ Connector Master A, 2 copies per student plus 1 copy per group and 1 transparency.
- ✔ Connector Masters B, C, and D (optional), 1 copy of each per group and 1 transparency of each.
- ✔ Cubical dice, 1 per student and 2 for use at the overhead.
- ✔ Game markers, 1 per student and 1 for use at the overhead.

FOCUS

OVERVIEW

Students use simulations to gather data and determine experimental probabilities for different strategies of playing a game called Monty's Dilemma. The process of carrying out these simulations provides insights about the theoretical probabilities of the various strategies.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activity 8.2, 1 copy per pair of students and 1 transparency.
- ✔ Focus Masters A, B, and D, 1 transparency of each.
- ✔ Focus Master C, 1 copy per pair of students and 1 transparency.
- ✔ Bobby pins or paper clips, 1 per pair of students.
- ✔ Coins, 1 per pair of students.

FOLLOW-UP

OVERVIEW

Students determine experimental probabilities of equally likely and unequally likely events. Simulations are used to determine the probabilities of strategies for several games.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 8.3, 1 copy per student.

LESSON IDEAS

QUOTE

Probability, the measure of the likelihood of an event, can be determined theoretically or experimentally. Students in the middle grades must actively participate in experiments with probability so that they develop an understanding of the relationship between the numerical expression of a probability and the events that give rise to these numbers (e.g., $\frac{2}{5}$ as it relates to the probability of choosing a red marble from a hat). Students must not only understand the relationship between the numerical expression and the probability of the events but realize that the measure of certainty or uncertainty varies as more data are collected.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

2. P(A) = $\frac{1}{27}$, P(B) = $\frac{6}{27}$, P(C) = $\frac{12}{27}$, P(D) = $\frac{8}{27}$

3.d) *Stick Strategy probability* is $\frac{1}{4}$, as there is an equal chance that the prize is behind any 1 of the 4 doors.

Flip Strategy probability is $\frac{1}{2}$, as there is an equal chance that the prize is behind 1 of the 2 unopened doors.

Switch Strategy probability is $\frac{3}{4}$, as the only way the contestant can lose with this strategy is if the initial selection was the door with the prize, and this probability is $\frac{1}{4}$. Or, once the contestant has selected a door, the probability the prize is behind 1 of the 3 remaining doors is $\frac{3}{4}$. After 2 of the doors without the prize are opened, the probability that the prize is behind the remaining door is $\frac{3}{4}$.

e) The probabilities for the extended versions of Monty's game:

5 doors: P(Stick) = $\frac{1}{5}$; P (Flip) = $\frac{1}{2}$; P(Switch) = $\frac{4}{5}$
6 doors: P(Stick) = $\frac{1}{6}$; P(Flip) = $\frac{1}{2}$; P(Switch) = $\frac{5}{6}$
n doors: P(Stick) = $\frac{1}{n}$; P(Flip) = $\frac{1}{2}$; P(Switch) = $\frac{(n-1)}{n}$

Connector Teacher Activity

OVERVIEW & PURPOSE

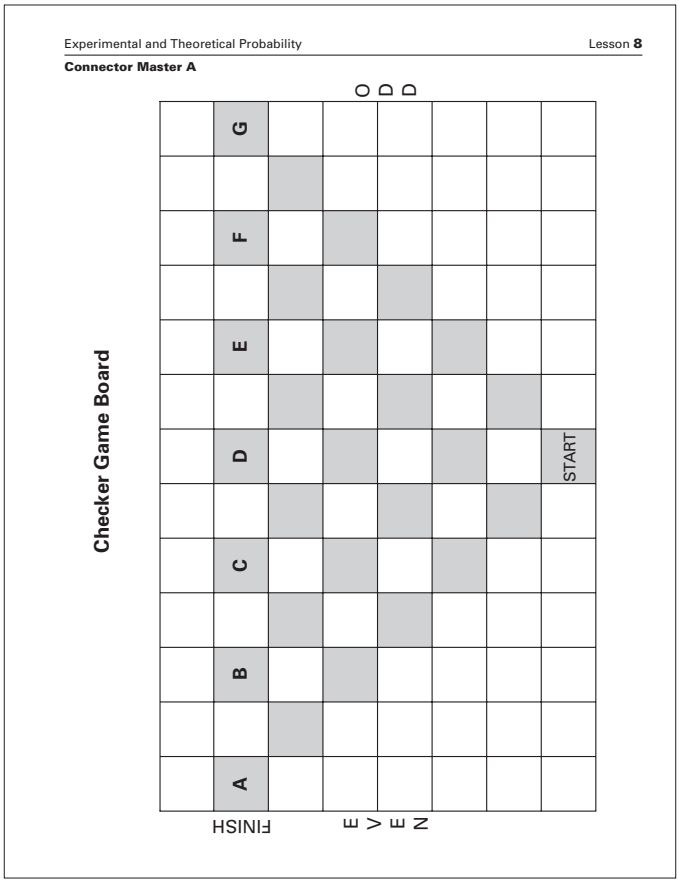
Students conduct binomial experiments with equally likely outcomes and with unequally likely outcomes, and compare the experimental and theoretical probabilities of such outcomes.

MATERIALS

- ✔ Connector Student Activity 8.1 (2 pages), 1 copy per group and 1 transparency.
- ✔ Connector Master A, 2 copies per student plus 1 copy per group and 1 transparency.
- ✔ Connector Masters B, C, and D (optional), 1 copy of each per group and 1 transparency of each.
- ✔ Cubical dice, 1 per student and 2 for use at the overhead.
- ✔ Game markers, 1 per student and 1 for use at the overhead.

ACTIONS

1 Arrange the students in groups and distribute a standard cubical die, a game marker, and a copy of Connector Master A to each student. Using a transparency of Connector Master A, illustrate procedures for moving markers in the Checker-A game. Discuss as needed.



COMMENTS

1 Procedures for the Checker-A game are as follows:

Step 1. Place a marker in the START square.

Step 2. Toss a die. If the upturned number is EVEN, the marker is moved 1 space forward diagonally to the LEFT. If the upturned number is ODD, the marker is moved 1 space forward diagonally to the RIGHT. Note: a marker always moves from a shaded square to a shaded square.

Step 3. Repeat until the marker is in 1 of the lettered squares A-G. Note: this requires 6 tosses of the die (see Comment 2).

Since each *trial*, or toss of the die, has only 2 possible outcomes, odd or even, performing these trials is called a *binomial experiment*.

Connector Teacher Activity (cont.)

ACTIONS

2 Ask the students to predict where they think the marker will land after 6 tosses of a die, and why they feel this way. Solicit predictions from the students without judging their responses. Then have the students each play one game. After playing one game, discuss any adjustments students wish to make in their predictions about the position of the marker after 6 tosses.

3 Ask the groups to imagine playing the Checker-A game 50 times and to discuss their intuitions about where the marker might land most often and least often. Out of the 50 games, how many times do they think the marker would be likely to land in square A? square B? C? D? E? F? G?

4 Discuss the students' ideas about the meanings of *experimental probability* and *theoretical probability,* clarifying as needed. Then give each group a copy of Connector Student Activity 8.1 to complete. When groups have completed all of the activity, have them post their bar graphs on the wall.

Discuss similarities and differences among the posted graphs. How do students account for these similarities and differences? What observations can they make about the overall shapes of the graphs? Would they make different predictions about the outcomes of 50 games now? Why or why not?

COMMENTS

2 Students may say it is impossible to tell, or they may pick a favorite square, anywhere on the board, with no other reason for choosing it. Some students may think the marker will land somewhere in the middle of the board. Some may predict the marker will land in the top row, but think that it is impossible to tell in which square (A-G). Still others may think that the marker will end in the top row and that some squares will be more likely than others (for a variety of reasons, some correct and some not). For example, students may reason that since there are as many even numbers as odd numbers on a die, the marker should land in square D, because it is in the center of the top row of the board. Rather than affirming or correcting the students' predictions, note there will be opportunities throughout the rest of this activity for them to discover errors in their reasoning or predictions.

It is the case that after 6 tosses a marker will always land in one of squares A-G.

3 Some students may suggest calculating the theoretical probability by determining all the possible paths of a marker, which is the objective of Actions 5 and 6. You might suggest that for now students make predictions based on their intuitions, so that later they can compare their predictions to experimental and theoretical probabilities they compute.

4 When the likelihood, or chance, of an event is expressed as a number that is based on the results of an experiment, that number is called the *experimental probability* that the event will occur. The experimental probability of an outcome is a number that tells the frequency of the outcome relative to the frequency of all possible outcomes for a given experiment. For example, if a marker lands in square C a total of 20 times out of 50 games, then the experimental probability of landing in square C is $P(C) = {}^{20}/_{50} = .4 = 40\%$.

The *theoretical probability* of an event occurring is based on what should happen under ideal conditions. The theoretical probabilities for outcomes of the Checker-A game are determined in Action 6. Experimental and theoretical probability were also explored in *Math Alive! Courses I* and *II*.

Connector Teacher Activity (cont.)

ACTIONS

Experimental and Theoretical Probability Lesson **8**

Connector Student Activity 8.1

NAME _____ DATE _____

1 Each group member is to play the Checker-A game several times and keep a record of the number of times the marker lands in each of squares A-G. Your group needs to play a total of 50 games. When everyone has finished, record your data and totals on the following chart:

Players' Names	Ending Square on Game Board							Total Games
	A	B	C	D	E	F	G	
1								
2								
3								
4								
5								
Group Totals								50

Based on your group's experimental results, determine the experimental probability of landing in each of squares A-G on the game board. Express each answer as a fraction, decimal, and percent. Note: P(A) stands for "the probability of landing in square A."

a) P(A) = b) P(B) = c) P(C) =

d) P(D) = e) P(E) = f) P(F) =

g) P(G) =

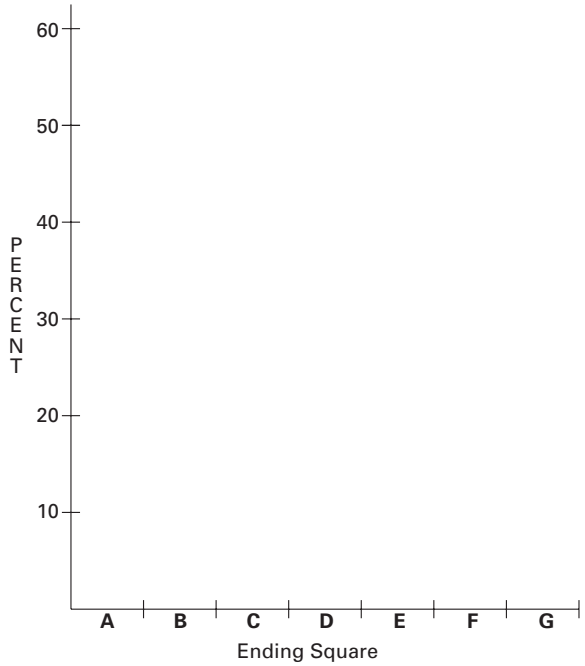
(Continued on back.)

COMMENTS

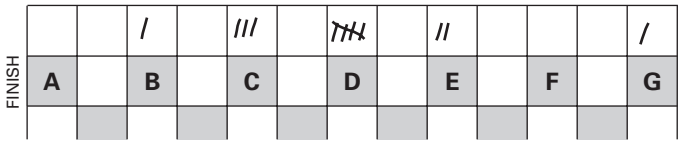
Lesson **8** Experimental and Theoretical Probability

Connector Student Activity 8.1 (cont.)

2 Make a bar graph showing the relative frequency of the marker ending in each column, based on your group's experimental results:



Students could keep track of the ending position of the marker by making tally marks on their game boards, as illustrated below:



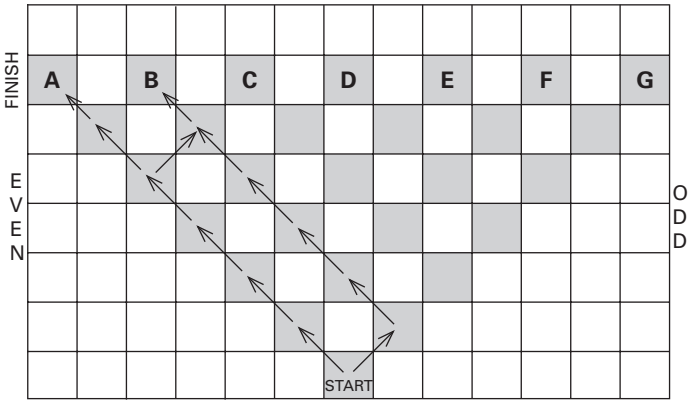
Once the graphs are posted, some students may suggest computing average (mean, median, and/or mode) values for the relative frequencies. They may point out a result that seems to be an *outlier*, i.e., a value that is widely separated from the rest of the data. Some may suggest graphing all of the A's from the class results together, all of the B's together, etc., and then make predictions based on the overall shape of this graph. It is important to encourage such discussion; pursue ideas based on student interest and needs.

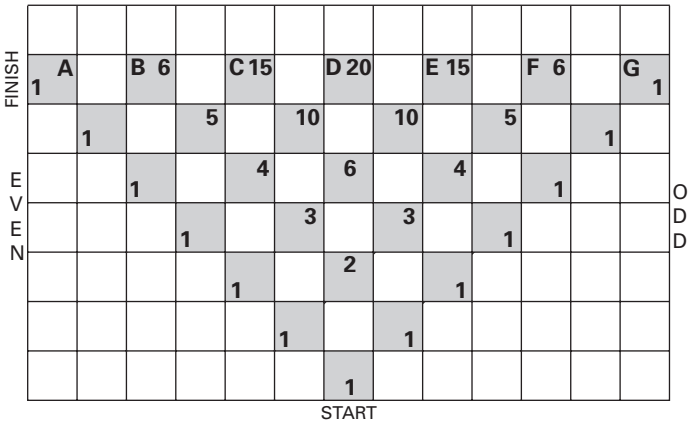
Note that these graphs will be compared to theoretical graphs during Action 7, so it is important not to discard these graphs now.

Connector Teacher Activity (cont.)

ACTIONS

5 Discuss the students' ideas about ways to determine all the Checker-A game paths possible on the game board. Have volunteers trace on the overhead a few paths that a marker could follow from start to finish. Then ask the groups to determine the number of paths that end in each of squares A-G, encouraging them to make note of any patterns they discover. Discuss their observations and results.



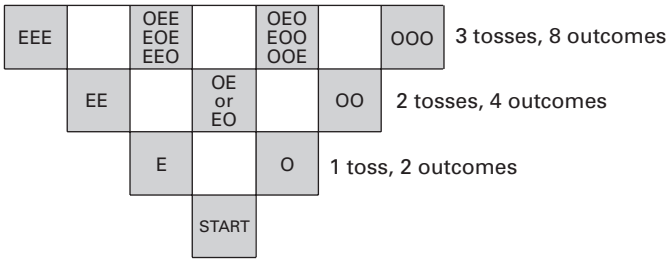


COMMENTS

5 Notice there is only 1 path to get to square A (see diagram at left), namely, the result of 6 even tosses in a row. Two of the 6 possible paths to square B are shown.

There are many patterns that students may notice. For example, they may note symmetry in the board—the 1 path to square A is the mirror image of the 1 path to square G; the number of paths to square B is the same as the number of paths to square F; and so forth. It may be helpful to have available extra copies of Connector Master A (the game board) for tracing and counting paths. Following are several ways that students have recorded and counted paths.

Method 1: *We let E stand for an even toss and O stand for an odd toss, and recorded the sequence of rolls needed to get to each of squares A-G. The following diagram shows that with 1 toss of the die there are 2 possible outcomes, E or O; with 2 tosses there are 4 possible outcomes, EE, OE, EO, or OO; and with 3 tosses there are 8 possible outcomes.*



Continuing in this manner, EOEEOE for example, represents an even-odd-even-even-odd-even sequence of tosses and a path that ends at square C. EEEOEE is 1 of the paths to square B. We found all 64 possible paths!

Method 2: *We kept track of the number of paths that land at each of the shaded squares* (see left). *We notice that the number of ways to reach a given shaded square is the sum of the number of ways to reach the shaded squares that touch the lower corner or corners of the given square. The numbers of paths to A, B, C, D, E, F, and G are 1, 6, 15, 20, 15, 6, and 1, respectively.*

Note: The numbers in the shaded squares in the diagram at the left are the numbers in the first 7 rows of Pascal's triangle which is explored in *Math Alive! Course IV*.

Method 3: *There are 2 possible paths from the START square to the 1st row of squares above START. From each of those paths, there are 2 possible paths to the 2nd row above*

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

START; so there are 2×2 ways to get to that row. From each of those paths, there are 2 possible ways to get to the 3rd row above START; so there are $2 \times (2 \times 2)$ paths to the 3rd row. From each of those paths, there are 2 possible paths to the 4th row above START; so there are $2 \times (2 \times 2 \times 2)$ ways to get to the 4th row. And so on. In general, the number of paths to each row is 2 times the number of paths to the previous row, and there are 2^n paths to the nth row above START. Therefore, the number of paths to the 6th row (the row with the lettered squares) is $2 \times (2 \times 2 \times 2 \times 2 \times 2) = 2^6 = 64$.

6 For each shaded square on the game board, ask the groups to determine the theoretical probability of landing in that square. Discuss their strategies and results.

Checker-A Game Theoretical Probabilities

FINISH	A $\frac{1}{64}$		B $\frac{6}{64}$		C $\frac{15}{64}$		D $\frac{20}{64}$		E $\frac{15}{64}$		F $\frac{6}{64}$		G $\frac{1}{64}$
		$\frac{1}{32}$		$\frac{5}{32}$		$\frac{10}{32}$		$\frac{10}{32}$		$\frac{5}{32}$		$\frac{1}{32}$	
			$\frac{1}{16}$		$\frac{4}{16}$		$\frac{6}{16}$		$\frac{4}{16}$		$\frac{1}{16}$		
EVEN				$\frac{1}{8}$		$\frac{3}{8}$		$\frac{3}{8}$		$\frac{1}{8}$			ODD
					$\frac{1}{4}$		$\frac{2}{4}$		$\frac{1}{4}$				
						$\frac{1}{2}$		$\frac{1}{2}$					
							1 START						

6 The theoretical probability of landing in a particular shaded square is the number of paths to that square divided by the total number of paths to the shaded squares in that row. Students may express probabilities as fractions, decimals, or percentages. In the following discussions, all probabilities are expressed as fractions.

The probability of landing in the START square is 1 since it is certain that the marker must begin there. Note: if it is impossible that an event can occur (e.g., landing in an unshaded square), the probability of that event is 0. All other probabilities vary from 0 (impossible) to 1 (certain).

Since there is 1 path to each square in the 1st row above START, and there are 2 possible paths to that row, the probability of reaching each square in that row is ½.

There are 3 shaded squares in the 2nd row above START and $2^2 = 4$ paths to that row (see Method 3 in Comment 5). Since the number of paths to these squares are 1, 2, and 1, respectively (reading left to right or right to left), the probabilities of reaching these squares are ¼, ²⁄₄, and ¼, respectively (see diagram at the left).

There are $2^3 = 8$ paths to the 3rd row above START, and there are, respectively, 1, 3, 3, and 1 paths to the shaded squares in that row. Therefore, the probabilities of reaching the shaded squares in the 3rd row above START are ⅛, ³⁄₈, ³⁄₈, and ⅛, respectively. Continuing in this manner, there are $2^6 = 64$ paths to the 6th row above START (the row with the lettered squares) and there are, respectively, 1, 6, 15, 20, 15, 6, and 1 paths to the squares in that row. So, the probability of landing on each of the squares A, B, C, D, E, F, and G, respectively, is ¹⁄₆₄, ⁶⁄₆₄, ¹⁵⁄₆₄, ²⁰⁄₆₄, ¹⁵⁄₆₄, ⁶⁄₆₄, and ¹⁄₆₄. Notice the sum of the probabilities in any row is 1, since landing in the row is certain during a Checker-A game.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

6 (continued.)
Another approach to determining the probability of landing in each shaded square is to associate each square with the even and odd outcomes of the die. Since there are only 2 possible outcomes on the first roll, E or O, and each outcome is equally likely, there is a $\frac{1}{2}$ chance of landing in each square in the 1st row above START. In general, once a marker is in a square, there is a $\frac{1}{2}$ chance that the next roll will be E and $\frac{1}{2}$ chance it will be O. Thus, the probability of getting each of the following outcomes, EE, EO, OE, or OO is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$, and the probability of landing in the 3 squares of the 2nd row above START are, respectively, $\frac{1}{4}$, $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $\frac{1}{4}$.

Based on the above reasoning, the chance of getting an EEE, for example, is $\frac{1}{2}$ the chance of getting an EE, or $\frac{1}{2} \times (\frac{1}{2} \times \frac{1}{2}) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$. Similarly, the probability of getting any one of the 8 possible outcomes for 3 rolls is $\frac{1}{8}$, and the probabilities of landing in the squares in the 3rd row above START are, respectively, $\frac{1}{8}$, $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$, $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$, and $\frac{1}{8}$.

Continuing in this manner, the chance of getting any one of the possible outcomes from 4 rolls of the die are $\frac{1}{2} \times (\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}) = \frac{1}{16}$. Since there are, respectively, 1, 4, 6, 4, and 1 outcomes that bring the marker to the squares in the 4th row above START, the probabilities of landing in these squares are, respectively, $\frac{1}{16}$, $\frac{4}{16}$, $\frac{6}{16}$, $\frac{4}{16}$, and $\frac{1}{16}$.

Notice the probability of any one outcome from n rolls of the die is $(\frac{1}{2})^n$. Hence, the probability of landing in a particular square in the nth row of the Checker Game Board is the product of $(\frac{1}{2})^n$ times the number of outcomes that bring the marker to that square.

7 Give each group a copy of Connector Master B and ask them to make bar graphs showing the theoretical probability of the marker landing in each lettered square. Discuss their observations about similarities and differences in this theoretical graph and their experimental graphs from Action 4. How does the theoretical graph compare to experimental graphs for the combined data from Action 4?

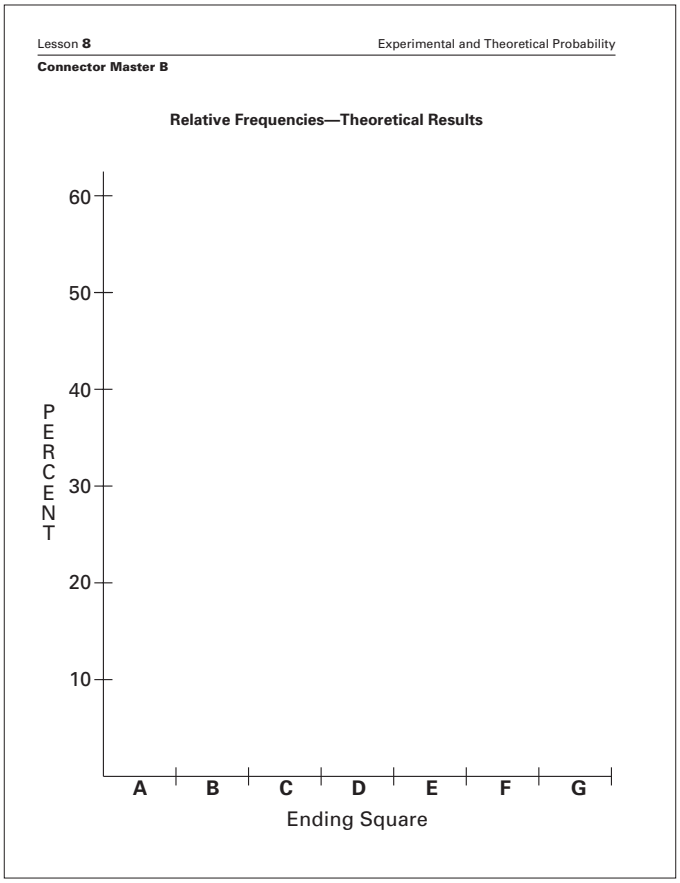
7 The theoretical probability, rounded to the nearest tenth of a percent, of landing in each lettered square is:

A, 1.6%
B, 9.4%
C, 23.4%
D, 31.2%
E, 23.4%
F, 9.4%
G, 1.6%

If the groups' experimental data from Action 4 was not combined and graphed previously, students can do so now to see the results as the number of games played increases. In general, as the number of Checker-A games

Connector Teacher Activity (cont.)

ACTIONS



8 Give each group a copy of Connector Master A and a copy of Connector Master C (see next page), and ask the groups to carry out the instructions. After the groups have made their presentations, play a class Checker-B game according to the instructions on Connector Master C. Discuss.

×	1	2	3	4	5	6
1	*1*	*2*	*3*	*4*	*5*	*6*
2	*2*	*4*	*6*	*8*	*10*	*12*
3	*3*	*6*	*9*	*12*	*15*	*18*
4	*4*	*8*	*12*	*16*	*20*	*24*
5	*5*	*10*	*15*	*20*	*25*	*30*
6	*6*	*12*	*18*	*24*	*30*	*36*

COMMENTS

increases, the experimental probability of landing in each lettered square should approach the corresponding theoretical probability. If you teach several classes, combining the class's experimental results may be useful for illustrating this.

8 Here are some issues to listen for as you observe while the groups work: Is *sample size* (i.e., the total number of games) a factor in planning for their experiments? Are they establishing procedures to assure random tosses of the dice? Do they have recording methods established so their data can be analyzed after the experiment? Are they devising methods that meaningfully involve everyone in the group? Are they showing evidence (theoretical and experimental) to support their conclusions? Are students attempting to make sense out of contradictions between experimental and theoretical outcomes? Are they utilizing/expanding upon methods from the Checker-A explorations?

Some students may use a grid such as the one at the left to show all the possible products of numbers from 2 dice. Since 27 of these 36 products are even numbers, the probability of tossing an even product is $^{27}/_{36} = {}^{3}/_{4}$, while the probability of an odd product is $^{9}/_{36} = {}^{1}/_{4}$.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Experimental and Theoretical Probability Lesson **8**

Connector Master C

Checker-B Game

Replace each of the letters A-G on your group's Checker Game Board with one of the numbers 1-7 (one number per square and no numbers can be repeated) so that your group will be likely to win the following game.

Each group places a marker on the START square of their game board. The teacher tosses a pair of dice, then computes and announces the product of the numbers on the upturned sides of the dice. If the product is an EVEN number, each group moves their marker forward 1 square diagonally to the LEFT. If the product is an ODD number, each group moves their marker 1 square diagonally to the RIGHT.

The winning group is the group (or groups) whose marker ends in the square with the highest number.

Your Group's Task

Record your responses to a)-c) on a poster. Plan to spend 2-3 minutes presenting your results to the class.

a) Without any data analysis or data collection, where do you *predict* you should place the numbers 1-7?

b) Now reason mathematically to verify or adjust the numbering system you predicted in a). Write a sound mathematical argument showing why this numbering is most likely to be a winning one. Your arguments should include the following evidence: experimental *and* theoretical probabilities, graphs, diagrams, and concise mathematical language. If your theoretical and experimental probabilities suggest different numberings, explain how you deal with that.

c) Suppose your teacher were to pick at random a shaded square on the game board. On your poster, explain how to determine the probability that a marker would land on that square during a Checker-B game.

Checker-B Game Theoretical Probabilities

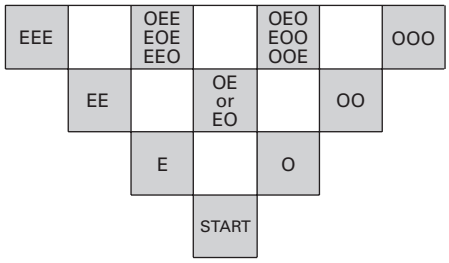
FINISH	A $\frac{729}{4096}$		B $\frac{1458}{4096}$		C $\frac{1215}{4096}$		D $\frac{540}{4096}$		E $\frac{135}{4096}$		F $\frac{18}{4096}$		G $\frac{1}{4096}$
		$\frac{243}{1024}$		$\frac{405}{1024}$		$\frac{270}{1024}$		$\frac{90}{1024}$		$\frac{15}{1024}$		$\frac{1}{1024}$	
EVEN			$\frac{81}{256}$		$\frac{108}{256}$		$\frac{54}{256}$		$\frac{12}{256}$		$\frac{1}{256}$		
				$\frac{27}{64}$		$\frac{27}{64}$		$\frac{9}{64}$		$\frac{1}{64}$			
					$\frac{9}{16}$		$\frac{6}{16}$		$\frac{1}{16}$				
						$\frac{3}{4}$		$\frac{1}{4}$					
							1						

ODD

START

COMMENTS

8 (continued.)
Combinations of even (E) and odd (O) products produce the same 64 paths for the Checker-B game as for the Checker-A game. Hence, there are 64 possible outcomes for the Checker-B game. The outcomes for the first 4 rows of the game board are shown here:



However, since the probabilities of rolling an even product and an odd product are not equal for the Checker-B game, the probabilities of landing in the shaded squares are different from those in the Checker-A game. That is, although the outcomes for a given row of the Checker-A game are the same as the outcomes for a given row of the Checker-B game (e.g., EE, OE, EO, and OO for the 2nd row above START), the outcomes for a given row of the Checker-A game are *equally likely* to occur, while the outcomes for a given row of the Checker-B game are *unequally likely*.

Just as for the Checker-A game, the probability of landing in the START square is 1, since it is certain that the marker begins there. However, the chance of rolling an even number (E) is $\frac{3}{4}$ and the chance of rolling an odd number (O) is $\frac{1}{4}$. Therefore, the probabilities of the marker landing in the left and right shaded squares of the 1st row above START are $\frac{3}{4}$ and $\frac{1}{4}$, respectively.

In general, once a marker is in a square, there is a $\frac{3}{4}$ chance that the next roll will be E, and a $\frac{1}{4}$ chance it will be O. Hence, the probability of rolling EE is $\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$ and the probability of rolling an EO is $\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$. Thus, the probabilities of getting EE, EO, OE, or OO, respectively, are $\frac{3}{4} \times \frac{3}{4}$, $\frac{3}{4} \times \frac{1}{4}$, $\frac{1}{4} \times \frac{3}{4}$, and $\frac{1}{4} \times \frac{1}{4}$, or $\frac{9}{16}$, $\frac{3}{16}$, $\frac{3}{16}$, and $\frac{1}{16}$. Therefore, the probabilities of landing in each of the 3 squares in the 2nd row above START are, respectively, $\frac{9}{16}$, $\frac{3}{16} + \frac{3}{16} = \frac{6}{16}$, and $\frac{1}{16}$.

Continuing with the above reasoning, the chance of getting an EEE, for example, is $(\frac{3}{4} \times \frac{3}{4}) \times \frac{3}{4}$ and the chance of getting an EEO is $(\frac{3}{4} \times \frac{3}{4}) \times \frac{1}{4}$. Thus, for any given outcome with m E's and n O's, the probability of that outcome is $(\frac{3}{4})^m \times (\frac{1}{4})^n$. The numbers in each shaded square of the game board shown at the left

Connector Teacher Activity (cont.)

ACTIONS

9 (Optional) Give each group a copy of Connector Master D and give each student another copy of Connector Master A. Have each student create a game board and then play a class Checker-C game according to the instructions on Connector Master D. Discuss.

Lesson **8** Experimental and Theoretical Probability

Connector Master D

Checker-C Game Procedures

Each player replaces the letters A-G on her/his Checker Game Board with the numbers 1-7 (one number per square, and no numbers can be repeated), positioning the numbers with the intent of winning the following game.

All players place a marker in the START square of their game boards. The teacher rolls a standard die, and announces the number obtained. If the number showing on the die is 1, 2, 3, or 4, all players move their markers forward 1 space diagonally to the LEFT. If the number showing on the die is 5 or 6, all players move their markers 1 space forward diagonally to the RIGHT. The winning player is the one whose marker ends in the square with the highest number.

COMMENTS

indicate the probability that the marker will land in that square during a Checker-B game.

As another approach to determining the probabilities of landing in any given square above START, some students may compute the following: (¾ times the probability for the square [if any] touching its lower right corner) plus (¼ times the probability for the square [if any] touching its lower left corner).

The theoretical probabilities (rounded to the nearest ten-thousandth) of reaching each of the lettered squares are:

A $(3/4)^6 \times 1 = .1780$ (1 path of 6 evens)
B $(3/4)^5 \times (1/4) \times 6 = .3560$ (6 paths of 5 even and 1 odd)
C $(3/4)^4 \times (1/4)^2 \times 15 = .2966$ (15 paths of 4 even, 2 odd)
D $(3/4)^3 \times (1/4)^3 \times 20 = .1318$ (20 paths of 3 even, 3 odd)
E $(3/4)^2 \times (1/4)^4 \times 15 = .0330$ (15 paths of 2 even, 4 odd)
F $(3/4)^1 \times (1/4)^5 \times 6 = .0044$ (6 paths of 1 even, 5 odd)
G $1 \times (1/4)^6 = .0002$ (1 path of 6 odd)

Based on the above theoretical probabilities, the following numbering would be most likely to win the Checker-B Game: A, 5; B, 7; C, 6; D, 4; E, 3; F, 2; G, 1.

9 You could also make this an individual assessment activity by giving each student a copy of Connector Master D and having them write mathematical arguments involving theoretical and experimental probabilities to support their assignments of the numbers 1-7 to squares A-G.

The probability of moving left is ⅔ and the probability of moving right is ⅓. The probabilities (to the nearest ten-thousandth) of landing on the squares A-G are:

$P(A) = (2/3)^6 \approx .0878 \approx 8.78\%$
$P(B) = 6(2/3)^5(1/3) \approx .2634 \approx 26.34\%$
$P(C) = 15(2/3)^4(1/3)^2 \approx .3292 \approx 32.92\%$
$P(D) = 20(2/3)^3(1/3)^3 \approx .2195 \approx 21.95\%$
$P(E) = 15(2/3)^2(1/3)^4 \approx .0823 \approx 8.23\%$
$P(F) = 6(2/3)(1/3)^5 \approx .0165 \approx 1.65\%$
$P(G) = (1/3)^6 \approx .0014 \approx .14\%$

Based on the above theoretical probabilities, the following numbering would be most likely to win the Checker-C game: A, 4; B, 6; C, 7; D, 5; E, 3; F, 2; G, 1.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use simulations to gather data and determine experimental probabilities for different strategies of playing a game called Monty's Dilemma. The process of carrying out these simulations provides insights about the theoretical probabilities of the various strategies.

MATERIALS

- ✔ Focus Student Activity 8.2, 1 copy per pair of students and 1 transparency.
- ✔ Focus Masters A, B, and D, 1 transparency of each.
- ✔ Focus Master C, 1 copy per pair of students and 1 transparency.
- ✔ Bobby pins or paper clips, 1 per pair of students.
- ✔ Coins, 1 per pair of students.

ACTIONS

1 Arrange the students in pairs. Place a transparency of Focus Master A on the overhead for the students to read. Ask them to discuss whether they think it is best to STICK or SWITCH.

Experimental and Theoretical Probability Lesson **8**

Focus Master A

Monty's Dilemma

On Monty's TV game show, there are 3 doors. Behind 1 of the doors is a valuable prize and behind the other 2 doors are gag prizes. To play Monty's game, a contestant is invited to choose (but not open) 1 of 3 doors.

After the contestant chooses a door, Monty reveals what is behind 1 of the other 2 doors, always showing a gag prize. Then Monty presents the contestant with the following dilemma:

"Would you like to STICK with the door you chose, or SWITCH to the other unopened door?"

COMMENTS

1 Note that, when a contestant selects a door, it is not opened. Rather, Monty opens 1 of the 2 remaining doors, and he always opens 1 that has a gag prize behind it. At that point, the contestant may STICK (i.e., stay with her/his original selection) or SWITCH to the other unopened door.

Some students may feel that it makes no difference whether the contestant switches or sticks. For example, they may argue that only 2 unopened doors are left after Monty reveals 1 gag prize, and the chance of winning a valuable prize is 1 out of 2, whether the contestant sticks or switches. Be careful not to give clues regarding correct or incorrect reasoning by students!

Focus Teacher Activity (cont.)

ACTIONS

2 Place a transparency of Focus Master B, on the overhead for the students to read. Clarify as needed.

Lesson **8** Experimental and Theoretical Probability

Focus Master B

Monty's Strategies

After 1 of the doors with a gag prize has been opened, which of these 3 strategies is most likely to lead the contestant to the winning door?

STICK strategy. Keep the door that was originally selected.

FLIP strategy. Choose again by randomly selecting a door from the remaining 2 closed doors.

SWITCH strategy. Switch from the original door to the other closed door.

3 Discuss the students' ideas regarding ways to conduct an experiment that involves playing Monty's game many times using each of the 3 strategies—stick, flip, and switch—to see the effect of each strategy on the outcome of the game. Use this discussion as a context for introducing the use of *simulations* for modeling experiments that are too cumbersome or unrealistic to carry out.

4 Place a transparency of Focus Master C on the overhead. Follow steps a)-d) below to briefly model the use of a spinner for simulating each of the strategies—stick, flip, and switch—to determine which of Monty's strategies is most likely to result in a contestant winning a valuable prize.

Step a) Ask the students to assume, for convenience's sake, that the prize is behind Door 1 and, when the spinner is spun, the pointer determines the door selected by the contestant.

COMMENTS

2 It is not intended that students answer the given question now; rather the intent is to assure that students notice the addition of a 3rd strategy—flip. Some students may feel that the chance of winning a valuable prize is 50% for all 3 strategies. Remember, no clues! In fact, you might encourage students to keep their predictions private for now, noting there will be opportunities to test their ideas in subsequent actions.

3 Some students may suggest using actual doors or boxes to represent the doors. Others may suggest methods such as using spinners or placing slips of paper in a hat to represent the process of choosing doors. While using real doors or boxes for the doors may help students become more familiar with the game, such experiments can be very time consuming. Using spinners and/or selecting slips of paper from a hat to model the real situation are examples of simulations. In general, a *simulation* is a process in which experiments that closely resemble the given situation are carried out repeatedly. See also Lesson 17 of this course.

4 The spinner provides one possibility for simulating each strategy. You may wish to use another method suggested by your students. Regardless the simulation method, it is important to clarify the use of each strategy before students begin their experiments.

The point here is to give just enough information so that students understand the procedures. In Action 5 students carry out a simulation to predict the best strategy—stick, flip, or switch.

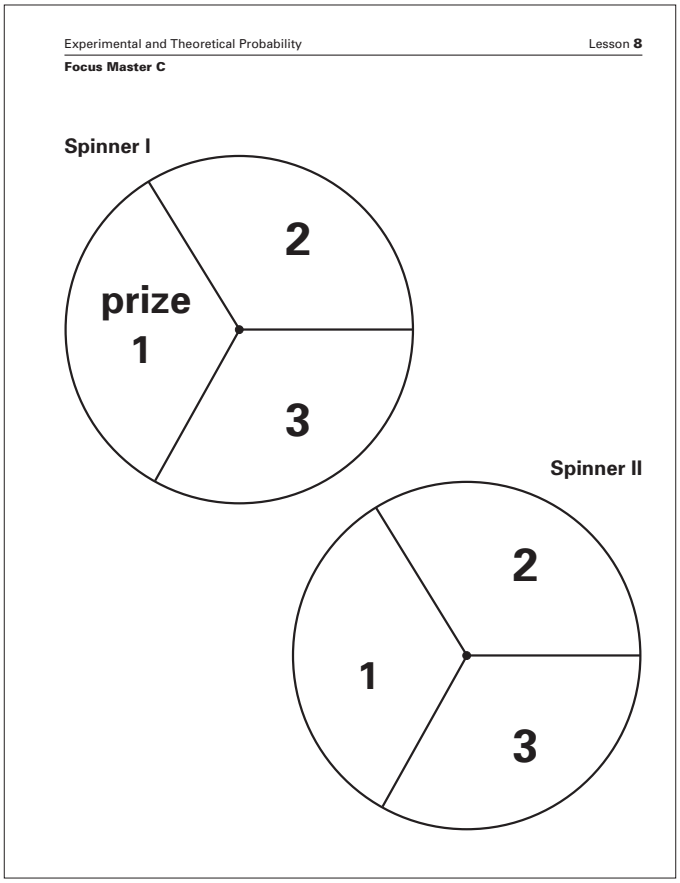
Focus Teacher Activity (cont.)

ACTIONS

Step b) Spin Spinner I and ask the students to determine the outcome using the STICK strategy. Discuss students' ideas and questions.

Step c) Spin Spinner I and ask the students to determine the outcome using the FLIP strategy. Discuss.

Step d) Spin Spinner I and ask the students to determine the outcome using the SWITCH strategy. Discuss.



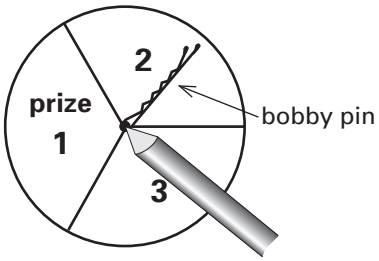
5 Give each pair of students a copy of Focus Master C, a copy of Focus Student Activity 8.2 (see next page), a bobby pin or paper clip, and a coin. Ask the pairs to complete 20 trials for each of Monty's 3 strategies and to record their results on Focus Student Activity 8.2. When completed, record the total class data on a transparency of Focus Student Activity 8.2. Ask the pairs to use this data as the basis for determining the winning strategy. Discuss their conjectures, conclusions, and reasoning.

COMMENTS

Step a) A bobby pin, or a paper clip with one end unfolded, held in place with a pencil point at the center of the spinner can be used as a pointer for the spinner. If the pointer lands on a line, the spin can be repeated.

The following discussion is intended to clarify use of the spinners and the 3 strategies for selecting a door:

Step b) The STICK strategy. Suppose the spinner lands on 2 and this means the contestant chooses Door 2. Monty opens door 3 (a gag prize), the contestant sticks with Door 2 and loses.



Step c) The FLIP strategy. Suppose the spinner lands on Door 2 and this means the contestant choses Door 2. Monty opens Door 3 to reveal a gag prize. The contestant then flips a coin to decide whether to (heads) stick with Door 2 or (tails) switch to Door 1. Thus, there are 2 possible outcomes: (1) the coin lands heads, the contestant sticks with Door 2 and loses; or (2) the coin lands tails, the contestant switches to Door 1 and wins.

Step d) The SWITCH strategy. Suppose the spinner lands on Door 2 and this means the contestant choses Door 2. Monty opens Door 3 to reveal a gag prize. The contestant then switches to Door 1 and wins.

Rather than illustrating with more examples for the class, you might move on to Action 5, circulating and clarifying questions about procedures while the pairs work.

5 It is important that students actually carry out the simulation, because in the process of doing so, they may develop insights regarding the actual chances of winning with each strategy.

To avoid confusion, you might suggest that students use 1 strategy for 20 consecutive spins, then a 2nd strategy for 20 consecutive spins, and finally, the 3rd strategy for 20 spins.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **8** Experimental and Theoretical Probability

Focus Student Activity 8.2

NAME _____ DATE _____

Tallies and Totals for Each Strategy

	WINS	LOSSES
STICK		
	Total:	**Total:**
FLIP		
	Total:	**Total:**
SWITCH		
	Total:	**Total:**

COMMENTS

5 (continued.)
Rather than designate one of the 3 doors as having the prize, as in step a) of Action 4, some pairs of students may find the game more interesting if played as follows:

• Student A spins Spinner II to obtain the number of the door with the valuable prize and conceals this number from Student B.
• Student B chooses Door 1, 2, or 3.
• Student A tells Student B the number of 1 door that contains a gag prize.
• Student B chooses a door based on either the stick, flip, or switch strategy.
• Student A announces whether Student B wins or loses, and records the result.

Depending on your time needs, this data collection could take place at home. The question of determining the best strategy in Monty's game may be an engaging one for families.

The data may point to the switch strategy as the winning strategy, showing less than 50% chance of winning with the stick strategy, about 50% chance with the flip strategy, and considerably more than 50% chance with the switch strategy.

Here are examples of a few observations made by students:

We can win with the stick strategy only if our first choice is the door with the prize. Our chance of doing that is 1 out of 3, so the probability of winning with the stick strategy is $\frac{1}{3}$.

With the flip strategy, we have to flip the coin to determine the choice of doors. Our chances of getting the door with the prize are 1 out of 2, so the probability of winning with the flip strategy is $\frac{1}{2}$.

The difference between the flip and stick strategies is that with the flip strategy we make a new choice, randomly, between 2 doors after 1 of the doors has been opened. With the flip strategy we use the information about the gag prize door; with the stick strategy we don't use that information, we just stay with our original choice.

The only way we can lose with the switch strategy is if our first choice is the door with the prize behind it. So, the probability of losing with the switch strategy is $\frac{1}{3}$, and the probability of winning is $\frac{2}{3}$.

Focus Teacher Activity (cont.)

ACTIONS

6 If it didn't come up during Action 5, ask the pairs to determine the theoretical probability of winning the game with each of the 3 strategies. Discuss.

7 Write the following on the overhead for each student to complete:

> **Write a letter to a friend explaining the chances of winning with each of Monty's strategies—stick, flip, and switch—so your friend will be confident about which strategy to use when playing Monty's game. Tell your friend what surprised you about Monty's game.**

8 Pose one or both of the following extensions of Monty's Dilemma for the pairs of students to investigate.

Monty's 4-Door Dilemma: A valuable prize is behind 1 of 4 doors and after the contestant selects 1 door, Monty opens a door with a gag prize. Which of the 3 strategies—stick, flip, or switch—is the best?

Monty's 5-Door Dilemma: Same as 4-door problem but the prize is behind 1 of 5 doors.

COMMENTS

6 The theoretical probabilities for the stick, flip, and switch strategies are $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{2}{3}$, respectively. See the sample student comments listed in Comment 5 for a discussion of the reasoning behind these probabilities.

7 This provides an opportunity for individual students to organize, clarify, and communicate their understandings regarding why each strategy has a different probability of winning.

You might suggest that the recipient of the letter already knows how to play the game. Thus, students' energy will be spent describing the reasoning behind the use of each strategy.

Note that more information about this game can be found in the April 1991 issue of the *Mathematics Teacher*, Vol. 84, No. 4, pp. 252-256.

8 Students may benefit from recalling the simulation with the 3-part spinners in Action 4 in order to design spinners and simulations for these extensions of Monty's Dilemma.

The stick strategy continues to be the best strategy for these extensions of Monty's Dilemma.

Monty's 4-Door Dilemma: *Stick Strategy*, probability $\frac{1}{4}$; *Flip Strategy*, probability $\frac{1}{3}$; *Switch Strategy*, probability $\frac{3}{8}$. With the Switch Strategy, once a door with a gag prize behind it is opened, the probability the prize is behind 1 of the remaining 3 doors is $\frac{3}{4}$, and by switching there are 2 doors to choose from. Thus, the $\frac{3}{8}$ probability is obtained from $(\frac{1}{2})(\frac{3}{4})$.

Monty's 5-Door Dilemma: *Stick Strategy*, probability $\frac{1}{5}$; *Flip Strategy*, probability $\frac{1}{4}$; *Switch Strategy*, probability $\frac{4}{15}$. With the Switch Strategy, once a door with a gag prize behind it is opened, the probability the prize is behind 1 of the remaining 4 doors is $\frac{4}{5}$, and by switching there are 3 doors to choose from. Thus the $\frac{4}{15}$ probability is obtained from $(\frac{1}{3})(\frac{4}{5})$.

Focus Teacher Activity (cont.)

ACTIONS

9 Place a transparency of Focus Master D on the overhead for the students to read. Discuss the students' ideas regarding possible simulation methods (not solutions). Then ask the pairs to complete Parts 1 and 2 on Focus Master D. When completed, discuss their results and reasoning.

Lesson **8** Experimental and Theoretical Probability

Focus Master D

Kelsey's Dilemma

The school tennis team is holding a special drawing for a new tennis racket. A total of 10 tickets were given out—1 to each of the 10 tennis team members. This morning Coach Ward said the following to Kelsey, "I have 2 ticket stubs in my hand, yours and another one. One of these 2 ticket stubs has the winning number. Would you like to STICK with your ticket number, or SWITCH for the other number I am holding?" What should Kelsey do?

1. Design and carry out a simulation to help you solve Kelsey's dilemma.

   a) Describe the step-by-step procedures of your simulation.
   b) Make an organized listing of all the data that you collect.
   c) Use experimental probabilities as the basis for solving Kelsey's dilemma.

2. Use theoretical probabilities to solve Kelsey's dilemma. Explain how you determine these probabilities.

COMMENTS

9 Initially, students may incorrectly predict that Kelsey has a 50% chance of winning with her ticket, since the coach is holding 2 tickets and 1 of them is the winning ticket. Avoid giving clues!

1a) It is important to discuss ways to simulate this situation since it is critical that students' simulations model the conditions of the problem. If students have difficulty designing an appropriate simulation, you might suggest the following approach which uses 10 numbered slips of paper in a hat (note that students investigate simulations in depth in Lesson 17 of this course).

Student A randomly selects a number from the hat and shows the number to Student B who records the number as Kelsey's and replaces it in the hat. Next Student B randomly selects and records the winning number without showing Student A. If that number is different from Kelsey's, then that number is both the winning number and the "other number" in Coach Ward's hand. If that number is Kelsey's, then Kelsey's number is the winning number, and Student B records a number which is 1 greater or 1 less than Kelsey's as the "other number" (note: choosing a number that is 1 more or 1 less than Kelsey's number assures randomness). Then, without seeing the "other number" recorded by Student B, Student A decides whether Kelsey sticks or switches, and Student B reports whether Kelsey wins or loses.

For example, suppose that Student A selects the number 1 as Kelsey's number. Then suppose that Student B selects the number 7 as the winning number. If Kelsey sticks with her number, she loses; if she switches to the 7, she wins. Or, as another example, suppose Student A selects the number 1 as Kelsey's number, and suppose that Student B also selects the number 1 as the winning number; hence, the "other number' recorded by Student B is 2. If Kelsey sticks with her number, she wins; if she switches to the 2 she loses.

1b)-c) Students' simulations should strongly support switching as the best choice.

2) The theoretical probability that Kelsey's original number will be the winning number is $1/10$, since there is only 1 way out of 10 possibilities that her number can be selected. Hence, if Kelsey sticks with her original number, the probability she will win is $1/10$. Notice that the only way Kelsey can lose by switching is if the coach selects Kelsey's number as the winning number, and there is only $1/10$ chance that he will select her number. Hence, the theoretical probability that Kelsey will win by switching is $9/10$!

Follow-up Student Activity 8.3

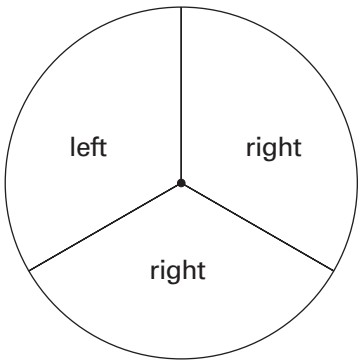
NAME _____ **DATE** _____

1 Explain in your own words the meaning of each of the following terms. Include an example to clarify each explanation.

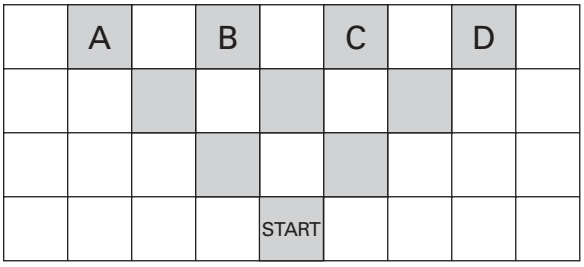
- Probability
- Experimental probability
- Theoretical probability
- Simulation
- Sample size
- Binomial experiment
- Equally likely outcomes
- Unequally likely outcomes

2 Replace the letters A-D on the Checker 1-4 Game Board shown below with the numbers 1-4 so that you are most likely to win the Checker 1-4 Game.

Checker 1-4 Game
Each player places a marker on the START square of their game board. One player spins a bobby pin or paper clip on the spinner shown below to determine the direction (left or right) of a forward diagonal move of every player's marker. Repeat until the markers each reach a numbered square. Each player records the number of the ending square for their marker. The winner is the player with the highest total after 20 games.



Checker 1-4 Game Board



Explain the mathematical reasoning behind your numbering system, using theoretical *and* experimental probabilities as supporting evidence. Show all data that you collect to determine experimental probabilities, and show all information that you use to determine theoretical probabilities.

(Continued on back.)

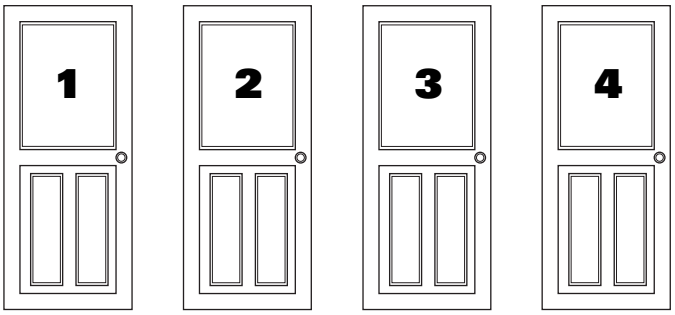
Follow-up Student Activity (cont.)

3 This version of the game, Monty's Dilemma, involves 4 doors—one with a valuable prize and 3 with gag prizes. After the contestant selects a door (but doesn't open it), Monty opens 2 of the remaining doors, both with gag prizes. A contestant may choose 1 of the following strategies:

i) STICK with the original choice

ii) FLIP a coin to determine which door to choose

iii) SWITCH from the original choice to the remaining door



a) Design a simulation of the game. Write a description of the step-by-step procedures of your simulation. Give several examples to show how each strategy works in your simulation.

b) Carry out your simulation at least 10 times for each of the 3 given strategies. Show an organized listing of all the experimental data that you collect.

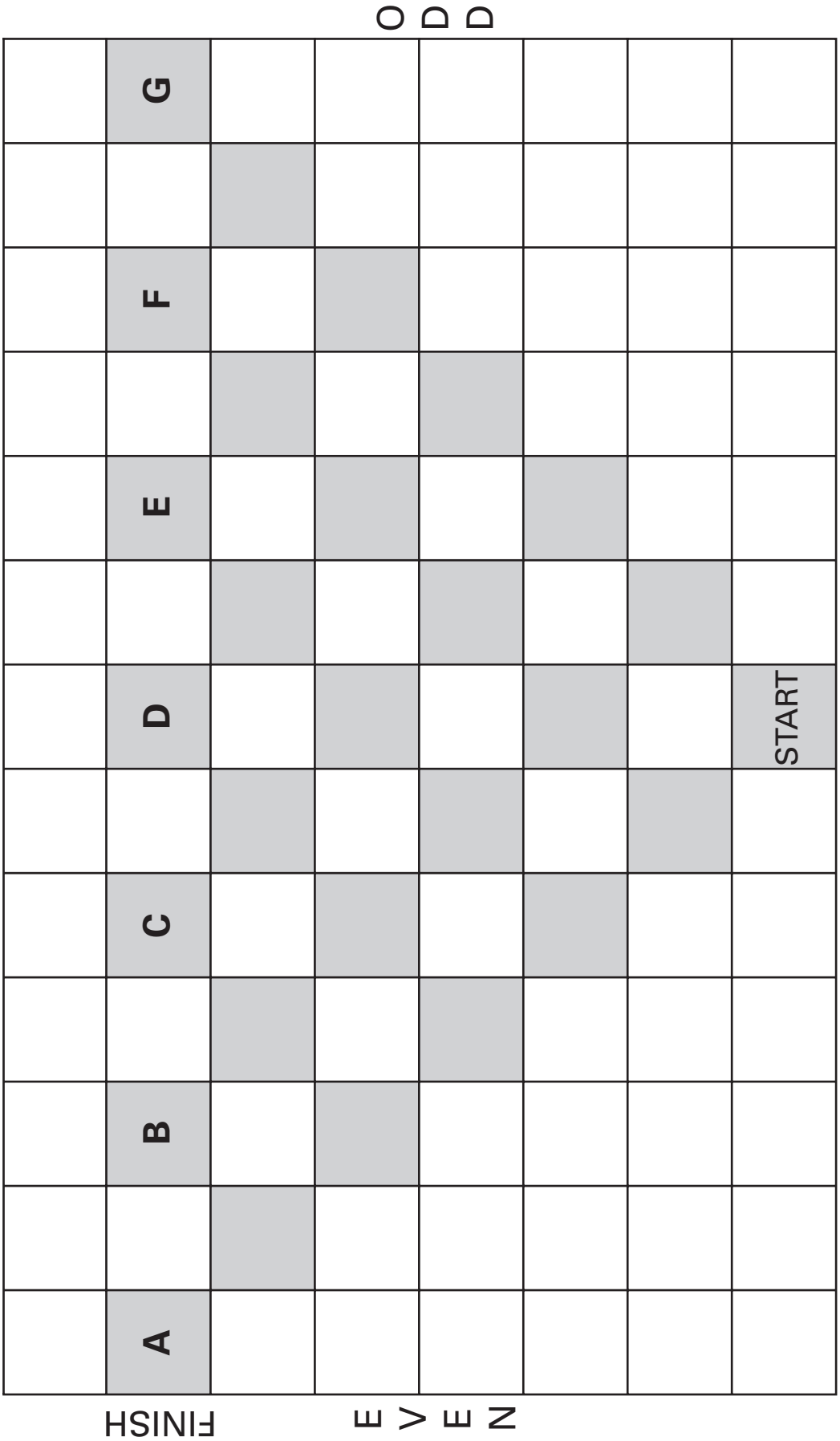
c) Based on an analysis of your experimental data, write a convincing argument telling which strategy is most likely to be a winning strategy for a contestant. Use experimental probabilities to support your position.

d) Based on theoretical probabilities, which strategy is most likely to be a winning strategy? Show how you determine each theoretical probability.

e) Determine the probabilities of winning and why with the stick, flip, or switch strategies for each of the following versions of Monty's game: there are 5 doors, 4 gag prizes, and Monty opens 3 doors; there are 6 doors, 5 gag prizes, and Monty opens 4 doors; there are n doors, $n - 1$ gag prizes, and Monty opens $n - 2$ doors.

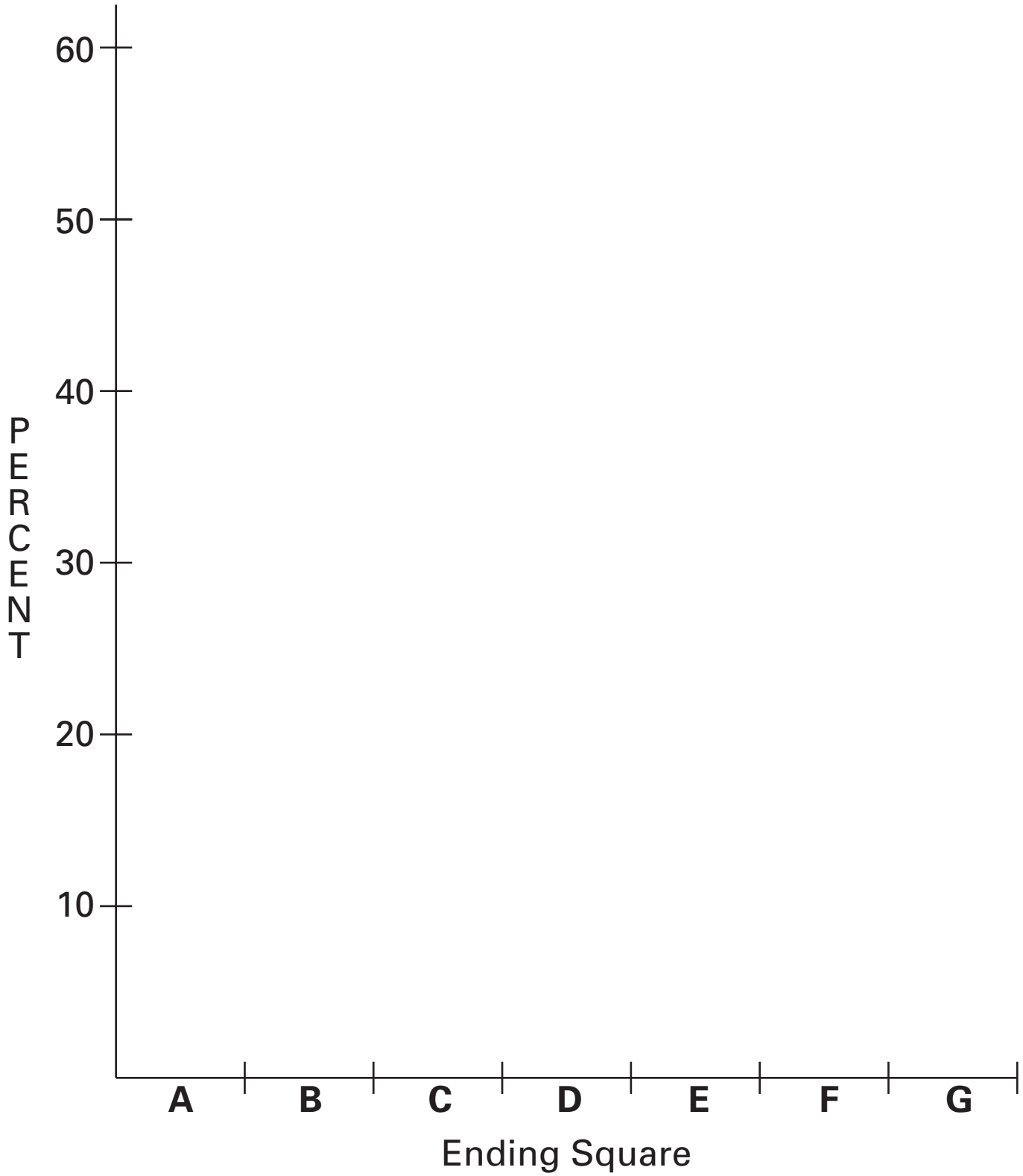
Connector Master A

Checker Game Board



Connector Master B

Relative Frequencies—Theoretical Results



Connector Master C

Checker-B Game

Replace each of the letters A-G on your group's Checker Game Board with one of the numbers 1-7 (one number per square and no numbers can be repeated) so that your group will be likely to win the following game.

Each group places a marker on the START square of their game board. The teacher tosses a pair of dice, then computes and announces the product of the numbers on the upturned sides of the dice. If the product is an EVEN number, each group moves their marker forward 1 square diagonally to the LEFT. If the product is an ODD number, each group moves their marker 1 square diagonally to the RIGHT.

The winning group is the group (or groups) whose marker ends in the square with the highest number.

Your Group's Task

Record your responses to a)-c) on a poster. Plan to spend 2-3 minutes presenting your results to the class.

a) Without any data analysis or data collection, where do you *predict* you should place the numbers 1-7?

b) Now reason mathematically to verify or adjust the numbering system you predicted in a). Write a sound mathematical argument showing why this numbering is most likely to be a winning one. Your arguments should include the following evidence: experimental *and* theoretical probabilities, graphs, diagrams, and concise mathematical language. If your theoretical and experimental probabilities suggest different numberings, explain how you deal with that.

c) Suppose your teacher were to pick at random a shaded square on the game board. On your poster, explain how to determine the probability that a marker would land on that square during a Checker-B game.

Connector Master D

Checker-C Game Procedures

Each player replaces the letters A-G on her/his Checker Game Board with the numbers 1-7 (one number per square, and no numbers can be repeated), positioning the numbers with the intent of winning the following game.

All players place a marker in the START square of their game boards. The teacher rolls a standard die, and announces the number obtained. If the number showing on the die is 1, 2, 3, or 4, all players move their markers forward 1 space diagonally to the LEFT. If the number showing on the die is 5 or 6, all players move their markers 1 space forward diagonally to the RIGHT. The winning player is the one whose marker ends in the square with the highest number.

Connector Student Activity 8.1

NAME _____ **DATE** _____

1 Each group member is to play the Checker-A game several times and keep a record of the number of times the marker lands in each of squares A-G. Your group needs to play a total of 50 games. When everyone has finished, record your data and totals on the following chart:

Players' Names	Ending Square on Game Board							Total Games
	A	B	C	D	E	F	G	
1								
2								
3								
4								
5								
Group Totals								50

Based on your group's experimental results, determine the experimental probability of landing in each of squares A-G on the game board. Express each answer as a fraction, decimal, and percent. Note: P(A) stands for "the probability of landing in square A."

a) P(A) =

b) P(B) =

c) P(C) =

d) P(D) =

e) P(E) =

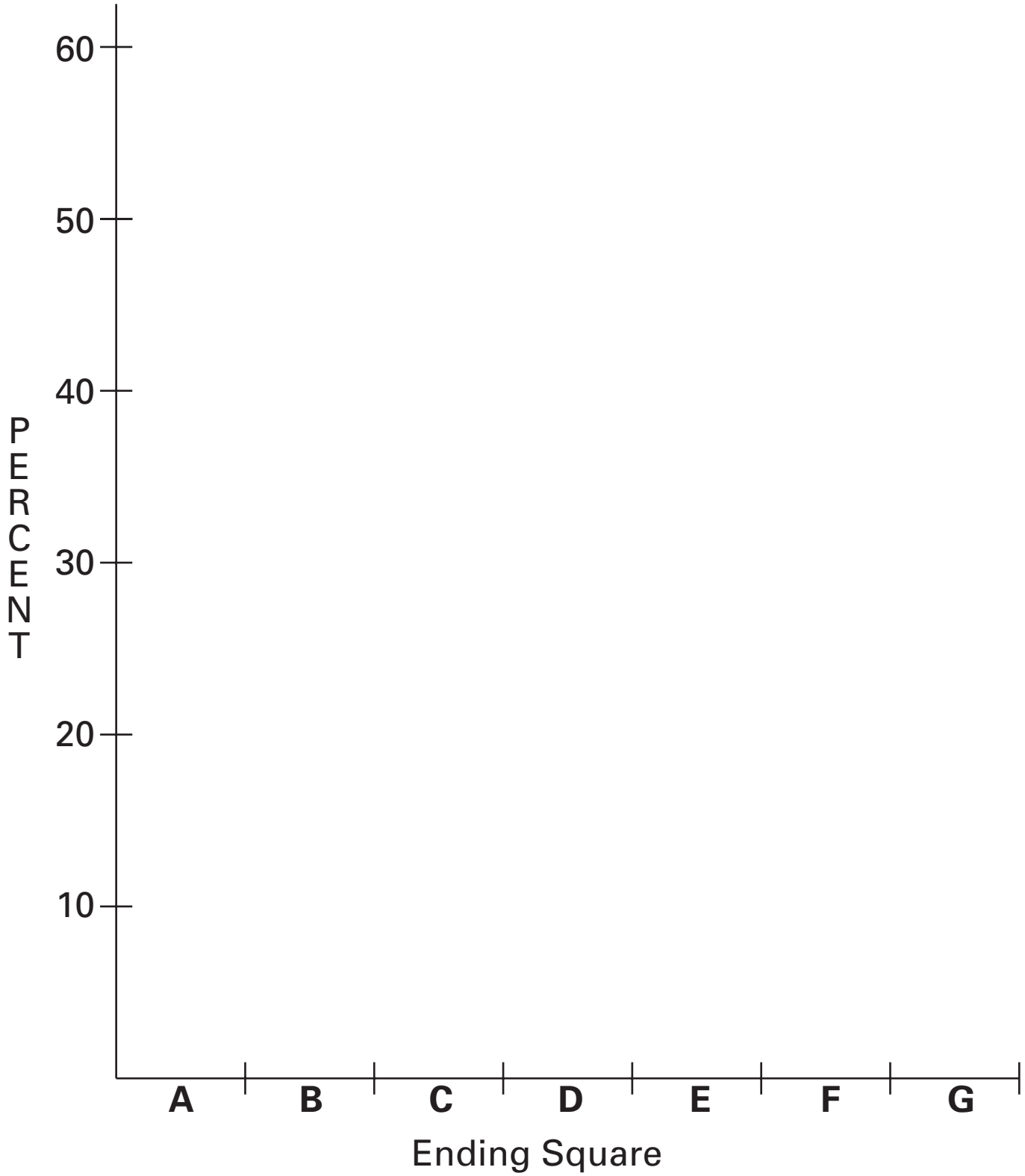
f) P(F) =

g) P(G) =

(Continued on back.)

Connector Student Activity 8.1 (cont.)

2 Make a bar graph showing the relative frequency of the marker ending in each column, based on your group's experimental results:



Focus Master A

Monty's Dilemma

On Monty's TV game show, there are 3 doors. Behind 1 of the doors is a valuable prize and behind the other 2 doors are gag prizes. To play Monty's game, a contestant is invited to choose (but not open) 1 of 3 doors.

After the contestant chooses a door, Monty reveals what is behind 1 of the other 2 doors, always showing a gag prize. Then Monty presents the contestant with the following dilemma:

"Would you like to STICK with the door you chose, or SWITCH to the other unopened door?"

Focus Master B

Monty's Strategies

After 1 of the doors with a gag prize has been opened, which of these 3 strategies is most likely to lead the contestant to the winning door?

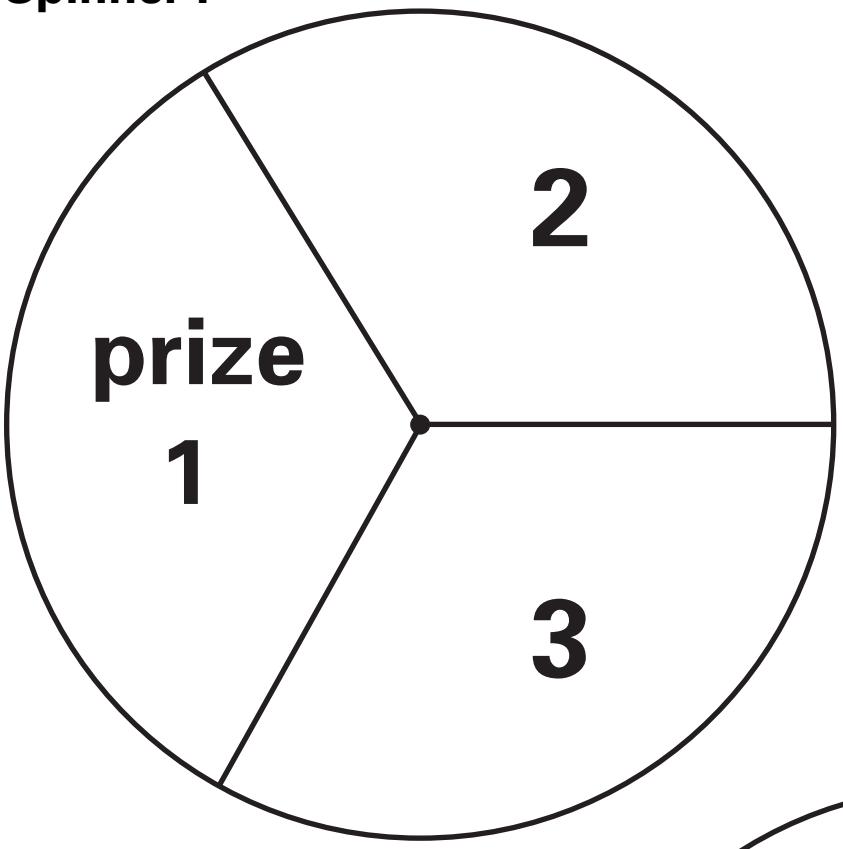
STICK strategy. Keep the door that was originally selected.

FLIP strategy. Choose again by randomly selecting a door from the remaining 2 closed doors.

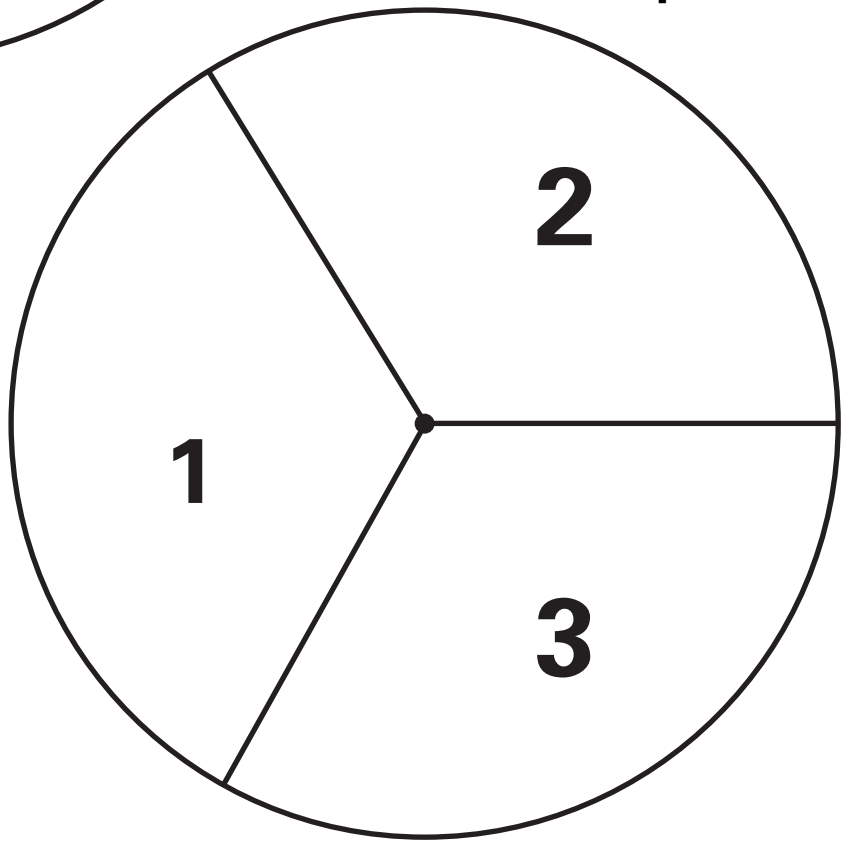
SWITCH strategy. Switch from the original door to the other closed door.

Focus Master C

Spinner I



Spinner II



Focus Master D

Kelsey's Dilemma

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Focus Student Activity 8.2

NAME _____ **DATE** _____

Tallies and Totals for Each Strategy

	WINS	LOSSES
STICK		
	Total:	**Total:**
FLIP		
	Total:	**Total:**
SWITCH		
	Total:	**Total:**

Follow-up Student Activity 8.3

NAME _____ **DATE** _____

1 Explain in your own words the meaning of each of the following terms. Include an example to clarify each explanation.

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- Theoretical probability
- Simulation
- Sample size
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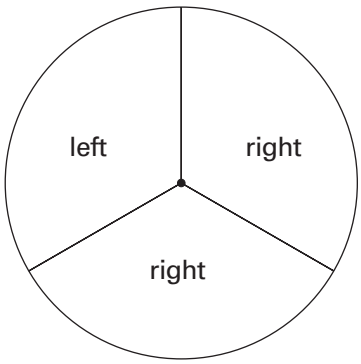
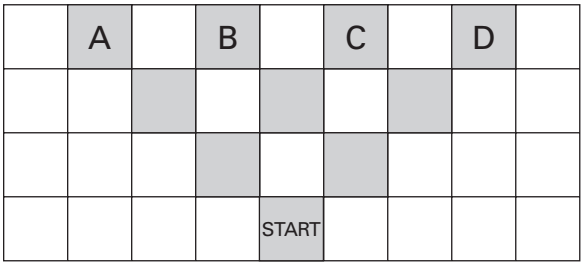


Checker 1-4 Game Board



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(Continued on back.)

Follow-up Student Activity (cont.)

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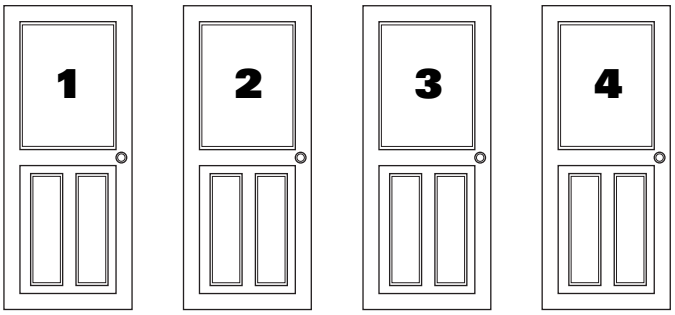

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b) Carry out your simulation at least 10 times for each of the 3 given strategies. Show an organized listing of all the experimental data that you collect.

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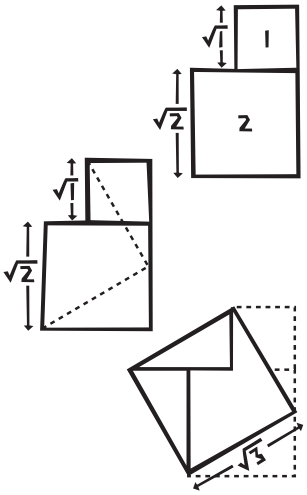
Reasoning & Radicals

Lesson 9

THE BIG IDEA

Examining relationships among squares and right triangles provides a meaningful context for developing inductive and deductive reasoning strategies and lays groundwork for understanding proof and the development of a mathematical system. Such investigations also reveal important relationships related to the Pythagorean Theorem and its applications and provide a conceptual basis for computing with radical expressions.



CONNECTOR

OVERVIEW

Students form squares on grids to illustrate the meaning of square root. They explore a visual proof of the Pythagorean Theorem.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Masters A, B, and C, 1 copy of each per pair of students and 1 transparency of each.
- ✔ Connector Student Activity 9.1, 1 copy per student and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Blank paper, 2 sheets per pair of students.
- ✔ Blank note card, 1 card per pair of students.
- ✔ Scissors and tape, 1 for each pair of students.

FOCUS

OVERVIEW

Students investigate another proof of the Pythagorean Theorem. They use inductive and deductive reasoning to demonstrate and prove mathematical relationships for right triangles and radical expressions. They apply these properties to 30°- 60°- 90° and 45°- 45°- 90° triangles.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Masters A and D, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Masters B and C, 1 transparency of each.
- ✔ Focus Master E, 2 copies per student and 1 transparency.
- ✔ Focus Student Activities 9.2 and 9.3, 1 copy of each per student and 1 transparency of each.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Ruler, protractor, and scissors, 1 of each per student.
- ✔ String, 1 18-inch length per group.
- ✔ Butcher paper strips (4"- 6" long) and marking pens, several of each per group.
- ✔ Blank paper, 1 sheet per student.

FOLLOW-UP

OVERVIEW

Students use their understanding of relationships among equivalent radical expressions to construct the square roots of selected whole numbers. They use the Pythagorean Theorem to find areas, perimeters, and lengths involving radicals expressions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 9.4, 1 copy per student.

LESSON IDEAS

LOOKING AHEAD
Rather than having students memorize rules and procedures that are more efficiently carried out by a calculator, the motive for having students explore and invent strategies for simplifying radicals is to promote understanding, to generate expressions that prompt generalizations about 30°- 60°- 90° and 45°- 45°- 90° triangles, and to prompt other insights that will be useful in more advanced work in algebra and trigonometry.

QUOTE
Reasoning is fundamental to the knowing and doing of mathematics. Although most disciplines have standards of evaluation by which new theories or discoveries are judged, nowhere are these standards as explicit and well formulated as they are in mathematics. Conjecturing and demonstrating the logical validity of conjectures are the essence of the creative act of doing mathematics. To give more students access to mathematics as a powerful way of making sense of the world, it is essential that an emphasis on reasoning pervade all mathematical activity.
NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

1. For example, eight of these square roots can be obtained from multiples of $\sqrt{2}$ and their opposites: $\sqrt{2}$, $-\sqrt{2}$, $2\sqrt{2} = \sqrt{8}$, $-\sqrt{8}$, $3\sqrt{2} = \sqrt{18}$, $-\sqrt{18}$, $4\sqrt{2} = \sqrt{32}$, $-\sqrt{32}$.

2. AF = FE = EG = GC = HJ = JI = JE = IG = $\sqrt{2}/4$;
   AH = HD = FJ = DI = IC = $1/2$; BE = $\sqrt{2}/2$

   Areas: HDI = $1/8$; AEB = BEC = $1/4$; GIC = FJE = $1/16$; EJIG = AHJE = $1/8$.

3. Each side length is 5 times its corresponding length in Problem 2 and each region has an area which is 25 times its corresponding area in Problem 2.

4. Determine the distance between each pair of points and compare the sum of the squares of the 2 shortest distances to the square of the greatest distance. If the sum is equal to the square of the greatest distance, the triangle is a right triangle; if the sum is greater than the square of the greatest distance, the triangle is acute; otherwise, the triangle is obtuse.

5. Yes, similar right triangles will be obtained.

6. The diagonals have length $\sqrt{l^2 + w^2 + h^2}$.

7. The diagonals have length $\sqrt{s^2 + s^2 + s^2} = s\sqrt{3}$.

8. a) For example, a square with area 5 and diagonal length $\sqrt{10}$ can be formed by dissecting and rearranging adjacent squares with areas 1 and 4. A square with area 7 and diagonal length $\sqrt{14}$ can be formed from adjacent squares with areas 5 and 2.

   c) A square with area 3, side length $\sqrt{3}$, and diagonal $\sqrt{6}$, can be formed by dissecting and rearranging adjacent squares with areas 1 and 2.

9. The lengths of the sides of square S and square T are $\sqrt{ab}$ and $a/2 + b/2$, respectively.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students form squares on grids to illustrate the meaning of square root. They explore a visual proof of the Pythagorean Theorem.

MATERIALS

- ✔ Connector Masters A, B, and C, 1 copy of each per pair of students and 1 transparency of each.
- ✔ Connector Student Activity 9.1, 1 copy per student and 1 transparency.
- ✔ 1-cm grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.
- ✔ Blank paper, 2 sheets per pair of students.
- ✔ Blank note card, 1 card per pair of students.
- ✔ Scissors and tape, 1 for each pair of students.

ACTIONS

1 Arrange the students in pairs and give each student a sheet of 1-cm grid paper. Write expression a) on the overhead, and ask the students to each sketch a geometric representation of the meaning of the expression. Ask for volunteers to demonstrate their thinking on a transparency of grid paper. Repeat for b)-e).

a) $\sqrt{16}$

b) $\sqrt{2}$

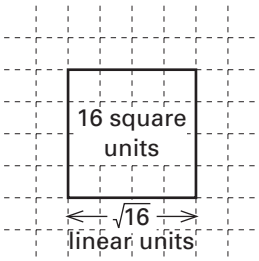
c) 3^2

d) $(\sqrt{5})^2$

e) $3\sqrt{2}$

COMMENTS

1 a) One can view $\sqrt{16}$ as the length of an edge of a square with area 16 square units. Since a 4×4 square has area 16, $4 = \sqrt{16}$. (See also *Math Alive! Course II*, Lesson 28.)



Some students may describe the meaning of $\sqrt{16}$, in nongeometric terms, as the positive solution of the equation $x^2 = 16$. The negative solution of the equation $x^2 = 16$ is negative square root of 16, written $-\sqrt{16}$. This can be verified by noting that $(\sqrt{16})^2 = (-\sqrt{16})^2 = 16$. Note that any positive real number n has two square roots—its *positive*, or *principal square root*, written $\sqrt{n}$, and its *negative square root*, written $-\sqrt{n}$. So, since lengths are only positive numbers, the length of the edge of a square of area 16 is the positive, or principal, square root of 16.

A square with *value* 16 has edges all of value $\sqrt{16}$, or all of value $-\sqrt{16}$. Hence, from a geometric perspective, finding all of the possible solutions for the equation $x^2 = 16$ is equivalent to finding the possible values of the edges of a square whose value is 16.

In general, for any square of value n, where n is a positive real number, the edges of the square have *length* $\sqrt{n}$ and there are two possibilities for the *value* of the edges: $+\sqrt{n}$ and $-\sqrt{n}$.

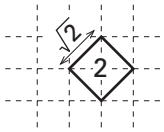
(Continued next page.)

Connector Teacher Activity (cont.)

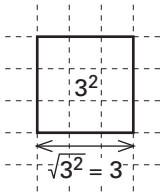
ACTIONS

COMMENTS

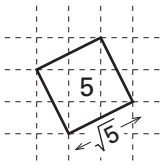
1 (continued.)
Students may be interested in the origin of the radical symbol, which was introduced in 1525 by Christoff Rudolff. He chose the symbol because it looks like a small *r*, the initial letter of the Latin word *radix*, meaning root.



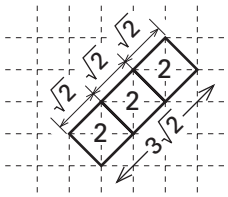
b) $\sqrt{2}$ may be viewed as the length of the edge of a square whose area is 2, or as one of the two possible values of the edge of a square whose value is 2 (the other possible value is $-\sqrt{2}$). Note that $\sqrt{2}$ is an *irrational number* since it is a *nonrepeating*, *nonterminating decimal*, and $\sqrt{2}$ is the *actual* length/value, whereas 1.4, 1.41, 1.414, etc., are decimal approximations of $\sqrt{2}$.



c) The expression 3^2, or "three squared," can be viewed as the value of a square whose edges have value 3 (or the area of a square whose edges have length 3). Some students may view "squaring a number *n*" literally as "forming a square on a segment whose length is *n*."

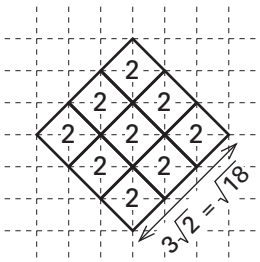


d) $\sqrt{5}$ is the length of the edge of a square whose area is 5. Since the area of a square with side length s is $s \times s = s^2$, the area of a square with side length $\sqrt{5}$ is $\sqrt{5} \times \sqrt{5} = (\sqrt{5})^2 = 5$.



e) Since $\sqrt{2}$ is the length of the edge of a square whose area is 2, then $3\sqrt{2}$ can be viewed as that length tripled, or $\sqrt{2} + \sqrt{2} + \sqrt{2}$.

Some students may point out that the length $3\sqrt{2}$ is also the length of the edge of a square whose area is 18. Hence, $3\sqrt{2}$ is equivalent to $\sqrt{18}$, as illustrated below.



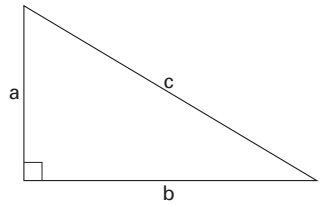
Connector Teacher Activity (cont.)

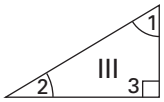
ACTIONS

2 Give each pair of students 2 sheets of blank paper, 1 copy of Connector Master A, 1 copy of Connector Master B, a blank note card, tape, and a pair of scissors. Ask the pairs to carry out the instructions on Master B. Discuss their observations and their ideas about completing the "Notice that …" statements in Steps C, D, and F on Connector Master B.

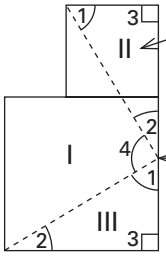
Reasoning and Radicals Lesson **9**

Connector Master A





$\angle 1 + \angle 2 + \angle 3 = 180°$
Since $\angle 3 = 90°$, then $\angle 1 + \angle 2 = 90°$.



Region II is congruent to Region III, so the corresponding angles are congruent.

$\angle 1 + \angle 2 + \angle 4 = 180°$ (they form a straight angle). Since $\angle 1 + \angle 2 = 90°$, then $\angle 4 = 90°$.

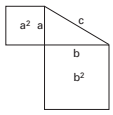
COMMENTS

2 It may be helpful to remind the students about common notation for *line segments* and their *lengths*. For example, when referring to the line segment with endpoints R and S, one typically uses the notation $\overline{RS}$. This refers to the set of points R and S and all the points between them on the line connecting R and S. Frequently, the length of a segment is represented by a single letter, such as *a* or *b*; other times the length is denoted by the letters of the endpoints of the segment with no bar over the letters. For example, the length of $\overline{RS}$ can be denoted as RS.

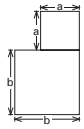
Lesson **9** Reasoning and Radicals

Connector Master B

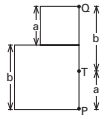
Step A Draw squares with edge lengths *a* and *b* on the legs of the right triangle on Connector Master A. Use a note card as a straightedge and guide for drawing right angles.



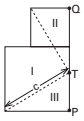
Step B Trace the squares from Step A to form 2 adjacent squares with right edges collinear.



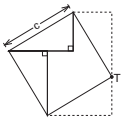
Step C Label points P and Q as shown at the right. Locate a point T on edge $\overline{PQ}$ so that PT = *a*. Notice that TQ = *b*, since…



Step D Draw the dotted lines shown to form the regions labeled I, II, and III. Notice that regions II and III are congruent. Notice also that the angle formed at the intersection of the dotted lines is a right angle since…



Step E Cut out regions I, II, and III. Then reassemble the regions to obtain a square.



Step F Tape the square formed in Step E on the hypotenuse of the triangle on Connector Master A. Notice that…

In Step D, the angle formed at the intersection of the dotted lines is a right angle because its measure is 180° minus the measures of 2 acute angles of a right triangle (see diagram at the left). Keep in mind that students' explanations may be less detailed than the one shown at the left.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

3 Give each pair of students 2 sheets of 1-cm grid paper and a copy of Connector Master C. Ask them to complete a). Discuss their results. Then repeat for b)-d).

Reasoning and Radicals Lesson **9**

Connector Master C

Use a note card as a straightedge and as a guide for drawing square corners.

a) From a geometric perspective, $\sqrt{n}$ is the length of the edge of a square whose area is n. On 1-cm grid paper and using a 1-cm square as 1 area unit, form all the actual lengths $\sqrt{n}$, for every integer n such that $1 \le n \le 12$. Tape your results, in order, on another sheet of paper, showing the square associated with each length. Label actual lengths.

b) On plain (no grid) paper, draw any 2 noncongruent squares so they are adjacent and their right edges are collinear. Then, dissect and rearrange these 2 adjacent squares to form a 3rd square whose area is equal to the sum of the areas of the 2 squares.

i) How many different 3rd squares are possible for a pair of adjacent squares?
ii) What if the 2 adjacent squares are congruent?

c) Challenge. On plain paper, draw a large square. Dissect and rearrange this large square to form 2 noncongruent adjacent squares whose total area equals the area of the large square.

i) How many different pairs of adjacent squares can be formed so the sum of their areas equals the area of the large square?
ii) What if the 2 adjacent squares must be congruent?
iii) What if the area of the large square and the area of 1 of the 2 adjacent squares are given?

d) List your conjectures, questions, and generalizations.

COMMENTS

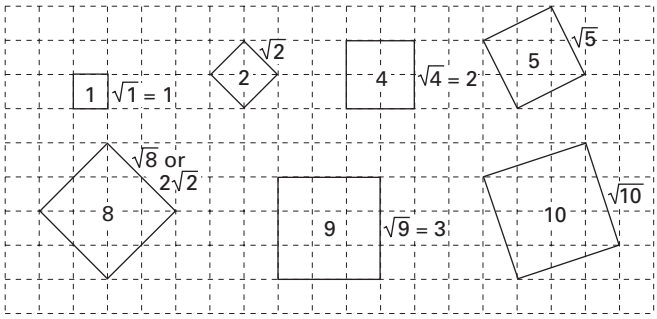
2 (continued.)
The procedure described on Master B demonstrates the *Pythagorean Theorem*, i.e., given a right triangle with legs of lengths a and b and hypotenuse of length c, squares of areas a^2 and b^2 can be dissected and rearranged to form a 3rd square of area c^2; hence, $a^2 + b^2 = c^2$. This procedure was introduced in Lesson 28 of *Math Alive! Course II*. You might reread that lesson for other discussion ideas.

The Pythagorean Theorem is named after the Greek philosopher, Pythagoras, who is considered the first person to have proven it. It is referred to as a theorem because it has been proven. Although Pythagoras proved this theorem during the 5th century B.C., there is evidence the relationship was used centuries earlier (see the Focus Activity, Action 4).

3 Some students may need help interpreting the language and notation on Connector Master C.

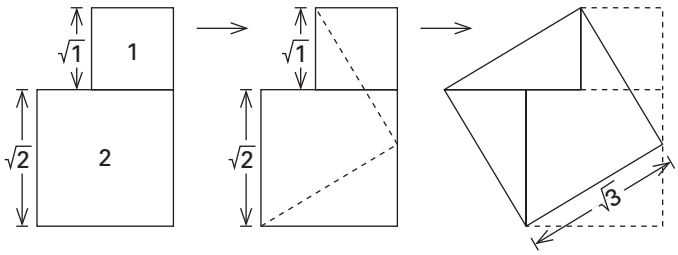
a) Since $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, … $\sqrt{12}$ are the lengths of the sides of squares with areas 1, 2, 3, … 12, the lengths $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, … $\sqrt{12}$ can be formed by constructing squares with areas 1-12.

The numbers 1, 4, and 9 are *perfect squares*. Hence, if 1 small square on a grid is 1 area unit and the length of an edge of an area unit is 1 linear unit, squares of areas 1, 4, and 9 area units have edge lengths $\sqrt{1} = 1$, $\sqrt{4} = 2$, and $\sqrt{9} = 3$ linear units, respectively. The edges of these squares coincide with grid lines, and vertices coincide with intersections of grid lines (see below). "Tilted squares" with edge lengths $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$, and $\sqrt{10}$ and with vertices that coincide with intersections of grid lines can also be formed, as illustrated below. In Lessons 11 and 28 of *Math Alive! Course II* students developed formula free strategies for finding the areas of such tilted squares.



Connector Teacher Activity (cont.)

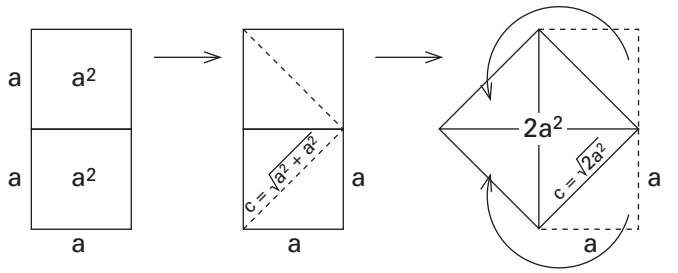
ACTIONS



Form adjacent squares of area 1 and 2

Dissect

Rearrange to form a square of area $(\sqrt{1})^2 + (\sqrt{2})^2 = (\sqrt{3})^2 = 3$



Form adjacent squares each of area a^2

Dissect

Rearrange to form a square of area $a^2 + a^2 = c^2$

COMMENTS

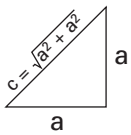
To form squares with areas 3, 6, 7, 11, and 12 it is necessary to combine pairs of squares. For example, using the dissection method from Connector Master B, a square of area 3 can be formed by dissecting and rearranging squares of areas 1 and 2 (see diagram at the left).

Similarly, a square of area 6 can be formed from squares of areas 4 and 2, or 5 and 1; area 7 from 5 and 2, or 6 and 1; area 11 from 9 and 2, or 10 and 1; and area 12 from 10 and 2, 8 and 4, 3 and 9, 5 and 7, or 11 and 1. Note: the vertices of these squares do not all coincide with the intersections of grid lines.

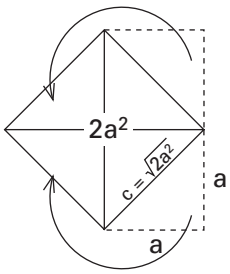
b) i) one

b) ii) If the 2 adjacent squares are congruent, a 3rd square can be constructed as shown at the left (students may use specific examples rather than the general case).

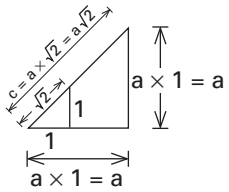
In this case, some students may "see" the length of c as the hypotenuse of an isosceles right triangle with legs of length a; hence, they may record c as $\sqrt{a^2 + a^2}$ (see diagram below). Others may "see" the length c as the edge of a square whose area is $2a^2$; hence they may record $c = \sqrt{2a^2}$ (see below). Still others may point out that expanding each side of an isosceles right triangle with sides of length 1 and hypotenuse $\sqrt{2}$ by a factor of a produces an isosceles right triangle with sides a, a, and $a\sqrt{2}$ (see below). Hence, $c = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$. Note: equivalent radicals are explored in the Focus activity of this lesson.



c as the hypotenuse of a right triangle with legs *a* and *a*.



c as the edge of a square with area $2a^2$.



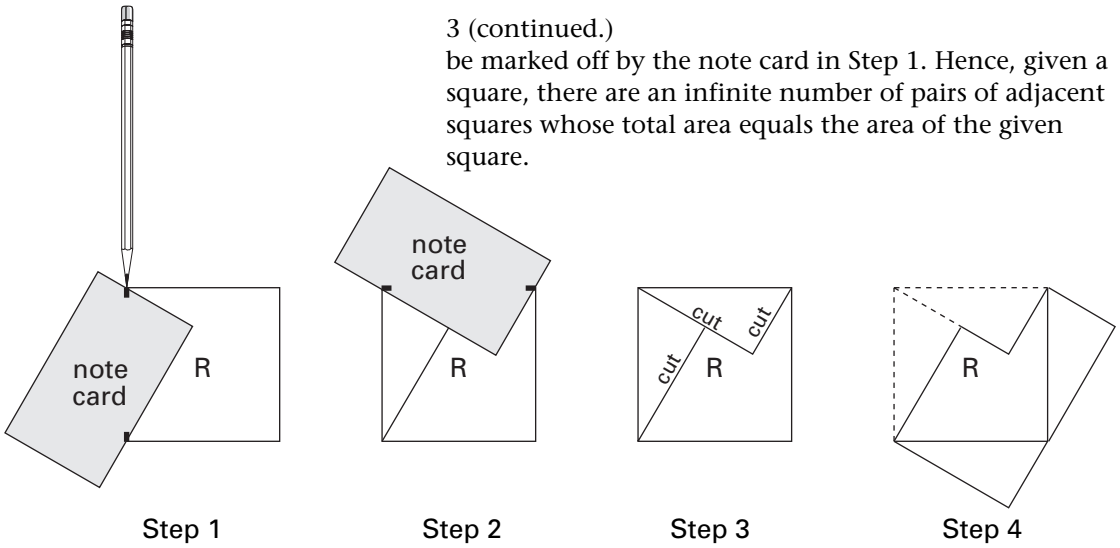
c as the result of expanding both legs of a right triangle by a factor of *a*, assuming the legs have length 1 unit.

c) i) Steps 1 and 2 on the next page illustrate a method of using a note card to mark congruent right triangles on adjacent sides of a square. Those triangles are cut and rearranged in Steps 3 and 4 to form adjacent squares. There are an infinite number of right triangles that can

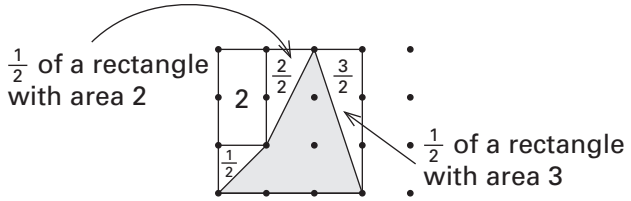
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



4 Give each student a copy of Connector Student Activity 9.1 and ask them to complete Problem 1. Invite volunteers to demonstrate their reasoning and results at the overhead. Repeat for Problems 2-5.

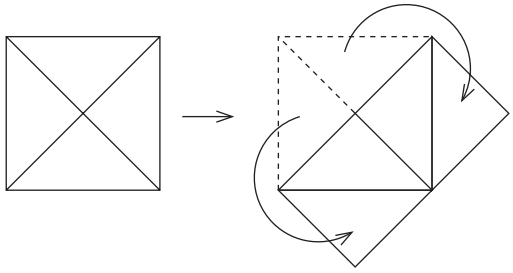


Area = $9 - 2 - 2/2 - 3/2 - 1/2$
= 4 square units

COMMENTS

3 (continued.)
be marked off by the note card in Step 1. Hence, given a square, there are an infinite number of pairs of adjacent squares whose total area equals the area of the given square.

c) ii) Dissect a square into 4 congruent right triangles and rearrange to form 2 congruent squares, as shown below.



c) iii) In this case, there is only one 3rd square possible.

4 The intent here is to illustrate the usefulness of knowing a variety of ways to determine areas and lengths. To ensure that students explore certain methods in Problems 1-3, restrictions are placed on the strategies to use. Then in Problems 4 and 5 students are free to use any strategy they wish. Notice that methods that are convenient in one setting, may be less convenient or not at all useful in another.

1) For this problem, students need to rely on "formula free" strategies (see Lessons 11 and 28 of *Math Alive! Course II* and Lessons 19 and 35 of *Course I*) that are based on the meanings of area and perimeter, together with their understanding of the meaning of $\sqrt{n}$ as the length of the edge of a square whose area is n. One strategy for finding the area of this polygon is to enclose it in a rectangle and subtract the area of the portion of the rectangle that is not part of the polygon, as shown at the left.

Connector Teacher Activity (cont.)

ACTIONS

Reasoning and Radicals Lesson **9**

Connector Student Activity 9.1

NAME _____ DATE _____

1 *Without* using any measuring tools, *without* using the Pythagorean Theorem, and *without* using area formulas, find the area and perimeter of Polygon A shown at the right. Mark the diagram to show your methods. Give *actual* measures rather than decimal approximations.

2 Use the Pythagorean Theorem where necessary to find all actual side lengths of Polygon B. Label each length on the diagram and show all of your calculations.

3 Use area formulas to compute the area of Polygon C. Label the diagram to show your calculations and formulas.

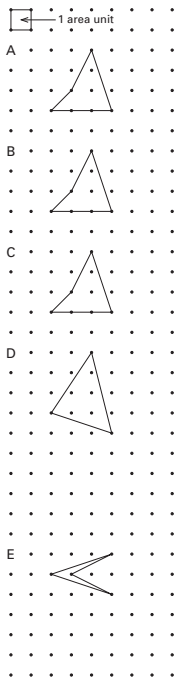
4 Complete a)-c) below for Polygon D. Show your methods.

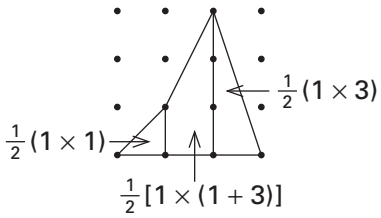
a) Determine its actual area.

b) Determine its actual perimeter.

c) Determine the actual and approximate lengths of its 3 altitudes.

5 Show the methods that you use to find the area and perimeter of Polygon E.

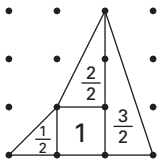




Area = ½ + 2 + ³⁄₂ = 4 square units
Perimeter = $\sqrt{2} + \sqrt{5} + \sqrt{10} + 3$ linear units

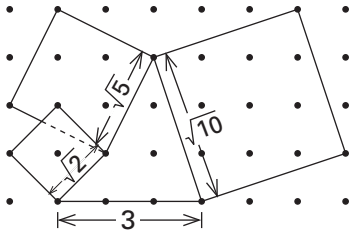
COMMENTS

Another method for finding the area is to subdivide the polygon into parts:



Area = ½ + 1 + ²⁄₂ + ³⁄₂
= 4 square units

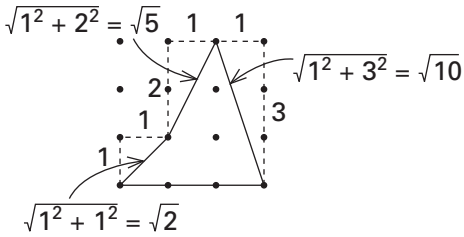
Computing the length of each "tilted segment" without using the Pythagorean Theorem requires constructing a square on the length and finding the area of the square. The length is the positive square root of the area of the square:



Perimeter = $\sqrt{2} + \sqrt{5} + \sqrt{10} + 3$

Note that some students may incorrectly conclude that $\sqrt{2} + \sqrt{5}$, for example, is $\sqrt{7}$. Computing $\sqrt{2} + \sqrt{5}$ on their calculators and comparing the result to $\sqrt{7}$ shows that $\sqrt{2} + \sqrt{5} \neq \sqrt{7}$, and in general that $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$. This idea is investigated further in Action 9 of the Focus activity.

2) Side lengths can be determined using the Pythagorean Theorem, as shown below.



3) Area formulas were developed in Lesson 3 of this course. To use formulas to compute the area of this polygon requires breaking it into parts. For example, the polygon can be divided into 2 triangles and 1 trapezoid, as shown at the left.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

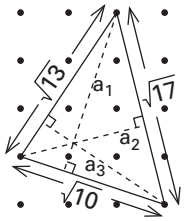
COMMENTS

4 (continued.)

4a) The area of the triangle is 5.5 square units.

4b) The perimeter is $\sqrt{13} + \sqrt{17} + \sqrt{10}$ linear units.

4c) Students may need to be reminded that an *altitude* of a triangle is a segment extending from a vertex perpendicular to the opposite side (the *base*) of the triangle; the term altitude is also used to represent the length of a segment that is an altitude. Every triangle has 3 altitudes.



Students may be challenged by the fact that the altitudes of this triangle—see a_1, a_2, and a_3 in the diagram at the left—do not intersect their corresponding bases at dots on the grid. However, since the area of a triangle is ½ the product of the base length times the corresponding altitude, and since the area is 5.5, the following equations can be used to determine the 3 altitudes, a_1, a_2, and a_3:

$5.5 = (\frac{1}{2})(\sqrt{10})(a_1)$,
$5.5 = (\frac{1}{2})(\sqrt{17})(a_2)$, and
$5.5 = (\frac{1}{2})(\sqrt{13})(a_3)$.

Note that students may reason in a variety of ways to solve the above equations (rather than using rote procedures). For example, some may reason that since the area of the triangle is $5.5 = (\frac{1}{2})(\sqrt{10})(a_1)$, then by combining 2 copies of the triangle, a parallelogram with area 11, base $\sqrt{10}$, and altitude a_1 can be formed. Hence, the area of the parallelogram is $11 = (\sqrt{10})(a_1)$, and so the exact value of a_1 is $\frac{11}{\sqrt{10}}$ (this is a correct response—it isn't expected that students simplify radicals here). Similarly, the values of the other 2 altitudes are: $a_2 = \frac{11}{\sqrt{17}}$ and $a_3 = \frac{11}{\sqrt{13}}$. The approximate lengths of a_1, a_2, and a_3 to 2 decimal places are 3.48, 2.67, and 3.05 linear units, respectively.

5) The area of this polygon is 1 square unit, and its perimeter is $2\sqrt{10} + 2\sqrt{5}$ linear units.

Focus Teacher Activity

OVERVIEW & PURPOSE

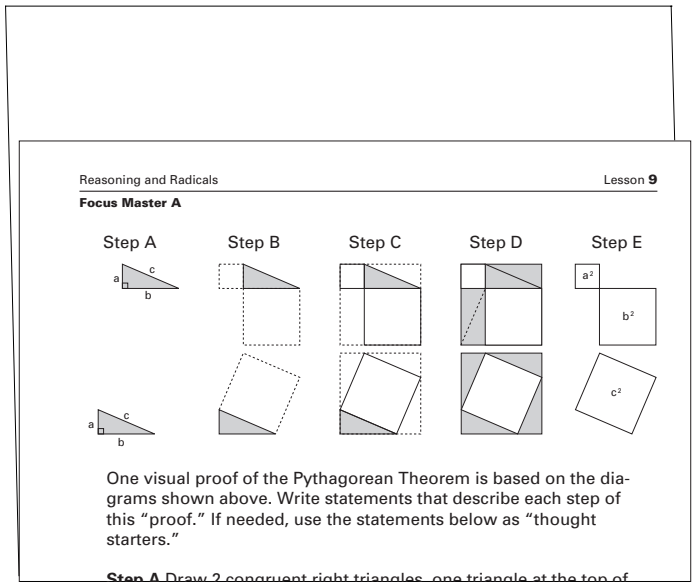
Students investigate another proof of the Pythagorean Theorem. They use inductive and deductive reasoning to demonstrate and prove mathematical relationships for right triangles and radical expressions. They apply these properties to 30°- 60°- 90° and 45°- 45°- 90° triangles.

MATERIALS

- ✔ Focus Masters A and D, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Masters B and C, 1 transparency of each.
- ✔ Focus Master E, 2 copies per student and 1 transparency.
- ✔ Focus Student Activities 9.2 and 9.3, 1 copy of each per student and 1 transparency of each.
- ✔ 1-cm grid paper (see *Blackline Masters*), 1 sheet per student and 1 transparency.
- ✔ Ruler, protractor, and scissors, 1 of each per student.
- ✔ String, 1 18-inch length per group.
- ✔ Butcher paper strips (4"- 6" long) and marking pens, several of each per group.
- ✔ Blank paper, 1 sheet per student.

ACTIONS

1 Arrange the students in groups and give each group a copy of Focus Master A, folded so the written statements associated with Steps A-E are concealed. Ask the groups to write their own statements that describe the sequence of diagrams shown for Steps A-E, unfolding Master A to read the "starters" if they get stuck or to prompt refinements of their ideas. Invite volunteers to share their group's statements. Use this as a context for discussing the differences between *inductive* and *deductive* reasoning and how *theorems* and *proofs* are used in building a *mathematical system*.


Reasoning and Radicals Lesson **9**

Focus Master A

One visual proof of the Pythagorean Theorem is based on the diagrams shown above. Write statements that describe each step of this "proof." If needed, use the statements below as "thought starters."

Step A Draw 2 congruent right triangles, one triangle at the top of

COMMENTS

1 Following are possibilities for Steps B-D (students may have other correct wordings):

Step B: Draw a square on each leg of the top triangle, and draw a square on the hypotenuse of the bottom triangle. (Note: the lengths of the edges of the triangles determine the lengths of the edges of the squares.)

Step C: Enclose the top figure and the bottom figure each in the smallest square possible. Notice the areas of the enclosing squares are equal because they each have side length $a + b$.

Step D: Draw a diagonal of the rectangle in the lower left corner of the top figure. Notice the 8 triangles (4 on the top figure and 4 on the bottom) are congruent because their corresponding sides and angles are all congruent.

Step E: If the 8 triangles are cut away, then the total area of the 2 squares that remain in the top figure equals the area of the remaining square in the bottom figure, because equal areas have been subtracted from equal areas.

In summary, if a right triangle has legs of length a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$. That is, the sum of the areas of the squares on the legs of a right triangle is equal to the area of the square on the hypotenuse.

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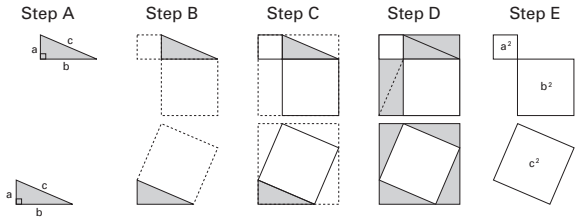
Focus Teacher Activity (cont.)

ACTIONS

Reasoning and Radicals Lesson **9**

Focus Master A



One visual proof of the Pythagorean Theorem is based on the diagrams shown above. Write statements that describe each step of this "proof." If needed, use the statements below as "thought starters."

Step A Draw 2 congruent right triangles, one triangle at the top of a sheet of paper, and the other at the bottom of the sheet. On each triangle, label the length of the short leg *a*, the long leg *b*, and the hypotenuse *c*.

Step B Draw a square on each leg of the top triangle, and draw a…

Step C Enclose the top figure and the bottom figure each in the smallest square possible. Notice that the areas of the enclosing squares are equal because…

Step D Draw a diagonal of the rectangle in the lower left corner of the top figure. Notice the 8 triangles (4 in the top figure and 4 in the bottom) are congruent because…

Step E If the 8 triangles are cut away, then… because…

In summary, if a right triangle has legs of length *a* and *b* and hypotenuse of length *c*, then…

COMMENTS

1 (continued.)
The Egyptians and Babylonians used geometry facts and relationships in surveying and architecture more than 5000 years ago. However, because they had no system for proving that formulas and procedures produced correct results, they frequently obtained incorrect results. It was the ancient Greeks, around 400-200 B.C., who shifted emphasis from computation to proving theorems and from reasoning *inductively* based on experimentation to reasoning *deductively* based on given information. While the Greeks used experimentation and observations (i.e., *inductive reasoning*) to formulate their ideas, they did not accept ideas without proof by *deductive reasoning*.

Perhaps the most famous example of Greek mathematical reasoning is *Euclid's Elements*, a series of 13 books written about 300 B.C. In the *Elements*, Euclid used 10 basic statements (called postulates) to deductively prove over 600 theorems. A *mathematical system* has a specified set of *undefined terms* (terms like point, line, and plane that are used to define other terms), *definitions*, and *postulates* (statements that are assumed to be true without proof). These undefined terms, definitions, and postulates provide the basis for proving statements called *theorems*. Once a theorem is proven, it can be used in proofs of other theorems.

Many of the conjectures that students have formulated throughout this course and earlier *Math Alive!* courses have involved reasoning inductively from observations. The visual proof presented on Focus Master A illustrates the process of reasoning deductively from facts that are already known or proven. Although creating formal proofs is not the focus of this lesson, regularly asking students to build arguments that show *why* their conjectures are true (as compared to only giving specific examples to show *that* the conjectures are true) lays important groundwork for the more formal study of proof by deduction.

The method from Focus Master A and the method from Connector Master B are both deductive proofs of the Pythagorean Theorem.

Focus Teacher Activity (cont.)

ACTIONS

2 Give each student a ruler, a protractor (see *Blackline Masters*), and a sheet of blank paper. Place a transparency of Focus Master B on the overhead and ask the groups to complete a). Discuss, using their findings as a context for introducing the *triangle inequality*. Then have them complete and discuss b) and c).

Lesson **9** Reasoning and Radicals

Focus Master B

a) Reason visually to determine which, if any, of the following statements are true about *all* triangles with side lengths *a*, *b*, and *c*:

$a + b = c$
$a + b < c$
$a + b > c$

b) Use a straightedge to draw 4 noncongruent acute triangles.

i) Use a protractor and ruler to measure all side lengths and angles. Label your drawings to show lengths to the nearest tenth of a centimeter and angles to the nearest degree.

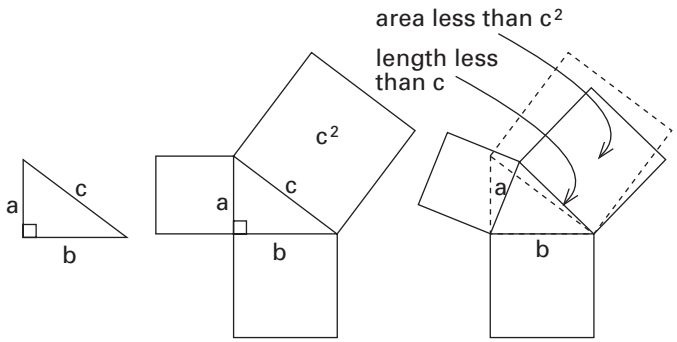
ii) Draw squares on each side of each triangle and record the areas of the squares.

iii) Using information from i)-ii) as evidence, reason inductively to complete Conjecture 1 below. Then give deductive arguments to show why your conjecture must always be true.

Conjecture 1 If a triangle is acute with side lengths *a*, *b*, and *c*, where *c* is the longest side length, then $a^2 + b^2$ ...

c) Complete Conjecture 2 below and give inductive and deductive arguments to support your conjecture:

Conjecture 2 If a triangle is obtuse with side lengths *a*, *b*, and *c*, where *c* is the longest length, then $a^2 + b^2$ ...

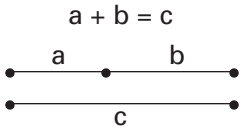


COMMENTS

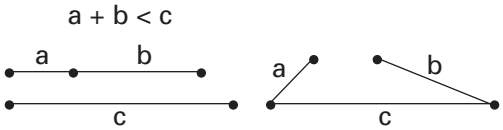
2 Acute and obtuse triangles are also referred to as *oblique* triangles.

a) Here is an informal argument that one group of students gave to support the idea that the sum of the lengths of any 2 sides of a triangle is always greater than the length of the 3rd side:

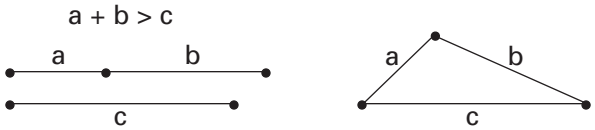
If a, b, and c are the lengths of 3 line segments, and a + b = c, then 2 congruent line segments can be formed, but not a triangle:



This diagram shows that if a + b < c, then all 3 segments couldn't meet:



Therefore, in order to form a triangle, a + b must be greater than c:



We can use the same reasoning to show that b + c must be greater than a, and a + c must be greater than b, in order to form a triangle.

The above relationship—the sum of the lengths of any 2 sides of a triangle must be greater than the length of the 3rd side—is called the *triangle inequality*.

b) iii) If students report that $a^2 + b^2 \neq c^2$, you might ask them to describe the inequality more specifically—i.e., can ≠ be replaced by < or by >? Here is one group's argument:

If we form a right triangle with a the shortest side length and c the longest, then $a^2 + b^2 = c^2$. However, to make the triangle acute, we can keep a and b constant and "reduce" the length of c. This also reduces c^2. Therefore, $a^2 + b^2 > c^2$, as shown at the left.

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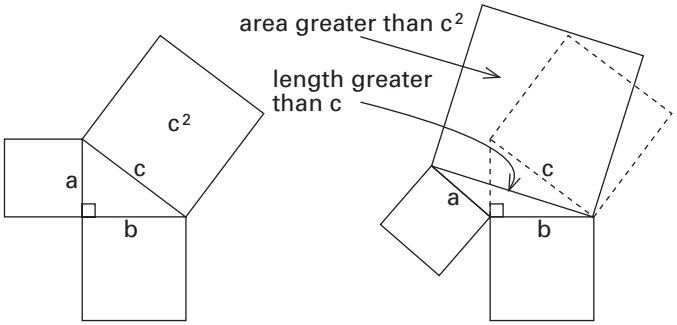
Focus Teacher Activity (cont.)

ACTIONS



3 If it hasn't come up previously, discuss the meaning of the *converse* of a *conditional statement*. Then, ask each group of students to complete a) below. Discuss their ideas, clarifying as needed. Repeat for b).

a) Write the converse of the Pythagorean Theorem.

b) Conjecture about whether the converse of the Pythagorean Theorem is true, and build arguments to show why it is or is not true.

COMMENTS

2 (continued.)
c) Using similar reasoning to that given above for b), if the right angle of a right triangle is increased to form an obtuse triangle and side lengths a and b are held constant, then side length c increases, and hence, the area of the square on side c increases. Thus, $a^2 + b^2 < c^2$, as shown at the left.

Hence, for any triangle with side lengths a, b, and c, where c is the longest side, $a + b > c$, and one of the following is true:

$a^2 + b^2 = c^2$ (the triangle is a right triangle)
$a^2 + b^2 > c^2$ (the triangle is acute)
$a^2 + b^2 < c^2$ (the triangle is obtuse)

3 A statement of the form "If_____, then _____" is called a *conditional statement*. The "if" portion of the statement is called the *hypothesis*, and the "then" portion is called the *conclusion*. For a true conditional statement, whenever the hypothesis is true, the conclusion is also true.

If the hypothesis and conclusion of a statement are interchanged, the new statement is called the *converse* of the original. The fact that a statement is true does not necessarily mean its converse is true. For example, the conditional statement, "If a polygon is a square, then it is a rectangle" is true. However, the converse, "If a polygon is a rectangle, then it is a square" is not necessarily true.

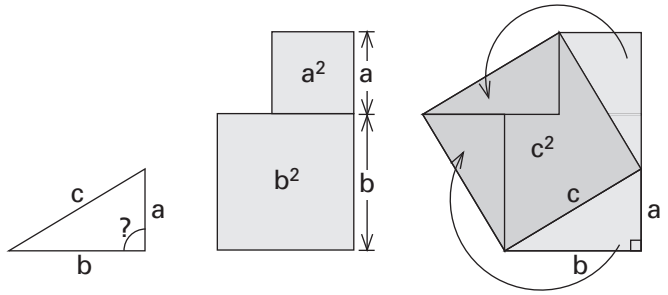
The Pythagorean Theorem states, "If a triangle is a right triangle with legs of length a and b and hypotenuse of length c, then $a^2 + b^2 = c^2$." Hence, the converse of the Pythagorean Theorem is: "If a triangle has sides of length a, b, and c, and $a^2 + b^2 = c^2$, then the triangle is a right triangle." Some students may reason inductively by finding examples that verify the converse of the Pythagorean Theorem. For example, they may draw several triangles for which the sum of the areas of the squares on 2 sides is equal to the area of the square on the 3rd side, and then measure to show that each contains a right angle. Such evidence leads to the conjecture that the converse holds, but it is not proof that it holds.

Some students may provide a visual proof that the converse of the Pythagorean Theorem is true. For example:

The statement $a^2 + b^2 = c^2$ about a triangle with sides of length a, b, and c, tells us there are squares with side lengths

Focus Teacher Activity (cont.)

ACTIONS



4 Give each group of students a piece of string (about 18") and a marking pen. Place a transparency of Focus Master C on the overhead, revealing part a) only for students to investigate. Suggest that they mark equal lengths on the string rather than tie knots. Then invite volunteers to share their group's ideas with the class. Repeat for parts b) and c). Use this as a context for discussing the meanings of these terms: *prime numbers, factors, multiples, composite numbers, relatively prime numbers,* and *Pythagorean triples*.

Reasoning and Radicals Lesson **9**

Focus Master C

a) Paintings in ancient Egyptian tombs from the 15th century B.C. show Egyptian geometers—called "rope stretchers," or surveyors—using long ropes with equally spaced knots. Historians believe these ropes were used to aid in the construction of right angles and right triangles. How do you think the rope stretchers did this?

b) What are all the different right triangles with integral side lengths that can be formed from all or part of a 35-knot rope?

c) Three positive integers that work in the Pythagorean Theorem are called a *Pythagorean triple*. A Pythagorean triple is called a *primitive* if the 3 integers have no common factors other than 1—that is, a triple is a primitive if the 3 integers are *relatively prime*. Enlarging a right triangle by a scale factor that is a whole number creates a new Pythagorean triple that is called a *multiple* of the original triple.

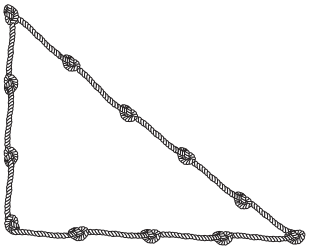
What are all the Pythagorean triples such that each number in the triple is less than 35? Which of these are primitives? Which are multiples of primitives?

COMMENTS

a and b whose total area equals the area of a square with side length c. In the Connector activity, we showed that 2 adjacent squares with areas a^2 and b^2 can be dissected and rearranged to form exactly one 3rd square with area $a^2 + b^2$.

To form this 3rd square, we cut off (from the 2 adjacent squares) and rearranged 2 congruent right triangles with legs of lengths a and b and hypotenuse c (see diagram at the left). We think there is only one possible triangle that can have side lengths a, b, and c; therefore, the triangle with side lengths a, b, and c must be a right triangle. Note that in Lesson 10 students explore in depth the notion that 3 side lengths determine a unique triangle.

4 a) Reasoning according to the converse of the Pythagorean Theorem, if one can form a triangle with side lengths *a*, *b*, and *c* such that $a^2 + b^2 = c^2$, then the triangle is a right triangle. For example, a 12-unit string length can be formed into a triangle with side lengths 3, 4, and 5 linear units (where the space between 2 consecutive knots is 1 linear unit). Since $3^2 + 4^2 = 5^2$, the triangle is a right triangle. The students may find it helpful to use pencil points to anchor the corners of the triangle.



Notice that, while there are other right triangles that can be formed from a string 12 units long, there is exactly 1 right triangle with integral side lengths and perimeter 12 units.

b) A 35-knot rope contains 34 linear units. Right triangles with the following integral side lengths can be formed using a 35-knot rope: 3-4-5; 6-8-10; and 5-12-13.

c) The terms *primitive* and *multiple*, as related to *Pythagorean triples*, are not intended for memorization. Rather, discussing them provides a context for determining and discussing Pythagorean triples and for recalling and clarifying the students' understandings of the meanings of factors and multiples of a number, prime and composite numbers, and relatively prime numbers. These terms were introduced in *Math Alive! Courses I and II*.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

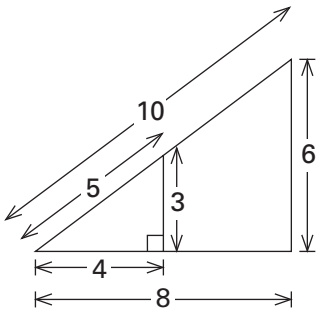
4 (continued.)
From a geometric perspective, the positive *factors* of a positive integer *n* are the *dimensions* of all possible rectangles that can be formed by *n* whole square tile, where the length of the edge of a tile is 1 linear unit. For example, the positive factors of 12 are 1, 12, 3, 4, 2, and 6 since the only rectangles that can be formed from 12 whole tile have dimensions 1 by 12, 3 by 4, and 2 by 6.

A *prime number* is a positive integer that has exactly 2 positive factors, itself and 1. Hence, the only rectangle that can be formed from a prime number of whole square tile is a non-square rectangle whose dimensions are 1 and the prime number. For example, 7 is prime because it has exactly 2 positive factors (1 and 7). The only rectangle that can be formed by 7 whole tile has dimensions 1 by 7.

A *composite number* is a positive integer with more than 2 positive factors; hence, more than 1 rectangle can be formed from a composite number of whole tile. Notice the number 1 is neither prime nor composite, since it has only 1 whole number factor.

Two or more integers are *relatively prime* if they have no common positive factors other than the number 1. Note that the numbers in a set of relatively prime numbers are not necessarily prime themselves. For example, neither 4 nor 15 are prime numbers, but they are relatively prime. The positive factors of 4 are 1, 4, and 2; the positive factors of 15 are 1, 15, 3, and 5. The only positive factor that 4 and 15 have in common is 1; hence, they are relatively prime.

c) Notice that this question asks the students to find triples such that *each number* is less than 35, whereas a) asks them to determine triples such that the *total* of the 3 numbers is less than 35.



In the example at the left, since the scale factor of enlargement, 2, is a positive integer, the Pythagorean triple 6-8-10 is called a *multiple* of the *primitive* 3-4-5.

The Pythagorean triples that contain integers less than 35 are listed below:

Primitives: 3-4-5, 5-12-13, 7-24-25, 8-15-17

Multiples: 6-8-10, 9-12-15, 12-16-20, 15-20-25, 18-24-30, 10-24-26, 16-30-34

Focus Teacher Activity (cont.)

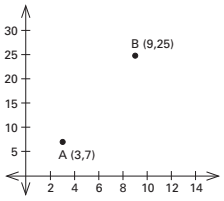
ACTIONS

5 Give each group of students a copy of Focus Master D, and ask them to complete Problem 1. Discuss their results. Repeat for Problem 2. Discuss the *distance formulas* and *midpoint formulas* invented by the students.

Lesson **9** Reasoning and Radicals

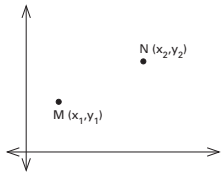
Focus Master D

1. Given 2 points A and B with coordinates (3,7) and (9,25):



i) Find the distance between A and B.
ii) Write the coordinates of the midpoint of $\overline{AB}$.

2. Given 2 points M and N with coordinates (x_1,y_1) and (x_2,y_2), respectively:

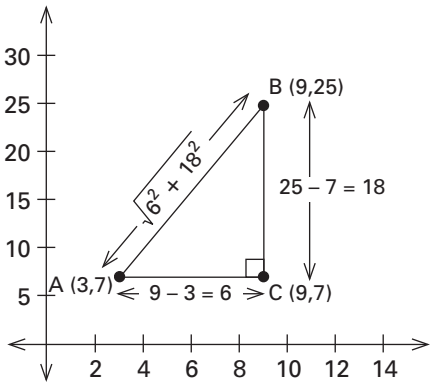


i) Write a formula for the distance between the 2 points.
ii) Write a formula for the coordinates of the midpoint of $\overline{MN}$.
iii) Do you think your results for i) and ii) work for any 2 points on the coordinate system? Explain your reasoning.

COMMENTS

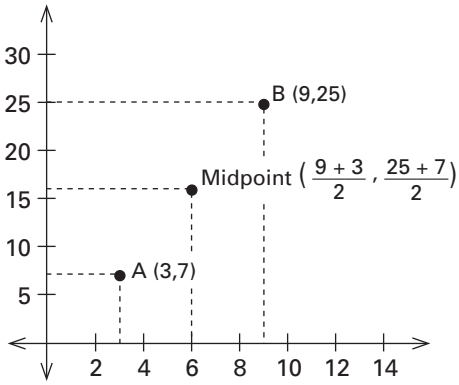
5 If students find these problems particularly challenging, you might give each student a sheet of 1-cm grid paper and ask them to sketch the coordinate axes, mark 2 points in the first quadrant, and determine the distance between those points and the coordinates of the *midpoint*. Then repeat for points in other quadrants.

1 i) Finding the distance between A and B is equivalent to finding the length of the segment with endpoints A and B. Drawing segments $\overline{AC}$ and $\overline{BC}$ so that $\overline{AC}$ is perpendicular to $\overline{BC}$, as shown below, creates a right triangle with right angle C. Hence, $AC^2 + BC^2 = AB^2$. Since AC = 9 – 3 = 6, and BC = 25 – 7 = 18, then $AB^2 = 6^2 + 18^2$ and AB = $\sqrt{6^2+18^2} \approx 18.97$.



1 ii) Some students may compute the *x*-coordinate of the midpoint of $\overline{AB}$ as $3 + {}^{6}\!/_{2} = 6$, and the *y*-coordinate as $7 + {}^{18}\!/_{2} = 16$.

Or, since the average (mean) of 2 numbers is midway between the numbers, some students may determine the coordinates of the midpoint of $\overline{AB}$ by finding the average, or "leveled-off," value of the *x*- and *y*-coordinates of points A and B. Thus, the midpoint of $\overline{AB}$ has *x*-coordinate, $\frac{(3+9)}{2}$ and *y*-coordinate $\frac{(7+25)}{2}$. So, the midpoint of $\overline{AB}$ is (6,16).



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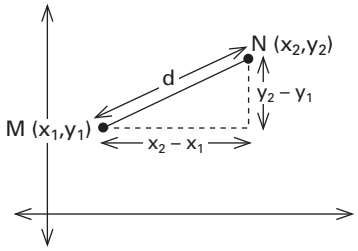
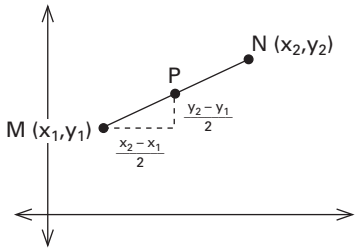
Focus Teacher Activity (cont.)

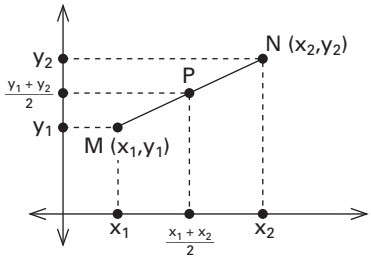
ACTIONS



$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance Formula



The coordinates of P, the midpoint of $\overline{MN}$ are $\left(x_1 + \frac{x_2 - x_1}{2}, y_1 + \frac{y_2 - y_1}{2}\right)$.



The coordinates of P, the midpoint of $\overline{MN}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Midpoint Formulas

6 Give each student a blank sheet of paper (no grids). Read aloud a)-g) below, pausing to allow time for students to complete each step before proceeding to the next one. Then discuss the students' observations about the squares. Point out *equivalent radical expressions* for the actual lengths of edges as they are shared during discussion of g). Then complete h).

a) Sketch a square and label its area 12 square units.

b) Label the length of an edge of the square.

c) Sketch a 2nd square so that the area of the 2nd square is 4 times the area of the original square.

COMMENTS

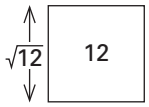
5 (continued.)
2 i) Constructing a right triangle, as in Problem 1, and applying the Pythagorean Theorem, the distance, *d*, between M and N is the length of $\overline{MN}$, or $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. This is commonly referred to as the *distance formula*, the formula for the distance between 2 points.

2 ii) Some students may compute the *x*-coordinate of the midpoint of $\overline{MN}$ as $x_1 + \frac{x_2 - x_1}{2}$, and the *y*-coordinate as $y_1 + \frac{y_2 - y_1}{2}$. Or, some may determine the midpoint of $\overline{MN}$ by averaging the *x*- and *y*-values of the endpoints, i.e., $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$. These formulas for determining the midpoint of a line segment are commonly referred to as *midpoint formulas* (see diagrams below).

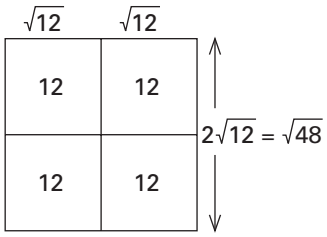
2 iii) To predict whether the formulas developed for i) and ii) apply for all pairs of points on the coordinate system, you might encourage students to explore several examples, including segments whose endpoints fall in any of the 4 quadrants. Note: the distance and midpoint formulas for 2 points apply regardless the position of the points on the coordinate system.

6 It is sufficient for students to make rough sketches rather than to use rulers to measure. Here are diagrams that illustrate a)-f) and show several equivalent radical expressions:

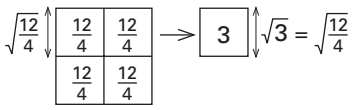
a)-b)



c)-d)



e)-f)



Focus Teacher Activity (cont.)

ACTIONS

d) Label the area and length of an edge of the 2nd square.

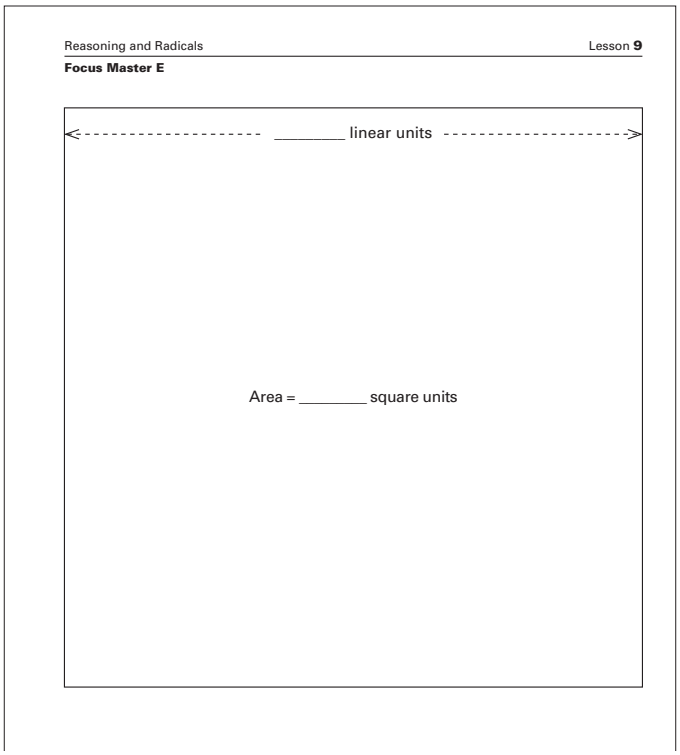
e) Sketch a 3rd square such that the area of the 3rd square is ¼ the area of the original square (i.e., the square from a).

f) Label the area and edge length of the 3rd square.

g) Record your observations about mathematical relationships among these squares and their edges.

h) (Optional) Repeat steps a)-g) for a square of area 8 square units.

7 Distribute 2 copies of Focus Master E and a pair of scissors to each student, and have them cut out the large square on each copy of Master E. Ask the students to assume that one of the cutout squares has area 108 square units, and to paperfold to determine several *equivalent radical expressions* for the actual length of the edge of the square. Invite volunteers to share their conclusions and reasoning. Then, if not already suggested by students, ask them to determine expressions that are equivalent to $\sqrt{108}$ but based on edges of squares that are *larger* than the square given on Master E.


Reasoning and Radicals Lesson **9**

Focus Master E

_____ linear units

Area = _____ square units

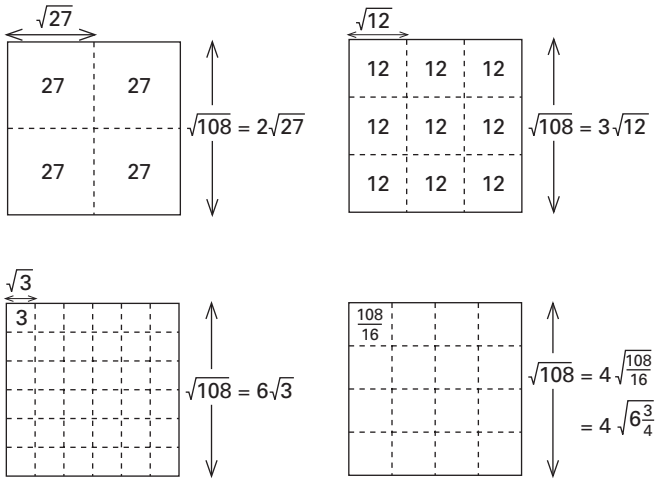
COMMENTS

g) Here are several *equivalent radical expressions* illustrated by the squares formed in a)-f): $\sqrt{12} = \frac{\sqrt{48}}{2} = 2\sqrt{3} = \sqrt{\frac{48}{4}}$; $\sqrt{3} = \frac{\sqrt{12}}{2} = \frac{\sqrt{48}}{4}$. Some other possible observations include: the length of the edge of a square is multiplied by 2 when the area of the square is multiplied by 4; the length of the edge of a square is multiplied by 4 when the area is multiplied by 16.

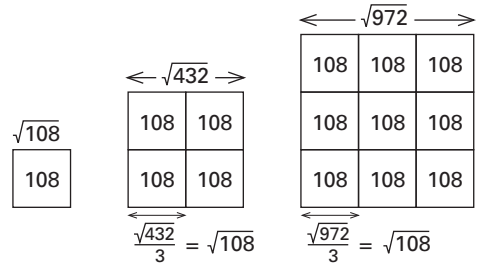
h) Here are some equivalent expressions that may be suggested by students' sketches: $\sqrt{8} = 2\sqrt{2} = \frac{\sqrt{32}}{2}$; $\sqrt{32} = 2\sqrt{8} = 4\sqrt{2}$; $\sqrt{2} = \frac{\sqrt{8}}{2} = \frac{\sqrt{32}}{4}$.

Note: these problems involve only positive, or principal, square roots (see Comment 1) since length is a positive measure.

7 Some possibilities determined by paperfolding to form smaller congruent squares are shown here:



Another method of obtaining equivalent expressions for lengths is to combine squares to form larger squares, as shown here:



This shows: $\sqrt{108} = \frac{\sqrt{432}}{3} = \frac{\sqrt{972}}{3}$,

and $\sqrt{432} = 2\sqrt{108}$,

and $\sqrt{972} = 3\sqrt{108}$.

Focus Teacher Activity (cont.)

ACTIONS

8 Ask the groups to determine equivalent radical expressions for several of the following, and to prove visually that the expressions are equivalent. Encourage efforts to generalize. Point out *simplified* radical expressions as they are shared and ask the students to approximate the decimal value of each expression. Discuss.

a) $\sqrt{40}$

b) $\sqrt{99}$

c) $\sqrt{180}$

d) $\sqrt{75}$

e) $\sqrt{21}$

f) $\sqrt{\frac{9}{4}}$

g) $\sqrt{\frac{17}{9}}$

h) $\sqrt{\frac{29}{2}}$

i) $\sqrt{18}$

j) $\sqrt{700}$

k) $\sqrt{\frac{64}{25}}$

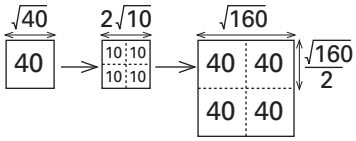
l) $\sqrt{\frac{16}{3}}$

m) $\sqrt{\frac{17}{8}}$

COMMENTS

8 As students formulate conjectures and generalizations, suggest they test them on several problems before sharing with the class. Here are a few equivalent expressions which can be determined by combining squares to form larger squares or by subdividing squares into smaller congruent squares:

a) $\sqrt{40} = 2\sqrt{10} = \frac{\sqrt{160}}{2}$

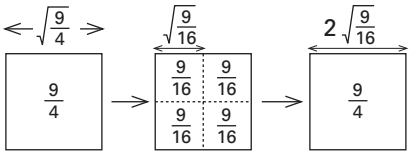


b) $\sqrt{99} = 3\sqrt{11} = \frac{\sqrt{396}}{2}$

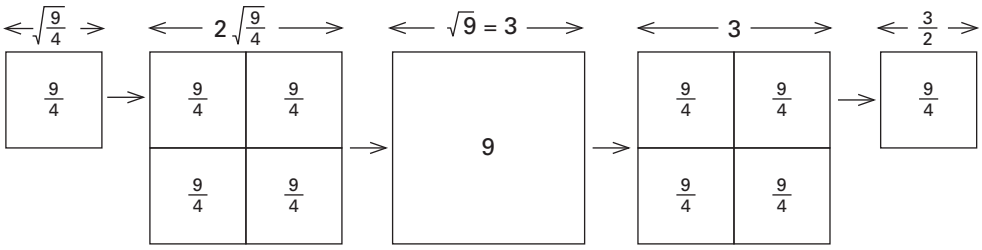
c) $\sqrt{180} = 6\sqrt{5}$

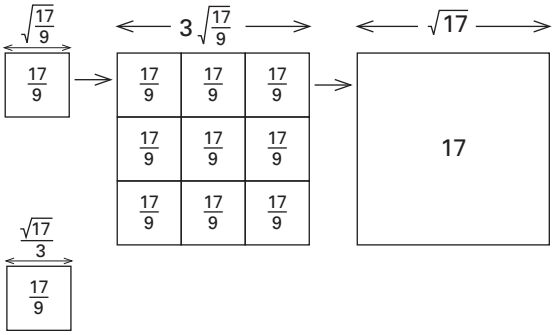
f) $\sqrt{\frac{9}{4}} = \frac{3}{2} = 2\sqrt{\frac{9}{16}}$

Some students may subdivide the square with area $\frac{9}{4}$ into smaller parts and find that the equivalent expressions for $\sqrt{\frac{9}{4}}$ become more complicated, as shown here:



The diagram below illustrates how 4 copies of the square with area $\frac{9}{4}$ can be combined to form a larger square with area 9 and edge length $\sqrt{9} = 3$, showing that $2\sqrt{\frac{9}{4}} = 3$, so $\sqrt{\frac{9}{4}} = \frac{3}{2}$.





g) Nine copies of the square with area $\frac{17}{9}$ can be arranged to form a larger square with area 17 and edge length $\sqrt{17}$, showing that $\sqrt{\frac{17}{9}} = \frac{\sqrt{17}}{3}$.

Students may observe that folding a square into 4 congruent squares produces an equivalent expression for the length of the edge of the square which is 2 times a radical. Similarly, folding a square into 9 congruent squares produces an equivalent expression for the length of the edge which is 3 times a radical. Notice that, in

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

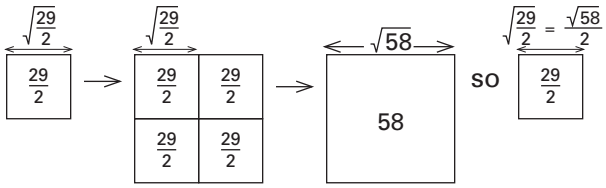
general, if a number has a perfect square as a factor, then the square root of the number can be written as the product of a whole number (the square root of the perfect square factor) and the square root of a whole number. That is, $\sqrt{a^2b} = a\sqrt{b}$ for $a > 0$.

One way to define the *simplest form* of the square root of a whole number is to say that a square whose area is the given number cannot be subdivided to form a square array of smaller squares with whole number areas greater than 1. That is, the *radicand*—the number under the radical symbol—is a whole number and has no factors greater than 1 which are perfect squares. A s*implified radical expression* also does not have a radical in its denominator (e.g., $\frac{4}{\sqrt{3}}$ is not simplified), or a fraction inside the radical (e.g., $\sqrt{\frac{3}{5}}$ is not simplified).

A calculator provides one method of determining decimal approximations for square roots. Some students may use the following procedure to approximate $\sqrt{21}$, for example: since $4^2 = 16$ and $5^2 = 25$ are the 2 perfect squares "closest" to 21, and since 21 is slightly closer to 25 than to 16, a reasonable approximation of $\sqrt{21}$ is 4.6.

Note: for any number n that is not a perfect square, $\sqrt{n}$ is an *irrational number*—a nonrepeating, nonterminating decimal. There is no exact decimal form of an irrational number.

It is important to note that the accessibility of calculators and computers have diminished the need for proficiency at simplifying complicated radical expressions; however, some experience with this process is useful for certain algebraic manipulations—those conveniently solved mentally or with minimal paper and pencil strategies. Emphasis here is on understanding the concept of equivalence of radical expressions.



h) $\sqrt{\frac{29}{2}}$ is equivalent to $\frac{\sqrt{58}}{2}$, as illustrated at the left.

While it may be tempting to provide students with rules for simplifying radicals, such as an algorithm for "rationalizing the denominator of a radical expression," we encourage you to allow time for students to invent their own rules that are based on relationships they can "see" and understand. In so doing, they are more likely to recall or be able to reinvent such procedures when needed.

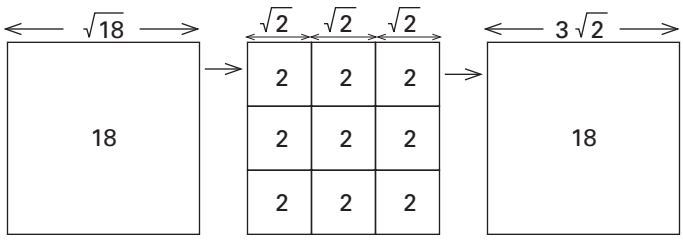
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

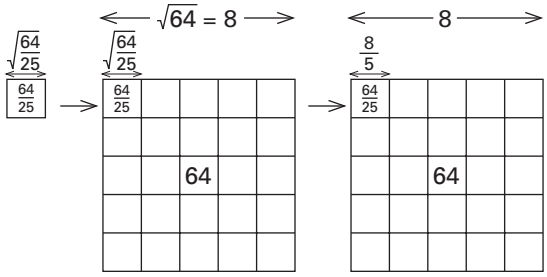
8 (continued.)
i) Here is one possible sequence of sketches, leading to the statement $\sqrt{18} = 3\sqrt{2}$ (see diagram below).



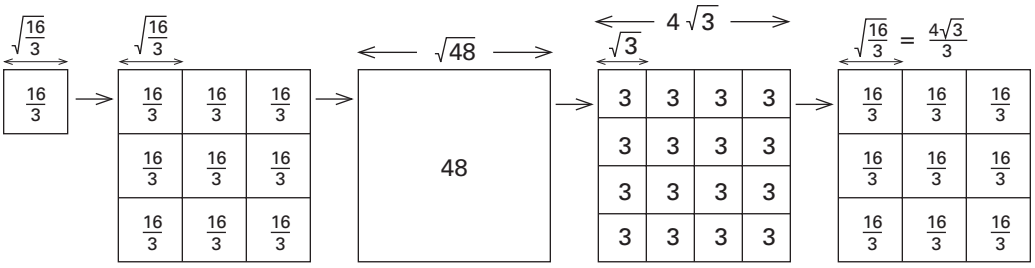
In general, if n is a perfect square factor of the radicand, r, then an equivalent radical expression can be determined by subdividing the square whose area is r into a $\sqrt{n}$ by $\sqrt{n}$ array of squares.

j) $\sqrt{700} = 10\sqrt{7}$

k) $\sqrt{\frac{64}{25}} = \frac{8}{5}$, as illustrated here:



l) Here is one possible sequence of sketches for eliminating the denominator from the radicand of $\sqrt{\frac{16}{3}}$:

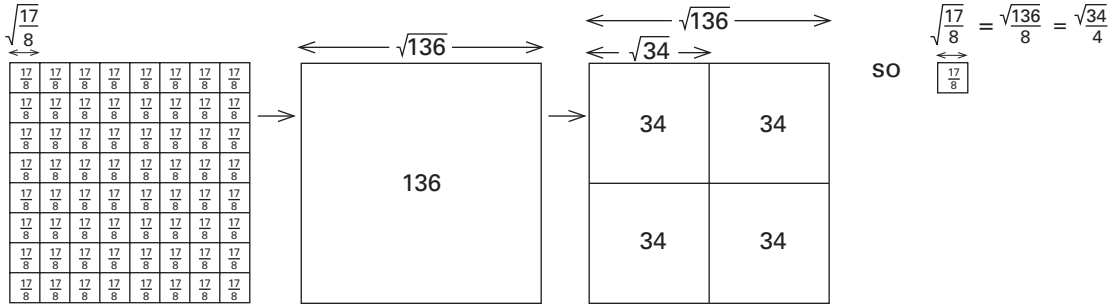


m) Following are two possible methods of obtaining $\sqrt{\frac{17}{8}} = \frac{\sqrt{34}}{4}$.

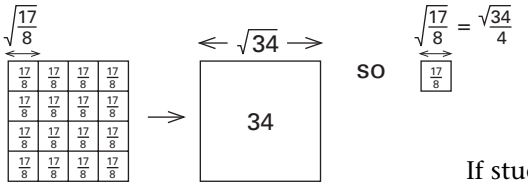
Focus Teacher Activity (cont.)

ACTIONS

Method 1

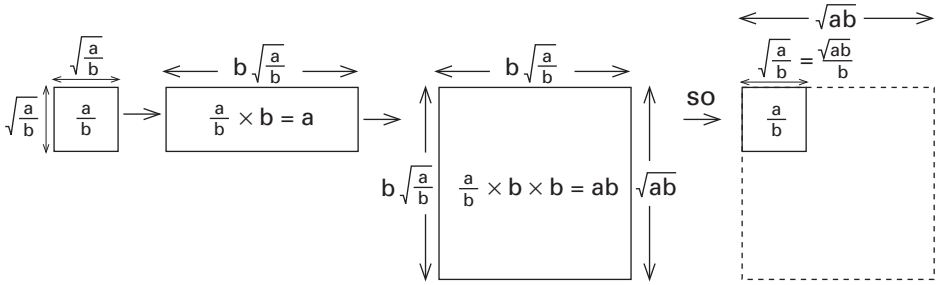


Method 2



COMMENTS

If students have conjectures regarding general procedures for simplifying radical expressions that contain a fraction in the radicand, you might invite them to share their ideas with the class. In general, if the radicand is a fraction of the form a/b, where a and b are positive integers, $b \neq 0$, and b is *not* a perfect square, then an equivalent radical expression with no denominator in the radicand can be determined by forming a larger square from b^2 copies of the square with area a/b. Hence, $\sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b}$, as illustrated here:



This shows that $b\sqrt{\frac{a}{b}} = \sqrt{ab}$

This shows that $\sqrt{\frac{a}{b}} = \frac{\sqrt{ab}}{b}$

9 Ask the groups to compare $\sqrt{S} + \sqrt{T}$ and $\sqrt{S+T}$ for positive S and T. Discuss their conclusions and reasoning. If students reason inductively from examples, encourage them to develop deductive arguments that show *why* their conclusions must be true.

9 In this action and Action 10, students generalize about the square root of sums, products, and quotients.

For positive S and T, $\sqrt{S} + \sqrt{T} > \sqrt{S+T}$. That is, the sum of the square roots of positive numbers is greater than the square root of their sum. The students may verify this theorem by comparing the values of $\sqrt{S} + \sqrt{T}$ and $\sqrt{S+T}$ for specific values of S and T. For example, when S = 4 and T = 9, $\sqrt{S} + \sqrt{T} = 5$ and $\sqrt{S+T} = \sqrt{13}$, which is less than 4.

(Continued next page.)

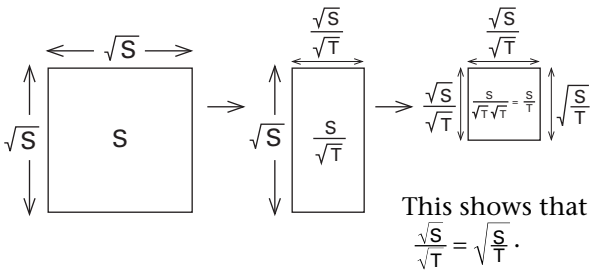
Focus Teacher Activity (cont.)

ACTIONS

10 Repeat Action 9 for:

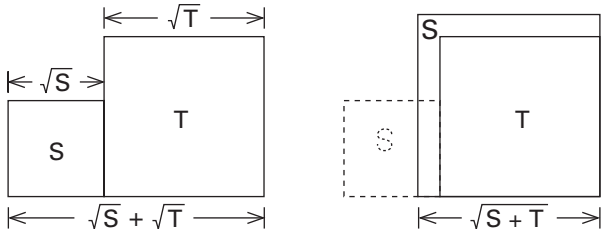
a) $\sqrt{ST}$ and $\sqrt{S}\sqrt{T}$

b) $\frac{\sqrt{S}}{\sqrt{T}}$ and $\sqrt{\frac{S}{T}}$



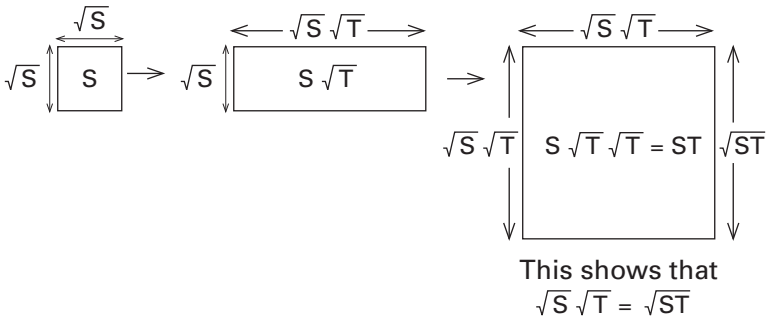
COMMENTS

9 (continued.)
The visual proof shown below, begins by noting that $\sqrt{S} + \sqrt{T}$ is the combined length of the edges of squares of areas S and T, respectively. If the square of area T is enlarged to a square of area S + T, the edge of the enlarged square is less than the combined length of the edges of the original squares:



10 a) The students may conjecture that the square root of the product of 2 numbers equals the product of their square roots by comparing the values of and $\sqrt{S}\sqrt{T}$ for specific values of S and T.

To visually prove the theorem that $\sqrt{S}\sqrt{T} = \sqrt{ST}$, multiply $\sqrt{T}$ times both dimensions of a square with area S. Notice this changes the area by a factor of $\sqrt{T} \times \sqrt{T}$, or T, as illustrated below. The result is a square of area ST whose side has length $\sqrt{S}\sqrt{T}$. On the other hand, the length of the side of a square is the square root of its area, in this case $\sqrt{ST}$. Thus, $\sqrt{ST} = \sqrt{S}\sqrt{T}$.



b) The situation for quotients is similar to that for products. If a new square is obtained from a square whose area is S by dividing both dimensions of its side, $\sqrt{S}$, by $\sqrt{T}$, the area of the new square is the area S of the original square divided by T, i.e., $\frac{S}{T}$. Hence, $\sqrt{\frac{S}{T}} = \frac{\sqrt{S}}{\sqrt{T}}$. That is, the square root of the quotient of 2 numbers is the quotient of their square roots. This argument is illustrated at the left.

Focus Teacher Activity (cont.)

ACTIONS

11 Tell the students that the results of a) and b) from Action 10 are sometimes used to simplify radicals. For example, $\sqrt{45} = \sqrt{9}\sqrt{5} = 3\sqrt{5}$; and $\sqrt{\frac{7}{5}} = \sqrt{\frac{35}{25}} = \frac{\sqrt{35}}{\sqrt{25}} = \frac{\sqrt{35}}{5}$.

Ask them to simplify the following expressions, keeping in mind the fact that simplified radicals contain no perfect square factors, no denominators in the radicand, and no radicals in the denominator:

a) $\sqrt{\frac{3}{4}}$

b) $\sqrt{44}$

c) $\frac{\sqrt{72}}{8}$

d) $\sqrt{\frac{64}{121}}$

e) $\sqrt{363}$

f) $4\sqrt{\frac{125}{2}}$

g) $\frac{3}{5}\sqrt{98}$

h) $\frac{7}{8}\sqrt{32} + \frac{1}{\sqrt{2}}$

12 Give each student a copy of Focus Student Activity 9.2 (see below and next page) and ask them to complete Problem 1. Discuss. Then repeat for Problems 2 and 3.

Reasoning and Radicals Lesson **9**

Focus Student Activity 9.2

NAME _____ DATE _____

Fill in the blanks below. Be prepared to provide sound mathematical arguments to support your answers. Write all radical expressions in simplified form. For each problem, use only the given variables without adding other variables. A = area, P = perimeter, h = altitude, and s = side length.

1 a)

h = _____
A = _____
P = _____

b)

h = _____
A = _____
P = _____

c)

h = _____
A = _____
P = _____

d)

h = _____
A = _____
P = _____

2 a)

A = _____
P = _____

b)

A = _____
P = _____

(Continued on back.)

COMMENTS

11 a) $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{\sqrt{4}} = \frac{\sqrt{3}}{2}$

b) $\sqrt{44} = \sqrt{4 \times 11} = \sqrt{4} \times \sqrt{11} = 2\sqrt{11}$

c) $\frac{6\sqrt{2}}{8} = \frac{3\sqrt{2}}{4}$

d) $\frac{8}{11}$

e) $11\sqrt{3}$

f) $4\sqrt{\frac{125}{2}} = 4\sqrt{\frac{250}{4}} = 4\frac{\sqrt{250}}{\sqrt{4}} = 4\frac{\sqrt{25}\sqrt{10}}{2} = 10\sqrt{10}$

g) $\frac{21}{5}\sqrt{2}$

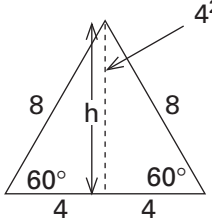
h) $4\sqrt{2}$

Note that removing a radical from the denominator, as in the example $\frac{1}{\sqrt{2}}$, is called *rationalizing the denominator*. Some students may invent algorithms for doing so.

12 The purpose in asking for simplified radical expressions is to reveal patterns in numerical relationships that lead to formulas for the area and perimeter of 30°- 60°- 90° and 45°- 45°- 90° triangles.

Since the triangles in 1a)-1d) are equilateral triangles, each angle measures $\frac{180°}{3} = 60°$, Based on explorations in Lessons 1 and 2 of this course, students may point out that each altitude of an equilateral triangle is a line of symmetry for the triangle. Hence, an altitude is the perpendicular bisector of a side of the triangle, and subdivides the triangle into 2 congruent right triangles, as illustrated below:

1a)



$4^2 + h^2 = 8^2$
so $h^2 = 64 - 16$
$h^2 = 48$
$h = 4\sqrt{3}$

$A = \frac{1}{2}(8)(4\sqrt{3})$
$= 16\sqrt{3}$ square units
$P = 8 + 8 + 8 =$ linear units

1b) $h = \frac{5}{2}\sqrt{3}$ linear units

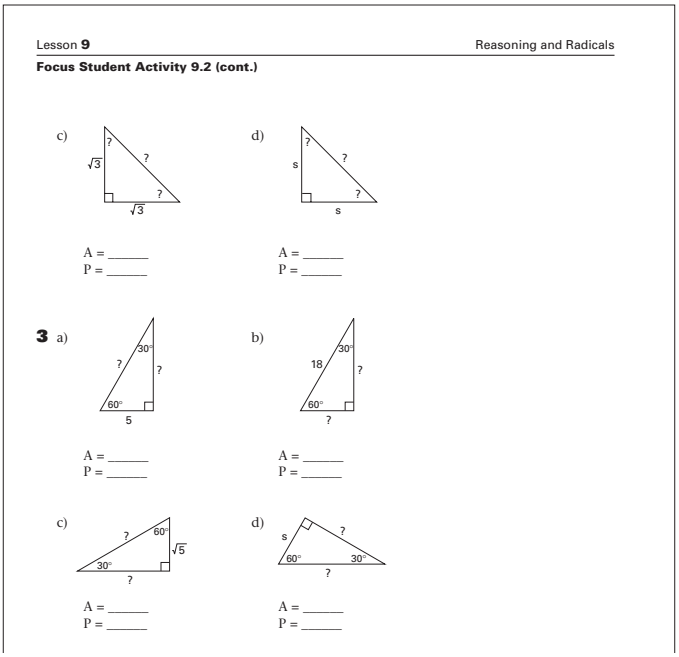
$A = \frac{1}{2} \times 5 \times \frac{5}{2}\sqrt{3}$ square units

$P = 3 \times 5 = 15$ linear units

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS


Lesson 9 Reasoning and Radicals

Focus Student Activity 9.2 (cont.)

c) $\sqrt{3}$? ? ? $\sqrt{3}$

A = _____
P = _____

d) s ? ? ? s

A = _____
P = _____

3 a) 30° ? ? 60° 5

A = _____
P = _____

b) 18 30° ? 60° ?

A = _____
P = _____

c) ? 60° $\sqrt{5}$ 30° ?

A = _____
P = _____

d) s ? 60° 30° ?

A = _____
P = _____

COMMENTS

12 (continued.)

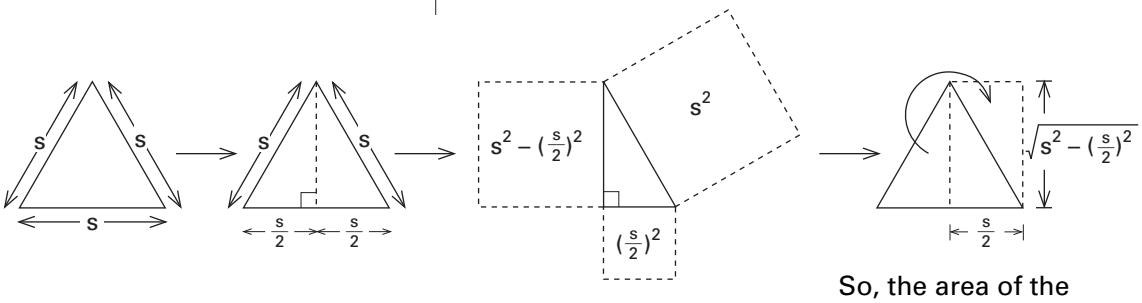
1c) $h = \frac{\sqrt{7}}{2} \times \sqrt{3} = \frac{\sqrt{21}}{2}$ linear units

Students may compute the area as follows:

$A = \frac{1}{2} \times \sqrt{7} \times \frac{\sqrt{21}}{2} = \frac{\sqrt{147}}{4}$ square units, or
$A = \frac{1}{2} \times \sqrt{7} \times \frac{\sqrt{21}}{2} = \frac{1}{2} \times \sqrt{7} \times \frac{\sqrt{7 \times \sqrt{3}}}{2} = \frac{1}{2} \times 7 \times \frac{\sqrt{3}}{2} = \frac{7}{4}\sqrt{3}$
$P = 3\sqrt{7}$ linear units

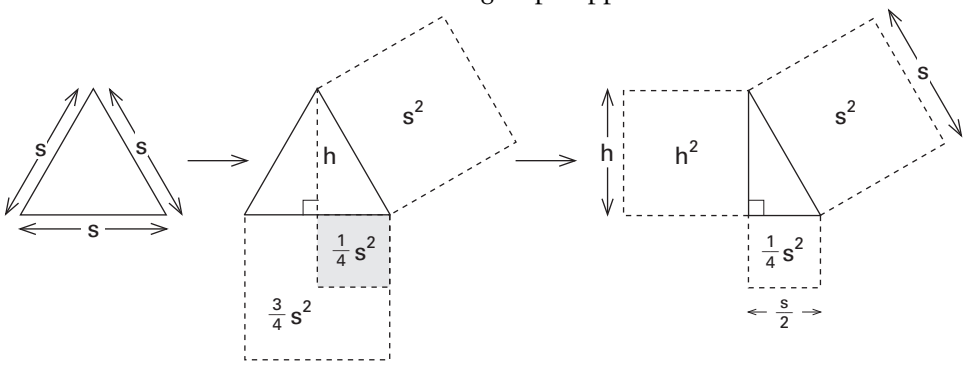
1d) $h = \frac{s}{2}\sqrt{3}$ linear units
$A = \frac{1}{2} \times s \times \frac{s}{2}\sqrt{3} = \frac{s^2}{4}\sqrt{3}$ square units
$P = 3s$ linear units

If some groups have difficulty, you might have others share "clues" for "seeing" a formula or relationship. For example, here is one group's visual reasoning and formula for the area of an equilateral triangle with side length *s*:



So, the area of the equilateral triangle is $\frac{s}{2}\sqrt{s^2 - \left(\frac{s}{2}\right)^2}$ square units.

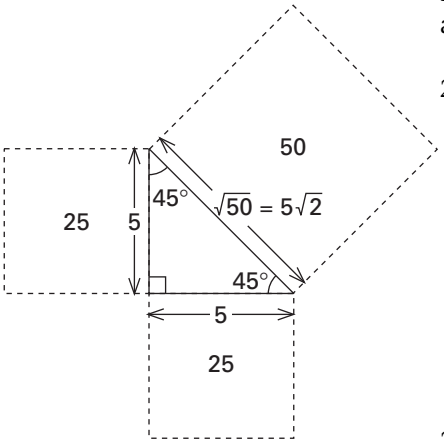
Here is another group's approach:



Since $\frac{1}{4}s^2 + h^2 = s^2$, then $h^2 = s^2 - \frac{1}{4}s^2 = \frac{3}{4}s^2$. So, $h = \sqrt{\frac{3}{4}s^2} = \frac{s}{2}\sqrt{3}$ linear units. Therefore, since the area of any triangle is ½ × base × height, the area of the equilateral triangle is $\frac{1}{2}(s)\left(\frac{s\sqrt{3}}{2}\right) = \frac{s^2\sqrt{3}}{4}$ square units.

Focus Teacher Activity (cont.)

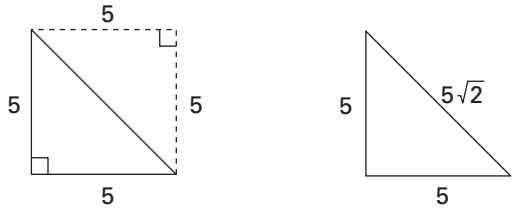
ACTIONS



COMMENTS

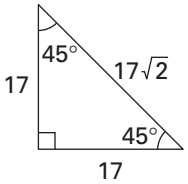
Each triangle in 2a)-2d) involves an isosceles right triangle, i.e., a 45°- 45°- 90° triangle.

2a) The following diagram illustrates one approach:



$A = \frac{5^2}{2} = \frac{25}{2}$ square units $\quad P = 10 + 5\sqrt{2}$ linear units

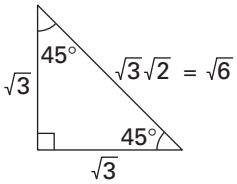
2b)



$A = \frac{1}{2} \times 17 \times 17 = 144.5$ square units

$P = 17 + 17 + 17\sqrt{2} = 34 + 17\sqrt{2}$ linear units

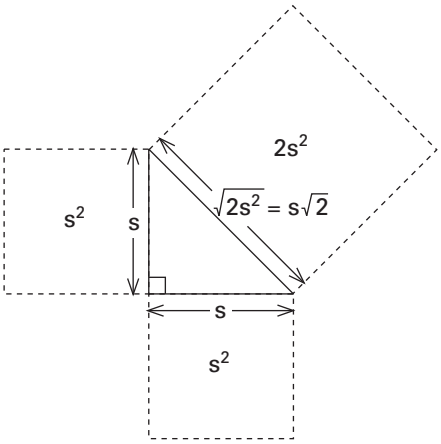
2c)



$A = \frac{1}{2}(\sqrt{3})^2 = \frac{3}{2}$ square units

$P = \sqrt{3} + \sqrt{3} + \sqrt{6} = 2\sqrt{3} + \sqrt{6}$ linear units

2d)



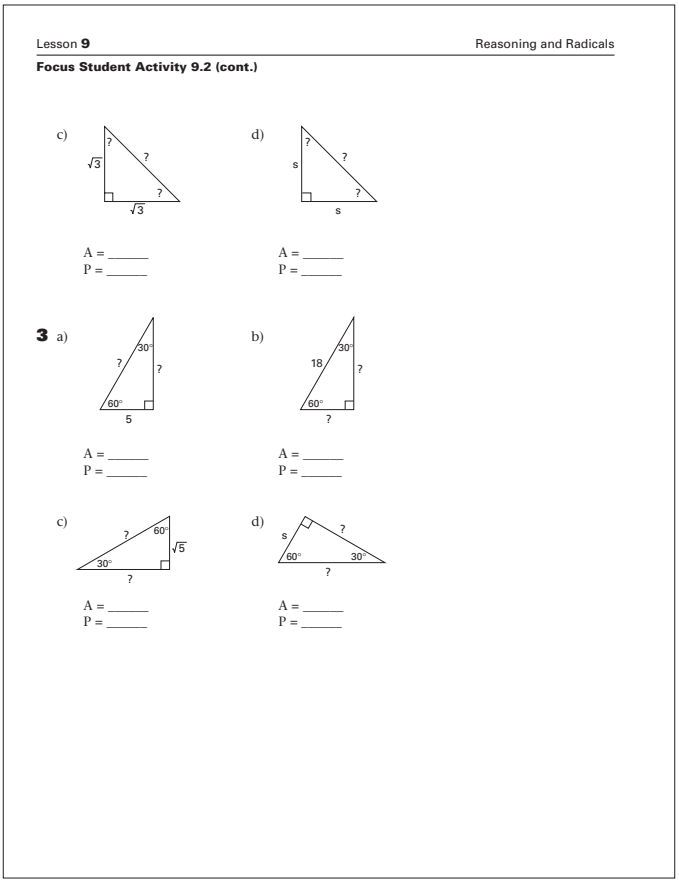
$A = \frac{1}{2}s^2 = \frac{s^2}{2}$ square units

$P = s + s + s\sqrt{2} = 2s + s\sqrt{2}$ linear units

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



Lesson **9** Reasoning and Radicals

Focus Student Activity 9.2 (cont.)

c) d)

A = _____ A = _____
P = _____ P = _____

3 a) b)

A = _____ A = _____
P = _____ P = _____

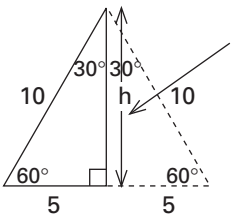
c) d)

A = _____ A = _____
P = _____ P = _____

COMMENTS

12 (continued.)
Problem 3 involves 30°- 60°- 90° triangles. Students may view these as "halves of equilateral triangles," as illustrated below.

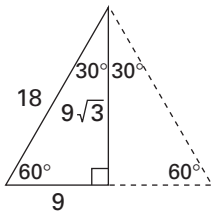
3a)



$h^2 = 10^2 - 5^2$
$h^2 = 100 - 25$
$h = \sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3}$
$h = 5\sqrt{3}$ linear units

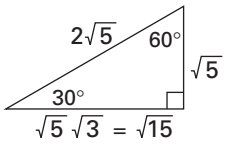
$A = \frac{1}{2} \times 5 \times 5\sqrt{3} = \frac{25\sqrt{3}}{2}$ square units
$P = 10 + 5 + 5\sqrt{3} = 15 + 5\sqrt{3}$ linear units

3b)



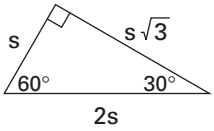
$A = \frac{9 \times 9\sqrt{3}}{2} = \frac{81\sqrt{3}}{2}$ square units
$P = 18 + 9 + 9\sqrt{3} = 27 + 9\sqrt{3}$ linear units

3c)



$A = \frac{1}{2} \times \sqrt{15} \times \sqrt{5}$
$= \frac{1}{2}\sqrt{75}$
$= \frac{5\sqrt{3}}{2}$ square units
$P = 3\sqrt{5} + \sqrt{15}$ linear units

3d)



$A = \frac{1}{2} \times s \times s\sqrt{3} = \frac{s^2\sqrt{3}}{2}$ square units
$P = s + 2s + s\sqrt{3} = 3s + s\sqrt{3}$ linear units

Focus Teacher Activity (cont.)

ACTIONS

13 Distribute several butcher paper strips and marking pens to each group of students. Ask the groups to write several "We conjecture…We wonder…" statements about equilateral triangles, 45°- 45°- 90° triangles, and 30°- 60°-90° triangles, and to identify those statements they can prove. Ask the groups to post their statements. Discuss the groups' conjectures, questions, and proofs. Then have the class help sort the posted statements and revise, refine, and/or combine statements as needed.

COMMENTS

13 If groups have difficulty getting started, you can prompt ideas by encouraging them to write statements that they think are true about the area, perimeter, and altitude of all equilateral triangles, all isosceles right triangles, and all 30°- 60°-90° triangles.

Following are several conjectures that students frequently suggest and support with visual proofs. If students' suggestions are limited, you could pose one or two statements and ask the groups to create visual proofs that show why the statements are true or not true.

Every 30°- 60°- 90° triangle is ½ of an equilateral triangle.

The hypotenuse of a 30°- 60°- 90° triangle is twice as long as its short leg.

The length of the long leg of a 30°- 60°- 90° triangle is √3 times the length of its short leg.

The altitude of an equilateral triangle with side length s is $\frac{s}{2}\sqrt{3}$.

The area of an equilateral triangle with side length s is $\frac{1}{2}(s)\left(\frac{s}{2}\sqrt{3}\right)=\frac{s^2}{4}\sqrt{3}$.

The area of a 30°- 60°- 90° triangle with short leg s is $\frac{1}{2}(s)(s\sqrt{3})=\frac{s^2}{2}\sqrt{3}$.

The area of a 30°- 60°- 90° triangle with hypotenuse s is ½ the area of an equilateral triangle with side lengths s, or $\frac{1}{2}\left(\frac{s^2}{4}\sqrt{3}\right)=\frac{s^2}{8}\sqrt{3}$.

Four copies of a 45°- 45°- 90° triangle fit together to form a square. If the hypotenuse of the 45°- 45°- 90° triangle is s, they fit together to form a square with side s and area s^2*. So the area of a 45°- 45°- 90° triangle with hypotenuse s is* $s^2/4$.

On a 45°- 45°- 90° triangle, the sides opposite the 45° angles, or the legs, are congruent.

If the legs of a 45°- 45°- 90° triangle have length s, then the hypotenuse has length $s\sqrt{2}$.

If the length of the hypotenuse of a 45°- 45°- 90° triangle is s, then the legs have length $\frac{s\sqrt{2}}{2}$.

The area of a 45°- 45°- 90° triangle with legs of length s is $\frac{s^2}{2}$.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

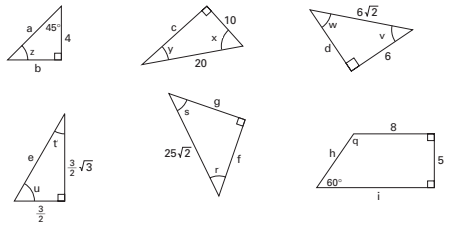
14 Give each student a copy of Focus Student Activity 9.3 and assign selected problems for completion. Discuss their results.

Reasoning and Radicals Lesson **9**

Focus Student Activity 9.3

NAME _____ DATE _____

1 Without measuring, find the value of each missing angle or length. Drawings are not necessarily to scale.



2 Draw diagrams to help you find the following values. Briefly explain and/or mark your diagrams to show your reasoning. Give actual measures rather than approximations.

a) The length of the diagonal of a square with area 225 cm^2.

b) The area of a regular hexagon with side length 12 inches.

c) The length of the diagonals of each face of a rectangular prism with dimensions 5 inches by 7 inches by 9 inches.

d) The length of the diagonals of the prism from Problem c). (Note: such diagonals extend corner to corner through the center of the prism.)

e) The area of an equilateral triangle with sides of length 17 feet.

f) The perimeter of an equilateral triangle whose area is $12\sqrt{3}$ square centimeters.

(Continued on back.)

COMMENTS

13 (continued.)
The area of a 45°- 45°- 90° triangle with hypotenuse of length s is $\frac{1}{2}\left(\frac{s\sqrt{2}}{2}\right)\left(\frac{s\sqrt{2}}{2}\right) = \frac{2s^2}{8} = \frac{s^2}{4}$.

All isosceles right triangles have angle measures 45°- 45°- 90°.

The hypotenuse is always the longest side of a right triangle.

The longest side of a triangle, if there is a longest side, is always opposite the largest angle and the shortest side (if there is one) is opposite the smallest angle.

14 You might have the students complete and discuss Problem 1 before proceeding to Problem 2. These problems could also be assigned as homework. Be sure to allow plenty of time for students to confer about their strategies and difficulties.

1) a) $4\sqrt{2}$ b) 4 c) $10\sqrt{3}$
d) 6 e) 3 f) 25
g) 25 h) $\frac{10\sqrt{3}}{3}$ i) $8 + \frac{10\sqrt{3}}{6}$
q) 120° r) 45° s) 45°
t) 30° u) 60° v) 45°
w) 45° x) 60° y) 30°
z) 45°

2) a) $15\sqrt{2}$ b) $216\sqrt{3}$ c) $\sqrt{74}, \sqrt{130}, \sqrt{106}$
d) $\sqrt{155}$ e) $\frac{289}{4}\sqrt{3}$ f) $12\sqrt{3}$

3) See Comments 9 and 10.

4) One possibility is to use the fact from Focus Student Activity 9.2, 1d) that the area of an equilateral triangle with side lengths of s is $\frac{\sqrt{3}s^2}{4}$. Since a regular hexagon is composed of 6 equilateral triangles, its area is $6\left(\frac{\sqrt{3}s^2}{4}\right) = \frac{3\sqrt{3}s^2}{2}$.

5) a) $2\sqrt{6}$ b) $\frac{4\sqrt{2}}{3}$ c) $\frac{\sqrt{3}}{3}$
d) $\frac{2\sqrt{5}}{5}$ e) $2\sqrt{5} + 2\sqrt{15}$ f) $3\sqrt{5} + 5\sqrt{3}$
g) $4\sqrt{10}$ h) $2\sqrt{3} - 6$

Focus Teacher Activity (cont.)

ACTIONS

Lesson **9** Reasoning and Radicals

Focus Student Activity 9.3 (cont.)

3 For each of the following, give an example to show evidence *that* the statement is true for positive values of x and y. Then demonstrate visually *why* each statement is true.

a) $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ b) $\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$ c) $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

4 Invent a formula for the area, A, of a regular hexagon with side length s. Be sure that A and s are the only variables in your formula.

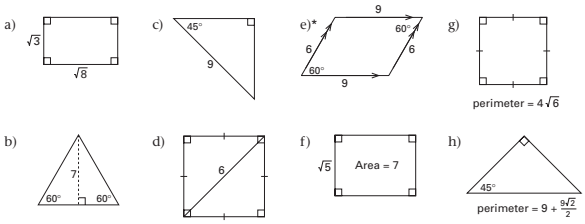
5 Simplify each of the following radical expressions.

a) $\sqrt{24}$ d) $\sqrt{\frac{4}{5}}$ g) $\sqrt{32} \times \sqrt{5}$

b) $\sqrt{\frac{32}{9}}$ e) $\sqrt{20} + \sqrt{60}$ h) $\sqrt{12} - \sqrt{36}$

c) $\frac{1}{\sqrt{3}}$ f) $\sqrt{45} + \sqrt{75}$

6 Sketch each of the following polygons on another sheet. Show the reasoning you use to find the area and perimeter of each (in some cases the area or perimeter is given). Drawings are not to scale.



*Note: > and >> markings on sides indicate pairs of parallel sides.

COMMENTS

6) a) $A = 2\sqrt{6}$, $P = 2\sqrt{3} + 2\sqrt{8}$

b) $A = \frac{49}{\sqrt{3}}$, $P = \frac{42}{\sqrt{3}}$

c) $A = \frac{81}{4}$, $P = 9 + \frac{18}{\sqrt{2}}$

d) $A = 18$, $P = \frac{24}{\sqrt{2}}$

e) $A = 27\sqrt{3}$, $P = 30$

f) $A = 7$, $P = 2\sqrt{5} + 2(\frac{7}{\sqrt{5}}) = 2\sqrt{5} + \frac{14}{\sqrt{5}} = 2\sqrt{5} + \frac{14\sqrt{5}}{5}$

g) $A = 6$, $P = 4\sqrt{6}$

h) $A = \frac{81}{8}$, $P = 9 + \frac{9\sqrt{2}}{2}$

TEACHER NOTES:

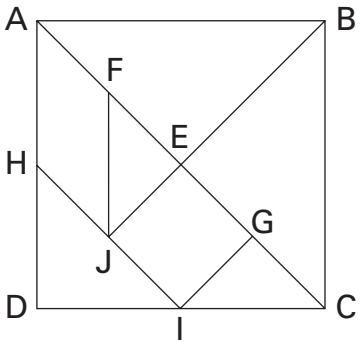
Follow-up Student Activity 9.4

NAME _____ **DATE** _____

1 One way to form the length $\sqrt{5}$ is to form a 1 by 2 rectangle and draw its diagonal. One way to form the length $\sqrt{20}$ is to double the length $\sqrt{5}$. The points $\sqrt{5}$, $-\sqrt{5}$, $\sqrt{20}$, and $-\sqrt{20}$ can be located on a number line by copying the length of a diagonal of a 1 by 2 rectangle. Determine how to locate points on the number line for the positive and negative square roots of all whole numbers less than 50, by constructing diagonals of the *minimum* number of rectangles. Make a chart to show the dimensions of the rectangles required, the length of the diagonals of each rectangle, and the points that can be located using each diagonal.

2 Tangrams, one of the oldest and most popular of the ancient Chinese puzzles, are made by constructing a geometric figure like the one below and then cutting it into the seven pieces shown.

In the square ABCD:
- Point E is the intersection of the diagonals.
- F and G are midpoints of $\overline{AE}$ and $\overline{EC}$, respectively.
- H and I are midpoints of $\overline{AD}$ and $\overline{DC}$, respectively.
- J is the midpoint of $\overline{HI}$.



If AB is 1 linear unit, find the side lengths and area of each different tangram piece. Show the calculations that you use to determine each measure.

3 Suppose that AB from Problem 2 is 5 linear units. Now find the side lengths and area of each different tangram piece. Show your reasoning.

4 Given the coordinates of any 3 points, without plotting the points on grid paper, how can you tell whether they form a right triangle? an obtuse triangle? an acute triangle? Show and explain your reasoning.

(Continued on back.)

Follow-up Student Activity (cont.)

5 If the lengths a, b, and c, form a right triangle, will $2a$, $2b$, and $2c$? $a/3$, $b/3$, and $c/3$? ka, kb, and kc, for k a positive integer? Explain.

6 Given a right rectangular prism with dimensions l, w, and h. What are the lengths of the diagonals of the prism. Justify your answer.

7 Given a cube with side length s. What are the lengths of its diagonals? Justify.

8 Draw each of the following quadrilaterals on 1-cm grid paper. Without using a calculator or a ruler to measure, determine and label the actual side lengths, diagonal lengths, and the area of each quadrilateral. Add comments as needed to communicate your methods and reasoning. Let 1 cm = 1 linear unit.

a) Squares with areas 1, 5, and 7 square units.

b) A rectangle with dimensions 1 by $\sqrt{2}$.

c) A square with diagonal length $\sqrt{6}$ linear units.

d) Two different nonsquare rectangles with diagonal lengths $\sqrt{6}$ linear units.

e) A square with side length $\sqrt{8}$ linear units and a nonsquare rectangle with diagonal length $\sqrt{8}$ linear units.

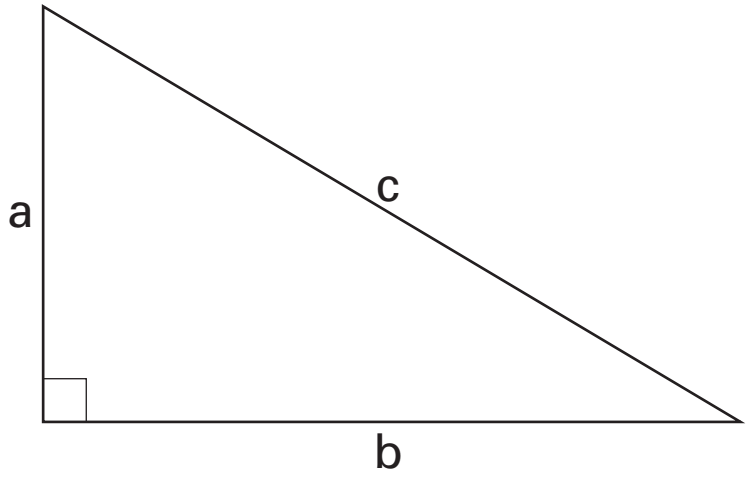
f) A rectangle with diagonal length $\sqrt{12}$ linear units.

9 Suppose that:

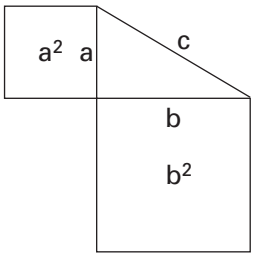
Nonsquare Rectangle R has sides of length a and b.
Square S has the same area as Rectangle R.
Square T has the same perimeter as Rectangle R.

Find the length of the sides of Square S and Square T. Show your reasoning.

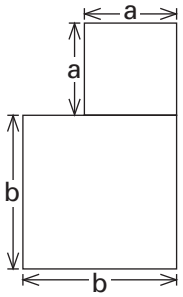
Connector Master A



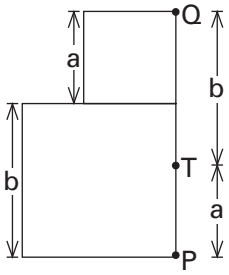
Connector Master B



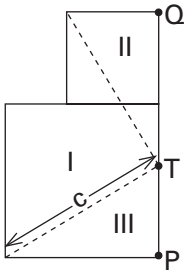
Step A Draw squares with edge lengths *a* and *b* on the legs of the right triangle on Connector Master A. Use a note card as a straightedge and guide for drawing right angles.



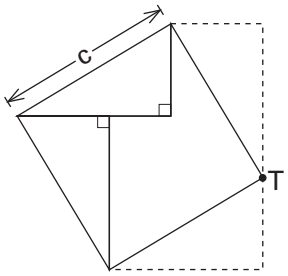
Step B Trace the squares from Step A to form 2 adjacent squares with right edges collinear.



Step C Label points P and Q as shown at the right. Locate a point T on edge $\overline{PQ}$ so that PT = *a*. Notice that TQ = *b*, since…



Step D Draw the dotted lines shown to form the regions labeled I, II, and III. Notice that regions II and III are congruent. Notice also that the angle formed at the intersection of the dotted lines is a right angle since…



Step E Cut out regions I, II, and III. Then reassemble the regions to obtain a square.

Step F Tape the square formed in Step E on the hypotenuse of the triangle on Connector Master A. Notice that…

Connector Master C

Use a note card as a straightedge and as a guide for drawing square corners.

a) From a geometric perspective, $\sqrt{n}$ is the length of the edge of a square whose area is n. On 1-cm grid paper and using a 1-cm square as 1 area unit, form all the actual lengths $\sqrt{n}$, for every integer n such that $1 \leq n \leq 12$. Tape your results, in order, on another sheet of paper, showing the square associated with each length. Label actual lengths.

b) On plain (no grid) paper, draw any 2 noncongruent squares so they are adjacent and their right edges are collinear. Then, dissect and rearrange these 2 adjacent squares to form a 3rd square whose area is equal to the sum of the areas of the 2 squares.

i) How many different 3rd squares are possible for a pair of adjacent squares?
ii) What if the 2 adjacent squares are congruent?

c) Challenge. On plain paper, draw a large square. Dissect and rearrange this large square to form 2 noncongruent adjacent squares whose total area equals the area of the large square.

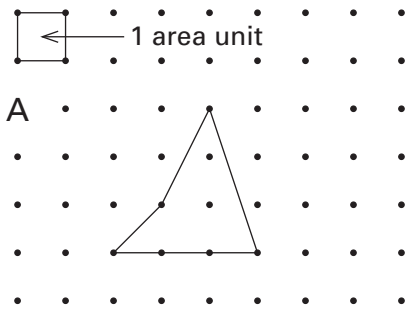
i) How many different pairs of adjacent squares can be formed so the sum of their areas equals the area of the large square?
ii) What if the 2 adjacent squares must be congruent?
iii) What if the area of the large square and the area of 1 of the 2 adjacent squares are given?

d) List your conjectures, questions, and generalizations.

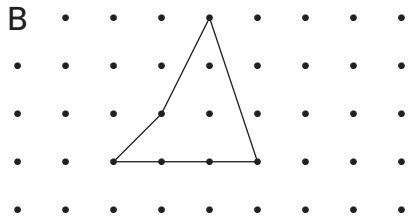
Connector Student Activity 9.1

NAME _____ **DATE** _____

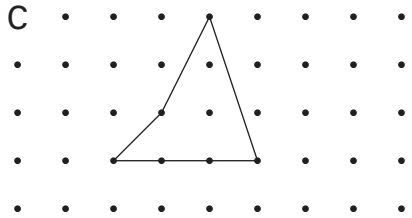
1 *Without* using any measuring tools, *without* using the Pythagorean Theorem, and *without* using area formulas, find the area and perimeter of Polygon A shown at the right. Mark the diagram to show your methods. Give *actual* measures rather than decimal approximations.



2 Use the Pythagorean Theorem where necessary to find all actual side lengths of Polygon B. Label each length on the diagram and show all of your calculations.



3 Use area formulas to compute the area of Polygon C. Label the diagram to show your calculations and formulas.

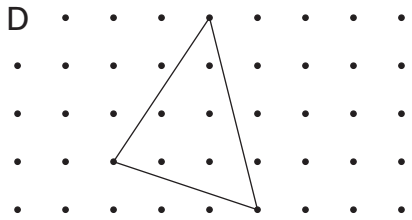


4 Complete a)-c) below for Polygon D. Show your methods.

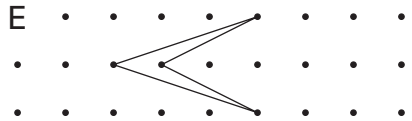
a) Determine its actual area.

b) Determine its actual perimeter.

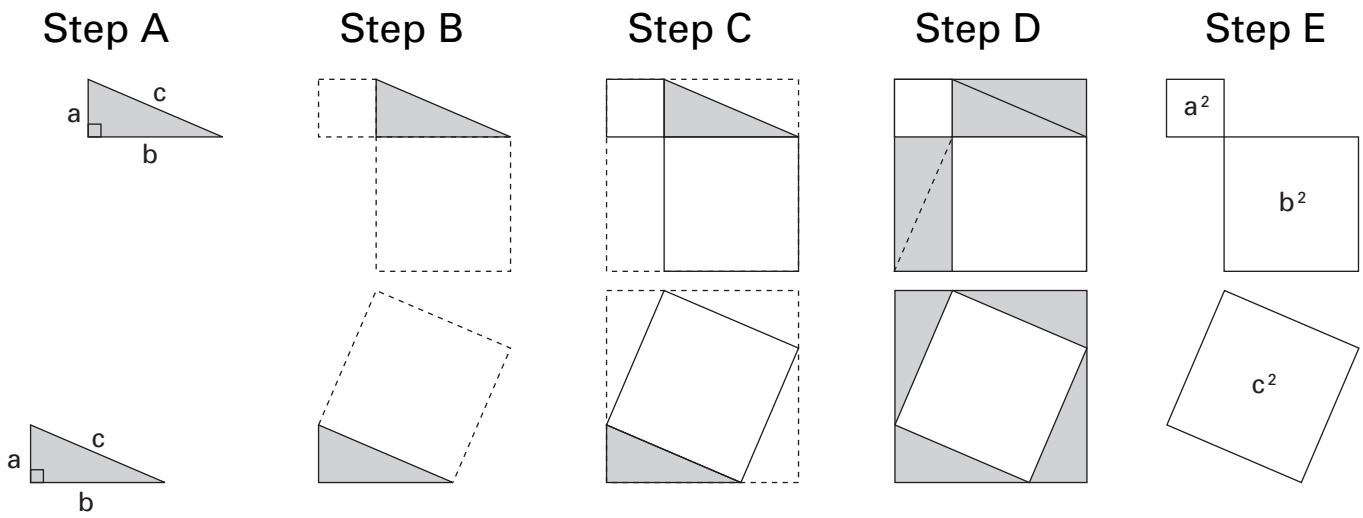
c) Determine the actual and approximate lengths of its 3 altitudes.



5 Show the methods that you use to find the area and perimeter of Polygon E.



Focus Master A



One visual proof of the Pythagorean Theorem is based on the diagrams shown above. Write statements that describe each step of this "proof." If needed, use the statements below as "thought starters."

Step A Draw 2 congruent right triangles, one triangle at the top of a sheet of paper, and the other at the bottom of the sheet. On each triangle, label the length of the short leg *a*, the long leg *b*, and the hypotenuse *c*.

Step B Draw a square on each leg of the top triangle, and draw a…

Step C Enclose the top figure and the bottom figure each in the smallest square possible. Notice that the areas of the enclosing squares are equal because…

Step D Draw a diagonal of the rectangle in the lower left corner of the top figure. Notice the 8 triangles (4 in the top figure and 4 in the bottom) are congruent because…

Step E If the 8 triangles are cut away, then… because…

In summary, if a right triangle has legs of length *a* and *b* and hypotenuse of length *c*, then…

Focus Master B

a) Reason visually to determine which, if any, of the following statements are true about *all* triangles with side lengths a, b, and c:

$a + b = c$
$a + b < c$
$a + b > c$

b) Use a straightedge to draw 4 noncongruent acute triangles.

i) Use a protractor and ruler to measure all side lengths and angles. Label your drawings to show lengths to the nearest tenth of a centimeter and angles to the nearest degree.

ii) Draw squares on each side of each triangle and record the areas of the squares.

iii) Using information from i)-ii) as evidence, reason inductively to complete Conjecture 1 below. Then give deductive arguments to show why your conjecture must always be true.

Conjecture 1 If a triangle is acute with side lengths a, b, and c, where c is the longest side length, then $a^2 + b^2$ ...

c) Complete Conjecture 2 below and give inductive and deductive arguments to support your conjecture:

Conjecture 2 If a triangle is obtuse with side lengths a, b, and c, where c is the longest length, then $a^2 + b^2$ ...

Focus Master C

a) Paintings in ancient Egyptian tombs from the 15th century B.C. show Egyptian geometers—called "rope stretchers," or surveyors—using long ropes with equally spaced knots. Historians believe these ropes were used to aid in the construction of right angles and right triangles. How do you think the rope stretchers did this?

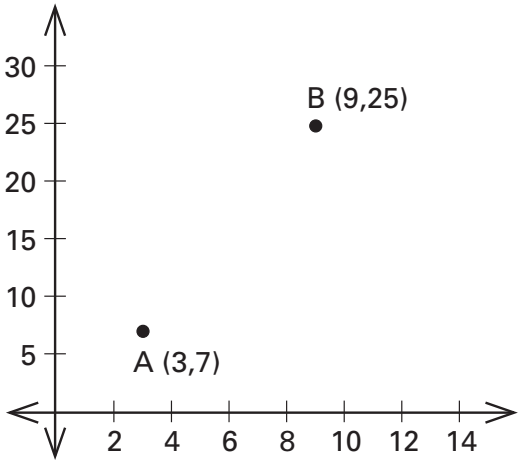
b) What are all the different right triangles with integral side lengths that can be formed from all or part of a 35-knot rope?

c) Three positive integers that work in the Pythagorean Theorem are called a *Pythagorean triple*. A Pythagorean triple is called a *primitive* if the 3 integers have no common factors other than 1—that is, a triple is a primitive if the 3 integers are *relatively prime*. Enlarging a right triangle by a scale factor that is a whole number creates a new Pythagorean triple that is called a *multiple* of the original triple.

What are all the Pythagorean triples such that each number in the triple is less than 35? Which of these are primitives? Which are multiples of primitives?

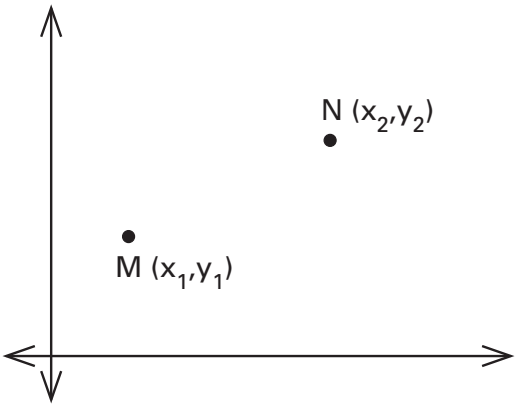
Focus Master D

1. Given 2 points A and B with coordinates (3,7) and (9,25):



i) Find the distance between A and B.
ii) Write the coordinates of the midpoint of $\overline{AB}$.

2. Given 2 points M and N with coordinates (x_1,y_1) and (x_2,y_2), respectively:



i) Write a formula for the distance between the 2 points. ___
ii) Write a formula for the coordinates of the midpoint of $\overline{MN}$.
iii) Do you think your results for i) and ii) work for any 2 points on the coordinate system? Explain your reasoning.

Focus Master E

<- _____ linear units ->

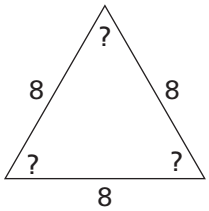
Area = _____ square units

Focus Student Activity 9.2

NAME _____ **DATE** _____

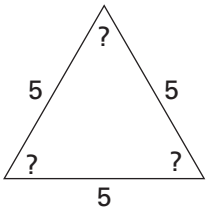
Fill in the blanks below. Be prepared to provide sound mathematical arguments to support your answers. Write all radical expressions in simplified form. For each problem, use only the given variables without adding other variables. A = area, P = perimeter, h = altitude, and s = side length.

1 a)



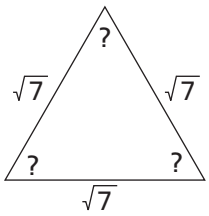
h = _____
A = _____
P = _____

b)



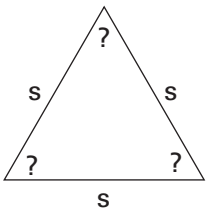
h = _____
A = _____
P = _____

c)



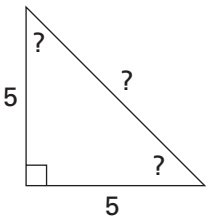
h = _____
A = _____
P = _____

d)



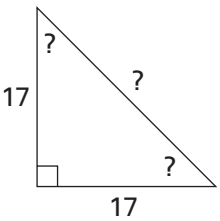
h = _____
A = _____
P = _____

2 a)



A = _____
P = _____

b)

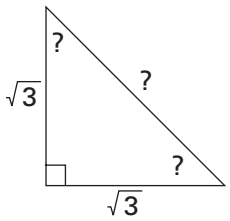


A = _____
P = _____

(Continued on back.)

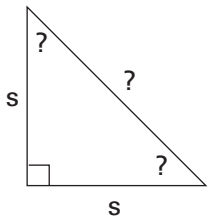
Focus Student Activity 9.2 (cont.)

c)



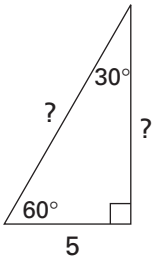
A = _____
P = _____

d)



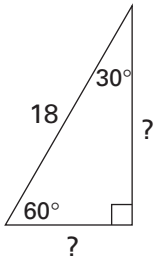
A = _____
P = _____

3 a)



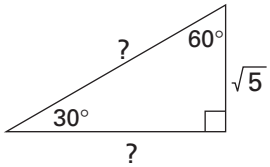
A = _____
P = _____

b)



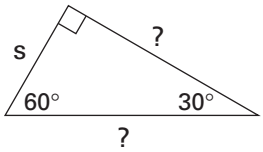
A = _____
P = _____

c)



A = _____
P = _____

d)

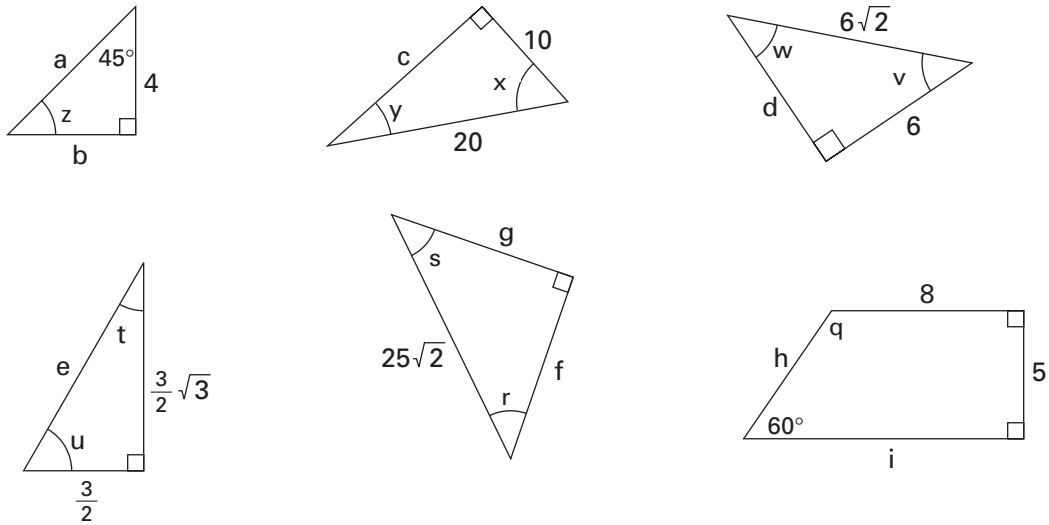


A = _____
P = _____

Focus Student Activity 9.3

NAME _____ **DATE** _____

1 Without measuring, find the value of each missing angle or length. Drawings are not necessarily to scale.



2 Draw diagrams to help you find the following values. Briefly explain and/or mark your diagrams to show your reasoning. Give actual measures rather than approximations.

a) The length of the diagonal of a square with area 225 cm^2.

b) The area of a regular hexagon with side length 12 inches.

c) The length of the diagonals of each face of a rectangular prism with dimensions 5 inches by 7 inches by 9 inches.

d) The length of the diagonals of the prism from Problem c). (Note: such diagonals extend corner to corner through the center of the prism.)

e) The area of an equilateral triangle with sides of length 17 feet.

f) The perimeter of an equilateral triangle whose area is $12\sqrt{3}$ square centimeters.

(Continued on back.)

Focus Student Activity 9.3 (cont.)

3 For each of the following, give an example to show evidence *that* the statement is true for positive values of x and y. Then demonstrate visually *why* each statement is true.

a) $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$ b) $\sqrt{x} + \sqrt{y} \neq \sqrt{x+y}$ c) $\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

4 Invent a formula for the area, A, of a regular hexagon with side length s. Be sure that A and s are the only variables in your formula.

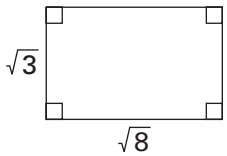
5 Simplify each of the following radical expressions.

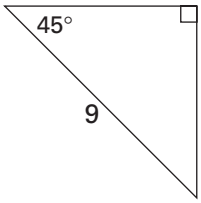
a) $\sqrt{24}$ d) $\sqrt{\frac{4}{5}}$ g) $\sqrt{32} \times \sqrt{5}$

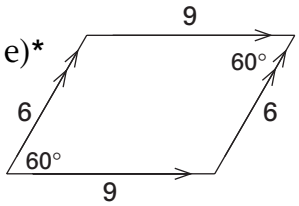
b) $\sqrt{\frac{32}{9}}$ e) $\sqrt{20} + \sqrt{60}$ h) $\sqrt{12} - \sqrt{36}$

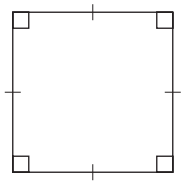
c) $\frac{1}{\sqrt{3}}$ f) $\sqrt{45} + \sqrt{75}$

6 Sketch each of the following polygons on another sheet. Show the reasoning you use to find the area and perimeter of each (in some cases the area or perimeter is given). Drawings are not to scale.

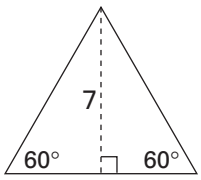
a)


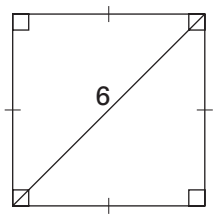
c)


e)*


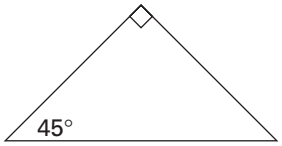
g)


perimeter = $4\sqrt{6}$

b)


d)


f)


h)


perimeter = $9 + \frac{9\sqrt{2}}{2}$

*Note: > and >> markings on sides indicate pairs of parallel sides.

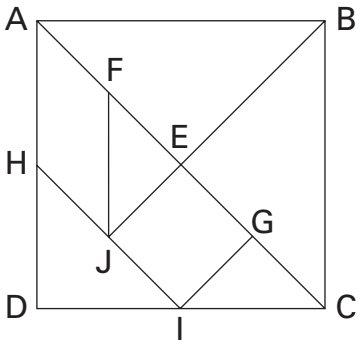
Follow-up Student Activity 9.4

NAME _____ **DATE** _____

1 One way to form the length $\sqrt{5}$ is to form a 1 by 2 rectangle and draw its diagonal. One way to form the length $\sqrt{20}$ is to double the length $\sqrt{5}$. The points $\sqrt{5}$, $-\sqrt{5}$, $\sqrt{20}$, and $-\sqrt{20}$ can be located on a number line by copying the length of a diagonal of a 1 by 2 rectangle. Determine how to locate points on the number line for the positive and negative square roots of all whole numbers less than 50, by constructing diagonals of the *minimum* number of rectangles. Make a chart to show the dimensions of the rectangles required, the length of the diagonals of each rectangle, and the points that can be located using each diagonal.

2 Tangrams, one of the oldest and most popular of the ancient Chinese puzzles, are made by constructing a geometric figure like the one below and then cutting it into the seven pieces shown.

In the square ABCD:
- Point E is the intersection of the diagonals.
- F and G are midpoints of $\overline{AE}$ and $\overline{EC}$, respectively.
- H and I are midpoints of $\overline{AD}$ and $\overline{DC}$, respectively.
- J is the midpoint of $\overline{HI}$.



If AB is 1 linear unit, find the side lengths and area of each different tangram piece. Show the calculations that you use to determine each measure.

3 Suppose that AB from Problem 2 is 5 linear units. Now find the side lengths and area of each different tangram piece. Show your reasoning.

4 Given the coordinates of any 3 points, without plotting the points on grid paper, how can you tell whether they form a right triangle? an obtuse triangle? an acute triangle? Show and explain your reasoning.

(Continued on back.)

Follow-up Student Activity (cont.)

5 If the lengths a, b, and c, form a right triangle, will $2a$, $2b$, and $2c$? $a/3$, $b/3$, and $c/3$? ka, kb, and kc, for k a positive integer? Explain.

6 Given a right rectangular prism with dimensions l, w, and h. What are the lengths of the diagonals of the prism. Justify your answer.

7 Given a cube with side length s. What are the lengths of its diagonals? Justify.

8 Draw each of the following quadrilaterals on 1-cm grid paper. Without using a calculator or a ruler to measure, determine and label the actual side lengths, diagonal lengths, and the area of each quadrilateral. Add comments as needed to communicate your methods and reasoning. Let 1 cm = 1 linear unit.

a) Squares with areas 1, 5, and 7 square units.

b) A rectangle with dimensions 1 by $\sqrt{2}$.

c) A square with diagonal length $\sqrt{6}$ linear units.

d) Two different nonsquare rectangles with diagonal lengths $\sqrt{6}$ linear units.

e) A square with side length $\sqrt{8}$ linear units and a nonsquare rectangle with diagonal length $\sqrt{8}$ linear units.

f) A rectangle with diagonal length $\sqrt{12}$ linear units.

9 Suppose that:

Nonsquare Rectangle R has sides of length a and b.
Square S has the same area as Rectangle R.
Square T has the same perimeter as Rectangle R.

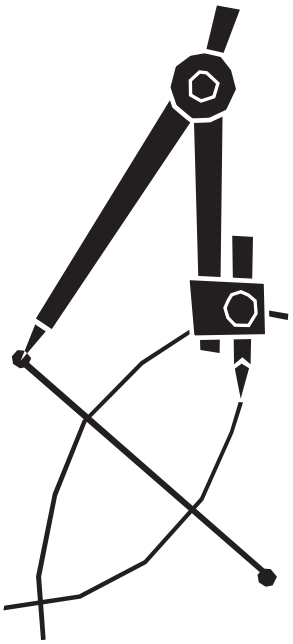
Find the length of the sides of Square S and Square T. Show your reasoning.

Constructions & Mappings

Lesson 10

THE BIG IDEA

Student-invented constructions provide a rich context for developing inductive and deductive reasoning skills. When the images of mappings, loci of points, and other constructions are obtained with straightedge and compass, triangle congruence properties and other important geometric properties arise.



CONNECTOR

OVERVIEW

Students paperfold to satisfy given conditions involving various types of angles and lines. They list observations and form conjectures about geometric relationships.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 transparency.
- ✔ Hamburger "patty paper" (or ¼-sheets of white copy or tracing paper), several sheets per student.
- ✔ Butcher paper, 1 large sheet per class.

FOCUS

OVERVIEW

Students invent methods of using a straightedge and compass to construct triangles, angle and segment bisectors, perpendicular and parallel lines, translations, rotations, and reflections. They form conjectures about triangle congruence, and develop deductive arguments to support their constructions and conjectures. They solve locus problems and investigate various geometric relationships for polygons and circles.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activities 10.1-10.4, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Masters A, C, and E, 1 transparency of each.
- ✔ Focus Masters B and D, 1 copy of each per pair of students and 1 transparency of each.
- ✔ Straightedge, 1 per student.
- ✔ Compass, 1 per student.
- ✔ Demonstration compass, 1 per teacher.
- ✔ Butcher paper and marking pens for each pair of students.
- ✔ Protractor (see *Blackline Masters*) and ruler (optional), 1 of each per student.

FOLLOW-UP

OVERVIEW

Students carry out compass and straightedge constructions, and write deductive arguments to explain several geometric relationships.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 10.5, 1 copy per student.
- ✔ Compass, 1 per student.
- ✔ Straightedge, 1 per student.

LESSON IDEAS

QUOTE
At this level, geometry should focus on investigating and using geometric ideas and relationships rather than on memorizing definitions and formulas.
NCTM Standards

LOOKING AHEAD
Throughout this course and all *Math Alive!* courses, students reason both inductively (i.e., they observe patterns and make conjectures that are based on the patterns) and deductively (i.e., they build arguments that a conjecture must be true as a direct consequence of other information that is known to be true). As students conjecture, debate, and build arguments to support their ideas they, in effect, develop informal "proofs."

QUOTE
Students discover relationships and develop spatial sense by constructing, drawing, measuring, visualizing, comparing, transforming, and classifying geometric figures. Discussing ideas, conjecturing, and testing hypotheses precede the development of more formal summary statements. In the process definitions become meaningful, relationships among figures are understood, and students are prepared to use these ideas to develop informal arguments.
NCTM Standards

FOLLOW-UP
These problems are intended to prompt experimentation and conjectures. It is helpful to allow periodic opportunities for students to confer with one another—to request clues (not answers) and to share places they are stuck. Based on your students' comfort with the lesson, you may wish to identify (or let the students identify) some problems as optional challenges.

SELECTED ANSWERS

1. a) One possibility: construct a square whose sides have a length of 1 linear unit. The diagonal of the square is $\sqrt{2}$ units. Then copy the diagonal length to construct an equilateral triangle having side lengths of $\sqrt{2}$.

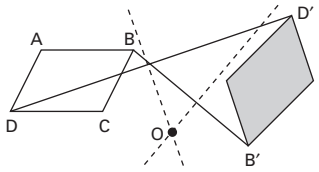
   c) Not possible because the triangle has two 45° angles and so it would be isosceles.

   e) Not possible because the diagonals of a rectangle have the same length.

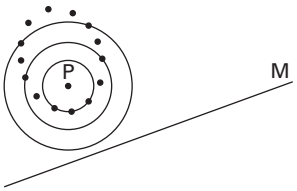
   g) One possibility: construct an equilateral triangle with side lengths of 2 units. Then construct the perpendicular bisector of 2 sides of the triangle and their intersection will be the center of the circle. Construct the circle whose radius is the segment from the center of the circle to a vertex of the triangle.

   i) One possibility: form a hexagon by constructing 6 adjacent non-overlapping equilateral triangles which all share a common vertex and whose side lengths are $2\sqrt{3}$ (i.e., the length of the diagonal of a $1 \times \sqrt{2}$ rectangle). The altitude of each triangle is 3 units. Construct a circle whose radius is 3 and whose center is the common vertex of the triangles.

3. Since point D′ is the image of D, DD′ is a chord of the circle that rotates D to D′. So, the perpendicular bisector of DD′ passes through the center of rotation. Similarly, the perpendicular bisector of BB′ passes through the center of rotation. These 2 perpendicular bisectors intersect in the center of rotation.



5. The locus of points is an ellipse. A few points of the locus are shown in the following figure. The construction of circles were used to locate points whose distance from line M is twice the distance from point P.



Connector Teacher Activity

OVERVIEW & PURPOSE

Students paperfold to satisfy given conditions involving various types of angles and lines. They list observations and form conjectures about geometric relationships.

MATERIALS

- ✔ Connector Master A, 1 transparency.
- ✔ Hamburger "patty paper" (or ¼-sheets of white copy or tracing paper), several sheets per student.
- ✔ Butcher paper, 1 large sheet per class.

ACTIONS

1 Arrange the students in groups and distribute several sheets of patty paper to each student. Write the following list of angle types on the overhead and ask the students to paperfold to obtain one or more examples of each type. Discuss their methods, observations, and conjectures. Encourage students to give informal arguments to support their conjectures. As needed, clarify other terminology that comes up, such as: *collinear*, *linear pair*, *opposite rays*, *straight angle*, etc.

Types of Angles:

a) Acute

b) Obtuse

c) Congruent

d) Adjacent

e) Vertical

f) Right

g) Supplementary

h) Complementary

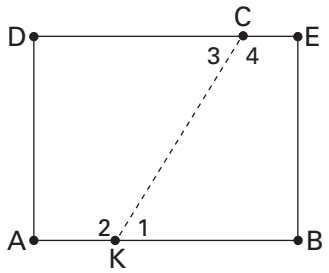
i) Straight

COMMENTS

1 Students were introduced to these types of angles in *Math Alive! Courses I* and *II*. Keep emphasis on understanding rather than memorizing. If students are keeping journal glossaries, you might remind them to include the terminology from this lesson as it comes up.

Some students may demonstrate more than one angle type on the same sheet. Following are examples of folds and observations students have made; there are other possibilities. Lines and angles are formed by making sharp creases in the paper.

Making 1 fold across a sheet, as illustrated below, produces 4 angles. ∠1 and ∠3 are *acute* angles because their measures are less than 90°; this can be verified by comparing the angles to a corner of the paper. ∠2 and ∠4 are *obtuse* angles because their measures are greater than 90° and less than 180°. ∠1 and ∠2 are *adjacent* angles, as are ∠3 and ∠4, since they have in common both a vertex and a side. ∠1 and ∠2 are *supplementary* because the sum of their measures is 180° (this is true since they combine to form a *straight angle*). ∠3 and ∠4 are supplementary for the same reason. ∠1 and ∠2 are also called a *linear pair* of angles because they are adjacent angles whose noncommon sides are *opposite rays* (i.e., the noncommon sides are 2 different *collinear* rays that have the same endpoint, or *origin*). Similarly, ∠3 and ∠4 are a linear pair.

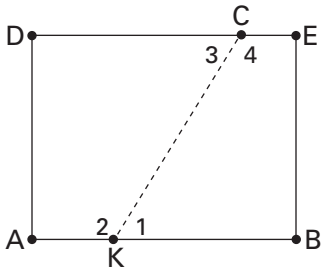


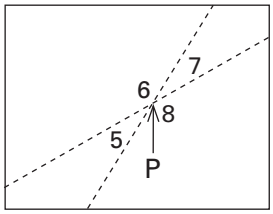
Given a fold such as the one shown above, students may correctly conjecture that ∠1 and ∠3 are *congruent* (i.e., they have the same measure) and that ∠2 and ∠4 are congruent. To support this conjecture, they may super-

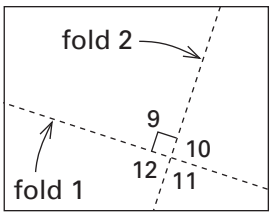
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS







2 Post a large poster entitled "We conjecture… We wonder…" on the classroom wall. Tell the students that throughout the remainder of the lesson you want them to post or suggest statements for the poster. Suggest that whenever the class reaches a high level of confidence that a conjecture is true, a star will be drawn next to the conjecture.

COMMENTS

1 (continued.)
impose a tracing of ∠1 with ∠3, for example. Or, they may demonstrate that a 180° rotation about the midpoint of $\overline{CK}$ causes the tracing of ∠1 to coincide with ∠3 and the tracing of ∠2 to coincide with ∠4. This is true if the opposite edges of the paper are parallel (see Comment 2).

Vertical angles are formed by two intersecting lines. Vertical angles have a common vertex (see point P at the left) and their sides are opposite rays. In the example shown at the left, ∠5 and ∠7 are vertical angles, as are ∠6 and ∠8.

Students may correctly conjecture that vertical angles are congruent, and verify this by folding ∠5 onto ∠7, and ∠6 onto ∠8, arguing that the new folds are lines of symmetry (see Lessons 1 and 2 of this course). Or, they may support their conjecture that vertical angles are congruent by rotating a tracing of the 4 angles 180° about point P to show that each angle maps onto its vertical angle. This is so because each line has 180° rotational symmetry about P.

One way to form a *right* angle is: crease to form a line; then fold the line onto itself, forming 4 congruent angles, each with measure 360° ÷ 4 = 90°. See ∠9, ∠10, ∠11, and ∠12 at the left. To form adjacent *complementary angles* (i.e., 2 angles whose measures total 90°), crease to form a line that passes through the vertex and between the sides of a right angle.

Note: Students may mistakenly assume that 2 angles must be adjacent in order to be supplementary or complementary. For 2 angles to be supplementary, the sum of their measures must be 180°, regardless their location. For example, since ∠1 + ∠2 = 180°, and since ∠1 and ∠3 are congruent and ∠2 and ∠4 are congruent, then ∠1 + ∠4 = 180° and ∠2 + ∠3 = 180°. So, ∠1 and ∠4 are supplementary, as are ∠2 and ∠3. Similarly, 2 complementary angles may be adjacent or nonadjacent angles.

2 The intent here is to encourage questions and conjectures and the use of reasoning to support conjectures. It is expected that students arguments are informal and intuitive. As the lesson progresses, students will develop tools for building stronger deductive arguments.

Throughout this lesson, you might provide students easy access to a supply of butcher paper strips and marking pens, so they can record conjectures or questions before posting them.

Connector Teacher Activity (cont.)

ACTIONS

3 Place a transparency of Connector Master A on the overhead, revealing a) only. Ask the students to each paperfold to form lines that satisfy the given conditions, to develop arguments that show why their methods work, and to make observations and conjectures about the results. Record their observations, questions, and conjectures on the class "We conjecture… We wonder…" poster for continued discussion throughout the lesson.

Constructions and Mappings Lesson **10**

Connector Master A

Paperfold **to satisfy each of the following conditions. Do not use a protractor or ruler.**

a) Form 2 lines which are perpendicular to each other and not parallel to the edges of the paper.

4 Repeat Action 3 for b)-h) on Connector Master A. Distribute additional patty paper as needed. Encourage students to make conjectures, whenever possible, in "if… then…" form. Use students' methods, observations, and conjectures as a context for clarifying relationships related to perpendicular and parallel lines and for clarifying terminology such as *transversal, corresponding angles, alternate interior angles, alternate exterior angles, angle bisector, perpendicular bisector,* and *isosceles triangle*.

COMMENTS

If you teach several classes, it is helpful to provide a poster for each class. During each class cover other classes' posters (or fold the bottom half of each poster to cover the top half) until the end of the class period. Then have the students compare their observations to those made by other classes.

3 Since 2 lines that intersect to form right angles are *perpendicular*, the students can use the method of forming right angles from Action 1 to form perpendicular lines. That is, fold the paper to form a crease that is a line; then select a point on this line and fold the line onto itself about the point. The crease of the latter fold is a line perpendicular to the original line at the selected point.

4 As discussed in Lesson 9 of this course, an "if-then" statement is called a *conditional statement*. The "if…" portion of the statement is called the *hypothesis* and represents the information that is given or assumed to be true. The "then…" portion is called the *conclusion*.

Following are some possible paperfolding strategies for b)-h), together with possible student conjectures.

b) Fold and crease the paper to form a line. Pick a point on the line and fold the line onto itself about that point. Crease this new fold to form a line perpendicular to the original line. Pick another point on the original line and repeat the process, forming a 2nd line that is perpendicular to the original line.

Students may correctly conjecture that if 2 lines are perpendicular to a 3rd line, then they are *parallel* to each other, because the 2 lines will never intersect. They may also conjecture that all perpendicular segments between 2 parallel lines are equal in length.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Constructions and Mappings Lesson **10**

Connector Master A

***Paperfold* to satisfy each of the following conditions. Do not use a protractor or ruler.**

a) Form 2 lines which are perpendicular to each other and not parallel to the edges of the paper.

b) Form 3 lines so that 2 of the lines are perpendicular to the 3rd line.

c) Form 2 parallel lines which are intersected by a 3rd line that is not perpendicular to the parallel lines.

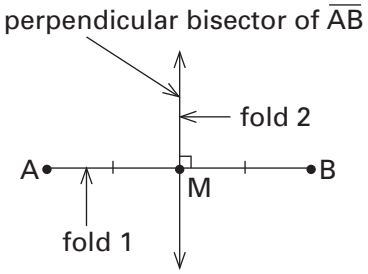
d) Form a line segment and locate its midpoint. Then form a line perpendicular to the segment and passing through the midpoint of the segment.

e) Create an acute angle; then form a line that bisects the angle.

f) Form a line *m* and mark a point P not on the line. Then form a new line which is perpendicular to line *m* and passes through point P.

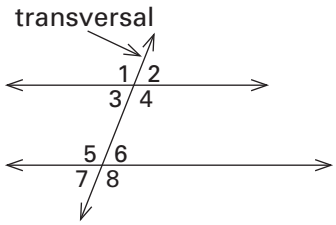
g) Form a line *r* and mark a point Q not on line *r*. Then form line *s* which passes through Q and is parallel to line *r*.

h) Form an isosceles triangle.



COMMENTS

4 (continued.)
c) Fold the paper so that the 2 parallel edges coincide. This produces a crease which is parallel to 2 edges of the paper. Next fold the paper to obtain a 2nd crease which is parallel to the 1st crease. Unfold the paper and then fold to obtain a 3rd crease which intersects the 2 parallel creases. Because the 3rd crease is a line that intersects 2 other lines, it is called a *transversal* of those lines.



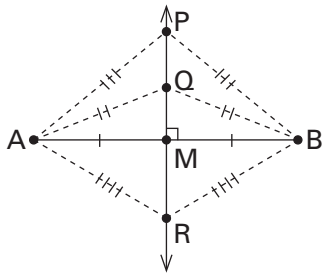
Students may notice that, at the intersections of a transversal with 2 parallel lines, there are 4 pairs of vertical angles, e.g., $\angle 1 \cong \angle 4$, $\angle 2 \cong \angle 3$, $\angle 5 \cong \angle 8$, and $\angle 6 \cong \angle 7$ in the diagram above. By using the midpoint of the segment of the transversal between the 2 parallel lines as the center of a 180° rotation, students may correctly conjecture that $\angle 1 \cong \angle 8$, $\angle 2 \cong \angle 7$, $\angle 3 \cong \angle 6$, and $\angle 4$ and $\angle 5$. Further, they may reason that, since $\angle 1 \cong \angle 8$, and $\angle 8 \cong \angle 5$, then $\angle 1 \cong \angle 5$. This illustrates the *transitive property* for congruence, i.e., if $a \cong b$ and $b \cong c$, then $a \cong c$. Similarly, it is possible to show that $\angle 2 \cong \angle 6 \cong \angle 3 \cong \angle 7$ and $\angle 4 \cong \angle 8 \cong \angle 5 \cong \angle 1$. Or, students may use a translation of one line onto the other, using the segment of the transversal between the lines as translation vector, to demonstrate that one line is a translation image of the other, and hence, the lines are parallel.

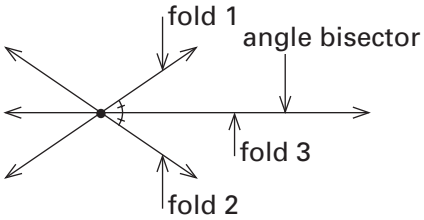
In general, if a transversal intersects parallel lines, then it forms several pairs of congruent angles: the *vertical angles* (see example above); *corresponding angles*, *alternate exterior angles*, and *alternate interior angles*. Corresponding angles are angles in corresponding positions relative to the transversal (e.g., see $\angle 1$ and $\angle 5$, $\angle 2$ and $\angle 6$, $\angle 3$ and $\angle 7$, and $\angle 4$ and $\angle 8$). *Alternate exterior angles* are angles on the exterior of the parallel lines and on alternate sides of the transversal (e.g., see $\angle 1$ and $\angle 8$, $\angle 2$ and $\angle 7$). *Alternate interior angles* are angles on the interior of the parallel lines and on alternate sides of the transversal (e.g., see $\angle 3$ and $\angle 6$, $\angle 5$ and $\angle 4$).

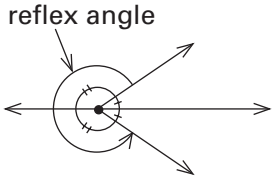
d) Crease the paper to form a line segment. Mark the endpoints. Fold the line segment onto itself so that the two endpoints coincide. This 2nd crease forms a line that passes through the *midpoint* of the segment and is perpendicular to the segment at that point. This line is called the *perpendicular bisector* of the segment.

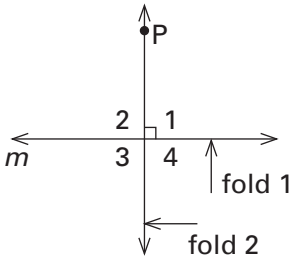
Connector Teacher Activity (cont.)

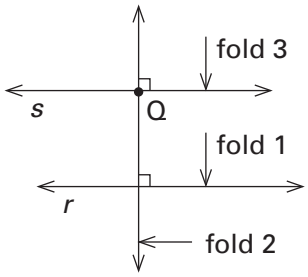
ACTIONS

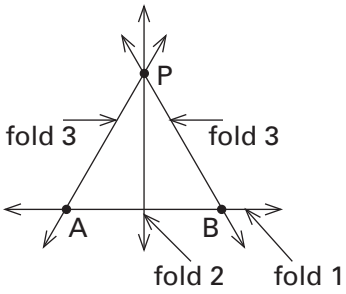












COMMENTS

Some students may observe that each point on the perpendicular bisector of a segment is the same distance from one endpoint of the segment as from the other endpoint. Some may support this observation by pointing out that the perpendicular bisector of $\overline{AB}$, for example, is a line of reflection across which point A maps onto point B (see Lessons 1 and 2 of this course). Hence, in the illustration at the left, since reflections preserve length, $\overline{AP} \cong \overline{PB}$, $\overline{AQ} \cong \overline{QB}$, and $\overline{AR} \cong \overline{RB}$.

e) First fold to create an acute angle. This may be done with 1 crease as in Comment 1, or by 2 creases as shown at the left. Fold the 2 sides of the angle onto each other and crease to form a line that bisects the angle (see fold 3 in the diagram). This line is called the *angle bisector*. Notice it is also a line of reflection that maps the sides of the angle onto each other.

Notice also that the crease bisects both the acute angle and the *reflex angle*. Note: a reflex angle is an angle whose measure is greater than 180° and less than 360°. In this case, the measure of the reflex angle is the difference between 360° and the measure of the acute angle.

f) Fold to form a line *m*. Mark a point P not on *m*. Fold the line *m* onto itself so that the crease contains the point P. This forms 4 congruent angles, and since the angles total 360°, each angle measures $^{360°}/_{4} = 90°$. Hence, each is a right angle. Notice there are also 4 different pairs of adjacent angles; each pair is a linear pair and supplementary.

g) Paperfold as for f) to obtain the line perpendicular to line *r* and passing through point Q not on *r*. Then fold this line onto itself about point Q. The result is line *s*, which passes through Q and is perpendicular to the line formed by fold 2 and parallel to line *r*.

This may prompt the conjecture that, if 2 lines are perpendicular to the same line, then they are parallel to each other.

h) Fold 1: paperfold and crease to form a line. Mark points A and B on the line. Fold 2: fold $\overline{AB}$ onto itself so that A coincides with B. Crease to form the perpendicular bisector of $\overline{AB}$. Keeping fold 2 in place, mark a point P on the perpendicular bisector. Fold 3: keeping fold 2 in place, fold and crease to form segments $\overline{AP}$ and $\overline{BP}$. Unfold to reveal ΔAPB which is isosceles since at least 2 of its sides are congruent.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
	4 (continued.) Students may pose several conjectures about isosceles triangles. For example: *If a triangle is isosceles, then the base angles are congruent.* *If a triangle is isosceles, then the altitude divides the triangle into 2 smaller congruent triangles.* *If an isosceles triangle is a right triangle, then the altitude divides it into 2 smaller isosceles right triangles.* *If a triangle is isosceles, then the altitude is a line of reflection of the triangle.*

Focus Teacher Activity

OVERVIEW & PURPOSE

Students invent methods of using a straightedge and compass to construct triangles, angle and segment bisectors, perpendicular and parallel lines, translations, rotations, and reflections. They form conjectures about triangle congruence, and develop deductive arguments to support their constructions and conjectures. They solve locus problems and investigate various geometric relationships for polygons and circles.

MATERIALS

- ✔ Focus Student Activities 10.1-10.4, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Masters A, C, and E, 1 transparency of each.
- ✔ Focus Masters B and D, 1 copy of each per pair of students and 1 transparency of each.
- ✔ Straightedge, 1 per student.
- ✔ Compass, 1 per student.
- ✔ Demonstration compass, 1 per teacher.
- ✔ Butcher paper and marking pens for each pair of students.
- ✔ Protractor and ruler (optional), 1 of each per student.

ACTIONS

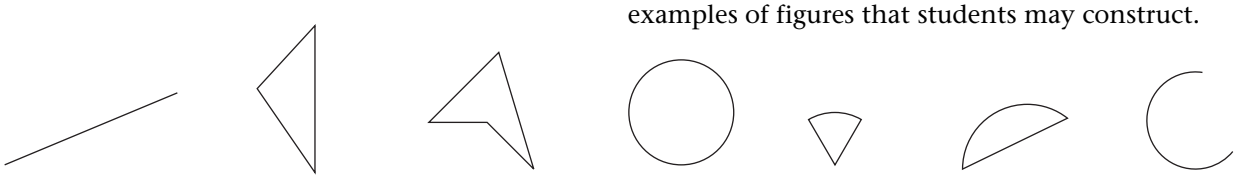
1 Arrange the students in pairs. Discuss the differences between *sketches*, *drawings*, and *geometric constructions*. Distribute a straightedge and compass to each student. Ask the students to investigate and determine several geometric figures that can be constructed using only a straightedge and/or compass. Discuss the students' results, clarifying the names of constructed figures as needed.

COMMENTS

1 Typically, a *sketch* is a freehand illustration; sketching may or may not include the use of any geometric tools. A *drawing* is generally more accurate; often it involves the use of a protractor to measure angles and a ruler to measure lengths. A *geometric construction* is a geometric figure that can be created using only a *straightedge* and *compass*. Marks (e.g., arcs, circles, points, etc.) made during the construction process are usually not erased. No measuring tools are allowed in geometric constructions (if a ruler is used as a straightedge, or a compass has markers to measure radii, the marks on the ruler and compass must not be used for measuring).

Some students may at first resist the limitations of using only a straightedge and compass for constructions. If so, you might suggest that working under such restrictions will lead to many discoveries, conjectures, and generalizations involving geometric shapes and their properties. In fact, over the centuries many important mathematical discoveries have happened in this context. The ancient Greeks were the first to explore geometric constructions. Over the ages, mathematicians have been intrigued and challenged by inventing geometric constructions; a goal of this lesson is for students to experience similar intrigue and challenge.

A demonstration-sized compass and straightedge may be useful at the board. Or, some standard compasses work well with an overhead pen at the overhead. Below are examples of figures that students may construct.



Focus Teacher Activity (cont.)

ACTIONS

2 Place a copy of Focus Master A on the overhead, revealing only Construction 1. Ask each pair of students to carry out the construction. Ask for illustrations and explanations at the overhead.

Constructions and Mappings Lesson **10**

Focus Master A

Construction 1

a) Use a straightedge to construct a line segment $\overline{AB}$.

b) Investigate ways to use a straightedge and compass only to construct a congruent copy of the line segment $\overline{AB}$ formed in a).

c) Devise a set of clear and concise, step-by-step instructions for *constructing* a copy of a line segment.

Construction 2

a) Use a straightedge to draw an angle with vertex X.

b) Investigate ways to use a straightedge and compass to construct ∠V congruent to ∠X formed in a).

c) Devise a clear set of instructions for *constructing* a copy of an angle.

3 Reveal Construction 2 on Focus Master A and repeat Action 2 for this construction.

COMMENTS

2 It may be helpful to remind students not to erase or remove their construction marks.

c) You might have the pairs exchange their completed instructions and provide feedback to each other. Here is a sample set of student instructions:

Step 1. *Using a straightedge, construct a line segment of any length and mark the endpoints A and B.*

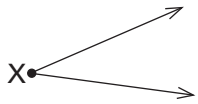
Step 2. *Open the compass so the compass point is at A and the pencil tip is at B. Pick a point not on $\overline{AB}$ and label it C. Keeping the compass open the distance from A to B, place the compass point at C and swing an arc with the pencil tip.*

Step 3. *Pick any point on the arc from Step 2 and label it D. Use the straightedge to connect C to D. This forms $\overline{CD}$ which is congruent to $\overline{AB}$.*

Note: If $\overline{AB}$ is longer than the opening of the compass, then $\overline{AB}$ can be divided into sections and the sections copied one at a time, endpoint to endpoint along a straight line.

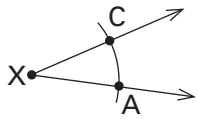
3 Again, you might have pairs exchange completed instructions and provide feedback to each other. Following is one possible set of instructions for constructing ∠V congruent to a given ∠X.

Given:



Step 1. *Pick a point on the paper and label it V. Using a straightedge, construct the beginning of a ray with endpoint V.*

Step 2. *Place the compass point at X, the vertex of the given angle. Swing an arc that passes through both sides of ∠X. Label the points A and C where the arc intersects the sides of ∠X.*

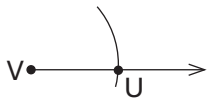


Focus Teacher Activity (cont.)

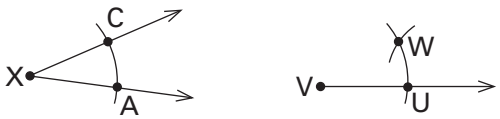
ACTIONS

COMMENTS

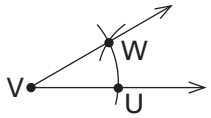
Step 3. *Without changing the opening of the compass from Step 2, place the compass point at point V and swing an arc that crosses the ray formed in Step 1. Label point U, the intersection of the arc and ray.*



Step 4. *Adjust the opening of the compass so that, when the compass point is on A, the pencil tip is on C. Without changing the compass opening, place the compass point at U and swing an arc that crosses the arc formed in Step 3. Label the intersection of the arcs W.*



Step 5. *Use a straightedge to construct ray* $\overrightarrow{VW}$. $\angle UVW \cong \angle AXC$.



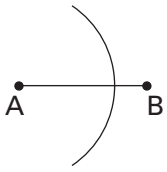
4 Ask the pairs to draw a line segment and then invent a method of constructing a line that is the *perpendicular bisector* of the segment. Have them write a set of step-by-step instructions for their constructions. Discuss the students' methods and reasoning. Encourage them to add conjectures and questions to the "We conjecture… We wonder…" poster created during the Connector activity.

4 It is important to allow plenty of time for experimentation and conferring among the pairs. Some students may find it helpful to reflect on their paperfolding strategies from the Connector and to base their constructions on observations and generalizations related to those strategies. For example, the following construction is based on the observation that any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment (see Connector Comment 4d):

Step 1. *Draw a line segment and label its endpoints A and B.*

Step 2. *Open the compass a distance greater than ½ AB.*

Step 3. *Without changing the compass opening, place the compass point at point A and swing an arc that extends above and below* $\overline{AB}$.



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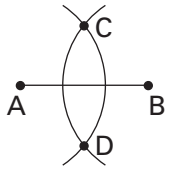
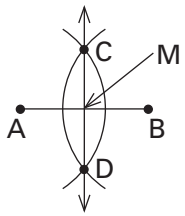
Focus Teacher Activity (cont.)

ACTIONS





5 Ask the pairs to invent a method of constructing a pair of parallel lines and to write a set of step-by-step instructions for their construction. Discuss their methods and reasoning.

6 If, in Action 5, students didn't bring up the method of forming parallel lines by constructing congruent corresponding angles on a transversal, ask them to do the following:

a) Draw a line *l* and label 2 points A and B on line *l*.

COMMENTS

4 (continued.)
Step 4. *Without changing the compass opening, place the compass point at B and swing an arc that intersects the arc formed in Step 3 in 2 points. Label the points of intersection C and D.*

Step 5. *Draw $\overleftrightarrow{CD}$, the perpendicular bisector of $\overline{AB}$ at point M, the midpoint of $\overline{AB}$.*

Note that in Action 14 students develop deductive arguments to show why this construction works.

5 Again, reflecting on their paperfolding strategies during the Connector may be helpful. If some pairs finish while others are still working, you might encourage the early finishers to see if they can devise more than one method of constructing parallel lines.

Many students may construct a line, and then based on the construction from Action 4, construct the perpendicular bisector of 2 different segments on the line, noting that 2 lines perpendicular to a 3rd line are parallel to each other (see Connector Comment 4).

Another way to construct parallel lines is to draw a line and label 2 points A and B on the line; then construct congruent corresponding (or alternate interior or alternate exterior) angles at points A and B. If this idea isn't suggested, notice it is explored in Action 6.

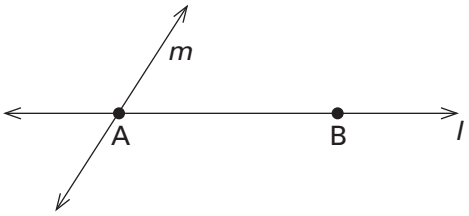
6 c) To construct a congruent corresponding angle at B, students must construct an angle identical to one of the angles formed at point A (i.e., one of the angles formed by the intersection of lines *l* and *m*)—identical in measure and identical in position relative to the lines. That is, both angles need to be on the same side of line *l* and both angles need to open in the same direction. The method of copying an angle described in Comment 3 can be used here.

Students may notice that by constructing congruent corresponding angles with transversal *l*, parallel lines are formed. Hence, both the statement—if lines are parallel, then they form congruent corresponding angles with a transversal—and its converse—if 2 lines form congruent corresponding angles with a transversal, then the lines are parallel—are true. These 2 statements can be re-

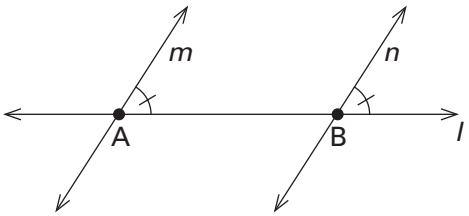
Focus Teacher Activity (cont.)

ACTIONS

b) Draw line *m* passing through A. Notice 4 angles are formed at A.



c) Using 1 of the 4 angles formed at A, construct a congruent corresponding angle at B.



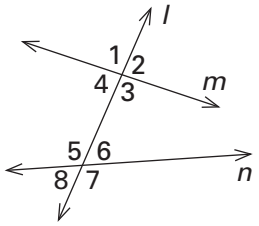
d) Discuss the students' observations and the use of *if and only if* when a statement and its converse are true.

COMMENTS

placed by the statement, 2 lines are parallel if and only if they form congruent corresponding angles with a transversal. In general, when the statement "if *a*, then *b*" and its *converse* "if *b*, then *a*" are both true, then one can say "*a* is true *if and only if b* is true." The statement "*a* if and only if *b*" is called a *biconditional statement*. Not every "if-then" statement can be written in "if and only if" format, because it is not always the case that if a statement is true, its converse is also true.

Students may also notice that by constructing congruent corresponding angles, congruent alternate interior and alternate exterior angles are automatically formed. You might have students determine whether "if and only if" statements can be written regarding alternate interior and alternate exterior angles forming parallel lines (they can).

Some students may conjecture that if 2 lines are cut by a transversal and a pair of corresponding angles (or alternate interior angles or alternate exterior angles) are *not* congruent, then the lines are not parallel. This is true, as is the converse.



In this lesson, many ideas are accepted as true based on informal arguments (e.g., using a 180° rotation to verify that, when parallel lines are cut by a transversal, alternate interior angles are congruent); whereas the same ideas may be postulated in a more formal approach.

Students' arguments and reasoning throughout this lesson will not necessarily be as detailed or well developed as ideas suggested in the Comments column. Information is provided here for teacher background, to bring up as deemed appropriate, based on students' comfort and needs. Many students may find it easier to argue orally from diagrams than to prepare written arguments. The intent here is to engage students in the process of thinking deductively about geometric relationships. Over time, with extensive opportunities to discuss their thinking and hear each other's ideas, students will gain confidence and written arguments will gain clarity.

Focus Teacher Activity (cont.)

ACTIONS

7 Ask each student to use a straightedge to construct a triangle and to label the vertices of the triangle A, B, and C. Next, ask them to exchange triangles with their partners, to devise a method of *constructing* (i.e., with a straightedge and compass only) a triangle congruent to their partner's triangle, and to label the vertices of this new triangle D, E, and F. Discuss their strategies and observations.

8 If it didn't come up in Action 7, ask the students for ideas regarding ways to record which *parts*, i.e., which sides, angles, and vertices, of the two triangles ABC and DEF are congruent. Use this as a context for discussing standard notation regarding congruent triangles.

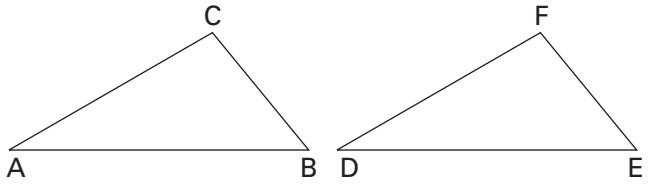
COMMENTS

7 Students may use a variety of strategies. If some devise strategies quickly, you might encourage continued investigation by posing questions such as: Is there more than one sequence of steps possible? Is there a minimal set of steps? Is there more than one minimal set?

Students may illustrate that their triangles are congruent by placing one on top of the other to show that they coincide. Students may make conjectures about constructing congruent triangles and relationships in general among congruent triangles. Rather than affirming or contradicting their conjectures now, suggest that students test, and refine their ideas during subsequent actions.

Notice that in Actions 9 and 10 the students generalize strategies for forming congruent triangles. See those comments for discussion ideas and conjectures that may come up here.

8 If students superimpose congruent triangles, the *corresponding parts* of the triangles coincide. Students may invent a number of ways to denote corresponding parts. For example, for the congruent triangles shown below, the corresponding vertices might be recorded as A,D; B,E; C,F. Note that the lettering of the correspondences for students' triangles may be different, based on the order of their labeling.



Double arrows are frequently used to indicate corresponding parts:

Corresponding vertices: $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$
Corresponding sides: $\overline{AB} \leftrightarrow \overline{DE}$, $\overline{BC} \leftrightarrow \overline{EF}$, $\overline{CA} \leftrightarrow \overline{FD}$
Corresponding angles: $\angle CAB \leftrightarrow \angle FDE$, $\angle ABC \leftrightarrow \angle DEF$, $\angle BCA \leftrightarrow \angle EFD$

The $\cong$ symbol denotes congruence of triangles, e.g., $\Delta ABC \cong \Delta DEF$, where the letters are placed in the order of their corresponding vertices. Similarly, the $\cong$ symbol denotes congruent sides and angles. For example, for the congruent triangles shown above, $\overline{AB} \cong \overline{DE}$ and $\angle ABC \cong \angle DEF$.

Focus Teacher Activity (cont.)

ACTIONS

9 Provide each pair of students a sheet of butcher paper, marking pens, and a copy of Focus Master B. Ask the pairs to carry out the instructions on Master B. When completed, have the pairs post and compare their results. Discuss. If there is consensus about conjectures and generalizations, record them on the class poster created during Action 2 of the Connector activity.

Constructions and Mappings Lesson **10**

Focus Master B

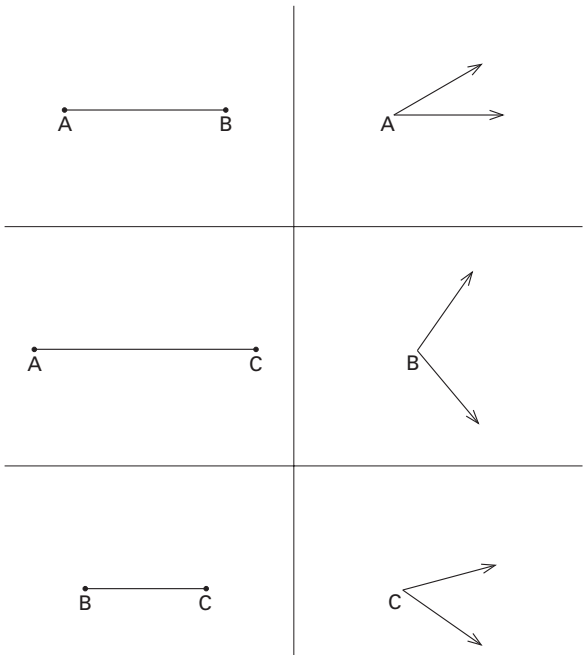
The 6 pieces of information shown on the next page make up the "maximal" set of information that you need to construct triangle ABC. What is a "minimal" set of information that you need to construct triangle ABC. How can you be sure you have all the information that you need? How can you be sure that you don't have more information than you need? Is there more than one minimal set of information?

Investigate. Test your ideas and build arguments to support your conclusions.

Make a poster of your conclusions. Show examples to support your ideas. List important observations, conjectures, and generalizations that you develop during the investigation.

Lesson **10** Constructions and Mappings

Focus Master B (cont.)



COMMENTS

9 It may be helpful to remind the students that angle A must be constructed at point A of segments $\overline{AB}$ and $\overline{AC}$, angle B at point B, and angle C at point C.

The intent here is to identify minimal combinations of measurements that determine a unique triangle. Students' investigations may be random at first; after a while, if needed, you might provide encouragement for approaching the problem systematically.

A minimal set of measurements is adequate and not redundant—i.e., the set is adequate if all triangles that can be formed using those measurements are congruent to ΔABC, and the set is not redundant if no measurements are unnecessary to determine the triangle.

Following are several observations that may be offered by students. Note that, if these generalizations do not arise here, rather than bringing them up for discussion now, you might wait until later in the lesson, since there will be several opportunities in subsequent actions.

We started to copy all 6 pieces of information, but discovered that, by the time we copied all 3 side lengths, we had "fixed" the triangle. So, the set of 3 side lengths is a minimal set of information.

Copying sides $\overline{AB}$ and $\overline{BC}$ and $\angle B$ is enough to "fix" ΔABC, and no other triangles are possible with these measurements. The same is true for sides $\overline{AB}$ and $\overline{AC}$ and $\angle A$, or sides $\overline{AC}$ and $\overline{BC}$ and $\angle C$. We notice that if we copy any 2 sides and the angle between the sides, then the only triangle we can construct is ΔABC.

By copying $\angle A$ and $\angle B$ and side $\overline{AB}$, we are able to form ΔABC. Similarly, $\angle B$ and $\angle C$ and side $\overline{BC}$ or $\angle A$ and $\angle C$ and side $\overline{AC}$ also determine ΔABC. So, any 2 angles and the side between the angles make a minimal set of information needed to form ΔABC.

There are an infinite number of triangles whose angles are all congruent to angles A, B, and C. These triangles are all enlargements or reductions of each other, but they are not necessarily congruent to each other. These triangles are all similar triangles. It isn't enough to copy the 3 angles; at least one side is needed.

An infinite number of triangles can be formed using only 2 side lengths and no other information about the triangle.

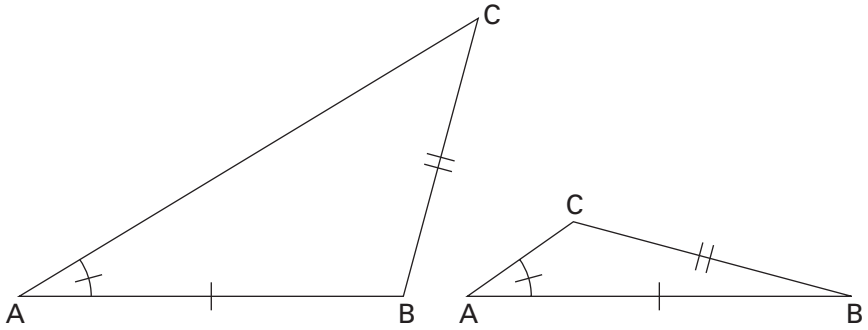
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

9 (continued.)
In the following diagram, we have shown that there are 2 triangles possible by copying sides $\overline{AB}$ and $\overline{BC}$ and angle A. So knowing 2 sides and an angle not between the 2 sides is not enough to determine a unique triangle.



If you pick any 2 angles and a side of the triangle that is not included by the angles, there is only 1 triangle that can be formed. Note: there would be 2 triangles possible if the side and 2 angles were not labeled to identify the position of the side relative to the angles.

Since the sum of the measures of the angles in a triangle is 180°, we can see that knowing 2 angles of a triangle "fixes" the 3rd angle.

10 Place a transparency of Focus Master C on the overhead and ask the pairs to carry out the instructions.

Discuss their results and reasoning. Use this as a context for introducing the *side-side-side* (SSS), *side-angle-side* (SAS), and *angle-side-angle* (ASA) congruence properties.

Constructions and Mappings Lesson **10**
Focus Master C

According to the definition of congruence:

If there is a correspondence between the vertices of 2 triangles such that all corresponding segments are congruent and all corresponding angles are congruent, then the triangles are congruent.

The above statement gives the *maximal* set of conditions needed to establish that 2 triangles are congruent. For any 2 triangles ABC and DEF, what are all the possible *minimal* sets of conditions for determining whether ΔABC is congruent to ΔDEF? Give convincing evidence to support your conclusions.

10 If students get "stuck," you might give each student a protractor and ruler and suggest they draw (out of sight of their partner) 3 triangles (ΔRST, ΔUVW, and ΔXYZ) so that at least 2 of the triangles are congruent, recording the side lengths and angle measures for the 3 triangles. Suggest that each student determine minimal sets of measurements they need to request from their partner in order to determine which of their partner's 3 triangles are congruent.

In Action 9, students determined that exactly 1 triangle could be constructed with the 3 side lengths. This is true for any 3 lengths, assuming they form a triangle. Hence, if 3 sides of 1 triangle are congruent to 3 sides of another triangle, the 2 triangles must be congruent. Thus, if the 3 side lengths of 2 triangles are known, it is possible to determine whether they are congruent without knowing any angle measurements.

The statement, *if 3 sides of 1 triangle are congruent to 3 sides of another triangle, then the 2 triangles are congruent*, is often referred to as the *side-side-side congruence property*. It is typically abbreviated by *SSS* to indicate that 3 pairs of congruent sides are sufficient to show that 2 triangles are congruent.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

As determined in Action 9, knowing 2 sides and the angle included by the 2 sides is also sufficient to construct a unique triangle. Hence, the *side-angle-side congruence property* is true: *If 2 sides and the included angle of 1 triangle are congruent to 2 sides and the included angle of another triangle, the 2 triangles are congruent*. This congruence property is abbreviated as SAS.

A 3rd congruence property is the *angle-side-angle congruence property: if 2 angles and the included side of 1 triangle are congruent to 2 angles and the included side of another triangle, the 2 triangles are congruent*. This congruence property is often abbreviated ASA.

If students don't raise the possibility of SSA (i.e., 2 sides and an angle not included by the sides of 1 triangle are congruent to the corresponding parts of a 2nd triangle), you might do so. As illustrated in Comment 9, given 2 sides and an angle not included by the sides, there is more than 1 triangle possible. Hence, SSA is *not* a congruence property.

Notice that, if one knows the measures of 2 angles of a triangle, then the measure of the 3rd angle is 180° minus the sum of the 2 given angles. Hence, knowing the measures of 2 angles and a side of a triangle is equivalent to knowing 3 angles and a side. Thus, if one knows 2 angles and a side of 1 triangle are congruent to the corresponding 2 angles and side of another triangle, then the triangles are congruent, for the reasons given in the discussion of the ASA congruence property. Some students may wish to refer to this as the AAS congruence property. Note that AAS holds as long as there is a correspondence so that, for both triangles, the given side is in the same position relative to the angles. If such a correspondence is not certain the triangles may or may not be congruent.

Note: Since there are an infinite number of triangles with any 3 given angle measures, AAA is not a congruence property. It does establish similarity, however.

Students may suggest congruence properties such as SASS. Note, however, if 3 sides and an angle of one triangle are congruent to 3 sides and an angle of another triangle, then the triangles are congruent by the SSS property. That is, there is more information than is needed to determine congruence, i.e., some information is redundant.

Focus Teacher Activity (cont.)

ACTIONS

11 Give each student a copy of Focus Student Activity 10.1 and ask them to carry out the instructions for the first 2 pairs of triangles. Discuss. Then repeat for the remaining pairs of triangles.

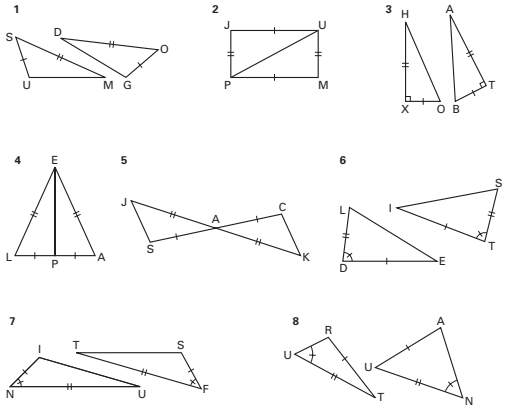
Constructions and Mappings Lesson **10**

Focus Student Activity 10.1

NAME _____ **DATE** _____

Using the information that is given (i.e., as indicated by the angle and hash marks) or that you can logically conclude from the given information, determine which of the following pairs of triangles are congruent. Note: the drawings may not be drawn to scale, so do not base your conclusions on measurements you make or on what appears to be true. For each of the following 12 pairs of triangles:

- If the 2 triangles are congruent tell which congruence property (SSS, SAS, or ASA) is the basis of your reasoning, and write a correct congruence statement (e.g., ΔABC ≅ ΔMNO).
- If the 2 triangles are not congruent, explain why.



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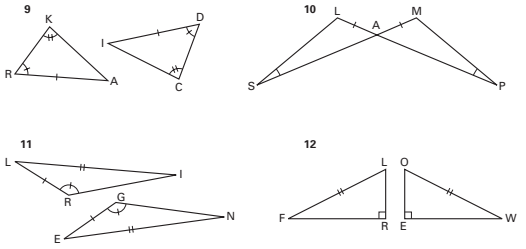
Lesson **10** Constructions and Mappings

Focus Student Activity 10.1 (cont.)



COMMENTS

11 The matching hash marks on the sides and angles of each pair of triangles denote congruent parts.

The following statements are based on the given information:

Pair 1: Not possible to tell. They look congruent, but knowing 2 sides of 2 triangles is not sufficient to determine congruence.

Pair 2: ΔPJU ≅ ΔUMP by SSS. Notice that side PU is contained in both triangles. Therefore, all 3 pairs of corresponding sides, PJ and UM, JU and MP, and UP and PU are congruent.

Pair 3: ΔHXO ≅ ΔATB by SAS

Pair 4: ΔEPL ≅ ΔEPA by SSS

Pair 5: ΔJAS ≅ ΔKAC by SAS

Pair 6: ΔLDE ≅ ΔSTI by SAS

Pair 7: ΔINU ≅ ΔSFT by SAS

Pair 8: Not necessarily congruent, not enough information

Pair 9: ΔKRA ≅ ΔCDI by ASA, since 3rd angles are congruent

Pair 10: ΔSAL ≅ ΔPAM by ASA

Pair 11: Not necessarily congruent, not enough information

Pair 12: Not necessarily congruent, not enough information

Focus Teacher Activity (cont.)

ACTIONS

12 Give each pair of students a copy of Focus Master D. Referring to the conditions given in a), ask the pairs whether knowing that a triangle is scalene and knowing its area are sufficient to determine a unique triangle, an infinite set of noncongruent triangles, a fixed number (greater than 1) of noncongruent triangles, or no triangle. Invite volunteers to explain their reasoning. Encourage students to post their conjectures and generalizations on the class poster started during the Connector activity. Repeat for one or more of b)-r).

Lesson **10** Constructions and Mappings

Focus Master D

Which, if any, of the following conditions determine a unique triangle? an infinite collection of noncongruent triangles? a fixed number (greater than 1) of noncongruent triangles? no triangles? Show or explain your reasoning. Give examples to support your conclusions. Write conjectures and generalizations in "if *x*, then *y*" form.

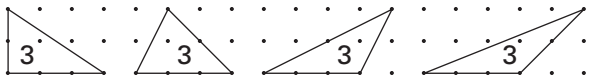
a) the area of a scalene triangle
b) the perimeter of an equilateral triangle
c) one side and the area of a scalene triangle
d) the perimeter and the noncongruent side of an isosceles triangle
e) one leg and the area of a right triangle
f) 2 sides of an isosceles (nonequilateral) triangle
g) the area of an equilateral triangle
h) 2 sides of a right triangle
i) the perimeter of a right triangle
j) 2 sides of a triangle and the altitude to the vertex between the 2 sides.
k) Challenge: 2 sides and an angle of a scalene triangle.
l) 2 angles and a side of a scalene triangle
m) Challenge: the area and an acute angle of a right triangle
n) angles 110°, 47°, 28°
o) side lengths 4, 8, 10
p) side lengths 3, 9, 14
q) angles 57°, 63°, 60°
r) the area of an isosceles triangle that is not equilateral

COMMENTS

12 This could be started in class and completed as homework. Be sure to provide sufficient time for investigations. In addition to a compass, straightedge, protractor, and ruler, students may find dot paper, tracing paper, grid paper, and geoboards (or geoboard paper) useful for investigating ideas and recording examples to support their conclusions. They may also find coffee stirrers or pieces of uncooked spaghetti useful for representing fixed lengths.

To encourage conjectures and generalizations, you might provide each pair of students a supply of butcher paper strips and a marking pen, reminding them to post their ideas on the class poster which was begun during the Connector. Following are examples of arguments students may use.

a) For any given area, there are an infinite number of noncongruent scalene triangles with that area. In previous *Math Alive!* courses, students noted that moving a vertex of a triangle along a line through that vertex and parallel to the opposite side generates an infinite set of triangles with the same area. For example, the following diagram shows 4 triangles with area 3 square units:



An "if *x*, then *y*" statement could be: If only the area of a scalene triangle is given, then an infinite number of noncongruent triangles with that area can be formed.

b) If a triangle is equilateral then all 3 sides are the same length. Hence, if the perimeter is p, the 3 side lengths are each $p/3$. There is exactly 1 triangle with 3 given side lengths. Therefore, if the perimeter of an equilateral triangle is given, then exactly 1 triangle can be formed.

You might encourage the students to extend conjectures to create new congruence statements. For example, based on the argument above: if 2 triangles are equilateral and their perimeters are equal, then the triangles are congruent.

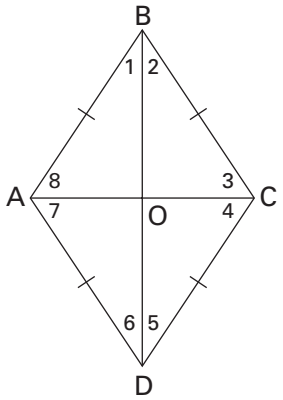
c) infinite number of noncongruent triangles
d) a unique triangle
e) a unique triangle
f) an infinite number of noncongruent triangles
g) a unique triangle

(Continued next page.)

Focus Teacher Activity (cont.)

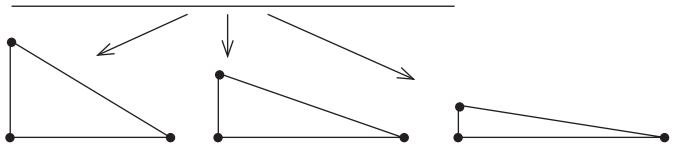
ACTIONS

13 Draw the following diagram on the overhead and ask the pairs of students to tell all the mathematical information they can about the figure by reasoning deductively from the given information. Invite volunteers to illustrate their results and reasoning at the overhead.



COMMENTS

12 (continued.)
h) A unique triangle if the 2 sides have the same length. Otherwise, 2 noncongruent triangles, 1 of which has the 2 given sides as legs and 1 which has the larger of the given sides as hypotenuse.
i) Infinite number of noncongruent triangles. One way to "see" this is to imagine a string of a given length, knotted to form a loop; stretch the string tautly around 3 pins positioned to form the vertices of a right triangle whose perimeter is the given string length; the pins can be repositioned to form an infinite number of noncongruent right triangles with perimeters equal to the given string length. Here are 3 examples:



j) a unique triangle
k) 4 noncongruent triangles
l) 3 noncongruent triangles
m) a unique triangle
n) no triangle is possible
o) a unique triangle
p) no triangle is possible
q) infinite number of noncongruent triangles
r) infinite number of noncongruent triangles

13 It may be helpful to begin by asking students to identify what information is "given" in the drawing, since they can build arguments based only on information that is provided (i.e., according to markings on the diagram) or information that can be deduced directly from the given information. Students may feel a little uncertain at first regarding what they can assume from a drawing and what information they can assume they already know. In general, they can assume: that segments that appear to be collinear are; that angles that appear to form a linear pair do so; that a point that appears to lie on a line does; and that angles and segments marked as congruent are so.

You might assure students that you will let them know if they are jumping to conclusions which they must first build arguments to support. And, once they have proven a relationship, they can use that relationship without reproving it—unless, of course, you ask specifically for justification.

Focus Teacher Activity (cont.)

ACTIONS

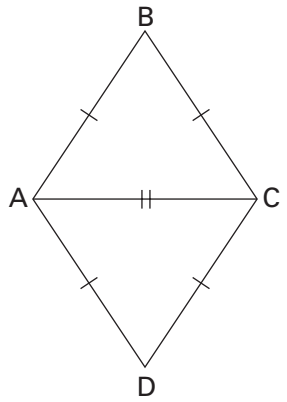


Figure 1
$\Delta ABC \cong \Delta ADC$ by SSS

$\overrightarrow{\text{so}}$

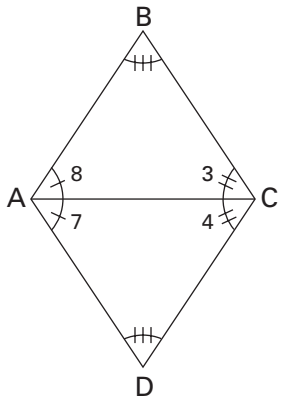


Figure 2
Corresponding angles of the congruent triangles are congruent

COMMENTS

For example, one can assume from the given drawing that ABCD is a quadrilateral and that the 4 sides are congruent. Hence, one can conclude that ABCD is a rhombus. Note: It is not possible to tell from the given information whether the rhombus is a square.

Students may also notice that ΔABC, ΔADC, ΔABD, and ΔCBD are all isosceles triangles because, if a triangle has at least 2 congruent sides, then it is isosceles.

If students have difficulty getting started, you could invite volunteers to make "I notice…" statements and have the class investigate arguments to support or refute the statements. Or, you could provide a thought starter such as, "I notice several isosceles triangles," or "I notice that $\Delta ABC \cong \Delta ADC$," or "I notice that ABCD is a rhombus." Then ask the students to investigate to determine how you can be sure.

Be sure to allow plenty of time and encouragement for "digging" out lots of information. If, after discussing some information, students need more "mulling time," you might post a list of their observations, proceed with Action 14, and then return periodically to this action for discussion. Even though it may be a little overwhelming at first, discovering how much information is revealed by a drawing provides motivation for searching in subsequent investigations.

Following are some conclusions that can be supported by reasoning deductively from the given information.

$\Delta ABC \cong \Delta ADC$ by SSS, since it is given that $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$, and since $\overline{AC} \cong \overline{AC}$ ($\overline{AC}$ is a side of both triangles). Note: the fact that a number is equal to itself is called the *reflexive property for equality*. Segments are congruent if their lengths are equal; hence, because of the reflexive property for equality any segment is congruent to itself.

Once it is established that $\Delta ABC \cong \Delta ADC$, one can conclude that $\angle ABC \cong \angle ADC$, $\angle 8 \cong \angle 7$, and $\angle 3 \cong \angle 4$ (see Figure 2 at the left). This is possible to deduce because corresponding angles of congruent triangles are congruent. It may be helpful to students to share with them how you word justifications. However, it is important to keep emphasis on reasoning and informal arguments—it is not intended that students create formal proofs here.

(Continued next page.)

Focus Teacher Activity (cont.)

COMMENTS

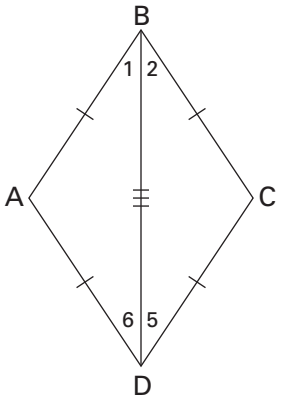


Figure 3
$\Delta ABD \cong \Delta CBD$ by SSS

$\overrightarrow{so}$

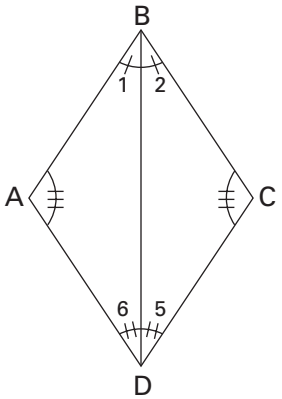


Figure 4
Corresponding angles of the congruent triangles are congruent

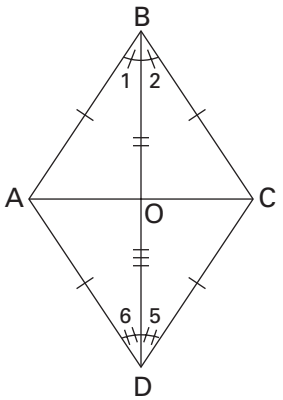


Figure 5
$\Delta ABO \cong \Delta CBO$ and $\Delta ADO \cong \Delta CDO$ by SAS

$\overrightarrow{so}$

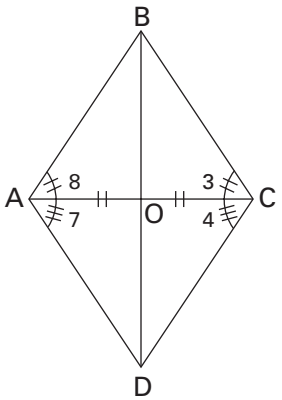


Figure 6
Corresponding angles and sides of the congruent triangles are congruent

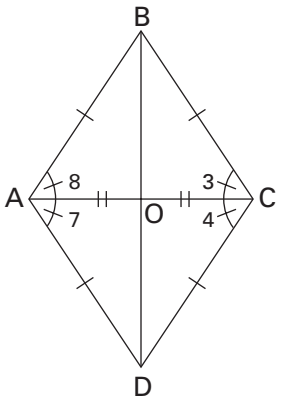


Figure 7
$\Delta ABO \cong \Delta CBO \cong \Delta CDO \cong \Delta ADO$ by SAS (see Figures 2 and 6) to show $\angle 8 \cong \angle 3 \cong \angle 4 \cong \angle 7$

$\overrightarrow{so}$

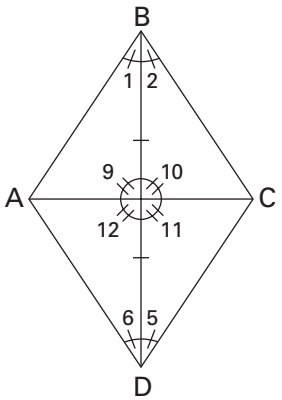


Figure 8
Corresponding angles and sides of the congruent triangles are congruent

13 (continued.)
Notice also that, since $\overline{AB} \cong \overline{BC} \cong \overline{AD} \cong \overline{CD}$ and since $\overline{BD} \cong \overline{BD}$, then $\Delta ABD \cong \Delta CBD$ by SSS (see Figure 3 at the left).

Therefore, one can conclude that $\angle 1 \cong \angle 2$, $\angle BAD \cong \angle BCD$, and $\angle 6 \cong \angle 5$ because they are corresponding angles of congruent triangles (see Figure 4 at the left).

The above information also can be used to prove that $\Delta ABO \cong \Delta CBO$ and $\Delta CDO \cong \Delta ADO$ (see Figure 5 at the left), because of the SAS congruence property. Therefore, $\angle 3 \cong \angle 8$, $\angle 4 \cong \angle 7$, and $\overline{AO} \cong \overline{OC}$, (see Figure 6) because corresponding parts of congruent triangles are congruent.

Since angles whose measures are equal are congruent, then the *transitive property for equality* (i.e., if $a = b$ and $b = c$, then $a = c$; see Lesson 7) can also be applied to congruence of angles. So, since $\angle 8 \cong \angle 3$ and $\angle 3 \cong \angle 4$, then $\angle 8 \cong \angle 4$. Further, since $\angle 8 \cong \angle 7$, then $\angle 7 \cong \angle 8 \cong \angle 3 \cong \angle 4$. Hence, $\Delta ABO \cong \Delta CBO \cong \Delta CDO \cong \Delta ADO$ by SAS (see Figure 7). Therefore, $\angle 1 \cong \angle 2 \cong \angle 5 \cong \angle 6$, $\overline{BO} \cong \overline{OD}$, and $\angle 9 \cong \angle 10 \cong \angle 11 \cong \angle 12$ (see Figure 8). Further, since $\angle 9$, $\angle 10$, $\angle 11$, and $\angle 12$ together total 360°, then each has measure $360°/4 = 90°$. Therefore, they are all right angles.

Note that in the above example we used the fact that $\angle 8 \cong \angle 7$, then $\angle 7 \cong \angle 8$. This is a result of the *symmetric property of equality*, i.e., $a = b$, then $b = a$.

Students may also correctly conclude that $\overline{BC} \parallel \overline{AD}$ since $\angle 7 \cong \angle 3$ (i.e., since transversal $\overline{AC}$ forms congruent alternate interior angles with $\overline{BC}$ and $\overline{AD}$. Similarly, $\overline{AB} \parallel \overline{CD}$ since $\angle 8 \cong \angle 4$. Note: The symbol $\parallel$ stands for "is parallel to."

There are several properties that are proven by the above arguments, for example: the diagonals of a rhombus are perpendicular bisectors of each other; the altitude of an isosceles triangle divides the triangle into 2 smaller congruent triangles; if a triangle is isosceles, the angles opposite the congruent sides are congruent; the opposite sides of a rhombus are parallel; if 2 triangles are congruent to a 3rd triangle, they are congruent to each other, etc.

Focus Teacher Activity (cont.)

ACTIONS

14 Place a transparency of Focus Student Activity 10.2 on the overhead revealing Problem 1 only. Ask the pairs of students to carry out the instructions for Problem 1. Discuss. Then give each student a copy of Focus Student Activity 10.2 and ask the students to complete problems 2-7. Discuss.

Constructions and Mappings Lesson **10**

Focus Student Activity 10.2

NAME _____ DATE _____

On separate sheets of paper, carry out each of the following constructions using only a straightedge and compass. Next to each construction, list the steps of your construction, and write an explanation that tells how you can be certain (without measuring) that your constructed figure meets the given conditions.

1 Construct an equilateral triangle ΔABC.

2 Construct an isosceles triangle ΔABC which is not an equilateral triangle.

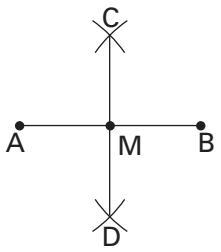
3 Construct the line that is the *perpendicular bisector* of a line segment AB. HINT: you must explain how you know the line is perpendicular to AB *and* you must explain how you know that the line divides AB into 2 congruent segments.

4 Construct a line that is the *angle bisector* of an angle A.

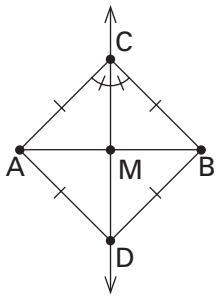
5 Construct a line perpendicular to a given line through a point P on the line.

6 Construct a line perpendicular to a given line through a point Q not on the line.

7 Construct a line parallel to a given line through a point not on the given line.



Construction

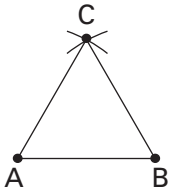


Justification

COMMENTS

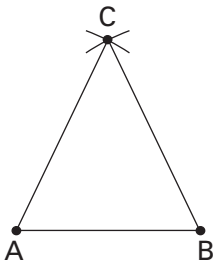
14 Students may have invented some of these constructions in previous actions. They may use those same methods here, or they may refine them. Students may find it easier to build arguments orally than to write them. However, you might ask them to write some arguments, as doing so helps to clarify their thinking and helps them see missing pieces in their arguments.

1) Construct a line segment $\overline{AB}$. Then with the compass open a distance equal to the length of $\overline{AB}$, construct an arc with A as center and an arc with B as center. Label the intersection of the arcs point C. Then ΔABC is equilateral.



Justification: Since the opening of the compass was set according to the length of $\overline{AB}$, and since $\overline{AC}$ and $\overline{BC}$ were constructed without changing the compass opening, then $\overline{AB} \cong \overline{AC} \cong \overline{BC}$. So, ΔABC is an equilateral triangle.

2) Construct $\overline{AB}$. Then open the compass to span a length greater that ½AB and different from the length AB. Using this length and with A and B as centers, construct intersecting arcs at point C. Hence, ΔABC is an isosceles triangle which is not equilateral.



Justification: $\overline{AC}$ and $\overline{BC}$ were constructed to be equal in length but different from the length of $\overline{AB}$. Therefore, exactly 2 sides of the triangle are congruent, so the triangle is isosceles, but not equilateral.

3) Construct $\overline{AB}$. Open the compass to span a length greater than ½AB. Using this length and with A and B as centers, construct 4 arcs intersecting in points C and D (see diagram at the left). Draw $\overleftrightarrow{CD}$, the perpendicular bisector of $\overline{AB}$ at point M.

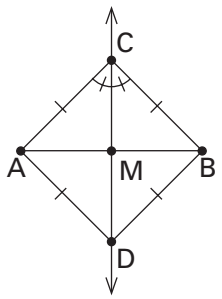
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

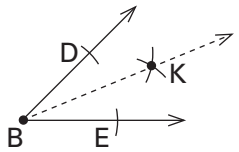
COMMENTS

14 (continued.)
Note: Writing justifications for this may be particularly challenging for some students. While it is important to keep emphasis on logical oral reasoning, encourage students to write their arguments since doing so can help organize and clarify their thinking. Avoid giving rules for formatting written arguments. However, some students may appreciate periodically seeing how you write an argument, and seeing abbreviations you use (e.g., $\perp$ for "is perpendicular to," $\therefore$ for "therefore," $\Rightarrow$ for "implies," and cpctc for "corresponding parts of congruent triangles are congruent"). Students may use arguments similar to those in Comment 13. Other arguments follow.

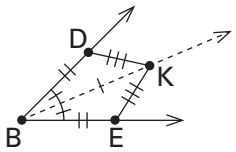


Justification: $\overline{AC} \cong \overline{BC} \cong \overline{AD} \cong \overline{BD}$ because the compass opening was not changed when each was constructed. $\overline{CD}$ is a side of ΔACD and ΔBCD, and $\overline{CD} \cong \overline{CD}$, so $\Delta ACD \cong \Delta BCD$ by SSS. So, $\angle ACD \cong \angle BCD$ because they are corresponding parts of congruent triangles. Notice $\overline{CM} \cong \overline{CM}$. Therefore, $\Delta ACM \cong \Delta BCM$ by SAS. So, $\overline{AM} \cong \overline{MB}$ because they are corresponding sides of congruent triangles. Hence, M must be the midpoint of $\overline{AB}$.

Also, $\angle CMA \cong \angle CMB$ because they are corresponding angles of congruent triangles. Notice that $\angle CMA$ and $\angle CMB$ form a linear pair, and so the sum of their measures is 180°. Since both angles are equal and they are supplementary then $\angle CMA = \angle CMB = {}^{180°}\!/_{2} = 90°$. Therefore, $\angle CMA$ and $\angle CMB$ are right angles. So, $\overline{CD}$ is perpendicular to $\overline{AB}$. Since $\overline{CD}$ is perpendicular to $\overline{AB}$ and passes through M the midpoint of $\overline{AM}$, then $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$.



Construction



Justification

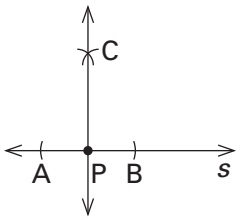
4) Construct $\angle B$. Using B as center, and without changing the compass opening, construct 2 arcs that intersect the sides of the angle at points D and E. Open the compass a distance greater than ½ the distance between D and E. Then, using this compass opening and using D and E as centers, construct arcs intersecting in point K. $\overline{BK}$ is the bisector of $\angle DBE$.

Justification: $\Delta BDK \cong \Delta BEK$ by SSS. So, by corresponding parts, $\angle DBK \cong EBK$ which shows that $\overline{BK}$ is the bisector of $\angle DBE$.

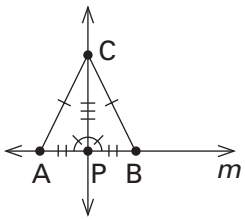
Note: Students may be interested in knowing that for over 2000 years, beginning with the ancient Greeks, mathematicians tried to trisect an angle (i.e., divide an angle into 3 congruent angles) by using only a straightedge and compass. Finally, in 1832 the 20 year-old Evariste Galois (pronounced gal-wah) developed alge-

Focus Teacher Activity (cont.)

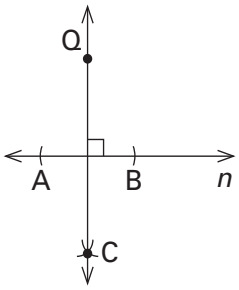
ACTIONS



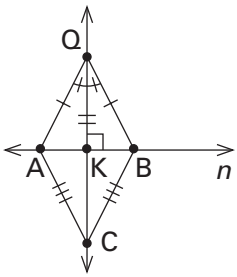
Construction



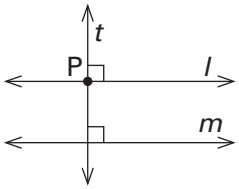
Justification



Construction



Justification



COMMENTS

braic theories which eventually lead to proving that this construction is not possible. Students may enjoy reading about the life of Galois and about the history of the trisection problem. They may also be interested in investigating the trisection problem.

5) Draw a line *s*. Pick a point on *s* and label that point P. Using P as center and without changing the compass opening, construct arcs which intersect line *s* at points A and B. Then using points A and B as centers and without changing the compass opening, construct arcs which intersect at point C. Draw $\overline{CP}$ which is perpendicular to line *s*.

Justification: Since $\overline{AP}$ and $\overline{BP}$ were constructed so $\overline{AP} \cong \overline{BP}$, since $\overline{AC}$ and $\overline{BC}$ were constructed so $\overline{AC} \cong \overline{BC}$, and since $\overline{PC} \cong \overline{PC}$, then $\Delta APC \cong \Delta BPC$ by SSS (see diagram at the left). So, $\angle CPA \cong \angle CPB$ because they are corresponding angles of congruent triangles. Since $\angle CPA$ and $\angle CPB$ are a linear pair, they are supplementary. And since $\angle CPA$ and $\angle CPB$ are supplementary and congruent, they are right angles. Therefore, $\overline{CP}$ is perpendicular to line *m*.

6) Draw line *n*. Mark a point Q not on *n*. Using Q as center and without changing the compass opening, construct arcs that intersect line *n* at points A and B. Then using A and B as centers and without changing the compass opening, construct arcs which intersect at point C. Draw $\overline{QC}$ which is perpendicular to line *n*.

Justification: $\Delta AQC \cong \Delta BQC$ by SSS. So, $\angle AQC \cong \angle BQC$ because they are corresponding parts of congruent triangles. If the intersection of $\overline{QC}$ and line *n* is K, then $\overline{KQ} \cong \overline{KQ}$ and $\Delta AQK \cong \Delta BQK$ by SAS. Thus $\angle QKA \cong \angle QKB$ because they are corresponding parts of congruent triangles. As stated in previous arguments, angles that are supplementary and congruent are right angles. Therefore, $\overline{QC}$ is perpendicular to line *n*.

7) Draw a line *m* and mark a point P not on *m*. Use construction 6) above to construct a line *t* through P so that line *t* is perpendicular to line *m*. Then use construction 5) above to construct line *l* through P which is perpendicular to line *t*. Lines *l* and *m* are parallel.

Justification: Lines *l* and *m* are cut by transversal *t* which is perpendicular to both. So, the alternate interior angles formed by *l*, *m*, and *t* are equal (all are right angles). If 2 lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel. Therefore, line *l* is parallel to line *m*.

Focus Teacher Activity (cont.)

ACTIONS

15 Give each student a copy of Focus Student Activity 10.3 and ask them to complete Problem 1a). Invite volunteers to demonstrate their methods on a transparency of the activity or using enlarged figures on the chalkboard. Discuss. Repeat for 1b) and 1c).

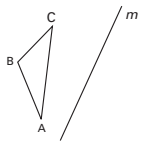
Constructions and Mappings Lesson **10**

Focus Student Activity 10.3

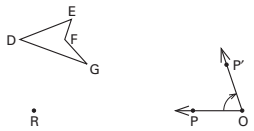
NAME _____ DATE _____

1 Use your straightedge and compass to construct the following transformations. Label the image of each point A as A′, the image of B as B′, etc. Don't remove your construction marks. Record your observations and conjectures.

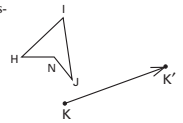
a) Reflect ΔABC across line *m*.



b) Rotate quadrilateral DEFG about point R, using ∠POP′ as the angle of rotation.



c) Translate quadrilateral HIJN, using translation vector $\overrightarrow{KK'}$.

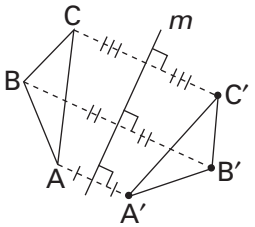


(Continued on back.)

COMMENTS

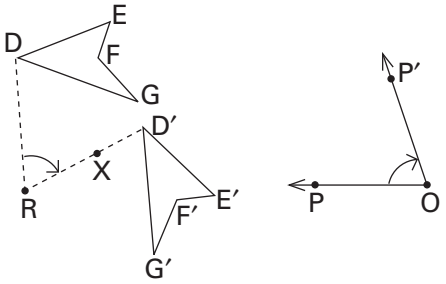
15 Students investigated the properties of transformations in Lessons 1 and 2 of this course.

1a) To obtain a *reflection* of ΔABC about line *m*, locate the images of each of the points A, B, and C. To construct C′, the image of C, for example, construct the perpendicular from C to line *m*. Then locate C′ on the perpendicular so that C and C′ are equidistant from line *m* and on opposite sides of *m*. Repeat this type of construction to locate A′ and B′, the images of A and B. Use a straightedge to draw the segments connecting A′, B′, and C′.



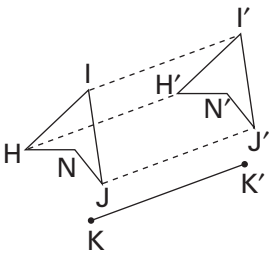
ΔABC ≅ ΔA′B′C′

1b) To obtain the *rotation* image of quadrilateral DEFG about point R, construct ∠DRX ≅ ∠POP′. Then locate D′ on $\overrightarrow{RX}$ by using a compass to mark RD ≅ RD′. Repeat this process to locate the images of E, F, and G. Use a straightedge to draw the segments connecting D′, E′, F′, and G′.



Quad DEFG ≅ Quad D′E′F′G′

1c) To obtain the *translation* image of quadrilateral HIJN using translation vector $\overrightarrow{KK'}$, construct a line through J parallel to KK′ and locate the image of J so that the distance from J to J′ is equal to the distance from K to K′. Repeat this process to locate the images of H, I, and N.



Quad HIJN ≅ Quad H′I′J′N′

Focus Teacher Activity (cont.)

ACTIONS

16 Repeat Action 15 for Problems 2-4 on Focus Student Activity 10.3.

Lesson **10** Constructions and Mappings

Focus Student Activity 10.3 (cont.)

2 Use your straightedge and compass for the following. Don't erase construction marks.

a) Draw a polygon with 4 or more sides. Construct the reflection of the polygon across the following lines. Then make observations and conjectures.

i) a line that intersects the shape at more than 1 point;

ii) a line that does not intersect the shape;

iii) a line that touches the shape at exactly 1 point.

b) Suppose you are given 2 shapes and 1 of the shapes is a reflection image of the other, but the line of reflection isn't given. Explain a method of constructing the line of reflection.

c) Suppose you are given 2 shapes and 1 is a translation image of the other, but the translation vector isn't given. Explain how to construct the translation vector.

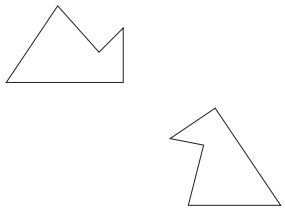
d) Draw a polygon with 4 or more sides. Elsewhere on the paper, draw an angle B. Construct the rotation of the polygon through ∠B and about each of the following points. Then make conjectures and observations.

i) a point P outside the shape;

ii) a point Q inside the shape;

iii) a point R on the perimeter of the shape.

3 Challenge. For the pair of congruent figures at the right, label the vertices and write a congruence statement to show the corresponding vertices. Determine the type of mapping (translation, reflection, or rotation) which maps one figure onto the other. Then locate the line of reflection, translation vector, or center of rotation.



4 Challenge. Investigate general relationships between the location of the center of a rotation and the location of points and their rotation images.

COMMENTS

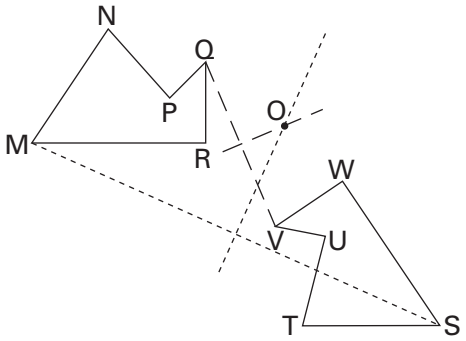
16 2a) Each intersection point of the line of reflection and the figure will coincide with its image for the reflection. If the line of reflection does not intersect the figure, it will not intersect the image for the reflection.

2b) The perpendicular bisector of the line segment whose end points are a point on the figure and its image is the line of reflection.

2c) A line segment which connects a point on the figure to its image may be used for the translation vector.

2d) If the center of a rotation is outside a figure, it is also outside the rotation image of the figure. If the center of rotation is inside a figure, it is also inside the rotation image. If the center of rotation is on the perimeter of a figure, it is also on the perimeter of the image.

3) As illustrated below, MNPQR ≅ STUVW by a rotation (i.e., a rotation preserves size and shape, see Lesson 2). To locate the center of rotation, some students may use tracings and repeated approximations. Others may discover that the center of the rotation is the intersection of the perpendicular bisectors of the segment connecting each point with its image [see also the discussion for 4) below]. Hence, the following procedure can be used to locate the center of rotation: first construct the perpendicular bisector of any segment connecting a point and its image. Repeat this process for a second point and its image. The intersection of the perpendicular bisectors is the center of rotation that maps MNPQR to STUVW. The rotation could be clockwise or counterclockwise (in this case, the clockwise angle of rotation is a reflex angle, and the counterclockwise angle of rotation is an obtuse angle).



4) Here is an example of one group's observations: *We notice that a point, its image, and the center of rotation form*

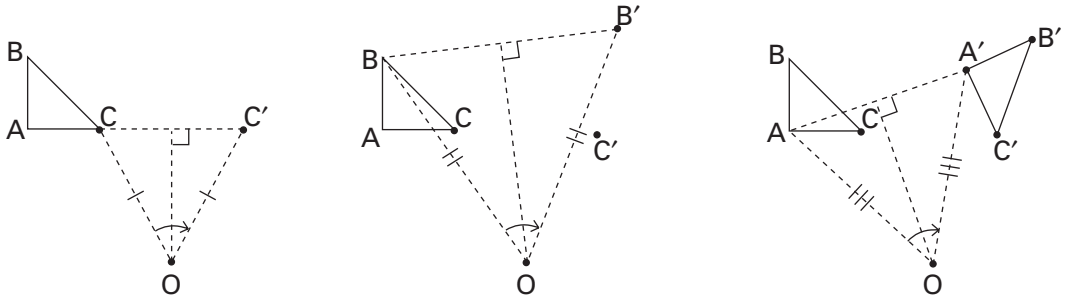
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

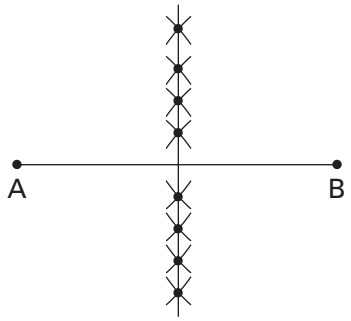
COMMENTS

the vertices of an isosceles triangle (e.g., see ΔCOC′, ΔBOB′ and ΔAOA′ below). The noncongruent side of such a triangle is the segment connecting a point to its image. The perpendicular bisector of this segment passes through the center of rotation (i.e., the vertex opposite the noncongruent side). Therefore, to locate the center of a rotation, locate the intersection of the perpendicular bisectors of 2 or more segments connecting points and their images.



17 Mark 2 points A and B on a blank transparency on the overhead. Ask the students to do the same on a sheet of paper. Then ask them to sketch or construct a set of points which are the same distance from A as from B, and to describe the set of *all* points which are equidistant from A and B. Ask for volunteers to demonstrate their methods and conclusions. Introduce the term *locus* to describe a figure consisting of all the points, and only those points, that satisfy a given condition.

17 A point that is equidistant from A and B can be located by using the same compass opening and constructing intersecting arcs centered at A and B. Other points can be located by changing the compass opening and repeating this process. All points that are equidistant from A and B lie on the perpendicular bisector of $\overline{AB}$. Hence, the locus of points equidistant from any 2 points A and B is the perpendicular bisector of $\overline{AB}$.



18 Place a transparency of Focus Master E on the overhead, revealing Problem a) only. Ask the pairs of students to sketch or construct and describe the given locus of points. Invite volunteers to demonstrate their conclusions and reasoning. Discuss. Repeat for locus Problems b)-l), or selected problems.

18 A method of determining a locus is to locate several points that satisfy the given conditions. Use a familiar or common mathematical term (e.g., triangle, perpendicular bisector, circle, etc.) to describe the figure created by all the points that satisfy the conditions. Include specific information in the description to identify both the shape and position of the locus (i.e., where possible identify lengths, center points, radii, intersection points, etc.)

Focus Teacher Activity (cont.)

ACTIONS

Constructions and Mappings Lesson **10**

Focus Master E

Sketch or construct, and describe, the locus of points in a plane which are:

a) equidistant from 2 intersecting lines.

b) equidistant from 2 parallel lines.

c) equidistant from 2 concentric circles.

d) equidistant from a given point A.

e) the midpoints of congruent chords of a circle with center A.

f) vertices of a right triangle with hypotenuse AB.

g) equidistant from the sides of a given angle B.

h) the midpoints of chords of a circle with center P such that all the chords have the same end point R on the circle.

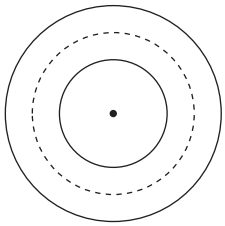
i) equidistant from a line and a point Q not on the line.

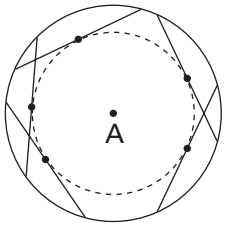
j) such that the distance from a point A is 2 times the distance from a 2nd point B.

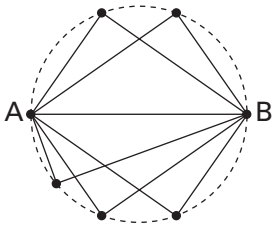
k) such that the sum of the distances to 2 fixed points A and B is the same number.

l) such that the difference of the distances from 2 fixed points C and D is the same number.

m) the centers of circles that are tangent to 2 intersecting lines.





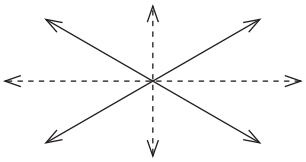


COMMENTS

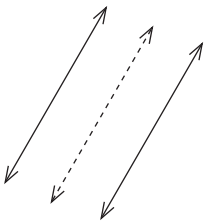
In the process of exploring these locus problems, students may make a number of conjectures and generalizations. You might remind them to write these statements in "if-then" form, when possible, and to post them on the class "We conjecture… We wonder…" poster started during the Connector.

Following are brief descriptions of the locus of points for conditions a)-m). Note that j), k), and l) may be challenging for some students. The dotted lines and curves in the diagrams represent each locus.

a) Perpendicular lines bisecting the 2 vertical angles formed by the given intersecting lines:



b) A line midway between the 2 parallel lines:



c) *Concentric circles* have the same center but different radii. The locus of points equidistant from 2 concentric circles is the circle concentric to the given circles and with radius equal to half the sum of the radii of the given circles, as shown at the left.

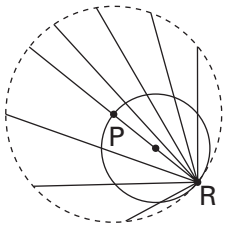
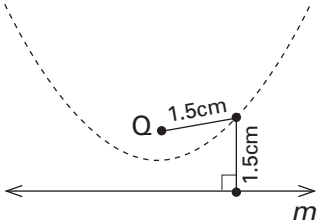
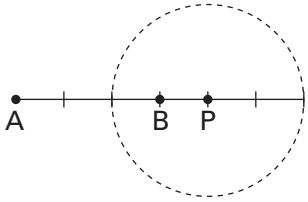
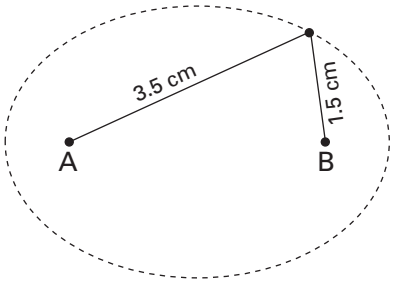
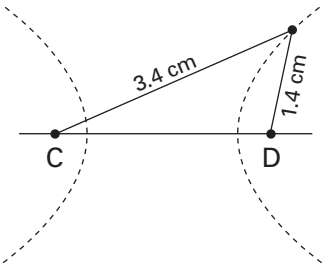
d) A circle whose center is A and whose radius is the given distance. Notice the instructions ask for a locus of points in a plane. In space, this locus would be a sphere with center A and radius the given distance.

e) A circle with center A and radius equal to the distance from A to the midpoints of the chords, as shown at the left.

f) A circle with diameter $\overline{AB}$, excluding points A and B. Note: An angle whose vertex is on a circle and whose sides intersect the circle is called an *inscribed angle*. A right angle that is inscribed in a circle intercepts a *semicircle*, as illustrated at the left. In *Math Alive! Course IV*,

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
	18 (continued.) students investigate relationships between the measure of other inscribed angles and their *intercepted arcs*.
	g) The bisector of angle B. h) A circle whose center is the midpoint of $\overline{PR}$ and whose diameter is $\overline{PR}$ (see diagram at the left).
	i) Students may need to be reminded that the distance between a point and a line is the length of the perpendicular segment from the point to the line. This locus is a curve called a *parabola*, as shown at the left. Note that point Q is called the *focus* and line M is called the *directrix* of the parabola. Students will investigate graphs of parabolas in Lessons 11, 12, and 14.
	j) A circle with center P and with point B inside the circle and point A outside the circle, where BP is ⅓ AB and the radius of the circle is ⅔ AB.
	k) An *ellipse* as shown at the left. Points A and B are called the *foci of the ellipse*. The sum of the distances from A and B to each point on this ellipse is 5 cm.
	l) A *hyperbola*. Points C and D are called the *foci of the hyperbola*. For the hyperbola shown at the left, the difference between the distances from each point on the hyperbola to C and D is 2 cm. m) Two perpendicular lines, (excluding the point of intersection), which bisect the angles formed by the intersecting lines.

Focus Teacher Activity (cont.)

ACTIONS

19 Give each student a copy of Focus Student Activity 10.4, and explain your expectations for completion of selected investigations. When the groups are finished, have them review one another's findings.

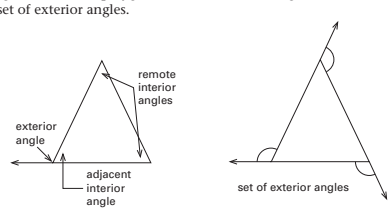
Focus Student Activity 10.4

NAME _____ DATE _____

Find out all that you can about each idea below that you investigate. Whenever possible state your conjectures in "if… then…" format. Support your conjectures with evidence. Tell when your evidence is based on inductive reasoning from examples that suggest your conclusions seem to be true, and tell when your evidence is based on deductive reasoning that shows why your conclusions are correct.

Investigate…

a) The *exterior angles* of a triangle; a quadrilateral; a pentagon; a polygon with *n* sides. Note: Extending 1 side of a polygon forms 1 exterior angle; extending each side of a polygon to form 1 exterior angle at each vertex forms a set of exterior angles.



b) Minimal conditions to prove a quadrilateral is a parallelogram. Note: by definition a parallelogram is a quadrilateral with exactly 2 pairs of parallel sides.

c) The diagonals of kites, rhombuses, rectangles, squares, parallelograms, trapezoids (isosceles and nonisosceles).

d) Minimal conditions for congruence of 2 kites; 2 rhombuses; 2 rectangles; 2 squares; 2 parallelograms; 2 trapezoids (isosceles and nonisosceles); and 2 quadrilaterals.

(Continued on back.)

Focus Student Activity 10.4 (cont.)

e) Relationships between inscribed angles and the arcs and chords they intercept on a circle. Note: The measure of an arc in degrees is equal to the measure of the central angle that intercepts the arc.

f) Perpendicular bisectors of chords of circles.

g) Triangles that are inscribed in a circle (i.e., the circle circumscribes the triangle); and triangles that circumscribe a circle (i.e., the sides of the triangle are tangent to the circle).

h) One midsegment of a triangle and the set of all midsegments of a triangle; a midsegment of a rectangle and the set of all midsegments of a rectangle; of a rhombus; of a parallelogram; and of a trapezoid. Note: A midsegment of a triangle is a segment that connects midpoints of 2 sides of the triangle; a midsegment of a quadrilateral connects opposite midpoints.

i) Connecting midpoints of consecutive sides of quadrilaterals.

j) Minimal conditions to prove a quadrilateral is a rectangle. Note: by definition a rectangle is a quadrilateral with 4 right angles.

COMMENTS

19 There are a number of ways to carry out these investigations. You might have groups select a subset to investigate; you might have individuals choose from the topics for investigation; or, you might select specific investigations for everyone to examine. These could be investigated and discussed one at a time, or several could be investigated before discussing. Base the number and types of investigations on your students needs and interest. Most of the ideas raised in these investigations will be investigated in more depth in *Math Alive! Course IV*.

Following are a few of many relationships that students may discover.

a) For a triangle, the measure of an *exterior angle* equals the sum of the measures of the 2 *remote interior angles*. The sum of the measures of a set of exterior angles of a polygon is 360°.

b) Two pairs of opposite congruent angles; or 1 pair of opposite congruent angles and 1 pair of opposite congruent sides.

c) The area of a *kite*, a quadrilateral with 2 pairs of congruent adjacent sides, as well as the areas of squares and rhombuses, which are special types of kites, is ½ times the product of the length of the diagonals. This relationship does not hold for rectangles, parallelograms, trapezoids or other quadrilaterals that are not kites, squares, or rhombuses. The 2 diagonals of a kite and a rhombus are perpendicular. The diagonals of a kite, rectangle, and isosceles trapezoid are congruent.

d) Congruence of 2 kites: 2 adjacent sides and the included angle of 1 kite congruent to 2 adjacent sides and the included angle of another kite.

Congruence of 2 rhombuses: 1 side and 1 angle of 1 rhombus congruent to 1 side and 1 angle of another.

Congruence of 2 rectangles: 2 adjacent sides of 1 rectangle congruent to 2 adjacent sides of another rectangle.

Congruence of 2 squares: 1 side of 1 square congruent to 1 side of another square.

Congruence of 2 parallelograms: same conditions as for 2 congruent kites.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

19 (continued.)
Congruence of 2 *isosceles trapezoids*: same conditions as for 2 congruent kites.

Congruence of 2 *nonisosceles trapezoids*: 2 bases and 1 side and an included angle of a base and the side congruent to 2 bases and 1 side and an included angle of a base and the side of another trapezoid.

Congruence of 2 quadrilaterals: 3 consecutive sides and the 2 included angles congruent to 3 consecutive sides and the 2 included angles of another quadrilateral.

e) Two congruent chords of a circle intercept congruent arcs; inscribed angles that intercept arcs of equal length in a circle have the same measure; inscribed angles whose sides intersect the ends of a diameter are right angles; the measure of an inscribed angle is half the measure of the central angle that intercepts the same arc.

f) The perpendicular bisectors of the chords of a circle pass through the center of the circle; the perpendicular bisector of a chord of a circle bisects the arc intercepted by the chord.

g) Triangles inscribed in a circle: The sum of the measures of the arcs intercepted by the 3 angles of the triangle is 360°; the perpendicular bisectors of any 2 of the sides of the triangle will intersect at the center of the circle; the measure of each angle of the triangle is ½ the measure of its intercepted arc.

Triangles that circumscribe a circle: Each side of the triangle is tangent to the circle; the bisectors of the 3 angles of the triangle intersect at the center of the circle.

h) Triangle: Each midsegment of a triangle is parallel to the 3rd side of the triangle and half its length; each midsegment creates a smaller triangle which is *similar to* the original triangle by a *scale factor* of 2; the 3 midsegments partition the triangle into 4 smaller similar triangles which are each similar to the original triangle by a scale factor of 2; the 3 midsegments partition the triangle into 4 smaller triangles which each have an area that is ¼ the area of the original triangle.

Rectangle: Each *midsegment* partitions the rectangle into 2 smaller congruent rectangles each having an area which is equal to ½ the area of the original rectangle; the 2 midsegments are perpendicular; the 2 midseg-

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
	ments partition the rectangle into 4 smaller rectangles which are each similar to the original rectangle by a scale factor of 2; each midsegment for 2 given sides of the rectangle is parallel to the remaining 2 sides of the rectangle. Rhombus: Each midsegment of 2 given sides of the rhombus is parallel to the remaining 2 sides of the rhombus; each midsegment partitions the rhombus into 2 parallelograms each having an area which is equal to ½ the area of the original rhombus; the 2 midsegments partition the rhombus into 4 smaller rhombuses which are each similar to the original rhombus by a scale factor of 2. Parallelogram: Each of the preceding statements for the midsegments of a rhombus will produce a statement for the midsegments of a parallelogram by replacing "rhombus" by "parallelogram." Trapezoid: The midsegment connecting the midpoints of the nonparallel sides of a trapezoid is parallel to the 2 bases and its length is ½ the sum of the lengths of the 2 bases; the midsegment connecting the midpoints of the parallel sides partitions the trapezoid into 2 trapezoids each having an area which is ½ the area of the original trapezoid. i) Connecting the midpoints of consecutive sides of a quadrilateral produces a parallelogram whose area is ½ the area of the original quadrilateral. j) Two pairs of opposite parallel sides and 1 right angle; 2 pairs of opposite sides equal and 1 right angle; 3 right angles.

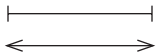
TEACHER NOTES:

Follow-up Student Activity 10.5

NAME _____ **DATE** _____

Complete the following on separate paper; write each problem next to your work.

1 Given the length at the right is 1 linear unit. Use a straightedge and compass only to construct each of the following, if possible. Show all of your construction marks and briefly describe your step-by-step procedures. Then write a brief explanation why your method works. If the construction is not possible, explain why.



a) An equilateral triangle with perimeter $3\sqrt{2}$ units, and an enlargement of the triangle by a factor of 3.

b) A kite with diagonals $\sqrt{5}$ units and $2\sqrt{5}$ units.

c) A scalene right triangle with one side 2 units and one angle 45°.

d) A circle with area 6π square units.

e) A rectangle with diagonals of length 5 units and 4.5 units.

f) 3 noncongruent rectangles with diagonals of length 3 units.

g) An equilateral triangle with perimeter 6 and inscribed in a circle.

h) An isosceles right triangle with hypotenuse 5 and inscribed in a semicircle.

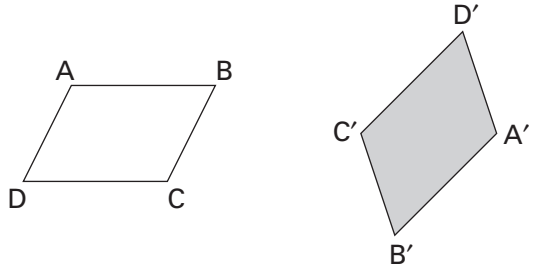
i) A regular hexagon whose perimeter is $12\sqrt{3}$ units and with an inscribed circle with radius 3 units.

2 Record, in "if… then…" format, 2 or more conjectures or generalizations that are based on your work for Problem 1. Give arguments to support each conjecture.

(Continued on back.)

Follow-up Student Activity (cont.)

3 A′B′C′D′ is the rotation image of ABCD. Trace these figures, construct the center of rotation, and briefly explain your steps.



4 A′B′C′D′ above is also the image of ABCD after 2 consecutive reflections across 2 different lines. Investigate and locate 2 such lines of reflection. Explain.

5 On a separate sheet of paper, draw line M and label point P not on M. Plot a few points whose distance from line M is twice the distance from point P. Form a conjecture about this locus of points.

6 For any 3 or more of the following statements, use a diagram and deductive arguments that are based on the diagram and properties that you know to argue why each of those statements is true.

a) The diagonals of a rhombus are perpendicular.

b) The area of a rhombus is ½ the product of its diagonals.

c) The altitude of an isosceles (nonequilateral) triangle bisects the vertex angle (i.e., the angle opposite the noncongruent side).

d) If a quadrilateral is a parallelogram, then a diagonal of the quadrilateral forms 2 congruent triangles.

e) If the diagonals of a quadrilateral bisect each other, then the quadrilateral must be a parallelogram.

f) The sum of the measures of the exterior angles of a polygon is 360°.

7 Find several possible angle measures that can be constructed by straightedge and compass only. Explain your methods.

Connector Master A

***Paperfold* to satisfy each of the following conditions. Do not use a protractor or ruler.**

a) Form 2 lines which are perpendicular to each other and not parallel to the edges of the paper.

b) Form 3 lines so that 2 of the lines are perpendicular to the 3rd line.

c) Form 2 parallel lines which are intersected by a 3rd line that is not perpendicular to the parallel lines.

d) Form a line segment and locate its midpoint. Then form a line perpendicular to the segment and passing through the midpoint of the segment.

e) Create an acute angle; then form a line that bisects the angle.

f) Form a line *m* and mark a point P not on the line. Then form a new line which is perpendicular to line *m* and passes through point P.

g) Form a line *r* and mark a point Q not on line *r*. Then form line *s* which passes through Q and is parallel to line *r*.

h) Form an isosceles triangle.

Construction 1

a) Use a straightedge to construct a line segment $\overline{AB}$.

b) Investigate ways to use a straightedge and compass only to construct a congruent copy of the line segment $\overline{AB}$ formed in a).

c) Devise a set of clear and concise, step-by-step instructions for *constructing* a copy of a line segment.

Construction 2

a) Use a straightedge to draw an angle with vertex X.

b) Investigate ways to use a straightedge and compass to construct $\angle V$ congruent to $\angle X$ formed in a).

c) Devise a clear set of instructions for *constructing* a copy of an angle.

Focus Master B

The 6 pieces of information shown on the next page make up the "maximal" set of information that you need to construct triangle ABC. What is a "minimal" set of information that you need to construct triangle ABC. How can you be sure you have all the information that you need? How can you be sure that you don't have more information than you need? Is there more than one minimal set of information?

Investigate. Test your ideas and build arguments to support your conclusions.

Make a poster of your conclusions. Show examples to support your ideas. List important observations, conjectures, and generalizations that you develop during the investigation.

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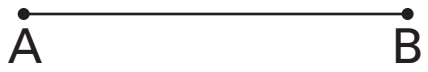
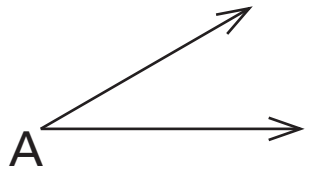
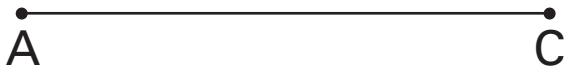
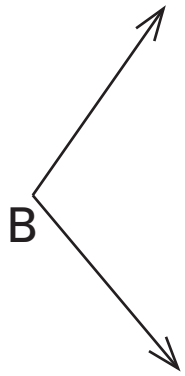
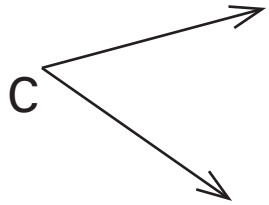
Focus Master B (cont.)











Focus Master C

According to the definition of congruence:

If there is a correspondence between the vertices of 2 triangles such that all corresponding segments are congruent and all corresponding angles are congruent, then the triangles are congruent.

The above statement gives the *maximal* set of conditions needed to establish that 2 triangles are congruent. For any 2 triangles ABC and DEF, what are all the possible *minimal* sets of conditions for determining whether ΔABC is congruent to ΔDEF? Give convincing evidence to support your conclusions.

Focus Master D

Which, if any, of the following conditions determine a unique triangle? an infinite collection of noncongruent triangles? a fixed number (greater than 1) of noncongruent triangles? no triangles? Show or explain your reasoning. Give examples to support your conclusions. Write conjectures and generalizations in "if *x*, then *y*" form.

a) the area of a scalene triangle

b) the perimeter of an equilateral triangle

c) one side and the area of a scalene triangle

d) the perimeter and the noncongruent side of an isosceles triangle

e) one leg and the area of a right triangle

f) 2 sides of an isosceles (nonequilateral) triangle

g) the area of an equilateral triangle

h) 2 sides of a right triangle

i) the perimeter of a right triangle

j) 2 sides of a triangle and the altitude to the vertex between the 2 sides.

k) Challenge: 2 sides and an angle of a scalene triangle.

l) 2 angles and a side of a scalene triangle

m) Challenge: the area and an acute angle of a right triangle

n) angles 110°, 47°, 28°

o) side lengths 4, 8, 10

p) side lengths 3, 9, 14

q) angles 57°, 63°, 60°

r) the area of an isosceles triangle that is not equilateral

Focus Master E

Sketch or construct, and describe, the locus of points in a plane which are:

a) equidistant from 2 intersecting lines.

b) equidistant from 2 parallel lines.

c) equidistant from 2 concentric circles.

d) equidistant from a given point A.

e) the midpoints of congruent chords of a circle with center A.

f) vertices of a right triangle with hypotenuse AB.

g) equidistant from the sides of a given angle B.

h) the midpoints of chords of a circle with center P such that all the chords have the same end point R on the circle.

i) equidistant from a line and a point Q not on the line.

j) such that the distance from a point A is 2 times the distance from a 2nd point B.

k) such that the sum of the distances to 2 fixed points A and B is the same number.

l) such that the difference of the distances from 2 fixed points C and D is the same number.

m) the centers of circles that are tangent to 2 intersecting lines.

Focus Student Activity 10.1

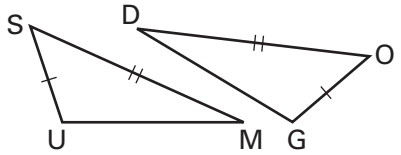
NAME _____ **DATE** _____

Using the information that is given (i.e., as indicated by the angle and hash marks) or that you can logically conclude from the given information, determine which of the following pairs of triangles are congruent. Note: the drawings may not be drawn to scale, so do not base your conclusions on measurements you make or on what appears to be true. For each of the following 12 pairs of triangles:

- If the 2 triangles are congruent tell which congruence property (SSS, SAS, or ASA) is the basis of your reasoning, and write a correct congruence statement (e.g., $\Delta ABC \cong \Delta MNO$).

- If the 2 triangles are not congruent, explain why.

1



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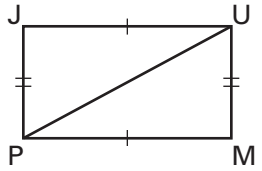

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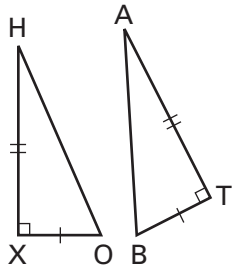

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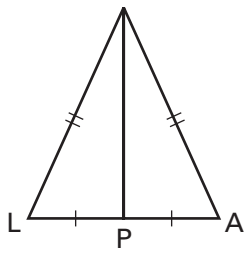

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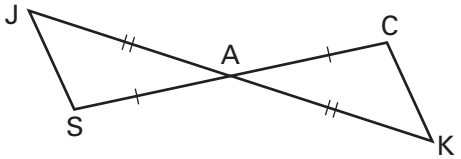

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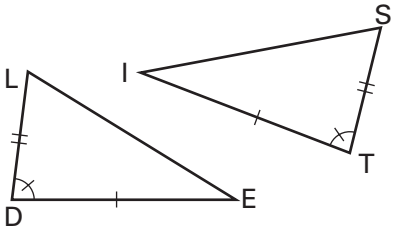

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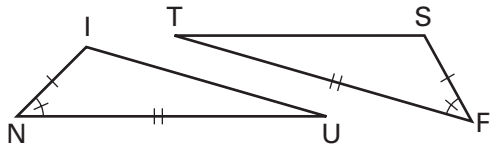

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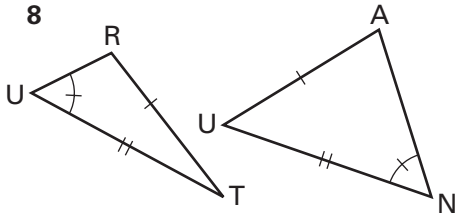

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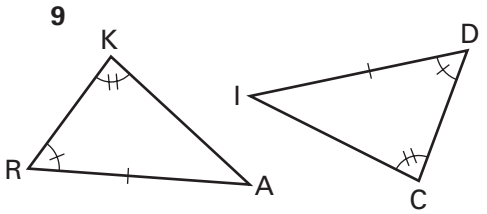
Focus Student Activity 10.1 (cont.)

9



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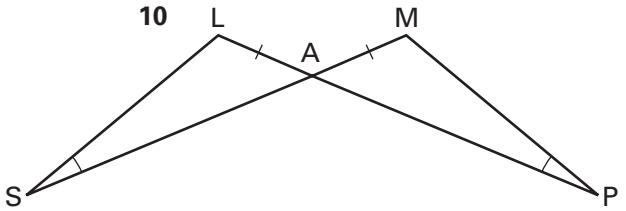

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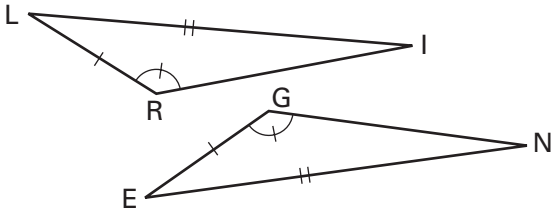

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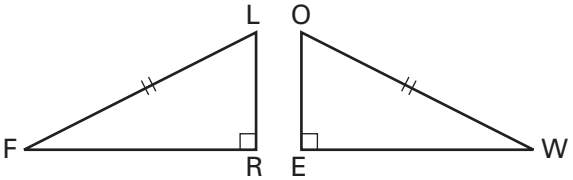


Focus Student Activity 10.2

NAME _____ **DATE** _____

On separate sheets of paper, carry out each of the following constructions using only a straightedge and compass. Next to each construction, list the steps of your construction, and write an explanation that tells how you can be certain (without measuring) that your constructed figure meets the given conditions.

1 Construct an equilateral triangle ΔABC.

2 Construct an isosceles triangle ΔABC which is not an equilateral triangle.

3 Construct the line that is the *perpendicular bisector* of a line segment AB. HINT: you must explain how you know the line is perpendicular to AB *and* you must explain how you know that the line divides AB into 2 congruent segments.

4 Construct a line that is the *angle bisector* of an angle A.

5 Construct a line perpendicular to a given line through a point P on the line.

6 Construct a line perpendicular to a given line through a point Q not on the line.

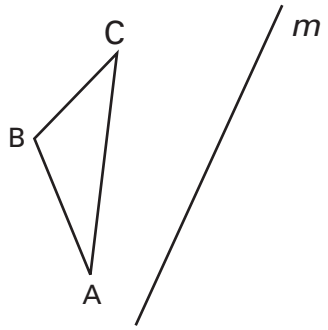
7 Construct a line parallel to a given line through a point not on the given line.

Focus Student Activity 10.3

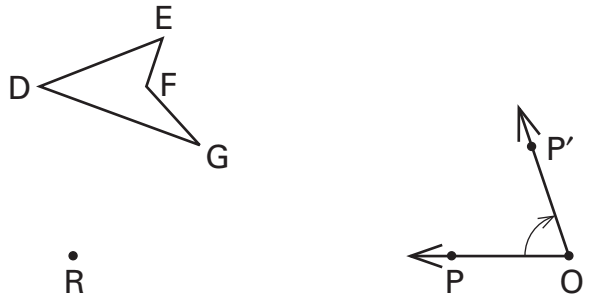
NAME _____ **DATE** _____

1 Use your straightedge and compass to construct the following transformations. Label the image of each point A as A′, the image of B as B′, etc. Don't remove your construction marks. Record your observations and conjectures.

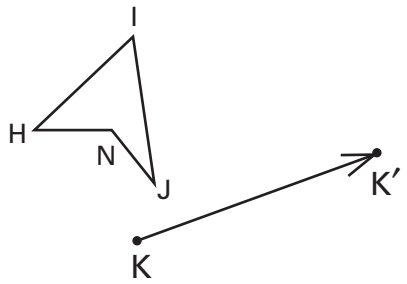
a) Reflect ΔABC across line *m*.



b) Rotate quadrilateral DEFG about point R, using ∠POP′ as the angle of rotation.



c) Translate quadrilateral HIJN, using translation vector $\overrightarrow{KK'}$.



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Focus Student Activity 10.3 (cont.)

2 Use your straightedge and compass for the following. Don't erase construction marks.

a) Draw a polygon with 4 or more sides. Construct the reflection of the polygon across the following lines. Then make observations and conjectures.

i) a line that intersects the shape at more than 1 point;

ii) a line that does not intersect the shape;

iii) a line that touches the shape at exactly 1 point.

b) Suppose you are given 2 shapes and 1 of the shapes is a reflection image of the other, but the line of reflection isn't given. Explain a method of constructing the line of reflection.

c) Suppose you are given 2 shapes and 1 is a translation image of the other, but the translation vector isn't given. Explain how to construct the translation vector.

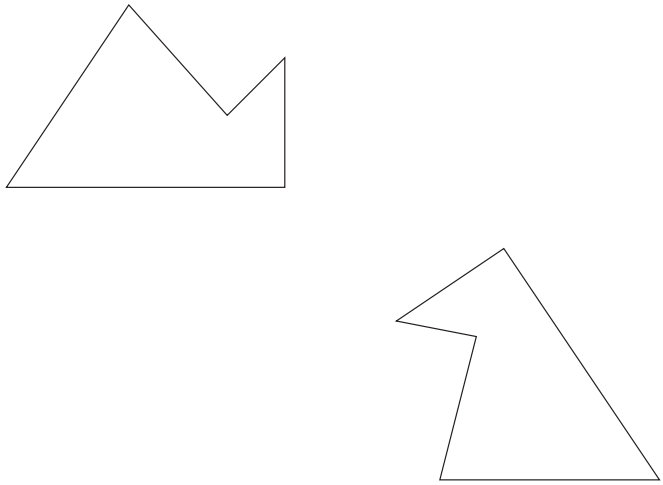
d) Draw a polygon with 4 or more sides. Elsewhere on the paper, draw an angle B. Construct the rotation of the polygon through $\angle B$ and about each of the following points. Then make conjectures and observations.

i) a point P outside the shape;

ii) a point Q inside the shape;

iii) a point R on the perimeter of the shape.

3 Challenge. For the pair of congruent figures at the right, label the vertices and write a congruence statement to show the corresponding vertices. Determine the type of mapping (translation, reflection, or rotation) which maps one figure onto the other. Then locate the line of reflection, translation vector, or center of rotation.



4 Challenge. Investigate general relationships between the location of the center of a rotation and the location of points and their rotation images.

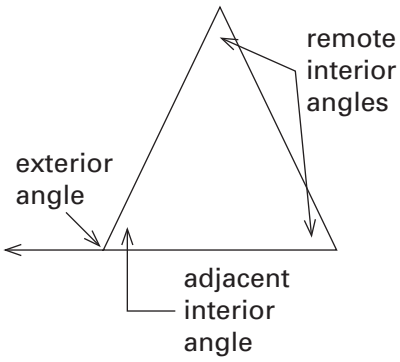
Focus Student Activity 10.4

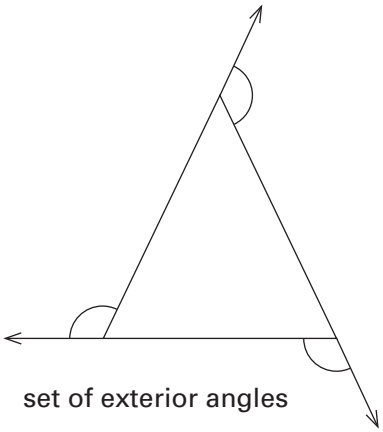
NAME _____ **DATE** _____

Find out all that you can about each idea below that you investigate. Whenever possible state your conjectures in "if… then…" format. Support your conjectures with evidence. Tell when your evidence is based on inductive reasoning from examples that suggest your conclusions seem to be true, and tell when your evidence is based on deductive reasoning that shows why your conclusions are correct.

Investigate…

a) The *exterior angles* of a triangle; a quadrilateral; a pentagon; a polygon with *n* sides. Note: Extending 1 side of a polygon forms 1 exterior angle; extending each side of a polygon to form 1 exterior angle at each vertex forms a set of exterior angles.





b) Minimal conditions to prove a quadrilateral is a parallelogram. Note: by definition a parallelogram is a quadrilateral with exactly 2 pairs of parallel sides.

c) The diagonals of kites, rhombuses, rectangles, squares, parallelograms, trapezoids (isosceles and nonisosceles).

d) Minimal conditions for congruence of 2 kites; 2 rhombuses; 2 rectangles; 2 squares; 2 parallelograms; 2 trapezoids (isosceles and nonisosceles); and 2 quadrilaterals.

(Continued on back.)

Focus Student Activity 10.4 (cont.)

e) Relationships between inscribed angles and the arcs and chords they intercept on a circle. Note: The measure of an arc in degrees is equal to the measure of the central angle that intercepts the arc.

f) Perpendicular bisectors of chords of circles.

g) Triangles that are inscribed in a circle (i.e., the circle circumscribes the triangle); and triangles that circumscribe a circle (i.e., the sides of the triangle are tangent to the circle).

h) One midsegment of a triangle and the set of all midsegments of a triangle; a midsegment of a rectangle and the set of all midsegments of a rectangle; of a rhombus; of a parallelogram; and of a trapezoid. Note: A midsegment of a triangle is a segment that connects midpoints of 2 sides of the triangle; a midsegment of a quadrilateral connects opposite midpoints.

i) Connecting midpoints of consecutive sides of quadrilaterals.

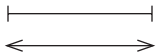
j) Minimal conditions to prove a quadrilateral is a rectangle. Note: by definition a rectangle is a quadrilateral with 4 right angles.

Follow-up Student Activity 10.5

NAME _____ **DATE** _____

Complete the following on separate paper; write each problem next to your work.

1 Given the length at the right is 1 linear unit. Use a straightedge and compass only to construct each of the following, if possible. Show all of your construction marks and briefly describe your step-by-step procedures. Then write a brief explanation why your method works. If the construction is not possible, explain why.



a) An equilateral triangle with perimeter $3\sqrt{2}$ units, and an enlargement of the triangle by a factor of 3.

b) A kite with diagonals $\sqrt{5}$ units and $2\sqrt{5}$ units.

c) A scalene right triangle with one side 2 units and one angle 45°.

d) A circle with area 6π square units.

e) A rectangle with diagonals of length 5 units and 4.5 units.

f) 3 noncongruent rectangles with diagonals of length 3 units.

g) An equilateral triangle with perimeter 6 and inscribed in a circle.

h) An isosceles right triangle with hypotenuse 5 and inscribed in a semicircle.

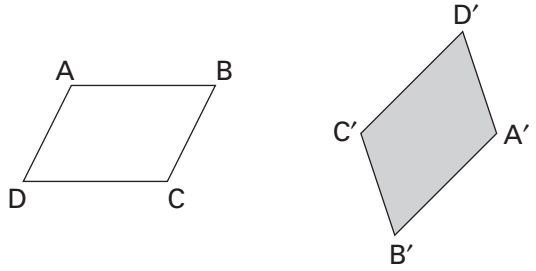
i) A regular hexagon whose perimeter is $12\sqrt{3}$ units and with an inscribed circle with radius 3 units.

2 Record, in "if… then…" format, 2 or more conjectures or generalizations that are based on your work for Problem 1. Give arguments to support each conjecture.

(Continued on back.)

Follow-up Student Activity (cont.)

3 A′B′C′D′ is the rotation image of ABCD. Trace these figures, construct the center of rotation, and briefly explain your steps.



4 A′B′C′D′ above is also the image of ABCD after 2 consecutive reflections across 2 different lines. Investigate and locate 2 such lines of reflection. Explain.

5 On a separate sheet of paper, draw line M and label point P not on M. Plot a few points whose distance from line M is twice the distance from point P. Form a conjecture about this locus of points.

6 For any 3 or more of the following statements, use a diagram and deductive arguments that are based on the diagram and properties that you know to argue why each of those statements is true.

a) The diagonals of a rhombus are perpendicular.

b) The area of a rhombus is ½ the product of its diagonals.

c) The altitude of an isosceles (nonequilateral) triangle bisects the vertex angle (i.e., the angle opposite the noncongruent side).

d) If a quadrilateral is a parallelogram, then a diagonal of the quadrilateral forms 2 congruent triangles.

e) If the diagonals of a quadrilateral bisect each other, then the quadrilateral must be a parallelogram.

f) The sum of the measures of the exterior angles of a polygon is 360°.

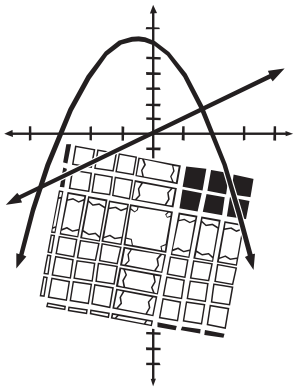
7 Find several possible angle measures that can be constructed by straightedge and compass only. Explain your methods.

Introduction to Quadratics

Lesson 11

THE BIG IDEA

Using Algebra Pieces to determine the values of *n* for which the *n*th arrangements of two sequences have the same value leads to the invention of strategies for solving linear and quadratic equations and systems of equations. Coordinate graphs of the values of the arrangements establish a relationship between algebra and geometry and illustrate solutions to systems of linear and quadratic equations.



CONNECTOR

OVERVIEW

Students relate graphs of points that lie along a linear path to sequences of counting piece arrangements and formulas for the *n*th arrangement of such sequences.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Algebra Pieces (including frames), 1 set per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Connector Student Activities 11.1 and 11.2, 1 copy of each per student and 1 transparency of each.
- ✔ Algebra Pieces for the overhead.

FOCUS

OVERVIEW

Students examine relationships between Algebra Piece, graphical, and symbolic representations of the *n*th arrangements of extended sequences of counting piece arrangements. They use Algebra Pieces and graphs to represent and solve linear and quadratic equations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Algebra Pieces (including frames), 1 set per student.
- ✔ Focus Masters A-C and E-G, 1 transparency of each.
- ✔ Focus Master D, 1 copy per group and 1 transparency.
- ✔ Focus Student Activities 11.3-11.6, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Student Activity 11.7, 1 copy per group and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ Algebra Pieces for the overhead.
- ✔ ¼" grid paper, 4 sheets per group and 1 transparency.

FOLLOW-UP

OVERVIEW

Students create sequences that satisfy specific conditions. They write formulas for, graph, and solve linear and quadratic equations. They complete the square to solve quadratic equations.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 11.8, 1 copy per student.
- ✔ Coordinate grid paper (see *Blackline Masters*), 8 sheets per student.

LESSON IDEAS

FOLLOW-UP
It may be helpful to remind the students that the graphs for this assignment are sets of discrete points. While the graphs follow the paths of lines and curves, the points are not connected since the only values for *n* in an extended sequence are the integers. Continuous functions whose domains are the real numbers are explored in Lessons 12 and 14.

LOOKING AHEAD
In this lesson, students use Algebra Pieces to explore the method of completing the square to solve quadratic equations. In Ma*th Alive! Course IV, stud*ents will generalize these methods to develop the quadratic formula.

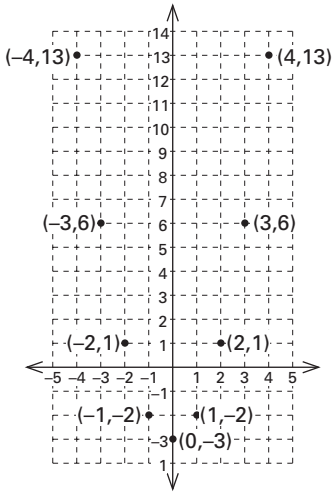
QUOTE
Formal equation-solving methods can be developed from, and supported by, informal methods. These informal methods, which may include actions on concrete materials that are paralleled by symbolic actions, can lead to more formal procedures. If students develop formal procedures from informal methods grounded in real-world contexts, they can validate their own formal thinking and develop a basis for extending these algebraic ideas.
NCTM Standards

SELECTED ANSWERS

1. g) One possibility:

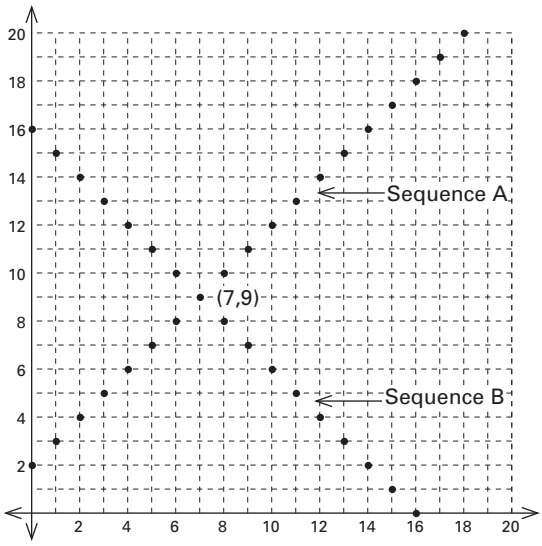
The value of the *n*th arrangement of the sequence shown above is $n^2 - 3$.

*n*th arrangement:

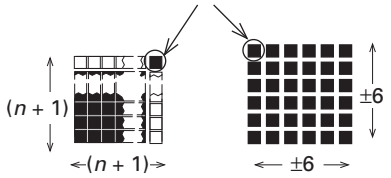


2. b) One possibility is Sequence A, $n + 2$; Sequence B, $-n + 16$.

c)



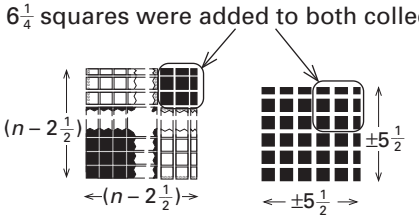
4. a) One square was added to both collections



Since $(n + 1)^2 = 36$, n must equal 5 or –7.

4. c) $\frac{1}{2}(2n^2 - 10n) = \frac{1}{2}(48)$, so $n^2 - 5n = 24$

$6\frac{1}{4}$ squares were added to both collections



Since $(n - 2\frac{1}{2})^2 = (\pm 5\frac{1}{2})^2$, $n - 2\frac{1}{2} = \pm 5\frac{1}{2}$ and so $n = 8$ or –3.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students relate graphs of points that lie along a linear path to sequences of counting piece arrangements and formulas for the nth arrangement of such sequences.

MATERIALS

- ✔ Algebra Pieces (including frames), 1 set per student.
- ✔ Connector Master A, 1 transparency.
- ✔ Connector Student Activities 11.1 and 11.2, 1 copy of each per student and 1 transparency of each.
- ✔ Algebra Pieces for the overhead.

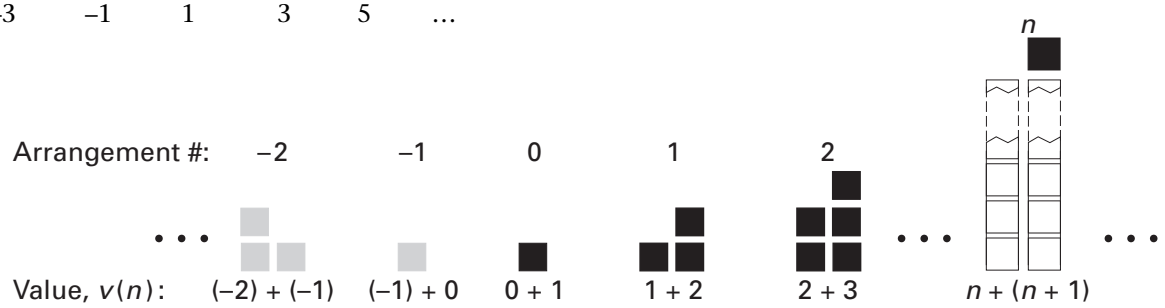
ACTIONS

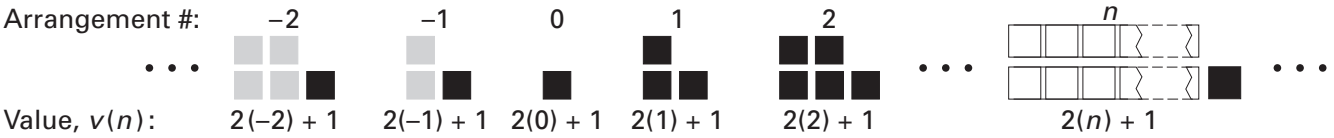
1 Arrange the students in groups and give Algebra Pieces to each student. Write the following chart on the overhead. Have the groups form the –2nd through 2nd and the *n*th arrangements of an extended sequence of counting piece arrangements which fits the data on the chart (note: *n* indicates the arrangement number and *v*(*n*) is the value of arrangement *n*). Ask the students to write a formula for *v*(*n*). Discuss.

n	…	–2	–1	0	1	2	…
v(*n*)	…	–3	–1	1	3	5	…

COMMENTS

1 Various extended sequences of arrangements (see Lesson 5 of this course) are possible. Shown below are two possibilities with formulas. Be sure the students indicate the arrangement numbers for their sequences.





2 Give each student a copy of Connector Student Activity 11.1 (see next page). Have the students form the –3rd through 3rd and the *n*th arrangements of an extended sequence of counting piece arrangements which fits the data displayed in graphical form on Student Activity 11.1. Ask them to determine *v*(–4), *v*(–3), *v*(3) and *v*(4) for the sequence and, if possible, add this information to their graph.

2 Shown on the next page is one extended sequence that fits the data.

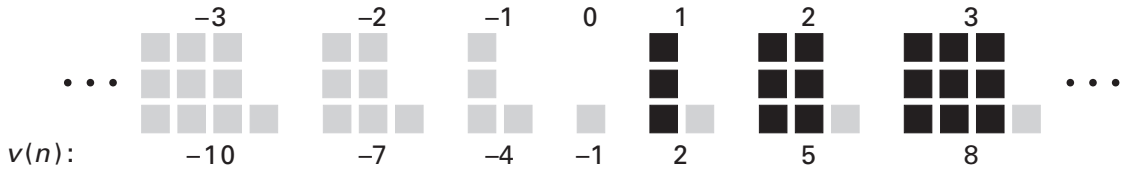
(Continued next page.)

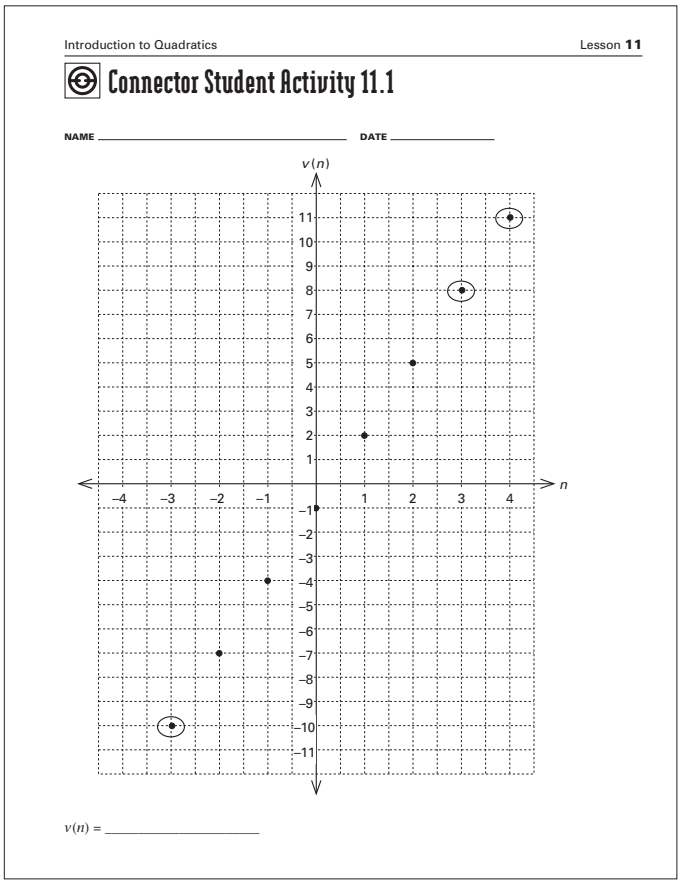
Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

2 (continued.)





For this sequence, $v(-3) = -10$, $v(3) = 8$ and $v(4) = 11$. Coordinate points for these 3 cases have been added and circled on the copy of Connector Student Activity 11.1 shown on the left. The ordered pair (–4,–13) lies off the graph.

It may be instructive here to review the use of terms such as *horizontal axis*, *vertical axis*, and *origin*. (Note: the origin is the coordinate (0,0) which is the point of intersection of the horizontal and vertical axes.)

3 Ask the students to:

a) record (in the space provided on Connector Student Activity 11.1) a formula for $v(n)$ for the sequence they constructed in Action 2;

b) label the coordinates of each point on the graph;

c) record 4 or 5 observations about the graph.

Discuss. Encourage observations about relationships between the numbers in the students' formulas for $v(n)$ and their graphs.

3 a) The arrangements shown in Comment 2 suggest the formula $v(n) = 3n - 1$. The students may have other equivalent formulas.

c) Students' observations will vary. If students make conjectures about relationships they think might be true for graphs of other sequences, rather than trying to confirm the conjectures now, you might suggest students test and refine them during the coming actions. Following are examples of observations that students have made about the graph.

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

The points of the graph lie along the path of a straight line.

The points are equally spaced.

Moving left to right, to get from one point to the next, go 1 unit to the right and 3 units up.

The increase in height from point to point is always the same.

There are only points on the graph where n is an integer.

Plotting points for $v(n) = 3n - 1$ is just like plotting points for $v(n) = 3n$ after shifting the coordinate axes down 1 unit.

In the formula, $v(n) = 3n - 1$, 3 is the *coefficient* of n and is the amount the value, $v(n)$, increases as n increases by 1. The *constant* term, –1, is the value of the 0th arrangement. It indicates where the graph intersects the vertical axis. Note that when the points on a graph lie along a straight line, the graph is called *linear*.

Some students may draw a line connecting the points of the graph, implying there are arrangements for non-integral values of n. The students may even suggest ways of constructing such arrangements (see Lesson 12); however, for this extended sequence, there are only points on the graph for integral values of n.

The intent throughout this lesson is to promote intuitions about relationships between graphs, formulas, and the sequences of arrangements the graphs and formulas represent. Terminology such as slope and x- or y-intercept are introduced in Lesson 12, after extended sequences of arrangements are augmented so their graphs are continuous.

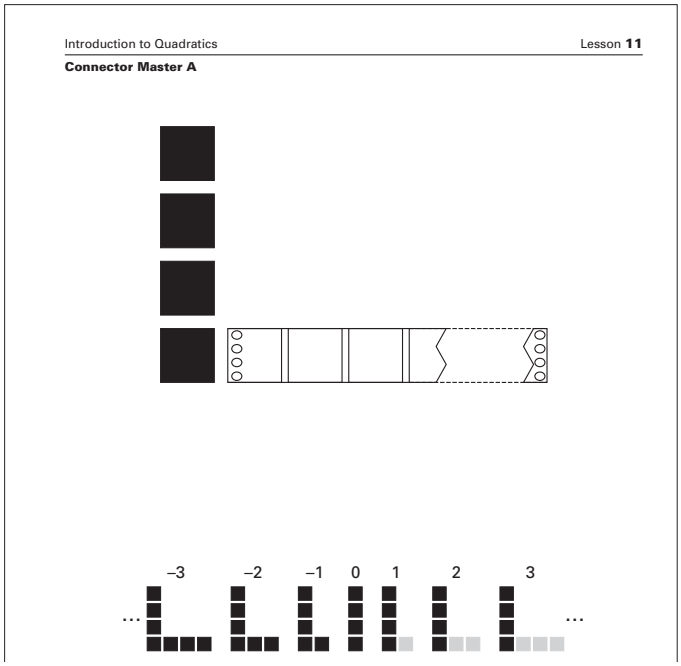
4 Place a transparency of Connector Master A on the overhead, revealing the top half only (see next page). Tell the students the arrangement shown is the nth arrangement of an extended sequence of counting piece arrangements. Ask the students to form the –3rd to 3rd arrangements of this sequence.

4 Arrangements numbered –3 through 3 are shown on the bottom half of Connector Master A. Recall that a –n-frame contains red tile if n is positive and black tile if n is negative. It contains no tile if n is 0.

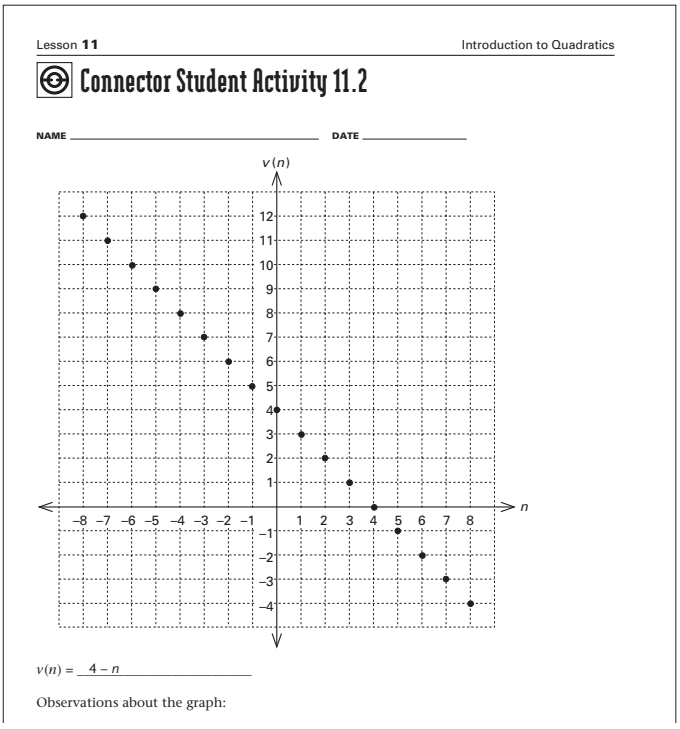
(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS



5 Distribute a copy of Connector Student Activity 11.2 to each student. For the sequence of Action 4, ask the students to record a formula for $v(n)$, construct its graph (see completed activity below), and record their observations about the graph. Discuss the students' ideas about the meaning of a *function*, whether $v(n) = 4 - n$ is a function, and the *domain* and *range* of $v(n)$.



COMMENTS

4 (continued.)

5 The formula for $v(n)$ can be written in various forms. One possibility is $v(n) = 4 - n$. Another is $v(n) = 4 + (-n)$.

In general, a *function* is a rule that relates 2 sets by assigning each element in the 1st set (called the *domain*) to exactly one element in the 2nd set (called the *range*). Hence, the relationship $v(n) = 4 + (-n)$ is a function that relates the variable n to $v(n)$ so that, for any arrangement number, n, there is exactly one value of the arrangement, $v(n)$. The set of all values for n—in this case, the integers—is the domain of the function $v(n) = 4 + (-n)$. The set of all possible values for $v(n)$—in this case, also the integers—is the range of the function. Note: The domain and range of the function $v(n) = 4 + (-n)$ can be all real numbers, but in this example only the integers are used for the arrangement (domain) numbers. In *Math Alive! Course II*, students worked with functions whose domains are the positive integers. In Lessons 12 and 14 of this course, students explore functions whose domains and ranges include all real numbers.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students examine relationships between Algebra Piece, graphical, and symbolic representations of the nth arrangements of extended sequences of counting piece arrangements. They use Algebra Pieces and graphs to represent and solve linear and quadratic equations.

MATERIALS

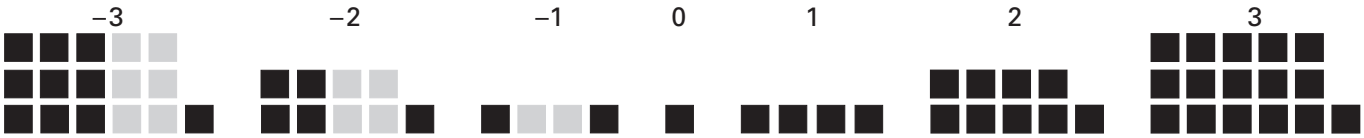
- ✔ Algebra Pieces (including frames), 1 set per student.
- ✔ Focus Masters A-C and E-G, 1 transparency of each.
- ✔ Focus Master D, 1 copy per group and 1 transparency.
- ✔ Focus Student Activities 11.3-11.6, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Student Activity 11.7, 1 copy per group and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 2 sheets per group and 1 transparency.
- ✔ Algebra Pieces for the overhead.
- ✔ ¼" grid paper, 4 sheets per group and 1 transparency.

ACTIONS

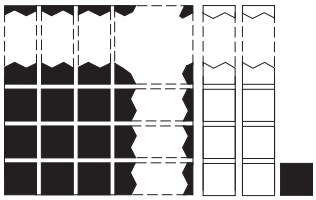
1 Arrange the students in groups and distribute Algebra Pieces to each student. Ask the groups to form the –3rd through 3rd and *n*th arrangements of an extended sequence of counting piece arrangements for which $v(n) = n^2 + 2n + 1$.

COMMENTS

1 Shown below is one possible set of arrangements.



One possible *n*th arrangement is shown below. Notice the use of the two frames to represent 2*n*. These frames are needed because they may represent a black *n*-strip or a red *n*-strip, depending on the value of *n*.



2 Tell the students there exists a sequence of square arrangements which fits the criterion of Action 1. Ask the groups to show how their arrangements from Action 1 can be formed into a sequence of squares, using edge pieces to show the values of the edges of the squares. Discuss.

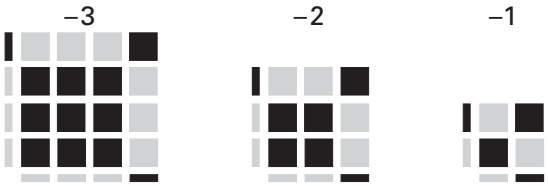
2 Some students may have formed square arrangements in Action 1. If so, you may call the other students' attention to these arrangements.

Below is a set of square arrangements with edge pieces.

(Continued next page.)

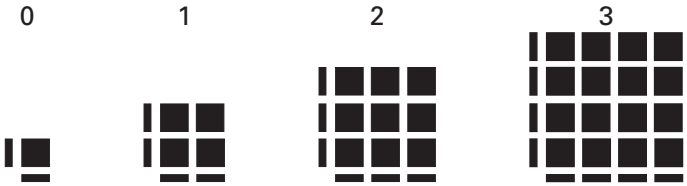
Focus Teacher Activity (cont.)

ACTIONS



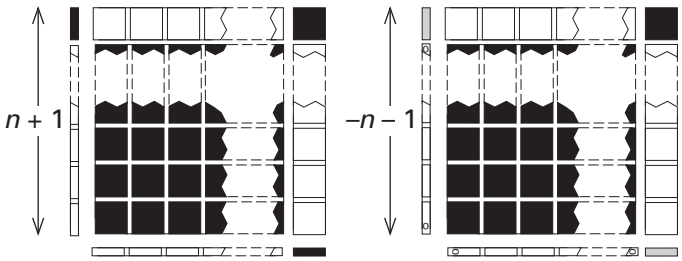
COMMENTS

2 (continued.)



Other edges are possible. For example, here is another possibility for the –3rd arrangement:

The nth arrangement formed in Action 1 can be rearranged to form a square with 2 possibilities for edges, as shown below. Notice that the figures show that $(n + 1)^2$ and $(-n - 1)^2$ are equivalent expressions for $v(n)$.



$v(n) = (n + 1)^2$ $v(n) = (-n - 1)^2$

If edge frames were not discussed during Lesson 5, you will need to do so now. Edge frames are edge pieces whose color, like that of frames, differs for positive and negative n. Edge frames are obtained by cutting frames into thirds as shown below.

Edge frames

Has value n for n positive, negative, or 0.

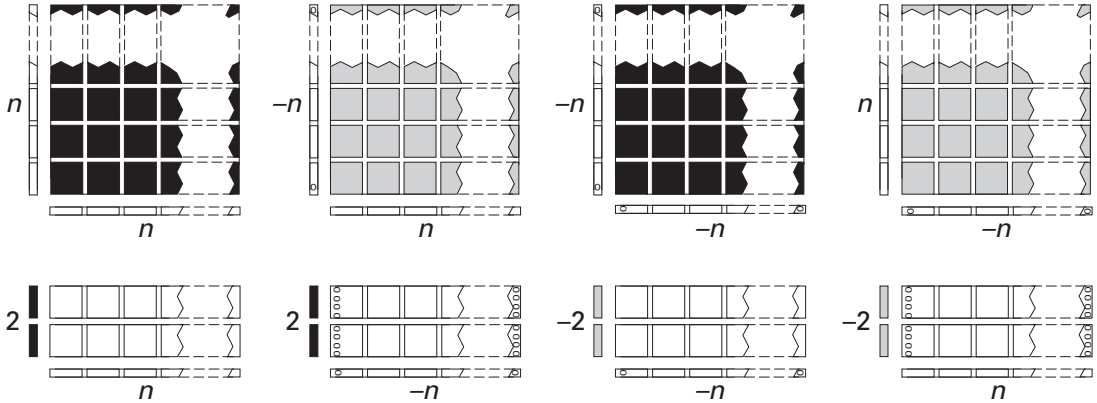
Has value $-n$ for n positive, negative, or 0.

The use of edge frames is illustrated on the next page. Recall that adjacent edges of black arrays have the same color and adjacent edges of red arrays have opposite color. Also, an array with a black edge has the same color as its other edge while an array with a red edge has color which is opposite the color of its other edge.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS



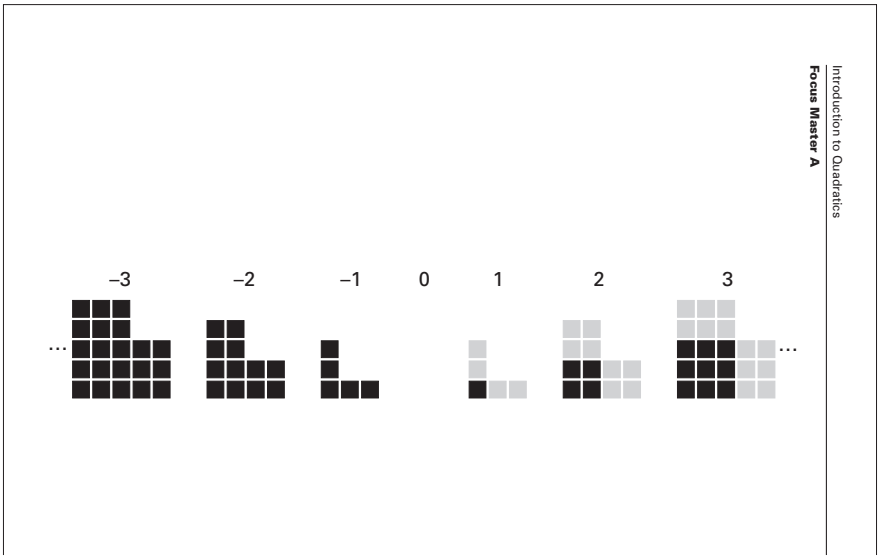
Note: the area of a square is always positive (or zero), while the value of a square can be positive, negative, or zero. Similarly, the *lengths* of the edges of a square are always identical and positive (or zero), while the *values* of the edges may be zero, both positive, both negative, or one positive and one negative.

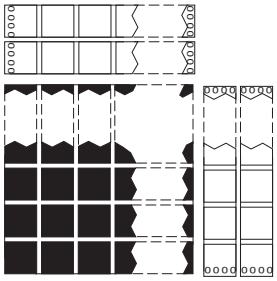
3 Ask the students to determine which arrangements in the extended sequence of Actions 1 and 2 have a value of 400. Discuss the students' methods. Ask them to identify the equation that has been solved.

3 A square has value 400 provided its edges all have value 20 or –20. Hence, the nth arrangement, viewed as a square whose edge has value $n + 1$, has value 400 provided $n + 1$ has value 20 or –20. Since $n + 1$ is 20 when n is 19 and $n + 1$ is –20 when n is –21, the 19th and –21st arrangements have value 400. Thus, the equation $(n + 1)^2 = 400$ has been solved. The solutions are 19 and –21. Note: $(-n - 1)^2 = 400$ also has solutions 19 and –21.

4 Place a transparency of Focus Master A on the overhead. Ask the students to form the nth arrangement of this sequence and to write an expression for $v(n)$.

4 The nth arrangement contains a black n^2-mat and 4 $-n$-frames:



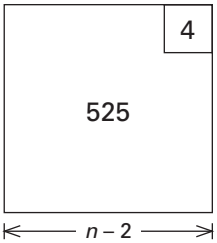


$v(n) = n^2 - 4n$

Focus Teacher Activity (cont.)

ACTIONS

5 Ask the students to determine for what n the extended sequence in Action 4 has $v(n) = 525$. Discuss their strategies. If it isn't suggested by students, introduce the method of solving the *quadratic equation* $n^2 - 4n = 525$ by *completing the square*.

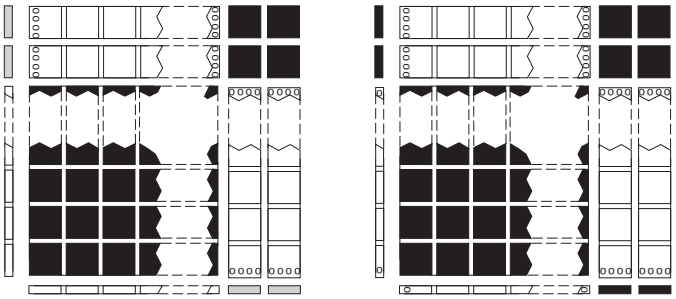


23 or –23

6 Ask the students to form the nth arrangement of an extended sequence for which $v(n) = n^2 + 4$ and then have them do the same for an extended sequence for which $v(n) = 2n^2 + 6n - 3$. Have the students use their Algebra Pieces to determine for which n the nth arrangements of these two extended sequences have the same value. Ask the students to identify the equation that has been solved and to verify their solutions.

COMMENTS

5 Adding 4 black tile to an nth arrangement results in a square array whose edges have value $n - 2$ or $-n + 2$, as shown below.



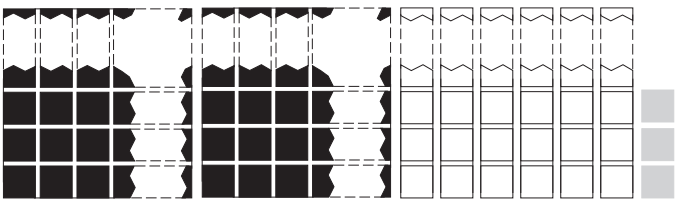
$v(n) + 4 = (n - 2)^2$ $\qquad$ $v(n) = (-n + 2)^2$

A square whose value is 525 + 4, or 529, has an edge whose value is 23 or –23 (a calculator with a square root key is helpful here). If $n - 2$ is 23, then n is 25, and if $n - 2$ is –23, then n is –21. Hence, the 25th and –21st arrangements have value 525. Similarly, if the edge has value $-n + 2$, then $-n + 2 = 23$ or –23; hence, $n = -21$ or 25.

Historically, the above method of solving a quadratic equation is called *completing the square*. A *quadratic equation* is an equation that can be written in the form $ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$. The word *quadratic* is derived from the Latin word, *quadratus*, meaning square. Sometimes it is necessary to rearrange and/or cut apart pieces to complete the square. This is illustrated in Comments 6 and 7. Students generalize this method in *Math Alive! Course IV*.

6 One way of representing the two nth arrangements is shown below:

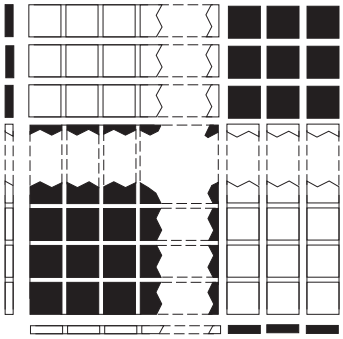
$n^2 + 4$



$2n^2 + 6n - 3$

Focus Teacher Activity (cont.)

ACTIONS



$n^2 + 6n + 9$

7 Write quadratic equation a) below on the overhead and ask the students to discuss their ideas about how the equation relates to sequences of counting piece arrangements. Then ask them to find all solutions of the equation. Repeat for one or more of b)-g). Discuss the students' generalizations or conjectures about the methods they use.

a) $n^2 - 6n = 40$

b) $2n^2 + 38 = 4n^2 - 12$

c) $(n - 1)(n + 3) = 165$

d) $4n^2 + 4n = 2600$

e) $n^2 - 5n + 6 = 0$

f) $n^2 + n = 6$

g) $n^2 + 3n - 10 = 0$

COMMENTS

These two arrangements have the same value if, after an n^2-mat has been removed from each of them, the remaining portions have the same value, i.e., if $n^2 + 6n - 3$ has value 4 or, equivalently, $n^2 + 6n$ has value 7.

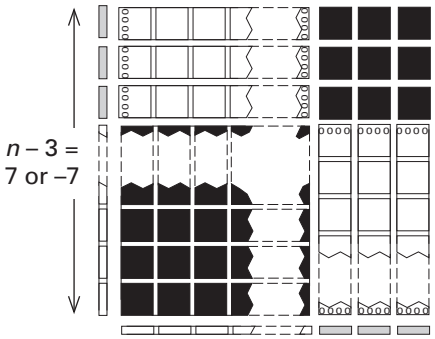
The method of completing the square is illustrated at the left. Adding 9 black tile to $n^2 + 6n$ produces a square array whose edge has value $n + 3$.

Thus, $n^2 + 6n$ has value 7 if the square array has value 16, i.e., if its edge has value 4 or –4. If $n + 3$ is 4, then n is 1; if $n + 3$ is –4, then n is –7. Hence, the 1st arrangements of the 2 sequences have the same value, as do the –7th arrangements.

7 Students' methods of solving these equations may vary.

a) The expression $n^2 - 6n$ may be viewed as the value of the nth arrangement of one sequence, and 40 as the value of the nth arrangement of another sequence. Hence, to solve the equation $n^2 - 6n = 40$ is to find the value of n for which the nth arrangements of the two sequences have the same value.

If 9 black tile are added to a collection for $n^2 - 6n$, the resulting collection can be formed into a square array with edge $n - 3$ (see below). If $n^2 - 6n$ has value 40, the square array has value 49 and its edge has value 7 or –7. If $n - 3$ is 7, then n is 10; if $n - 3$ is –7, then n is –4. So the equation has two solutions: 10 and –4. Note: the square could also have edge $-n + 3 = 7$ or –7, in which case, the solutions are still 10 and –4.



$(n^2 - 6n) + 9 = 40 + 9 = 49$

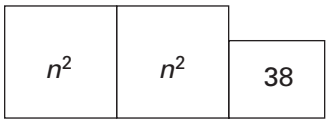
(Continued next page.)

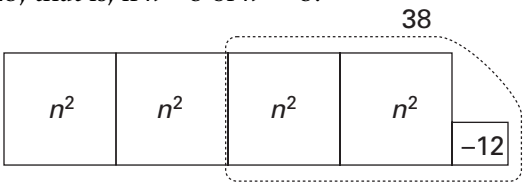
Focus Teacher Activity (cont.)

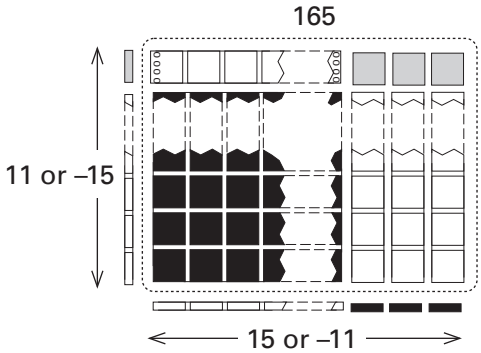
ACTIONS

COMMENTS

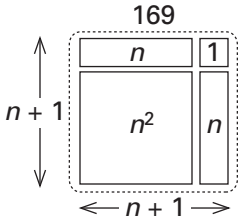
7 (continued.)
b) Comparing sketches for $2n^2 + 38$ and $4n^2 - 12$, note they have the same value if $2n^2 - 12$ is 38. This is the case if n^2 is 25, that is, if $n = 5$ or $n = -5$.



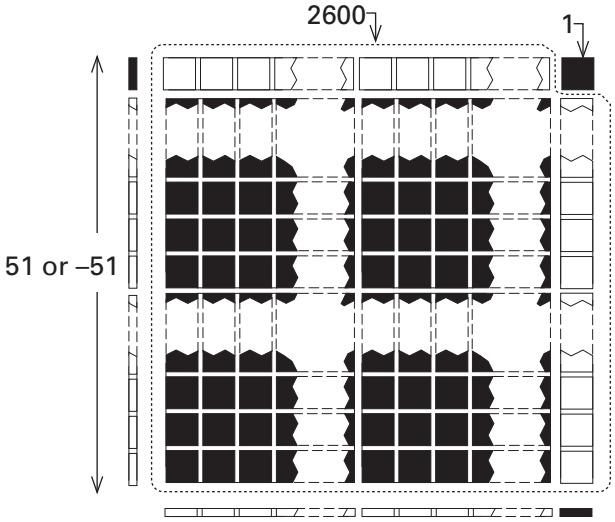




c) Shown at the left is a representation of $(n - 1)(n + 3)$. Note the values of the edges differ by 4 and their product is 165. Since 11 and 15 differ by 4 and $11 \times 15 = 165$, and since –11 and –15 differ by 4 and $-11 \times -15 = 165$, the array will have value 165 if the edges have values 11 and 15 or –11 and –15. The edges have values 11 and 15 if n is 12; they have values –11 and –15 if n is –14. (Finding the pair 11 and 15 is facilitated by noting that one of the pair should be smaller and one larger than $\sqrt{165} \approx 13$.)



Alternatively, adding 4 black tile to the array shown in the preceding paragraph and removing collections whose values are 0 leaves a collection of pieces that can be arranged in a square array that has value 169 and edge $n + 1$. Hence $n + 1$ is 13 or –13, in which case $n = 12$ or $n = -14$. Note: the edge could also be $-(n + 1) = 13$ or –13.

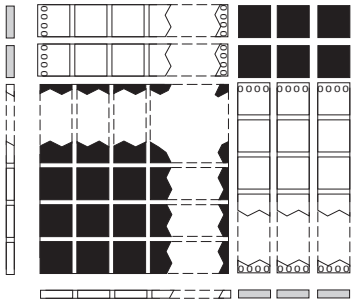


d) If 1 black tile is added to a collection for $4n^2 + 4n$, a square array with edge $2n + 1$ (or $-2n - 1$) can be formed, as shown at the left. If the value of the original collection is 2600, the value of the square array is 2601. Using a calculator, one finds $\sqrt{2601} = 51$. Hence $2n + 1$ is 51 or –51. Thus $n = 25$ or $n = -26$.

Using another approach, dividing a collection for $4n^2 + 4n$ by 4 results in the collection $n^2 + n$ with value 650. From this collection, a rectangle with value 650 and edges n by $(n + 1)$ can be formed. Since $25 \times 26 = 650$ and $-25 \times -26 = 650$, $n = 25$ or –26.

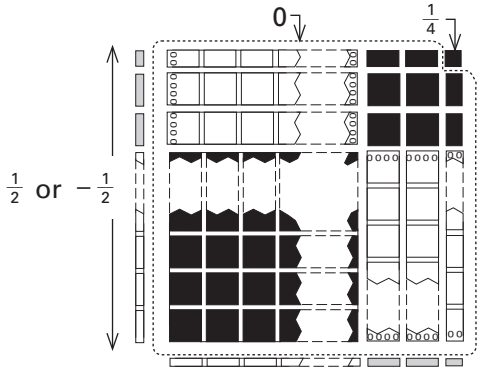
Focus Teacher Activity (cont.)

ACTIONS

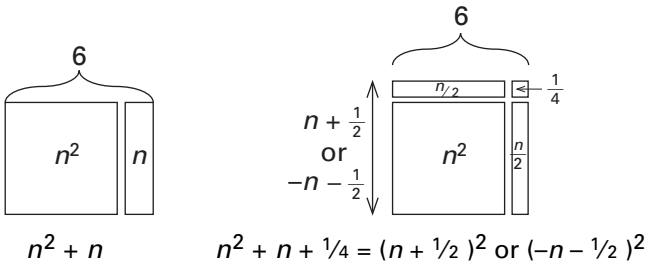


COMMENTS

e) A collection for $n^2 - 5n + 6$ can be formed into a rectangular array with edges $n - 2$ and $n - 3$. The array has value 0 if an edge has value 0. This is the case if $n = 2$ or $n = 3$.



Alternatively, by cutting a $-n$-frame and 2 black tile in halves and adding ¼ of a black tile to a collection for $n^2 - 5n + 6$, the resulting collection can be formed into a square with edge $n - 2\frac{1}{2}$. If the original collection has value 0, the square array has value ¼ and its edge has value ½ or –½. Now, $n - 2\frac{1}{2} = \frac{1}{2}$ when $n = 3$ and $n - 2\frac{1}{2} = -\frac{1}{2}$ when $n = 2$.



$n^2 + n$ $\qquad$ $n^2 + n + \frac{1}{4} = (n + \frac{1}{2})^2$ or $(-n - \frac{1}{2})^2$

f) Beginning with a collection for $n^2 + n$, if one cuts the n–frame in halves and adds ¼ of a black tile, a square array with edge $n + \frac{1}{2}$ (or $-n - \frac{1}{2}$) can be formed. This square array has value $6\frac{1}{4}$ and its edge has value $2\frac{1}{2}$ or $-2\frac{1}{2}$. If $n + \frac{1}{2} = 2\frac{1}{2}$, then $n = 2$; if $n + \frac{1}{2} = -2\frac{1}{2}$, then $n = -3$.

g) $n = 2$ or $n = -5$

Here are other equations you might have students solve: $(n - 4)(n + 2) = 0$; $n^2 + 4n - 5 = 0$; $n^2 + 6n = -8$; $n^2 = 7n - 6$; $2n^2 - 2n = 112$. If you create others, be sure they have integer solutions (in Lessons 12-14 students explore sequences with nonintegral arrangement numbers).

8 Place a transparency of Focus Master B (see next page) on the overhead, revealing the top half only. Tell the students that the arrangement shown is the nth arrangement of an extended sequence of counting piece arrangements. Ask them to form the –3rd through 3rd arrangements (with edges) of this sequence. Then distribute a copy of Focus Student Activity 11.3 to each student and have the students write a formula for $v(n)$, construct its graph (see completed graph on the next page), and record their observations about the graph.

8 Arrangements numbered –3 through 3 are shown on the bottom half of Focus Master B.

Here are three possibilities for $v(n)$: $v(n) = n^2 - 2n - 3$; $v(n) = (n + 1)(n - 3)$; and $v(n) = (-n - 1)(-n + 3)$.

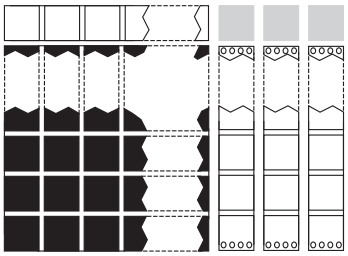
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **11** Introduction to Quadratics

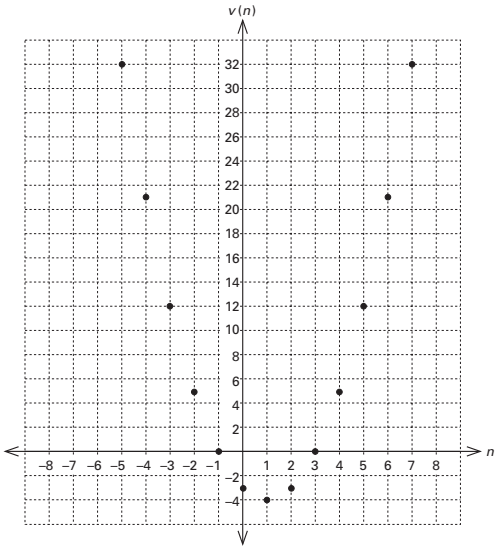
Focus Master B



Introduction to Quadratics Lesson **11**

Focus Student Activity 11.3

NAME _____ DATE _____

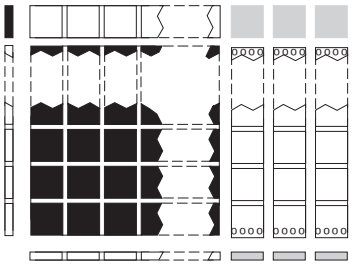


$v(n) = n^2 - 2n - 3$

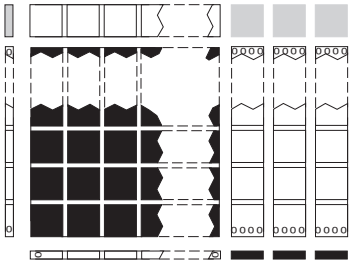
Observations about the graph:

COMMENTS

8 (continued.)
Edge pieces help to illustrate the latter two formulas:



$v(n) = (n + 1)(n - 3)$



$v(n) = (-n - 1)(-n + 3)$

The coordinate points associated with these arrangements are shown on the completed graph below.

As students make observations, they may use descriptors such as turning point, minimum or smallest value, increasing or rising, decreasing or falling, u-shaped or cup-shaped, etc. Encourage this. You may wish to model ways of using algebraic notation to record their observations (e.g., for $n \geq 1$, $v(n) \geq -4$). Here are examples of observations students have made about the graph:

The points of the graph do not lie along the path of a straight line; they lie along the path of a U-shaped curve.

The graph is symmetric about the vertical line that passes through n = 1. That is, if the graph is folded along the vertical line that goes through n = 1, the points to the right of the fold coincide with those to the left of the fold.

The point (1,–4) is a turning point where the graph stops falling and starts to rise (looking from left to right).

The smallest value for v(n) is –4. It occurs when n = 1.

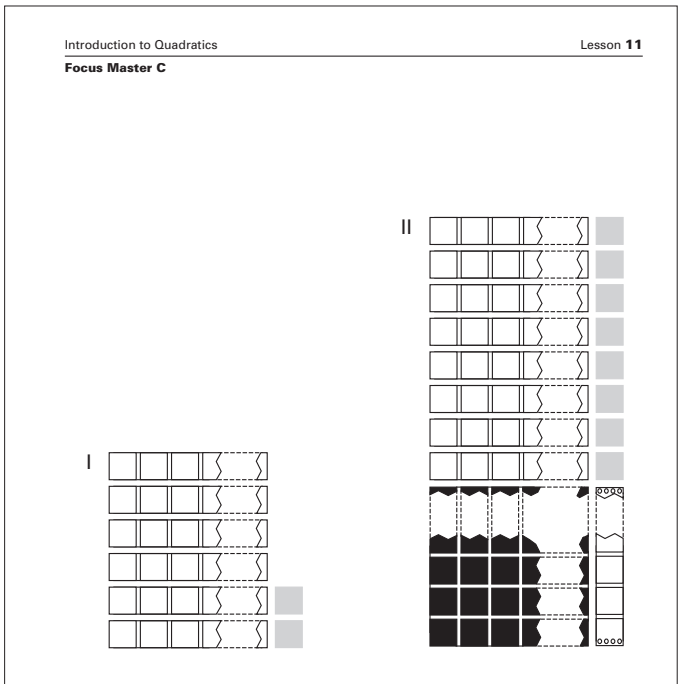
When n is greater than 1, as n increases so does v(n). When n is less than 1, as n decreases v(n) increases.

$v(n) = n^2 - 2n - 3$ is a function with domain the integers and range the integers greater than or equal to –4.

Focus Teacher Activity (cont.)

ACTIONS

9 Place a copy of Focus Master C on the overhead, and tell the students that Arrangements I and II are the *n*th arrangements of two different extended sequences. Ask them to write formulas for $v_1(n)$, the value of the *n*th arrangement of the first sequence and $v_2(n)$, the value of the *n*th arrangement of the second sequence.



10 Give each student a copy of Focus Student Activity 11.4 and ask them to do the following in reference to Sequences I and II from Focus Master C.

a) record their formulas for $v_1(n)$ and $v_2(n)$;

b) graph $v_1(n)$ and $v_2(n)$ on the coordinate grid, indicating the points on the graph of $v_1(n)$ with an x and those on the graph of $v_2(n)$ with an o (see completed graph on the next page);

c) examine the graphs and record observations about relationships between the graphs.

Discuss, using their observations to motivate discussion of *inequality* notation.

COMMENTS

9 Here are 2 possible formulas for the given *n*th arrangements:

$v_1(n) = 6n - 2$
$v_2(n) = n^2 + 7n - 8$

Other formulas are possible. For example, $v_2(n) = (n + 8)(n - 1)$; edge pieces may help the students see this formula.

10 The completed graphs are shown on a copy of Focus Student Activity 11.4 on the next page.

Following are some observations that students have made about the graphs. If students don't bring these up, you might prompt discussion by posing questions such as: What points, if any, do the 2 graphs have in common? What do the common points on the graphs tell about the 2 sequences of counting piece arrangements? When is $v_1(n) > v_2(n)$? How do the shapes of the graphs compare? How could you change the equation of v_1 so that it doesn't intersect v_2? Are v_1 and v_2 functions? How do the domains and ranges of v_1 and v_2 compare?, etc.

Two points, (2,10) and (–3,–20), are on both graphs. This tells us that the 2nd arrangements of the 2 sequences have the same value and the –3rd arrangements also have the same value. It also tells us the equation $6n - 2 = n^2 + 7n - 8$ *has 2 solutions,* $n = 2$ *and* $n = -3$. Note: you might have

(Continued next page.)

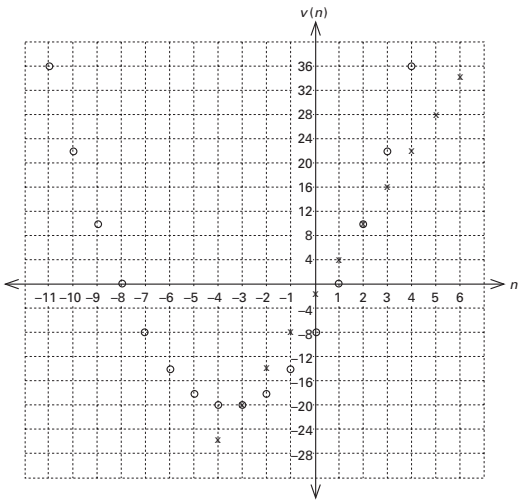
Focus Teacher Activity (cont.)

ACTIONS

Lesson **11** Introduction to Quadratics

Focus Student Activity 11.4

NAME _____ DATE _____



x $v_1(n) = 6n - 2$ _____ o $v_2(n) = n^2 + 7n - 8$ _____

Observations about the graph:

COMMENTS

10 (continued.)
the students form the 2nd arrangement of each sequence and verify that they have the same value. Likewise for the –3rd arrangements.

When n is between –3 and 2, the values of the arrangements in Sequence I are greater than the values of the arrangements in Sequence II. Using *inequality* notation: if $-3 < n < 2$, then $v_1(n) > v_2(n)$.

When n is less than –3 and when n is greater than 2, the value of Sequence II is greater than the value of Sequence I. Using inequality notation, if $n < -3$ or $n > 2$, then $v_2(n) > v_1(n)$.

The graph of $v_1(n)$ follows the path of a straight line, and the graph of $v_2(n)$ follows the path of a U-shaped curve.

The graph of Sequence I always rises as values of n increase from left to right. Looking at the graph of Sequence II from left to right, the graph falls as n increases, until $n = -4$; $v(-4) = v(-3)$; then after $n = -3$, as n increases the graph rises.

Once Sequence II starts to rise, it rises faster than Sequence I.

Sequence II has line symmetry about a vertical line that passes midway between $n = -3$ and $n = -4$.

If the U-shaped curve that the graph of Sequence II follows is traced, we think the turning point is $(-3\frac{1}{2}, -20\frac{1}{4})$. Since the curve is symmetric, we think the turning point is half way between $n = -3$ and $n = -4$, or at $n = -3\frac{1}{2}$. We found the y-coordinate by finding $(3\frac{1}{2})^2 + 7(-3\frac{1}{2}) - 8 = -20\frac{1}{4}$.

Note: an equation such as $v_1(n)$ whose graph lies on a straight line is called *linear*; and an equation such as $v_2(n)$ whose graph lies on a *parabola* (i.e., a U-shaped curve) is called *quadratic*.

11 Ask the students to use Algebra Pieces to build the *n*th arrangements of both extended sequences, $v_1(n)$ and $v_2(n)$, given in Action 9. Then have them solve the equation $6n - 2 = n^2 + 7n - 8$ using the pieces. Ask for volunteers to share their solutions.

11 The students now have two different ways to solve an equation like $6n - 2 = n^2 + 7n - 8$: using Algebra Pieces and using a graph they construct by hand. You might ask the students to discuss their ideas about the advantages and disadvantages of each method. (Note: in Lesson 14 students use the graphing calculator to determine solutions of equations and they develop symbolic strategies that represent actions with Algebra pieces.)

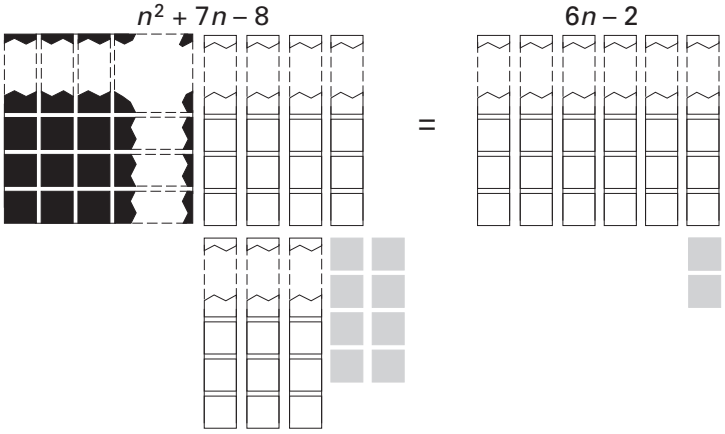
Focus Teacher Activity (cont.)

ACTIONS

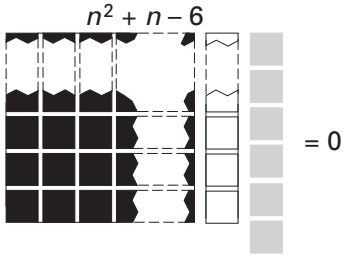
COMMENTS

Following is one method of using Algebra Pieces to solve the equation $6n - 2 = n^2 + 7n - 8$:

Step 1: Model the equation.

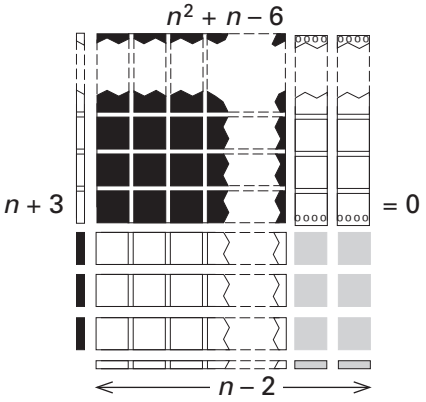


Step 2: Remove 6 n-strips and 2 red counting pieces from each arrangement to obtain the equivalent equation: $n^2 + n - 6 = 0$.



Step 3: Complete a rectangle by rearranging the pieces and adding 2 $-n$-frames and 2 n-frames (i.e., adding 0 to the net value of the original collection). This rectangle has edges $n - 2$ and $n + 3$.

Step 4: Since $(n - 2)(n + 3) = 0$ when $n - 2 = 0$ or when $n + 3 = 0$, then $n = 2$ and $n = -3$ are solutions to the equation $6n - 2 = n^2 + 7n - 8$.



Some students may invent ways of using algebra symbols to record their Algebra Piece methods. For example, the above procedures could be recorded as follows:

$$
\begin{aligned}
6n - 2 &= n^2 + 7n - 8 \\
6n - 2 - (6n - 2) &= n^2 + 7n - 8 - (6n - 2) \\
0 &= n^2 + n - 6 \\
0 &= n^2 + n - 6 + 2n + {}^{-}2n \\
0 &= (n - 2)(n + 3) \\
\text{so, } n - 2 = 0 &\quad \text{or} \quad n + 3 = 0 \\
n = 2 &\quad \text{or} \quad n = {}^{-}3
\end{aligned}
$$

Focus Teacher Activity (cont.)

ACTIONS

12 Give each group 2 sheets of coordinate grid paper (see *Blackline Masters*) and a copy of Focus Master D. Have them carry out the instructions. When groups are finished, invite volunteers to share their questions, observations, and conjectures.

Lesson **11** Introduction to Quadratics

Focus Master D

1. Formulas for the values of the *n*th arrangements of 3 pairs of extended sequences are given below. For each pair of sequences, please do the following:

a) Sketch the –3rd through 3rd arrangements and the *n*th arrangement of both sequences.

b) Make a table showing $v_1(n)$ and $v_2(n)$ for n from –3 to 3. Then graph $v_1(n)$ and $v_2(n)$ on the same coordinate axes.

c) Make mathematical observations about similarities, differences, and relationships you notice in the graphs of $v_1(n)$ and $v_2(n)$.

d) Use pictures or algebraic symbols to show your step-by-step methods of using Algebra Pieces to determine when $v_1(n) = v_2(n)$.

Pair 1
$v_1(n) = -3n + 2$ $v_2(n) = 4n - 12$

Pair 2
$v_1(n) = -7$ $v_2(n) = -n^2 - 2n + 8$

Pair 3
$v_1(n) = n^2 + 2$ $v_2(n) = -n^2 + 4$

2. Review your completed graphs for Problem 1 above and then record your general observations and conjectures about graphing equations.

COMMENTS

12 This could also be completed by individuals as homework and then discussed in class.

A table of values for $v_1(n)$ and $v_2(n)$ is useful for generating and organizing ordered pairs to plot. For example:

Pair 1

n	$v_1(n)$	$v_2(n)$
–3	11	–24
–2	8	–20
⋮	⋮	⋮

If students' observations and conjectures are limited you might encourage them to plot additional points to get a better feel for the behavior of the graphs. Following are some conjectures posed by students (if these ideas do not come up, you could bring some up; however, note they will be addressed in Lesson 12).

We think a graph that is linear rises from left to right if the coefficient of n is positive and falls if the coefficient is negative.

For an equation whose graph follows a linear path, we think the constant moves the graph up or down from the horizontal axis ("the zero line"). We think the coefficient of the n-term determines the "steepness" of the graph.

We think the graph of an equation with an n^2-term and with or without an n-term or constant is U-shaped.

We are pretty sure that if the coefficient of the n^2-term is negative the U opens down (the U would spill water). If the coefficient of the n^2-term is positive the U opens up (the U would hold water).

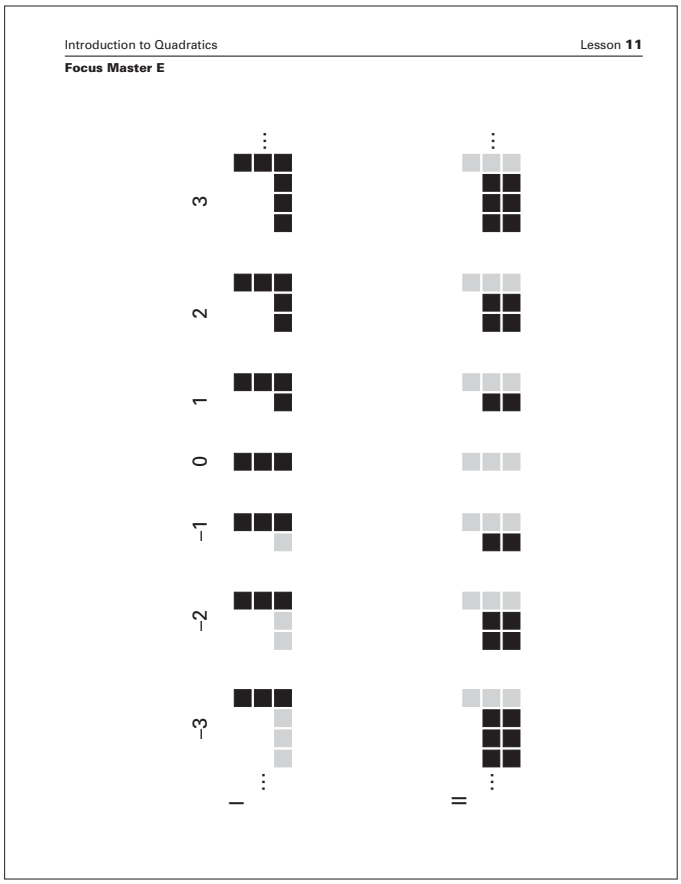
When the nth arrangement of a sequence contains no n^2-mats and no n-frames, then the values of all arrangements are identical. The graph follows a horizontal line that crosses the y-axis at the constant value of the arrangements.
We think that a line and a U-shaped curve can only intersect in 0, 1, or 2 points; 2 different lines can intersect in 0 or 1 point; 2 different U-shaped curves can intersect in 0, 1, or 2 points.

The domain of each of the functions that are given is the integers but the ranges vary. When the value of the nth arrangement is a constant, there is only one number in the

Focus Teacher Activity (cont.)

ACTIONS

13 Place a transparency of Focus Master E on the overhead. Ask the students to write formulas for $v_1(n)$, i.e., the value of the nth arrangement of Sequence I, and $v_2(n)$, i.e., the value of the nth arrangement of Sequence II. Discuss. Then give each student a copy of Focus Student Activity 11.5; have them graph $v_1(n)$ and $v_2(n)$ and record their observations (see completed graph on the next page). Discuss.



COMMENTS

range (e.g., $v_1(n)$ from pair 2). The ranges of the given functions are listed below:

Pair 1: v_1 and v_2, all integers
Pair 2: v_1, –7; v_2, all integers ≤ 9
Pair 3: v_1, all integers ≥ 2; v_2, all integers ≤ 4.

13 From Sequence I, one sees that $v_1(n) = n + 3$.

The students may readily see the pattern of Sequence II, but have difficulty writing a formula for $v_2(n)$. You might suggest that students write the formula in two parts, one part for nonnegative arrangement numbers and one part for negative arrangement numbers:

$$v_2(n) = \begin{cases} 2n - 3, \text{ for } n \text{ nonnegative} \\ 2(-n) - 3, \text{ for } n \text{ negative} \end{cases}$$

Notice that the case $n = 0$ is included in the nonnegative requirement.

You may want to tell the students about the mathematical symbol $|n|$, read *absolute value of n,* which is defined to be n, if n is nonnegative (e.g., $|3| = 3$) and $-n$ if n is negative (e.g., $|-3| = -(-3) = 3$). Using this symbol, one has $v_2(n) = 2|n| - 3$ for all n. Note that absolute value is explored further in Lessons 12 and 14.

The graphs of $v_1(n)$ and $v_2(n)$ are shown on the next page on a copy of Focus Student Activity 11.5. Some observations that may come up include:

The points of the graph of $v_1(n)$ lie on a straight line. The points of the graph of $v_2(n)$ lie on a V-shaped graph whose vertex is the point (0,–3). Note that a V-shaped graph is not a parabola.

The points (–2,1) and (6,9) lie on both graphs.

$v_2(n) < v_1(n)$ when $-2 < n < 6$.

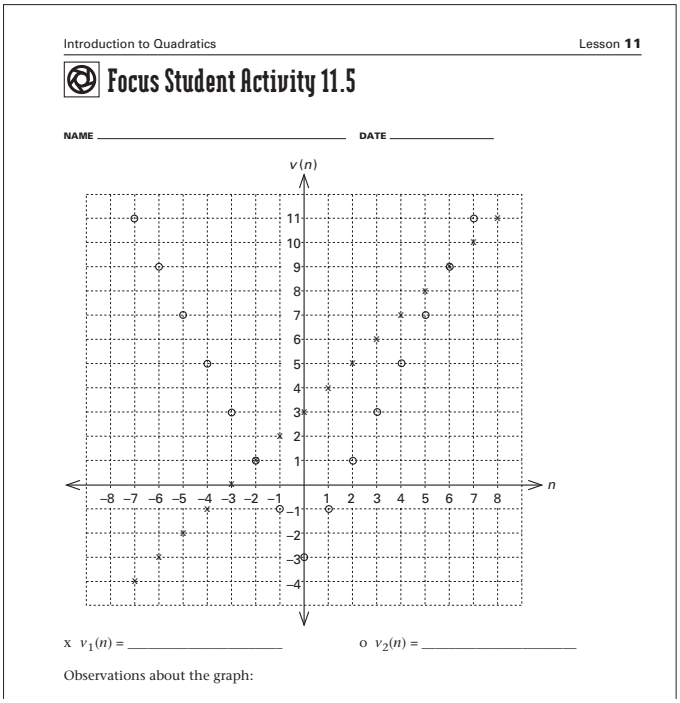
The minimum value of $v_2(n)$ is –3 and it has no maximum value since both "sides of the V" increase indefinitely.

$v_1(n)$ has neither a minimum nor a maximum value since it extends indefinitely—there are no upper or lower limits on the values of the arrangements.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



Introduction to Quadratics Lesson **11**

Focus Student Activity 11.5

NAME _____ DATE _____

x $v_1(n)$ = _____ o $v_2(n)$ = _____

Observations about the graph:

14 Give each student a copy of Focus Student Activity 11.6. Place a transparency of Focus Master F on the overhead, and repeat Action 13 for the two sequences shown. Have the students sketch the graphs on Focus Student Activity 11.6 (see completed graph on the next page).

COMMENTS

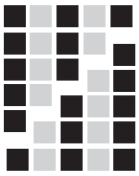
13 (continued.)
Because v_1 and v_2 are extended sequences whose arrangement numbers are the integers, the domain of each function is the integers. The range of v_1 is the integers and the range of v_2 is the integers greater than or equal to –3.

14 Notice that both of these sequences represent functions whose domains are the positive integers since the sequences are nonextended. The range of Sequence I is the integers 0 and 3. The range of Sequence II is the positive and negative integers.

Formulas for the *n*th arrangement of these 2 sequences can be written in 2 parts, 1 part for *n* odd and 1 part for *n* even.

$$v_1(n) = \begin{cases} 3, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \qquad v_2(n) = \begin{cases} (n+1)/2, & n \text{ odd} \\ -n/2, & n \text{ even} \end{cases}$$

One way to see the above formula for $v_2(n)$ is to notice that, if *n* is odd, 2 copies of the *n*th arrangement form a rectangle whose value is $n + 1$ (an example for $n = 5$ is shown below) and, if *n* is even, 2 copies of the *n*th arrangement form a rectangle whose value is $-n$ (an example for $n = 4$ is shown below).

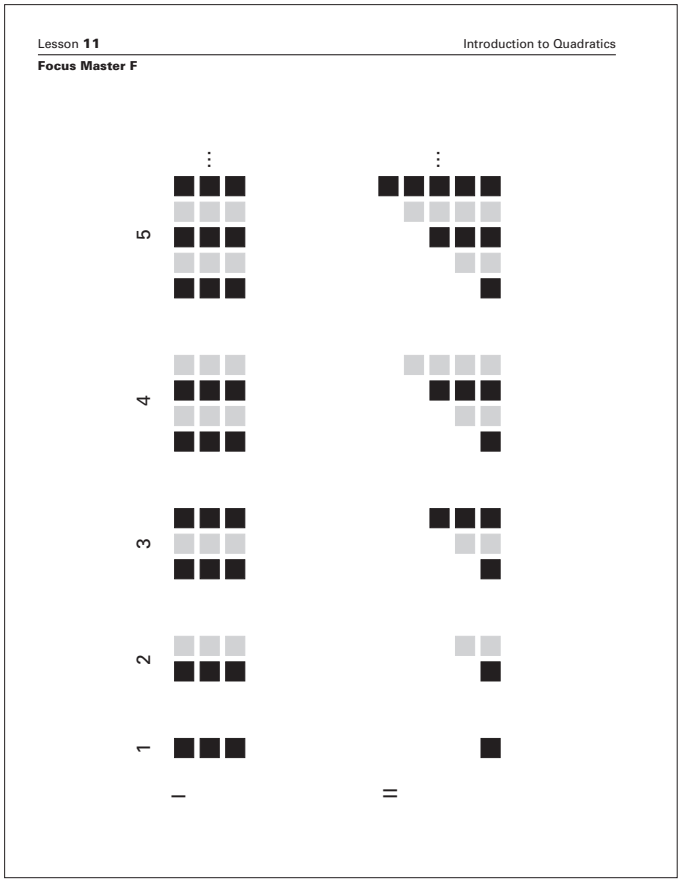


2 copies of
5th arrangement

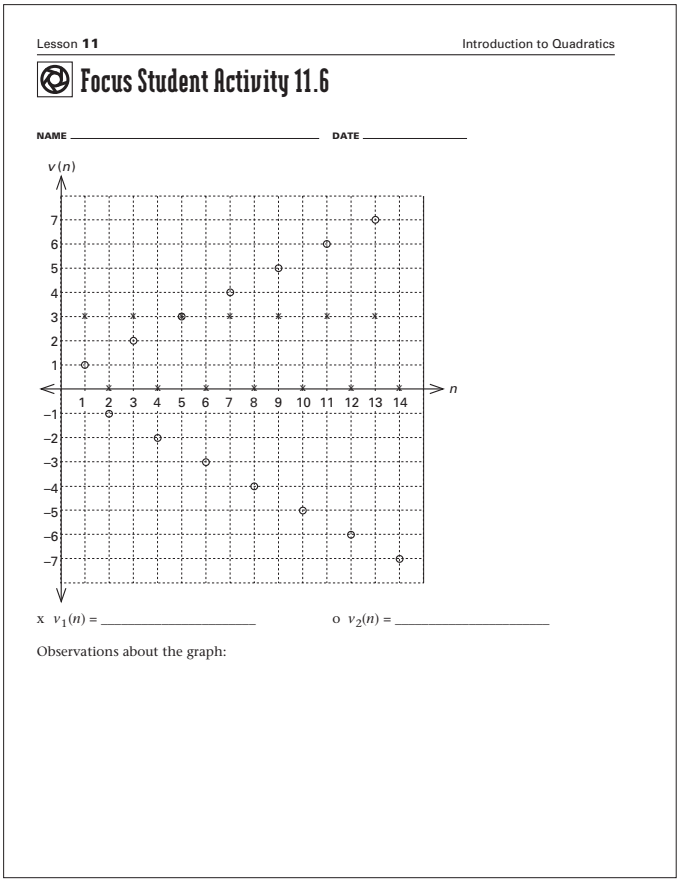
2 copies of
4th arrangement

Focus Teacher Activity (cont.)

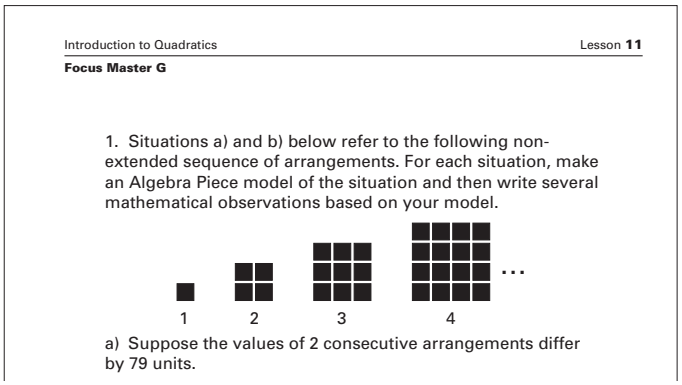
ACTIONS



COMMENTS

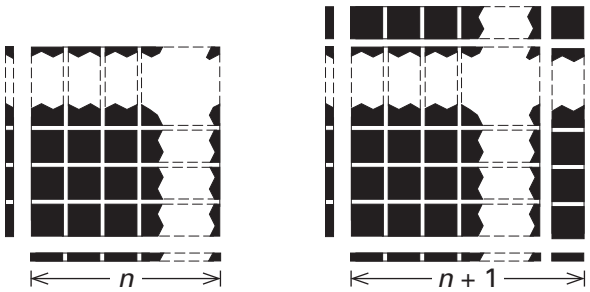


15 Place a transparency of Focus Master G on the overhead, revealing Situation 1a) only. Ask the groups to use their Algebra Pieces to form a model of the situation and to make several mathematical observations about the situation. Discuss the reasoning behind their observations. Repeat for Situations 1b) and 2a)-d).



15 Groups' mathematical observations will vary.

1a) Following are diagrams of two consecutive squares:



Notice the difference between the values of the two squares is $2n + 1 = 79$. Hence, $n = 39$ and the squares have edges 39 and 40.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Introduction to Quadratics Lesson **11**

Focus Master G

1. Situations a) and b) below refer to the following non-extended sequence of arrangements. For each situation, make an Algebra Piece model of the situation and then write several mathematical observations based on your model.

a) Suppose the values of 2 consecutive arrangements differ by 79 units.

b) Suppose the difference between 2 arrangement numbers is 3 and the difference between the values of the 2 arrangements is 111 units.

2. Suppose Mystery Sequence X is a nonextended sequence of arrangements of black counting pieces. To form the *n*th arrangement of Sequence X: form a rectangle that is *n* units wide and twice as long as it is wide; surround the rectangle with a 2-unit wide border (note: the outer perimeter of the border is a rectangle). Use Algebra Piece models as a basis for making mathematical observations about each of the following:

a) The 1st, 2nd, and *n*th arrangements of Sequence X.

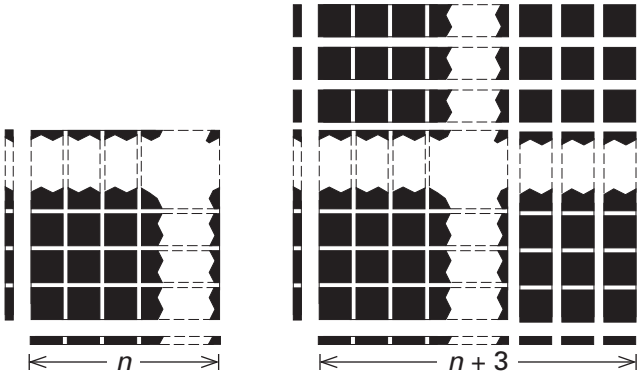
b) An arrangement whose border contains 160 units.

c) An arrangement that contains a total of 126 units.

d) Challenge. Two consecutive arrangements whose values differ by 62 units.

COMMENTS

15 (continued.)
1b) The difference between the values of the 2 squares shown below is $6n + 9$. If $6n + 9 = 111$, then $6n = 102$, $n = 17$, and the squares have edges 17 and 20.



2a) Here is the *n*th arrangement:

*n*th arrangement

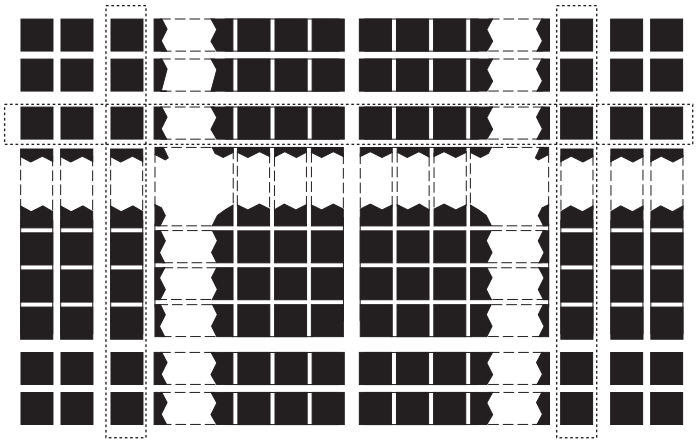
2b) The value of the border is $12n + 16 = 160$ when $n = 12$. Hence, the 12th arrangement has value 448 and its edges have values 16 and 28.

2c) If the value of an arrangement is 126, removing 4 units from each corner leaves 2 n^2-mats and 12 n-strips with total value 110. Half of this collection is $n^2 + 6n$ with value 55. If 9 black units are added to a collection for $n^2 + 6n$, the resulting collection can be formed into a square array with edge $n + 3$ (see diagram at the left). If $n^2 + 6n$ has value 55, the square array has value 64 and its edge has value 8. Hence, $n + 3 = 8$ so $n = 5$. The case in which the edge has value –8 is not considered here since this sequence is nonextended. Note: rather than halving a collection of $2n^2 + 12n$, some students may double the collection and complete the square on a collection of 4 n^2-mats and 24 n-strips with value 220.

Focus Teacher Activity (cont.)

ACTIONS

$$
\begin{aligned}
2n^2 + 12n + 16 &= 126 \\
2n^2 + 12n + 16 - 16 &= 126 - 16 \\
2n^2 + 12n &= 110 \\
(2n^2 + 12n)/2 &= 110/2 \\
n^2 + 6n &= 55 \\
n^2 + 6n + 9 &= 64 \\
(n + 3)^2 &= 64 \\
n + 3 &= 8 \\
n &= 5
\end{aligned}
$$



(n + 1)st arrangement

16 Discuss the students' ideas about the meaning of *factors of a number*. Relate this discussion to the meaning of *factoring a quadratic*. Give each group 4 sheets of ¼" grid paper, and a copy of Focus Student Activity 11.7 (see next page). Ask the groups to complete Problems 1i)-1v) for collections a) and b) listed on page 3 of the activity. Discuss, clarifying as needed. Then repeat for c)-m).

COMMENTS

Some students may use symbols to record their Algebra Piece actions. For example, the above method could be represented as shown at the left.

2d) Consecutive arrangements are the nth and (n + 1)st arrangements. To form the (n + 1)st arrangement each n-strip must increase by 1 unit, and both dimensions of each n^2-mat must increase by 1, as shown at the left (the circled pieces indicate the difference between the nth and (n + 1)st arrangements).

The difference between the nth and (n + 1)st arrangements is 62 when $4n + 14 = 62$, i.e., when $n = 12$. Thus, the values of the 12th and 13th arrangements differ by 62.

You might ask students to write and compare equivalent expressions for the values of the nth and (n + 1)st arrangements. For example, the nth arrangement could be viewed as $(n + 4)(2n + 4) = 2n^2 + 12n + 16 = 2(n + 4)(n + 2)$ and the (n + 1)st arrangement could be viewed as $(n + 5)(2n + 6) = [(n + 1) + 4][2(n + 1) + 4] = 2n^2 + 16n + 30 = 2(n + 1)^2 + 12(n + 1) + 16$.

16 The intent here is to prompt intuitions and conjectures about factors of quadratic expressions. It is not expected that students memorize or be shown rote procedures for factoring. Rather, it is hoped that students look for generalizations and formulate questions to investigate further in Lessons 12 and 14 of this course and in *Math Alive! Course IV*.

In *Math Alive! Courses I* and *II*, students were introduced to the meaning of the *factors* of an integer, n, as the values of the edges of rectangles with value n. For example, a rectangle with value 15 could have edges 3×5, 1×15, -3×-5, and -1×-15. Hence, the integral factors of 15 are 3, 5, 1, 15, –3, –5, –1, and –15. (The whole number factors of 15 are 3, 5, 1, and 15.) Similarly, *factoring a quadratic over the integers* can be viewed as finding the values of the edges of Algebra Piece rectangles that represent the expression, assuming the edges include only whole Algebra Pieces. If such a rectangle cannot be formed then the expression does not factor over the integers.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Introduction to Quadratics Lesson **11**

Focus Student Activity 11.7

NAME _____ DATE _____

1 See the chart on page 3. Suppose that each Algebra Piece collection listed in a)-m) on the chart forms the *n*th arrangement of an extended sequence of counting piece arrangements. For each of a)-m):

i) Form the given collection of Algebra Pieces. In Column VII, write the standard quadratic form of the equation that represents the collection. Note: standard quadratic form is $v(n) = an^2 + bn + c$, where a, b, and c are integers and $a \neq 0$.

ii) If possible, form a rectangle without cutting any pieces and without changing the net value of the collection. In Column VIII, record the factored form of the quadratic equation. If no rectangles are possible, write NP (not possible) in Column VIII. Hint: it's okay to add zeros.

iii) In Column IX of the chart, record the values of n for which $v(n) = 0$.

iv) Graph $v(n)$. Be sure to plot enough points to show the shape of the graph. Label the coordinates of any zeroes of the graph.

v) Next to your graph and/or in the last column of the chart, record your observations, conjectures, and questions about factoring and graphing quadratic equations.

Introduction to Quadratics Lesson **11**

Focus Student Activity 11.7 (page 3)

	I	II	III	IV	V	VI	VII	VIII	IX	X
	n^2 mats	$-n^2$ mats	n frames	$-n$ frames	black units	red units	standard quadratic form	factored form/s	n for $v(n) = 0$	observations and conjectures
a)	1	0	8	0	7	0				
b)	1	0	0	8	7	0				
c)	1	0	8	0	0	0				
d)	1	0	5	0	6	0				
e)	0	1	0	5	0	6				
f)	1	0	7	0	6	0				
g)	1	0	0	5	0	6				
h)	1	0	5	0	0	8				
i)	2	0	0	0	0	18				
j)	1	0	0	0	0	25				
k)								$(-3n-3)(n+4)$ or $(3n+3)(-n-4)$		
l)								$(n+3)(n-3)$ or $(-n-3)(-n+3)$		
m)	1					15		$(n-3)(\quad)$ or $(\quad)(\quad)$		

COMMENTS

16 (continued.)
1ii) In several cases, it is necessary to add zero (i.e., to add equal numbers of n-frames and $-n$-frames) to form a rectangle. Two possible pairs of factors for each of collections a)-j) follow. The students may find other possibilities for some of these. Notice that for every pair of factors, $(n + r)$ and $(n + s)$, their opposites, $-(n + r)$ and $-(n + s)$, are also factors.

a) $(n + 1)(n + 7)$ or $(-n - 1)(-n - 7)$
b) $(n - 1)(n - 7)$ or $(-n + 1)(-n + 7)$
c) $(n)(n + 8)$ or $(-n)(-n - 8)$
d) $(n + 3)(n + 2)$ or $(-n - 3)(-n - 2)$
e) $(-n - 3)(n + 2)$ or $(n + 3)(-n - 2)$
f) $(n + 6)(n + 1)$ or $(-n - 6)(-n - 1)$
g) $(n - 6)(n + 1)$ or $(-n + 6)(-n - 1)$
h) does not factor
i) $(2n - 6)(n + 3)$ or $(-2n + 6)(-n - 3)$
j) $(n - 5)(n + 5)$ or $(-n + 5)(-n - 5)$

Notice that collections k)-m) are given in factored form. Students must determine a collection of Algebra Pieces that can be arranged to form a rectangle with edges whose values are the factors.

Collections k)-m) are:
k) $-3n^2 - 15n - 12$
l) $n^2 - 9$
m) $n^2 + 2n - 15 = (n - 3)(n + 5)$

iii) A rectangle has value zero when either of its edges has value zero. Hence, collection a) for example, has value zero when $n + 1 = 0$ or when $n + 7 = 0$, i.e., when $n = -1$ or $n = -7$. Notice the edges $(-n - 1)$ and $(-n - 7)$ also have value zero when $n = -1$ or when $n = -7$, respectively.

iv-v) The graph of any quadratic equation is a parabola. Note that graphing quadratic equations is explored in more depth in Lessons 12 and 14 of this course.

The *zeroes* of a graph are the points where the graph intersects the horizontal axis. Hence, the graph of the sequence whose nth arrangement is formed from collection a), for example, crosses the horizontal axis at (–1,0) and (–7,0). The zeroes of the graph of an equation are also called the *roots* of the equation.

Focus Teacher Activity (cont.)

ACTIONS

17 Have the groups complete Problem 2a) on Focus Student Activity 11.7. Discuss their results and observations. Then have them finish Problem 2 and Problem 3. Discuss.

Introduction to Quadratics Lesson **11**

Focus Student Activity 11.7

NAME _____ DATE _____

1 See the chart on page 3. Suppose that each Algebra Piece collection listed in a)-m) on the chart forms the *n*th arrangement of an extended sequence of counting piece arrangements. For each of a)-m):

i) Form the given collection of Algebra Pieces. In Column VII, write the standard quadratic form of the equation that represents the collection. Note: standard quadratic form is $v(n) = an^2 + bn + c$, where a, b, and c are integers and $a \neq 0$.

ii) If possible, form a rectangle without cutting any pieces and without changing the net value of the collection. In Column VIII, record the factored form of the quadratic equation. If no rectangles are possible, write NP (not possible) in Column VIII. Hint: it's okay to add zeros.

iii) In Column IX of the chart, record the values of n for which $v(n) = 0$.

iv) Graph $v(n)$. Be sure to plot enough points to show the shape of the graph. Label the coordinates of any zeroes of the graph.

v) Next to your graph and/or in the last column of the chart, record your observations, conjectures, and questions about factoring and graphing quadratic equations.

2 For each set of conditions a)-g) below, determine 3 different collections of Algebra Pieces which meet that set of conditions. For each collection, record the standard quadratic form and the factored form of the quadratic equation that represents the collection.

a) Each of these collections forms a square and contains both n-frames and $-n$-frames.

b) Each of these collections forms a square and contains no $-n$-frames.

c) Each of these collections forms a square and contains no n-frames.

(Continued on back.)

18 (Optional) Have the students complete Problem 4 on Focus Student Activity 11.7, either by having individuals select an idea to investigate, or by having the class determine the focus for an investigation. Discuss their results.

COMMENTS

17 To prompt thinking while students work on Problem 3, you might circulate the room, posing questions such as: Are there relationships among the pieces in a collection that could help you predict the edges of a rectangle? What are several collections of pieces that do not form a rectangle and what do they have in common? In general, what can you conjecture about collections that form rectangles and have no n-frames? What conditions do you think are necessary for a collection that forms a rectangle but contains no red or black units? What generalizations can you make about collections that form squares?, etc.

You might have the students post their observations, conjectures, and questions on butcher paper for consideration during upcoming lessons.

Lesson **11** Introduction to Quadratics

Focus Student Activity 11.7 (page 2)

d) These collections each form a rectangle that is twice as long as it is wide.

e) These collections each form a nonsquare rectangle with one edge whose value is n.

f) These collections each form a nonsquare rectangle that contains no n-frames or $-n$-frames.

g) These collections each form a nonsquare rectangle that contains both n-frames and $-n$-frames.

3 Record any new conjectures, questions, generalizations, and "I wonder…" statements that you have about quadratic equations and their factors.

4 Choose one idea that you recorded for Problem 1v) or for Problem 3 above. Investigate that idea further. Report your examples, reasoning, and results.

18 It is helpful to provide the students with a few guidelines for such an investigation. For example, you might let them know the amount of time that you expect they devote to the investigation. Letting them know the criteria that you will use to assess the investigation is also helpful (see the assessment chapter of *Starting Points*).

TEACHER NOTES:

Follow-up Student Activity 11.8

NAME _____ **DATE** _____

Do all work for this assignment on separate paper. Attach each problem to your work for the problem.

1 For each of a)-i) below, create an extended sequence of counting piece arrangements whose graph meets the given conditions. Then on coordinate grid paper, do the following:

i) sketch the –3rd to 3rd and *n*th arrangements of the sequence;
ii) write a formula for the value of the *n*th arrangement;
iii) graph the sequence (plot enough points that the shape of the graph is evident).

a) The graph of this sequence is linear (i.e., the points follow the path of a line) and it rises from left to right.

b) The graph of this sequence is linear, falls from left to right, and contains the point (0,2).

c) The points (1,5) and (2,8) lie on the graph of this sequence.

d) The graph of this sequence is U-shaped and the U "opens up."

e) The graph of this sequence is U-shaped and the U "opens down."

f) The graph of this sequence is a horizontal line.

g) The value of the 0th arrangement of this sequence is –3; the graph of the sequence is U-shaped, and (0,–3) is the turning point of the graph.

h) This sequence meets all of the criteria of g), but the graph of this sequence is not identical to the graph of the sequence you created for g).

i) The formula for values of negative numbered arrangements of this sequence differs from the formula for nonnegative numbered arrangements.

(Continued on back.)

Follow-up Student Activity (cont.)

2 Create 2 extended sequences whose graphs are both linear and have in common only the point (7,9). On coordinate grid paper, do the following:

a) sketch the –3rd through 3rd and *n*th arrangements of both sequences;

b) write formulas for the values of the *n*th arrangements of the sequences;

c) graph the sequences (on the same coordinate axes);

d) record pictures or algebra symbols that show your step-by-step Algebra Piece procedures for determining the value of *n* for which the *n*th arrangements of the sequences are equal.

3 Repeat Problem 2 a)-d) for 2 sequences whose graphs are parabolic (U-shaped); the turning point of each graph is (0,5) and they have no other points in common.

4 Show how to use the method "completing the square" to determine the values of *n* for which each of the following quadratic equations is true:

a) $n^2 + 2n = 35$

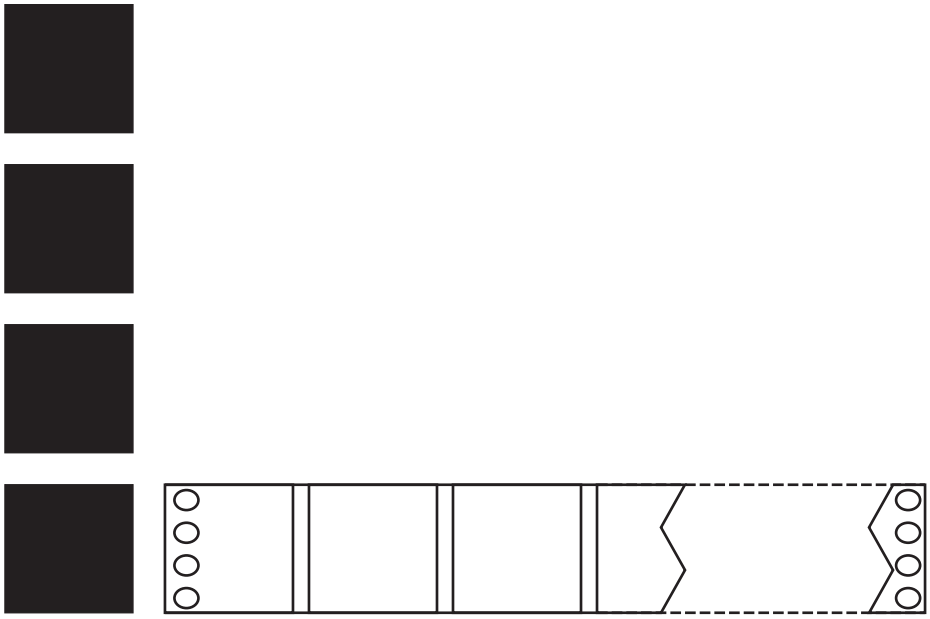
b) $(n - 8)(n + 2) = 0$

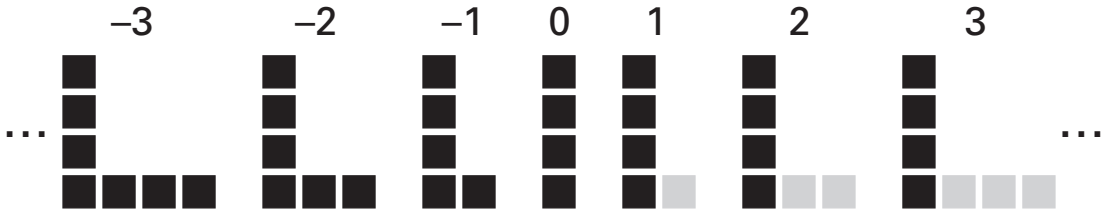
c) Challenge. $2n^2 - 10n = 48$

5 Use algebra symbols to record each step of your Algebra Piece methods in Problem 4b).

6 Using Algebra Pieces and using graphs are 2 methods of determining when the *n*th arrangements of 2 extended sequences of arrangements have the same value. Discuss your ideas about the advantages and disadvantages of each method.

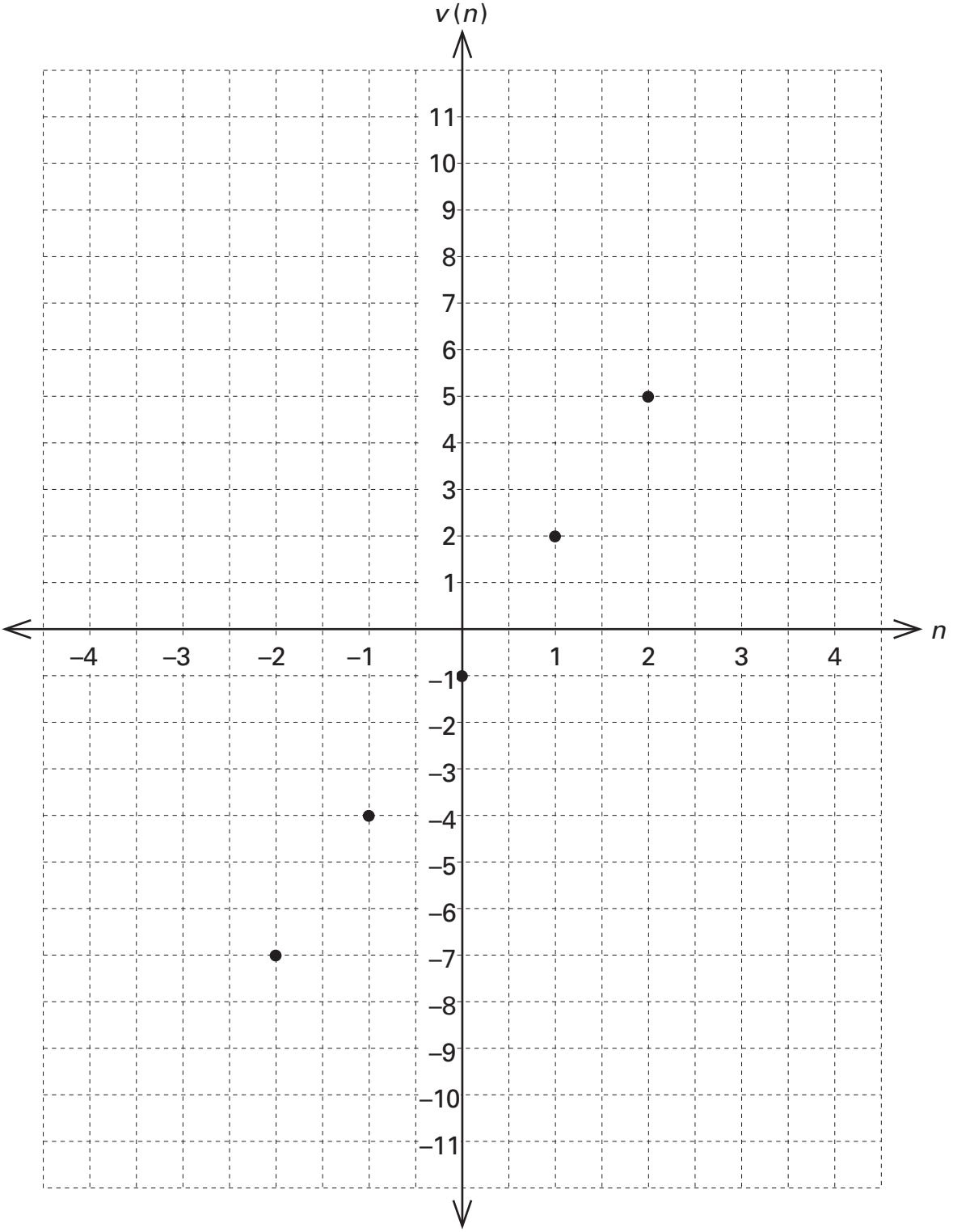
Connector Master A





Connector Student Activity 11.1

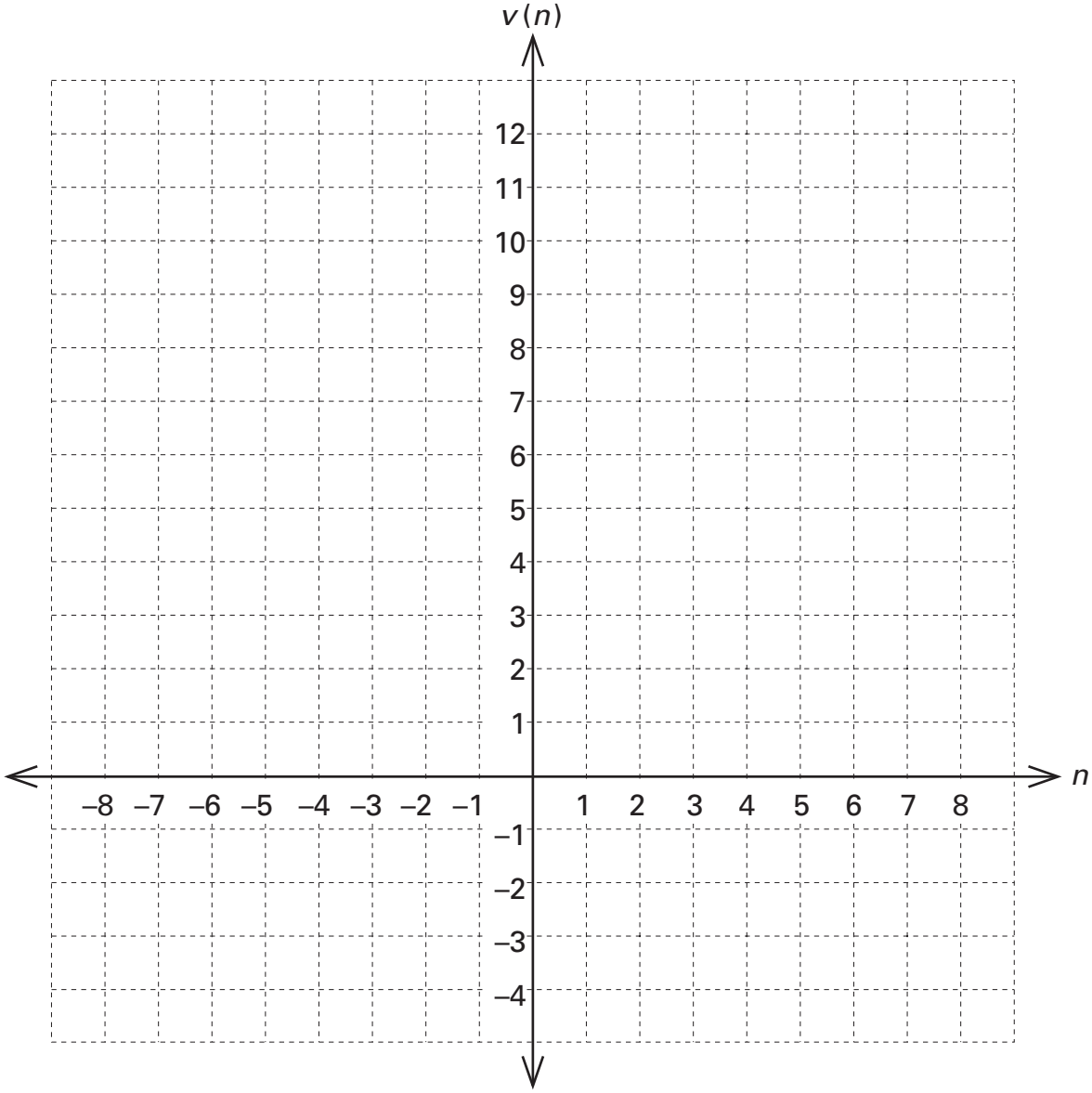
NAME _____ **DATE** _____



$v(n) =$ _____

Connector Student Activity 11.2

NAME _____ **DATE** _____

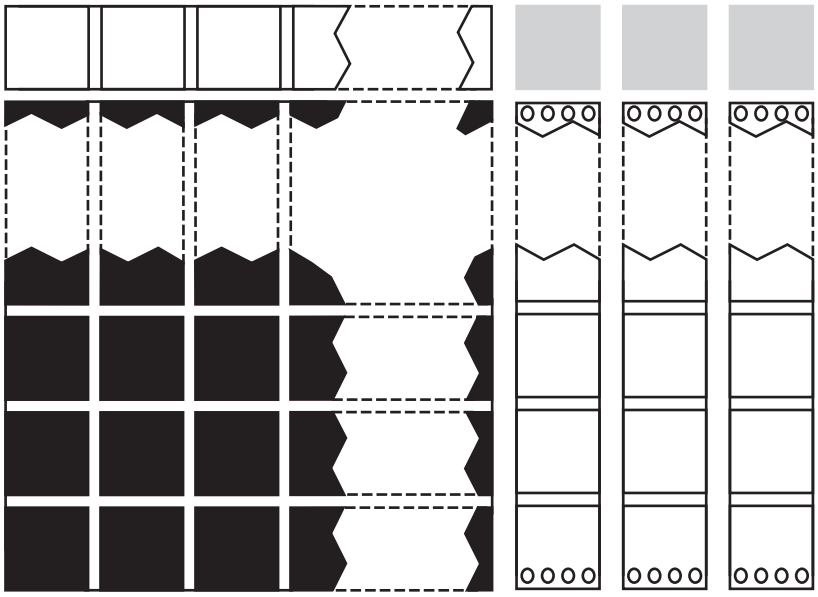


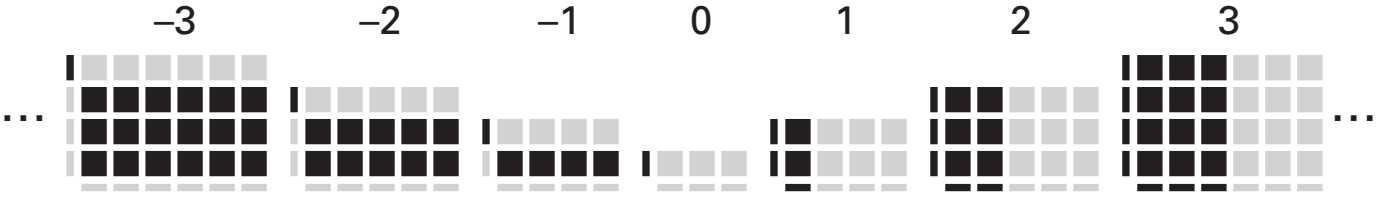
$v(n)$ = _____

Observations about the graph:

Focus Master A

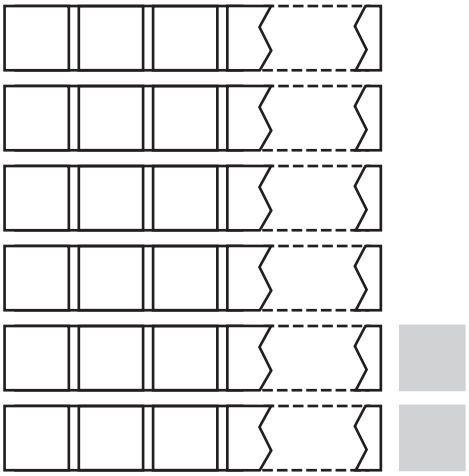
Focus Master B



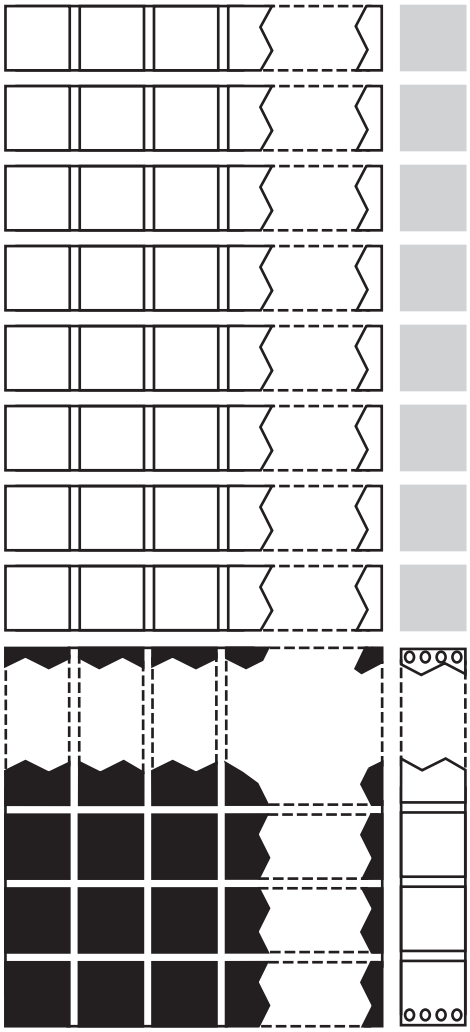


Focus Master C

I



II



Focus Master D

1. Formulas for the values of the *n*th arrangements of 3 pairs of extended sequences are given below. For each pair of sequences, please do the following:

a) Sketch the –3rd through 3rd arrangements and the *n*th arrangement of both sequences.

b) Make a table showing $v_1(n)$ and $v_2(n)$ for n from –3 to 3. Then graph $v_1(n)$ and $v_2(n)$ on the same coordinate axes.

c) Make mathematical observations about similarities, differences, and relationships you notice in the graphs of $v_1(n)$ and $v_2(n)$.

d) Use pictures or algebraic symbols to show your step-by-step methods of using Algebra Pieces to determine when $v_1(n) = v_2(n)$.

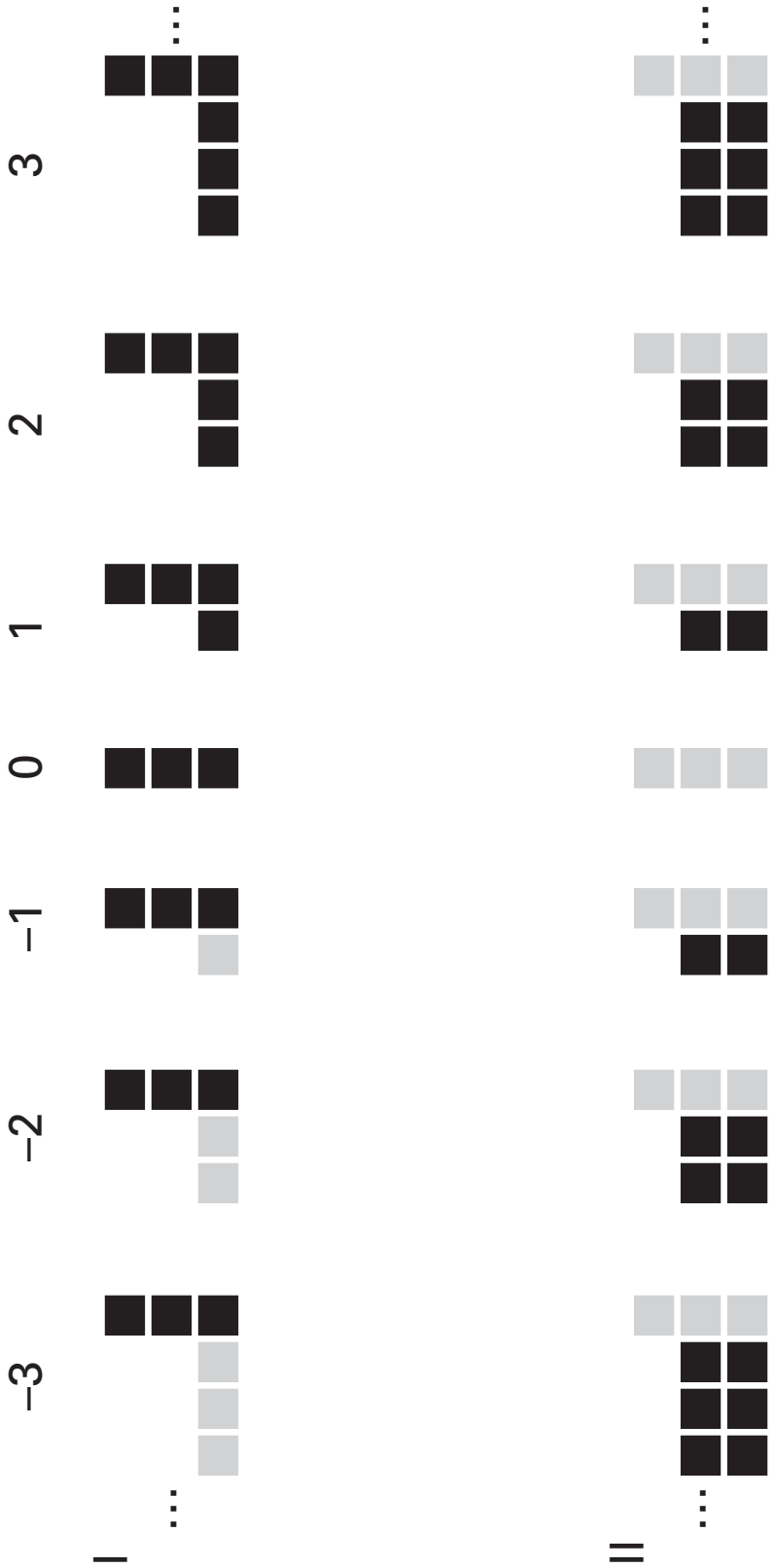
Pair 1
$v_1(n) = -3n + 2$ $\quad$ $v_2(n) = 4n - 12$

Pair 2
$v_1(n) = -7$ $\quad$ $v_2(n) = -n^2 - 2n + 8$

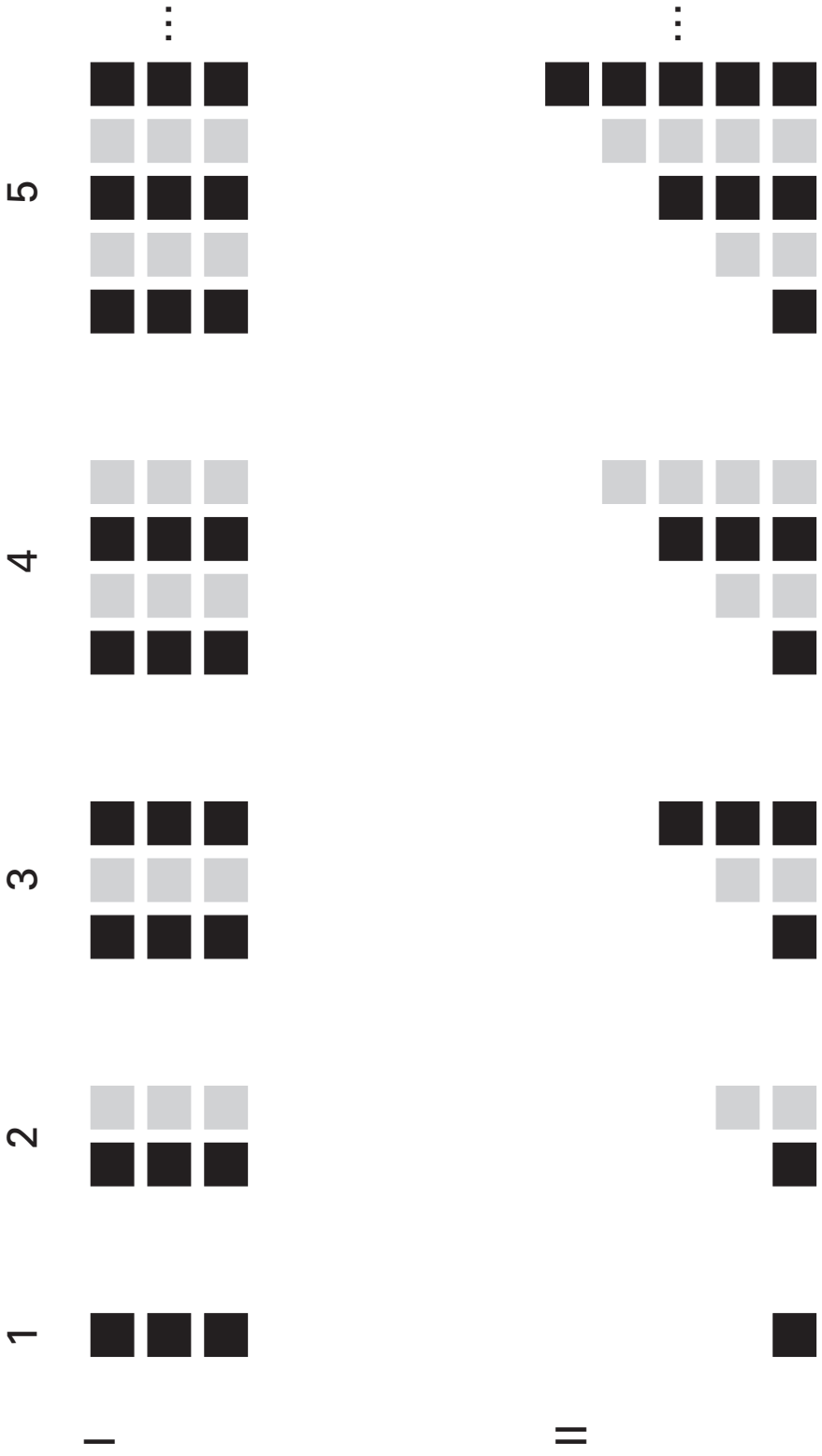
Pair 3
$v_1(n) = n^2 + 2$ $\quad$ $v_2(n) = -n^2 + 4$

2. Review your completed graphs for Problem 1 above and then record your general observations and conjectures about graphing equations.

Focus Master E

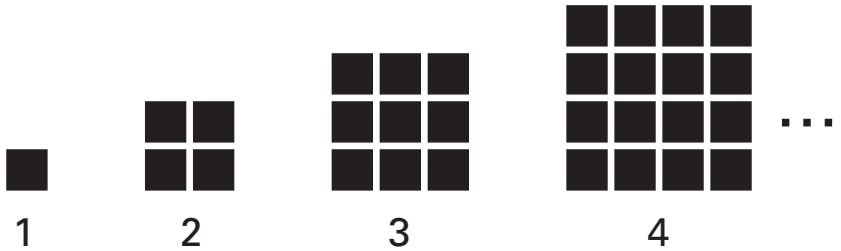


Focus Master F



Focus Master G

1. Situations a) and b) below refer to the following non-extended sequence of arrangements. For each situation, make an Algebra Piece model of the situation and then write several mathematical observations based on your model.



a) Suppose the values of 2 consecutive arrangements differ by 79 units.

b) Suppose the difference between 2 arrangement numbers is 3 and the difference between the values of the 2 arrangements is 111 units.

2. Suppose Mystery Sequence X is a nonextended sequence of arrangements of black counting pieces. To form the *n*th arrangement of Sequence X: form a rectangle that is *n* units wide and twice as long as it is wide; surround the rectangle with a 2-unit wide border (note: the outer perimeter of the border is a rectangle). Use Algebra Piece models as a basis for making mathematical observations about each of the following:

a) The 1st, 2nd, and *n*th arrangements of Sequence X.

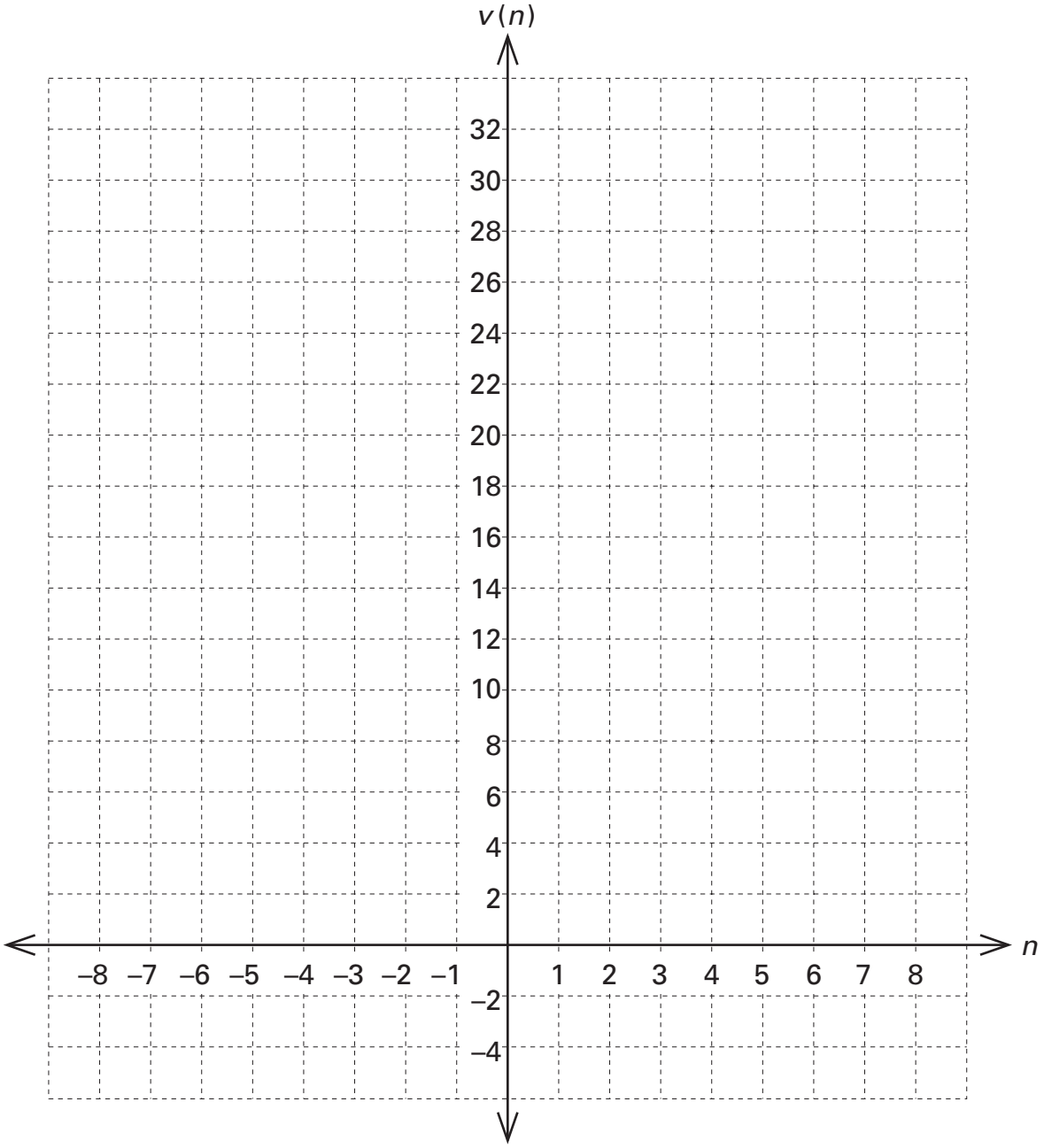
b) An arrangement whose border contains 160 units.

c) An arrangement that contains a total of 126 units.

d) Challenge. Two consecutive arrangements whose values differ by 62 units.

Focus Student Activity 11.3

NAME _____ **DATE** _____

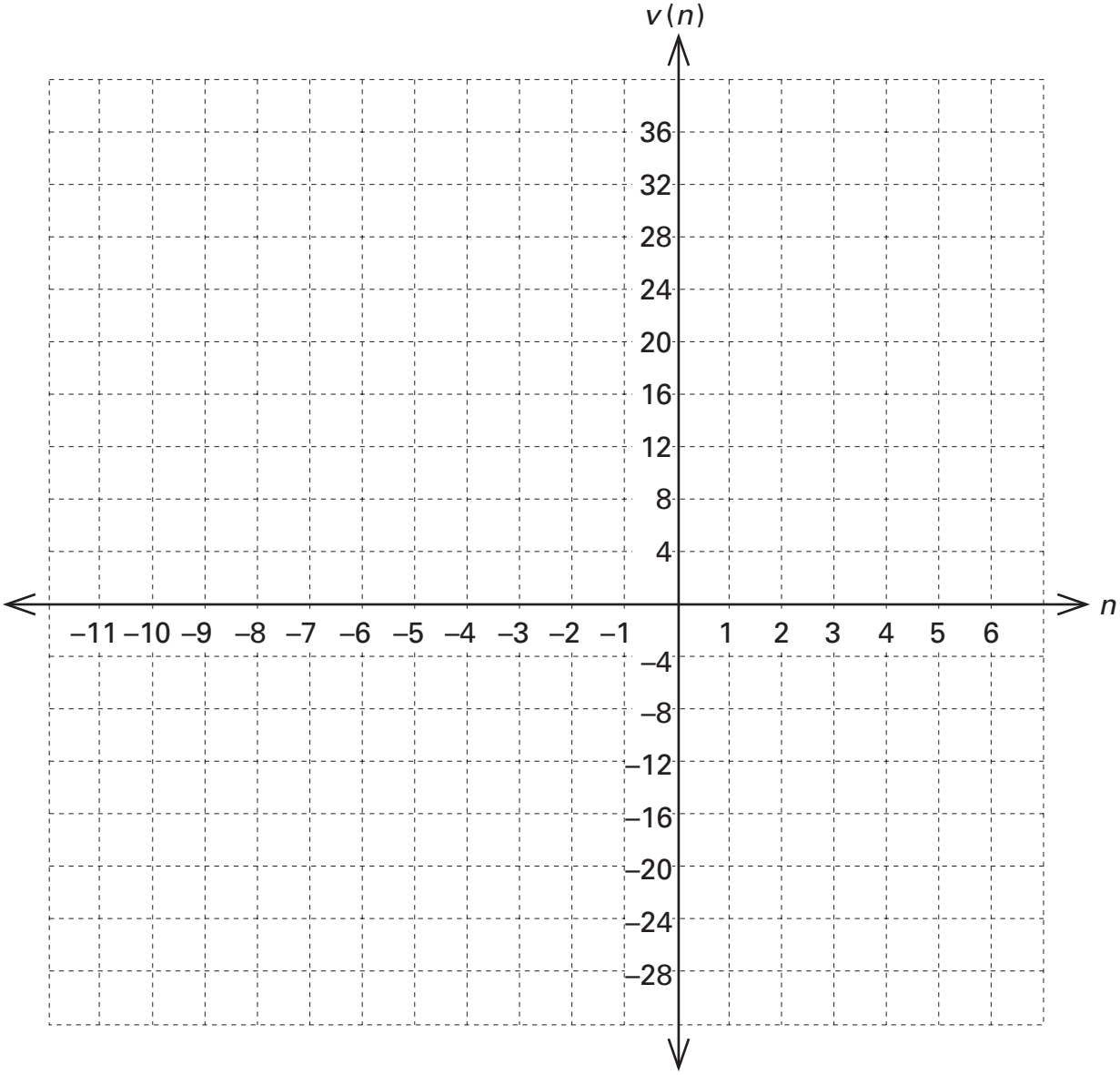


$v(n)$ = _____

Observations about the graph:

Focus Student Activity 11.4

NAME _____ **DATE** _____

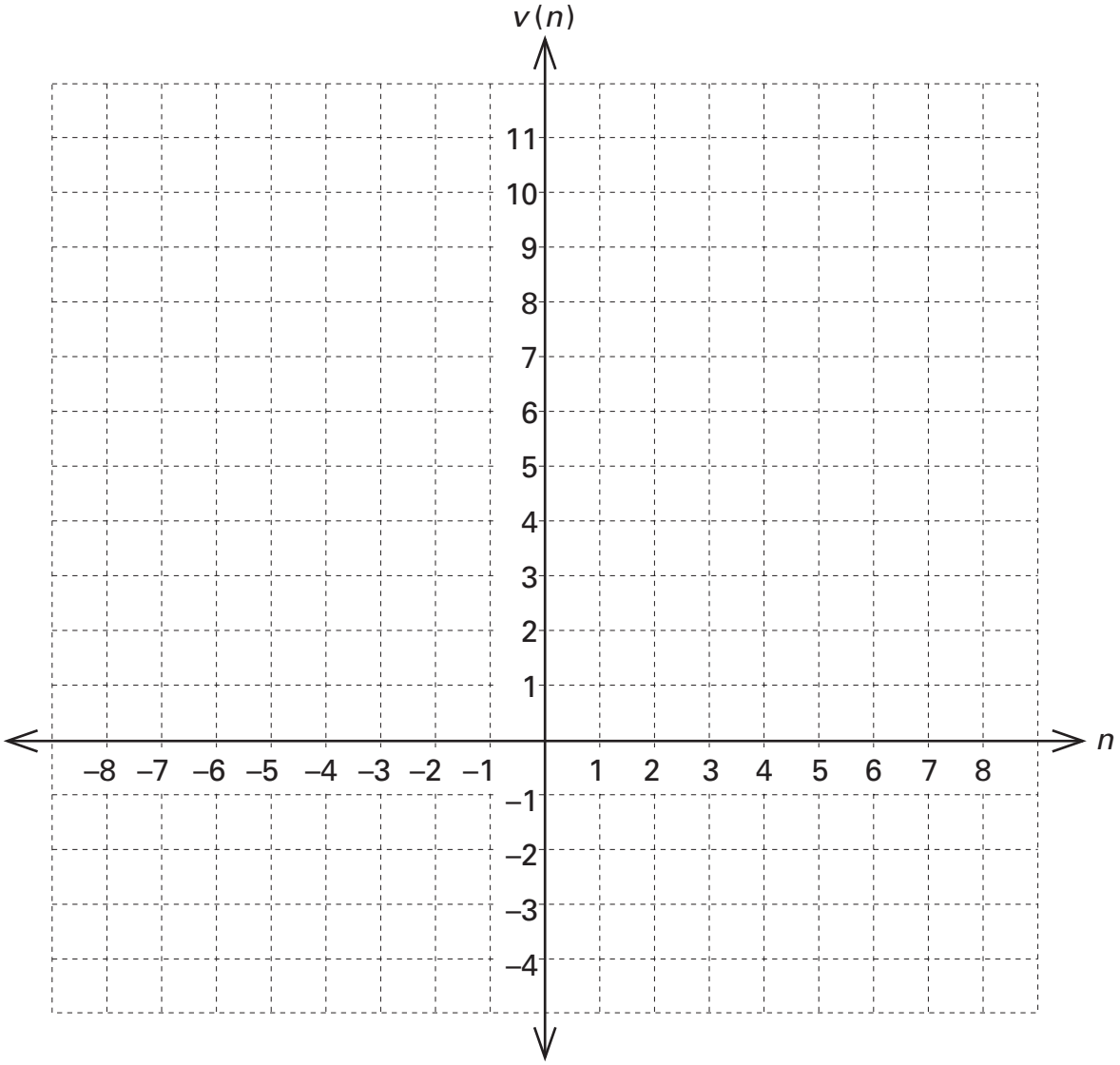


x $v_1(n)$ = _____ o $v_2(n)$ = _____

Observations about the graph:

Focus Student Activity 11.5

NAME _____ **DATE** _____

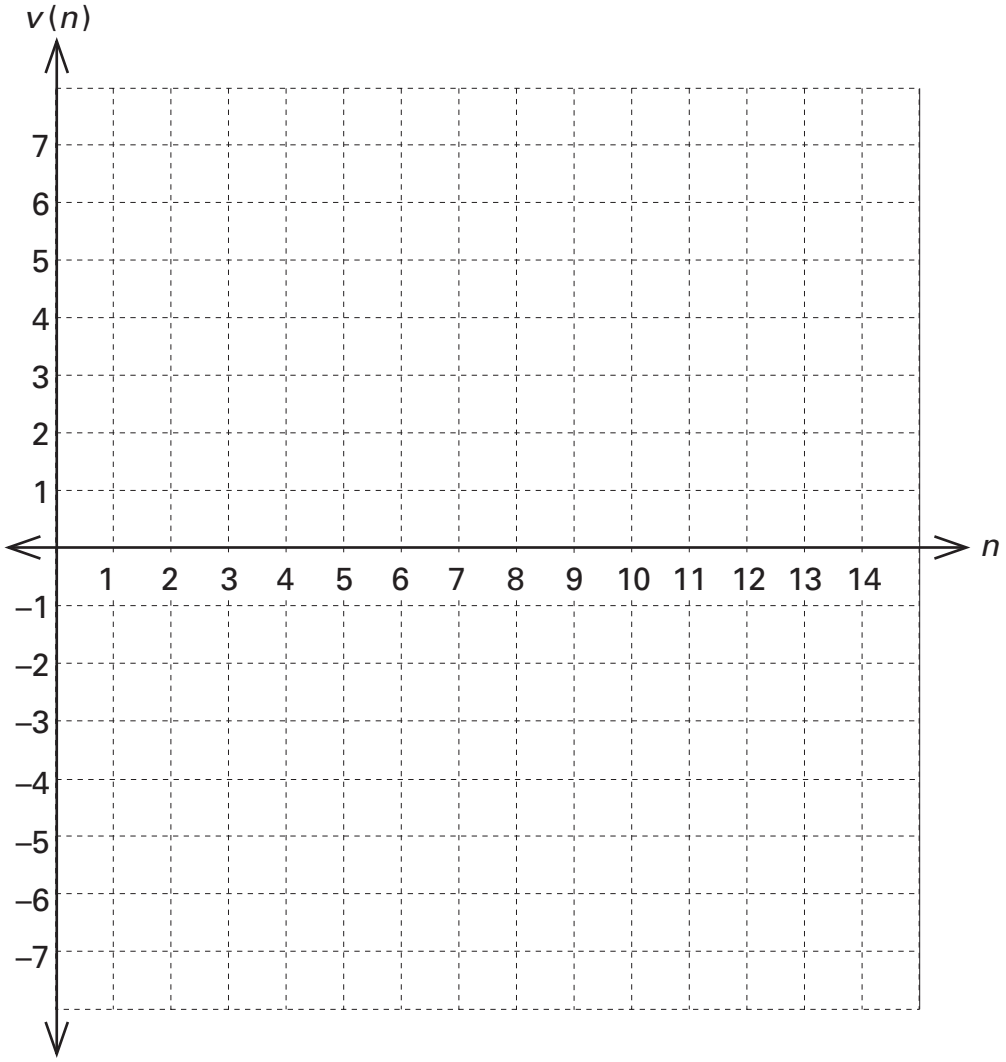


x $v_1(n)$ = _____ o $v_2(n)$ = _____

Observations about the graph:

Focus Student Activity 11.6

NAME _____ **DATE** _____



x $v_1(n)$ = _____ o $v_2(n)$ = _____

Observations about the graph:

Focus Student Activity 11.7

NAME _____ **DATE** _____

1 See the chart on page 3. Suppose that each Algebra Piece collection listed in a)-m) on the chart forms the *n*th arrangement of an extended sequence of counting piece arrangements. For each of a)-m):

i) Form the given collection of Algebra Pieces. In Column VII, write the standard quadratic form of the equation that represents the collection. Note: standard quadratic form is $v(n) = an^2 + bn + c$, where a, b, and c are integers and $a \neq 0$.

ii) If possible, form a rectangle without cutting any pieces and without changing the net value of the collection. In Column VIII, record the factored form of the quadratic equation. If no rectangles are possible, write NP (not possible) in Column VIII. Hint: it's okay to add zeros.

iii) In Column IX of the chart, record the values of n for which $v(n) = 0$.

iv) Graph $v(n)$. Be sure to plot enough points to show the shape of the graph. Label the coordinates of any zeroes of the graph.

v) Next to your graph and/or in the last column of the chart, record your observations, conjectures, and questions about factoring and graphing quadratic equations.

2 For each set of conditions a)-g) below, determine 3 different collections of Algebra Pieces which meet that set of conditions. For each collection, record the standard quadratic form and the factored form of the quadratic equation that represents the collection.

a) Each of these collections forms a square and contains both n-frames and $-n$-frames.

b) Each of these collections forms a square and contains no $-n$-frames.

c) Each of these collections forms a square and contains no n-frames.

(Continued on back.)

Focus Student Activity 11.7 (page 2)

d) These collections each form a rectangle that is twice as long as it is wide.

e) These collections each form a nonsquare rectangle with one edge whose value is n.

f) These collections each form a nonsquare rectangle that contains no n-frames or $-n$-frames.

g) These collections each form a nonsquare rectangle that contains both n-frames and $-n$-frames.

3 Record any new conjectures, questions, generalizations, and "I wonder…" statements that you have about quadratic equations and their factors.

4 Choose one idea that you recorded for Problem 1v) or for Problem 3 above. Investigate that idea further. Report your examples, reasoning, and results.

Focus Student Activity 11.7 (page 3)

	I	II	III	IV	V	VI	VII	VIII	IX	X
	n^2 mats	$-n^2$ mats	n frames	$-n$ frames	black units	red units	standard quadratic form	factored form/s	n for $v(n) = 0$	observations and conjectures
a)	1	0	8	0	7	0				
b)	1	0	0	8	7	0				
c)	1	0	8	0	0	0				
d)	1	0	5	0	6	0				
e)	0	1	0	5	0	6				
f)	1	0	7	0	6	0				
g)	1	0	0	5	0	6				
h)	1	0	5	0	0	8				
i)	2	0	0	0	0	18				
j)	1	0	0	0	0	25				
k)								$(-3n-3)(n+4)$ or $(3n+3)(-n-4)$		
l)								$(n+3)(n-3)$ or $(-n-3)(-n+3)$		
m)	1					15		$(n-3)(\quad)$ or $(\quad)(\quad)$		

Follow-up Student Activity 11.8

NAME _____ **DATE** _____

Do all work for this assignment on separate paper. Attach each problem to your work for the problem.

1 For each of a)-i) below, create an extended sequence of counting piece arrangements whose graph meets the given conditions. Then on coordinate grid paper, do the following:

i) sketch the –3rd to 3rd and *n*th arrangements of the sequence;
ii) write a formula for the value of the *n*th arrangement;
iii) graph the sequence (plot enough points that the shape of the graph is evident).

a) The graph of this sequence is linear (i.e., the points follow the path of a line) and it rises from left to right.

b) The graph of this sequence is linear, falls from left to right, and contains the point (0,2).

c) The points (1,5) and (2,8) lie on the graph of this sequence.

d) The graph of this sequence is U-shaped and the U "opens up."

e) The graph of this sequence is U-shaped and the U "opens down."

f) The graph of this sequence is a horizontal line.

g) The value of the 0th arrangement of this sequence is –3; the graph of the sequence is U-shaped, and (0,–3) is the turning point of the graph.

h) This sequence meets all of the criteria of g), but the graph of this sequence is not identical to the graph of the sequence you created for g).

i) The formula for values of negative numbered arrangements of this sequence differs from the formula for nonnegative numbered arrangements.

(Continued on back.)

Follow-up Student Activity (cont.)

2 Create 2 extended sequences whose graphs are both linear and have in common only the point (7,9). On coordinate grid paper, do the following:

a) sketch the –3rd through 3rd and *n*th arrangements of both sequences;

b) write formulas for the values of the *n*th arrangements of the sequences;

c) graph the sequences (on the same coordinate axes);

d) record pictures or algebra symbols that show your step-by-step Algebra Piece procedures for determining the value of *n* for which the *n*th arrangements of the sequences are equal.

3 Repeat Problem 2 a)-d) for 2 sequences whose graphs are parabolic (U-shaped); the turning point of each graph is (0,5) and they have no other points in common.

4 Show how to use the method "completing the square" to determine the values of *n* for which each of the following quadratic equations is true:

a) $n^2 + 2n = 35$

b) $(n - 8)(n + 2) = 0$

c) Challenge. $2n^2 - 10n = 48$

5 Use algebra symbols to record each step of your Algebra Piece methods in Problem 4b).

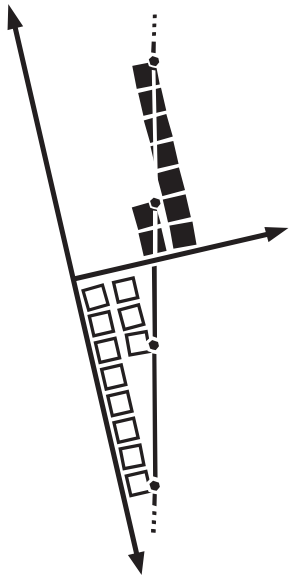
6 Using Algebra Pieces and using graphs are 2 methods of determining when the *n*th arrangements of 2 extended sequences of arrangements have the same value. Discuss your ideas about the advantages and disadvantages of each method.

Continuous Grap s

Lesson 12

THE BIG IDEA

Sequences of counting piece arrangements provide a concrete and meaningful context for developing intuitions about the meaning of continuity. Examining relationships between continua of arrangements and their graphs promotes conceptual understanding regarding the real number system, variables, functions, and graphing.



CONNECTOR

OVERVIEW

Given specific arrangement numbers and the corresponding values of the arrangements, students sketch several arrangements in a sequence, determine the value of the *n*th arrangement, and sketch a graph of the sequence.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Student Activity 12.1, 1 copy per pair of students and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.

FOCUS

OVERVIEW

Extended sequences of arrangements are augmented so their graphs become continuous. Slopes, intercepts, and values of arrangements are used to plot continuous graphs of linear and quadratic equations. Graphs, Algebra Pieces, and algebraic symbols are used to represent inequality relationships and solve systems of equations.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Algebra Pieces for each student.
- ✔ Focus Student Activities 12.2-12.4, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Masters A, C, and E, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Master D, 1 copy per student and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 11 sheets per student and 1 transparency.
- ✔ Red and black pencils or marking pens (optional).
- ✔ Algebra Pieces for the overhead.

FOLLOW-UP

OVERVIEW

Students graph linear and quadratic equations and determine intercepts and coordinates of points for intersecting graphs. They graph situations and answer questions that can be determined from the graphs.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 12.5, 1 copy per student.
- ✔ Coordinate grid paper (see *Blackline Masters*), 6 sheets per student.

LESSON IDEAS

GRAPHING CALCULATORS
Note that at the end of the Focus activity students begin getting acquainted with graphing calculators.

Emphasis throughout this lesson is on the concepts of graphing as preliminaries to using the graphing calculator. If students already have calculators, some may wish to use them during this lesson. One way to deal with this is to allow calculator use but, to decrease distractions from the conceptual discussion, periodically ask the students to set aside the calculators. And, always ask the students to think critically about the relative advantages and disadvantages of both calculator and "by-hand" methods for graphing and solving equations.

FOLLOW-UP
It is helpful to have students complete the Follow-up prior to assigning Focus Student Activity 12.4 in Action 21.

QUOTE
When students make graphs, data tables, expressions, equations, or verbal descriptions to represent a single relationship, they discover that different representations yield different interpretations of a situation. In informal ways, students develop an understanding that functions are composed of variables that have a dynamic relationship. Changes in one variable result in change in another. The identification of the special characteristics of a relationship, such as minimum or maximum values or points at which the value of one of the variables is 0 (x- and y-intercepts), lays the foundation for a more formal study of functions in grades 9-12.
NCTM Standards

SELECTED ANSWERS

1. a)

x	$v(x)$
$6\frac{1}{2}$	17
$-3\frac{1}{2}$	−13
$67\frac{1}{2}$	200
31	90.5
x	$3x - \frac{5}{2}$

b)

x	$v(x)$
55	−108.5
$115\frac{3}{4}$	−230
$61\frac{1}{4}$	−121
$-88\frac{1}{4}$	178
x	$\frac{3}{2} - 2x$

2. a) $v(x) = 3x - 2$
   b) $v(x) = 4x - 1$
   c) $v(x) = -x + 3$
   d) $v(x) = -7$

3. a) x-intercepts, 3 and −4, y-intercept, −12; turning point at $(-\frac{1}{2}, -\frac{49}{4})$.

   b) x-intercepts, 2 and −5, y-intercept, 10; turning point at $(-\frac{3}{2}, \frac{49}{4})$.

4. a) $v_1(x) = v_2(x)$ for $x = 3\frac{5}{7}$
   b) $v_1(x) = v_2(x)$ for $x = 3$ and $x = -5$
   c) $v_1(x) = v_2(x)$ for $x = 3$ and $x = -2$

5. a) The graph of v_1 is above the graph of v_2 for $x < 3\frac{5}{7}$ and below for $x > 3\frac{5}{7}$.

   b) The graph of v_1 is above the graph of v_2 for $x < -5$ or $x > 3$ and below for $-5 < x < 3$.

   c) The graph of v_1 is above the graph of v_2 for $x < -2$ or $x > 3$ and below for $-2 < x < 3$.

6. a) x-intercepts are −7 and 1, y-intercept is −7
   b) $x = 2\frac{1}{2}$
   c) $-8 < x < 2$
   d) $x < -8$ and $x > 2$
   e) (−3,−25)

Connector Teac er Activity

OVERVIEW & PURPOSE

Given specific arrangement numbers and the corresponding values of the arrangements, students sketch several arrangements in a sequence, determine the value of the nth arrangement, and sketch a graph of the sequence.

MATERIALS

- ✔ Connector Student Activity 12.1, 1 copy per pair of students and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 2 sheets per student and 1 transparency.

ACTIONS

1 Arrange the students in pairs and give each pair a copy of Connector Student Activity 12.1 and 2 sheets of coordinate grid paper. Ask the pairs to carry out the instructions. Discuss their results.

Continuous Graphs Lesson **12**

Connector Student Activity 12.1

NAME _____ DATE _____

The 1st column of each table below lists, in order, the arrangement numbers of a sequence of counting piece arrangements, and the 2nd column lists the values of the arrangements. For each sequence:

a) Sketch 4 consecutive arrangements. Label the number and value of each arrangement.

b) Sketch the *n*th arrangement of the sequence.

c) On a coordinate grid, plot ordered pairs that represent several arrangements.

d) Fill in the blanks in the table.

e) Write a concise mathematical description of the set of all arrangement numbers in the sequence. Then describe mathematically the set of all numbers that are the values of the arrangements.

1

n	$v(n)$
1	4
2	9
3	14
4	19
5	24
⋮	⋮
13	__
14	__
⋮	⋮
__	154
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

2

n	$v(n)$
⋮	⋮
–3	18
–2	8
–1	2
0	0
1	2
2	8
⋮	⋮
__	450
⋮	⋮
43	__
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

3

n	$v(n)$
–1	1
–2	2
–3	3
–4	4
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

4

n	$v(n)$
1	3
2	–4
3	9
4	–8
5	15
6	–12
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

COMMENTS

1 It may be helpful to remind students that each point shown on their graphs represents one arrangement. Graphs should be sets of discrete points, since connecting the points on the graph would imply there are arrangements for every point on the line or curve. For these sequences, the only arrangement numbers are from the set of integers. In the Focus activity, sequences are augmented to include nonintegral arrangement numbers.

Note that Tables 1 and 4 do not have an ellipsis that precedes $n = 1$. This implies there are no arrangement numbers less than 1. Hence, Tables 1 and 4 do not represent an extended sequence and their graphs should include points associated with only the positive integers on the n-axis.

Table 3 represents a sequence whose arrangement numbers are the negative integers (note that zero is considered neither negative nor positive) and so the graph should include points associated with only the negative integers on the n-axis.

Notice that, for each sequence represented by the tables on Connector Student Activity 12.1, for every arrangement number n, there is exactly one $v(n)$. Hence, each of the tables represents a set of ordered pairs $[n,v(n)]$ that is a function. The set of all arrangement numbers for each sequence is the *domain* of the function. The set of values of all arrangements is the *range* of the function.

Students' expressions for $v(n)$ may depend on how they "see" the nth arrangements of each sequence. However, all expressions should be equivalent to those given below.

1d) $v(n) = 5n - 1$

1e) The set of arrangement numbers is the set of positive integers; the set of values of the arrangements is the set of positive integers that are 1 less than 5 times the value of a positive integer.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)

2d) $v(n) = 2n^2$

2e) The arrangement numbers are the set of integers. The values of the arrangements are the set of positive integers that are 2 times the square of an integer.

3d) $v(n) = -n$

3e) The arrangement numbers are the set of negative integers. The values of the arrangements are the set of positive integers.

4d) $v(n) = -2n$, when n is even, and $v(n) = 3n$, when n is odd.

4e) The arrangement numbers are the set of positive integers. The values of the arrangements are every other positive integer that is a multiple of 3, together with the set of negative integers that are multiples of 4.

2 Announce that students will use graphing calculators at the end of this lesson and they will be considered everyday classroom tools thereafter. If you haven't discussed your preferences regarding brand and model (we use the TI-83) and recommendations for best places to purchase a calculator, do so now.

2 Announcing this now allows students time to purchase calculators prior to needing them for the last action of the Focus. Conceptual work in this lesson is intended to provide students with insights about graphing that will be useful later when they use a graphing calculator.

Focus Teac er Activity

OVERVIEW & PURPOSE

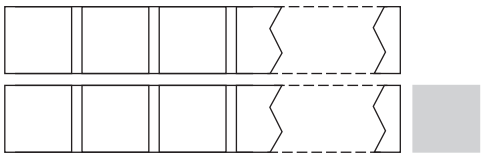
Extended sequences of arrangements are augmented so their graphs become continuous. Slopes, intercepts, and values of arrangements are used to plot continuous graphs of linear and quadratic equations. Graphs, Algebra Pieces, and algebraic symbols are used to represent inequality relationships and solve systems of equations.

MATERIALS

- ✔ Algebra Pieces for each student.
- ✔ Focus Student Activities 12.2-12.4, 1 copy of each per student and 1 transparency of each.
- ✔ Focus Masters A, C, and E, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master B, 1 transparency.
- ✔ Focus Master D, 1 copy per student and 1 transparency.
- ✔ Coordinate grid paper (see *Blackline Masters*), 11 sheets per student and 1 transparency.
- ✔ Red and black pencils or marking pens (optional).
- ✔ Algebra Pieces for the overhead.

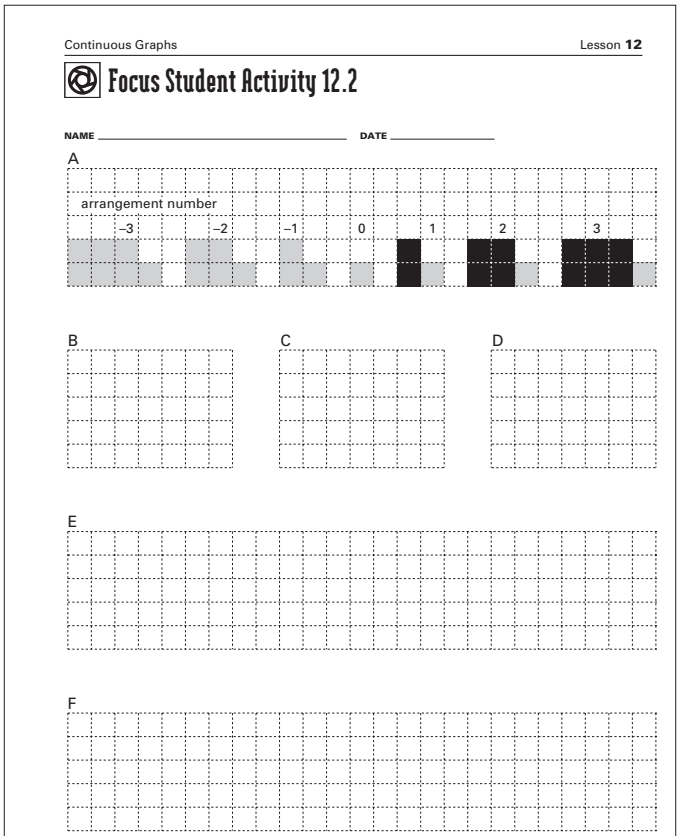
ACTIONS

1 Arrange the students in groups and give red and black pencils or marking pens, Algebra Pieces, and 1 copy of Focus Student Activity 12.2 to each student. Show them the arrangement of 2 *n*-frames and 1 red counting piece shown below. Tell them it is the *n*th arrangement of an extended sequence of counting piece arrangements. Ask them to sketch (in Section A of Focus Student Activity 12.2, see completed copy at the right) the –3rd through 3rd arrangements of the sequence, representing counting pieces by grid squares.



COMMENTS

1 Some students may wish to use red and black counting pieces to form the arrangements; if so, ask them to also draw sketches. If red and black pencils or marking pens are unavailable, students can devise ways of indicating red and black. In the copy of Focus Student Activity 12.2 shown below, part A is completed with the lighter-shaded squares representing red counting pieces.



Continuous Graphs Lesson **12**

Focus Student Activity 12.2

NAME _____ DATE _____

A

arrangement number

–3 –2 –1 0 1 2 3

B C D

E

F

Focus Teacher Activity (cont.)

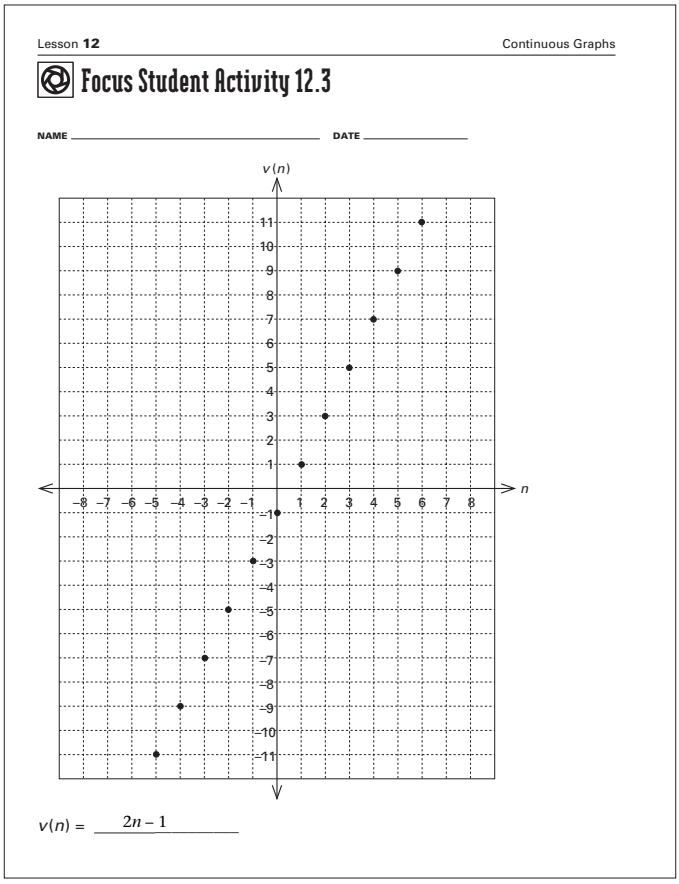
ACTIONS

2 Give each student a copy of Focus Student Activity 12.3. For the extended sequence introduced in Action 1, ask the students to record a formula for $v(n)$ in the space provided at the bottom of the sheet and then construct the graph of $v(n)$. Discuss and invite volunteers to graph the sequence on a transparency of Focus Student Activity 12.3 (see completed copy at the right).

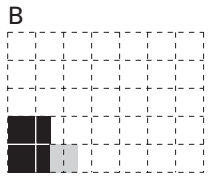
3 Mention to the students that there is no point on the graph for $n = 1\frac{1}{2}$ since there is no $1\frac{1}{2}$th arrangement. Ask the students to imagine that the sequence has been augmented to contain such an arrangement. Ask them to sketch, in Section B of Focus Student Activity 12.2, what they think the $1\frac{1}{2}$th arrangement looks like. Have them compute the value of that arrangement and add the corresponding point to their graph. Discuss their ideas and reasoning. Repeat for $n = 3\frac{1}{2}$ and $n = -2\frac{3}{4}$ in Sections C and D of Focus Student Activity 12.2.

COMMENTS

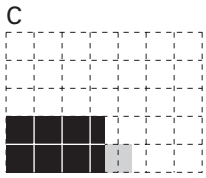
2 The completed graph is shown below on a copy of Focus Student Activity 12.3. A formula is $v(n) = 2n - 1$. Note that the set of ordered pairs $(n, 2n - 1)$ is a function whose domain is the set of all integers and the range is the set of all odd integers and their opposites.



3 Below on the left is a sketch of a $1\frac{1}{2}$th arrangement, based on the pattern of the arrangements in the original sequence. Its net value is 2. Thus, $(1\frac{1}{2},2)$ is the point on the graph corresponding to this arrangement. Similarly, $(3\frac{1}{2},6)$ is the ordered pair associated with the $3\frac{1}{2}$th arrangement.



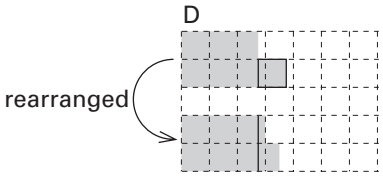
$v(1\frac{1}{2}) = 2(1\frac{1}{2}) - 1 = 2$



$v(3\frac{1}{2}) = 2(3\frac{1}{2}) - 1 = 6$

Focus Teacher Activity (cont.)

ACTIONS



$v(-2\frac{3}{4}) = 2(-2\frac{3}{4}) - 1 = -6\frac{1}{2}$

4 Have the students each do the following on Focus Student Activities 12.2 and 12.3:

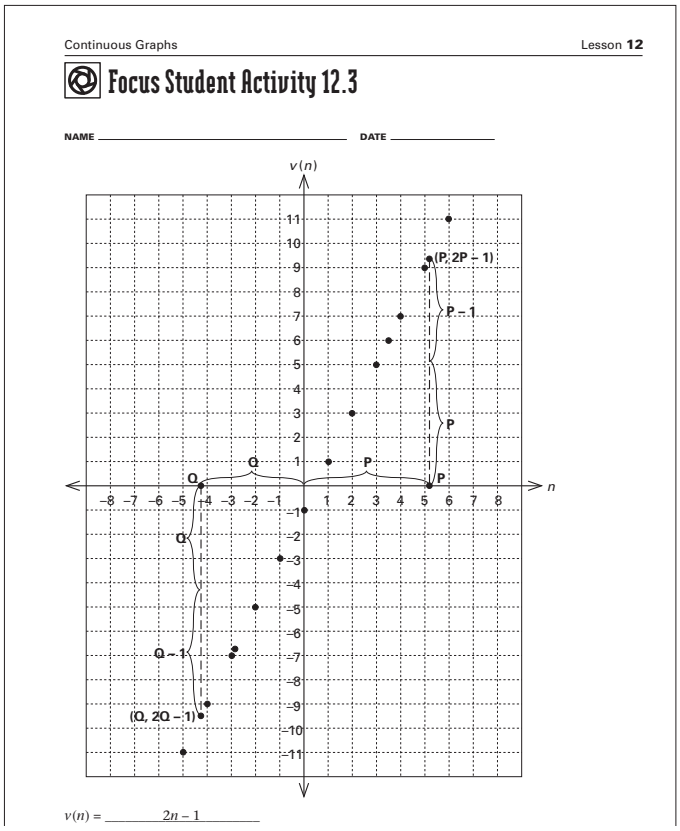
a) on 12.3 choose some noninteger number on the positive part of the *n*-axis and label it *P* (students' numbers will vary),

b) on 12.3 choose a noninteger number on the negative part of the *n*-axis and label it *Q*,

c) in Sections E and F of 12.2 sketch approximations of the *P*th and *Q*th arrangements and determine the values of these arrangements,

d) on 12.3 add the points associated with the *P*th and *Q*th arrangements to the graph,

e) discuss the students' methods of determining the location of the points for d) above.


Continuous Graphs Lesson 12

Focus Student Activity 12.3

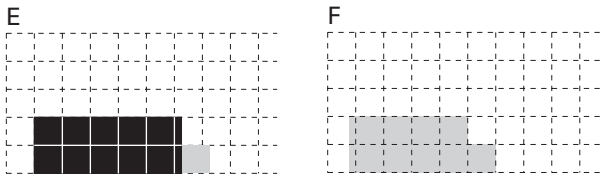
NAME _____ DATE _____

$v(n) =$ 2n – 1

COMMENTS

The net value of the $-2\frac{3}{4}$th arrangement is $-6\frac{1}{2}$, as illustrated at the left. Its corresponding point on the graph is $(-2\frac{3}{4}, -6\frac{1}{2})$.

4 Shown below are sketches of the *P*th and *Q*th arrangements for the choices of *P* and *Q* shown on the graph at the left below. The corresponding points [P,*v*(P)] and [Q,*v*(Q)] are shown on the graph.



$v(P) = 2P - 1$ $v(Q) = 2Q - 1$

The location of the points on the graph can be determined by measuring. For example, one can mark off on the edge of a piece of paper a segment whose length is the distance between 0 and *P*, and adjoin to it a segment whose length is $P - 1$. The sum of these two lengths will be the distance of the point $[P, v(P)]$ above the *n*-axis. This is illustrated at the left. Some of the students may locate the points by noting that all the points of the graph are *collinear* (i.e., they all lie on the same line) and locate $[P, v(P)]$ and $[Q, v(Q)]$ so that collinearity is maintained.

Focus Teacher Activity (cont.)

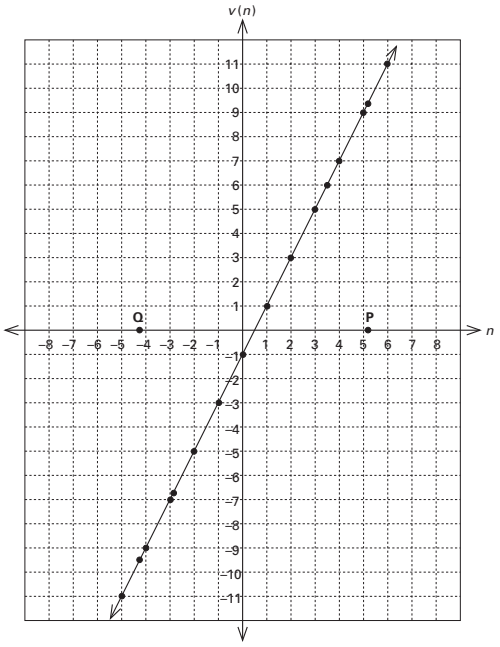
ACTIONS

5 Ask the students to imagine that the sequence of arrangements discussed in Actions 1-4 has been augmented so there is an arrangement corresponding to every point on the *n*-axis. Ask them how they could show this on their graphs. Discuss. Use this as a context for a brief discussion of the *real number system*.

Lesson **12** Continuous Graphs

Focus Student Activity 12.3

NAME _____ DATE _____



$v(n) =$ $2n - 1$

COMMENTS

5 The resulting graph is a straight line, only a portion of which shows in the graph the students have constructed. The actual graph extends indefinitely in both directions, as indicated by the arrowheads.

A sequence of arrangements for which there is an arrangement corresponding to every point on the horizontal axis, will henceforth be referred to as a *continuum* of arrangements, or a *continuous sequence of arrangements*.

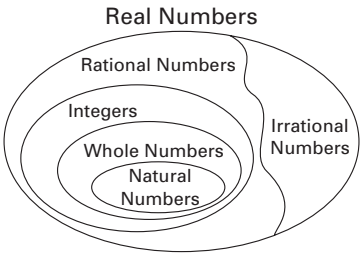
A sequence of arrangements whose arrangement numbers are the integers or the natural numbers (see below) is a *discrete* sequence and its graph is *discontinuous* because it is a set of discrete (disconnected) points.

The graph of a continuum of arrangements may be a *continuous graph*. An informal "test" for *continuity* of a graph over an interval on the *x*-axis is: can the graph be traced over the interval without ever picking up one's pencil?

The *x*- and *y*-axes are called *real number lines* because there is a *one-to-one correspondence* between all the points on a line and the real numbers. That is, for every point on a line, there is a real number that can be assigned to the point, and conversely.

The set of *real numbers* contains the set of *irrational numbers* (nonrepeating, nonterminating decimals such as $\sqrt{2}$) together with the set of *rational numbers* (terminating or repeating decimals). Every real number is either rational or irrational.

Any number that is *rational* can be written in the form a/b, where a and b are integers and $b \neq 0$. *Irrational numbers* cannot be written in this form (this idea is explored further in *Math Alive! Course IV*). The sets of *natural numbers* {1, 2, 3, 4, 5,…}, *whole numbers* {0, 1, 2, 3, 4,…}, and *integers* {…–3, –2, –1, 0, 1, 2, 3…} are contained in the set of rational numbers. You might ask the students to create a Venn diagram to show the relationships among these *subsets* of the real numbers. For example:

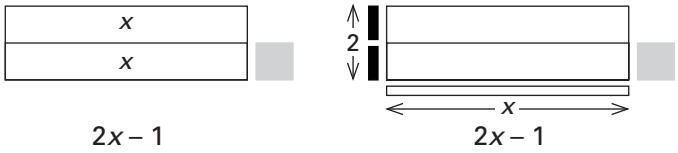


Focus Teacher Activity (cont.)

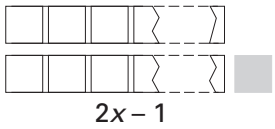
ACTIONS

6 Ask the students to suppose a generic point x on the n-axis is selected and to create a representation of the xth arrangement. Discuss their representations.

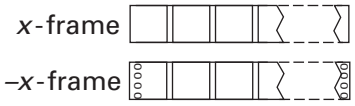
Sketches:



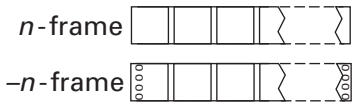
Pieces:



When arrangement numbers are real numbers:



When arrangement numbers are integers:



COMMENTS

Notice that every natural number is a whole number, an integer, a rational number, and a real number, but not an irrational number. The natural numbers are used to number the arrangements for nonextended, noncontinuous sequences of arrangements. The integers number the arrangements for extended, noncontinuous sequences of arrangements. The reals number the arrangements of a continuous sequence of arrangements.

6 The students may suggest a variety of ways to represent the xth arrangement. It might be represented by a sketch. For example, the 1st sketch on the left consists of 2 unshaded strips, each labeled x, and a single red counting piece. It is understood that each strip is to be filled with a collection of counting pieces (and/or parts of pieces) whose value equals its label. Thus, if x is positive, the strip is filled with black counting pieces; if x is negative it is filled with red counting pieces; and if x is 0, it is empty. The 2nd sketch on the left uses edge pieces to represent $2x$ as a rectangle with edge values 2 and x.

Alternatively, Algebra Pieces might be used to form a representation (see left). The frames are to be thought of as x-frames rather than n-frames; that is, each frame represents a strip of counting pieces whose value is x rather than n.

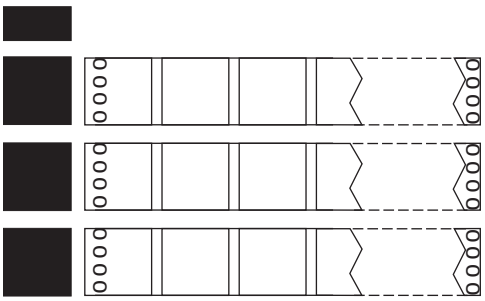
Henceforth, the letter x will refer to a generic arrangement from a continuous sequence of arrangements, i.e., the domain is the real numbers. The letter n will be used for a discrete sequence of arrangements, i.e., the domain is the integers. In the former case, frames will be designated as x-frames or $-x$-frames and represent strips of counting pieces whose values are x and $-x$, respectively. In the latter case, frames will be referred to as n-frames or $-n$-frames. It may help to call $-x$-frames, *opposite x-frames*.

The above usage follows the customary, but not universal, practice of using letters like x, y, and z to represent quantities that can take on any value, integral or not, (i.e., *continuous variables*) and using letters like k, m, and n to represent quantities that have integral values (i.e., *discrete variables*). The choice of a letter to represent a generic arrangement is arbitrary. For example, one might refer to the zth arrangement and write $v(z) = 2z - 1$. In this case, if frames were used to represent the zth arrangement, they would be referred to as z-frames or $-z$-frames and have values z or $-z$, respectively.

Focus Teacher Activity (cont.)

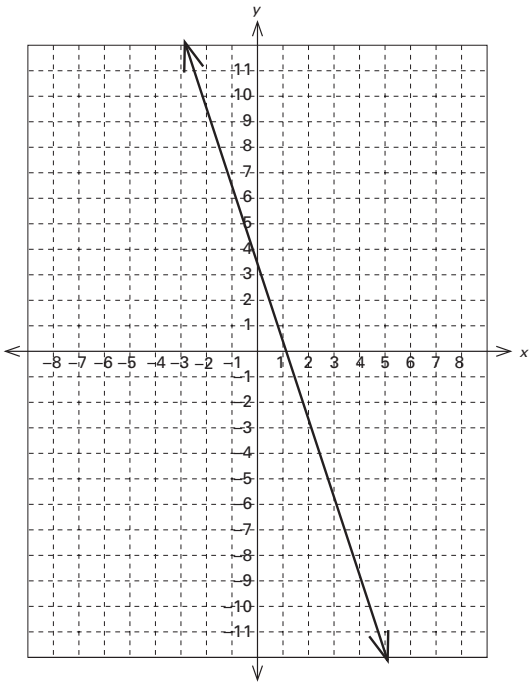
ACTIONS

7 Show the students the following xth arrangement from a continuum of arrangements. Ask them to write a formula for $v(x)$. Then give 1 sheet of coordinate grid paper (see *Blackline Masters*) to each student and have them construct a graph consisting of all points (x,y) such that $y = v(x)$. Discuss their ideas about whether $y = v(x)$ is a function, and if so, what are the domain and range of the function. If it hasn't already come up, introduce use of the *vertical line test* for a function.



COMMENTS

7 You may want to clarify that the partial counting piece is ½ of a black counting piece, so $y = v(x) = \frac{7}{2} - 3x$. Note that, in the graph shown below, the horizontal axis is labeled x. The vertical axis is labeled y and could also be labeled $v(x)$.



$y = \frac{7}{2} - 3x$

Notice that for this graph, the values for x included in the graph are all of the real numbers and the values for y are all of the real numbers. Hence, the domain and range of $y = \frac{7}{2} - 3x$ are the real numbers. To verify that $y = \frac{7}{2} - 3x$ is a function, one needs to show that any point in the domain is associated with exactly 1 point in the range. This can be done by sliding a vertical line along the x-axis (keeping the line perpendicular to the x-axis) and noting that the vertical line always crosses the graph of $v(x) = \frac{7}{2} - 3x$ in exactly one point (this is often referred to as the *vertical line test* for a function). In terms of a sequence of arrangements, this means there is exactly 1 value for each arrangement. If $y = \frac{7}{2} - 3x$ were not a function, there would be at least one value of x for which the vertical line would cross the graph in more than 1 point.

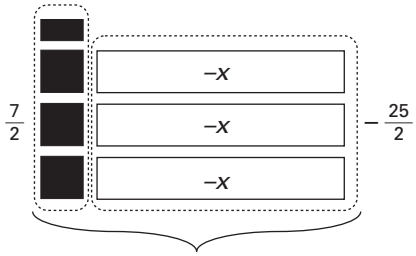
Focus Teacher Activity (cont.)

ACTIONS

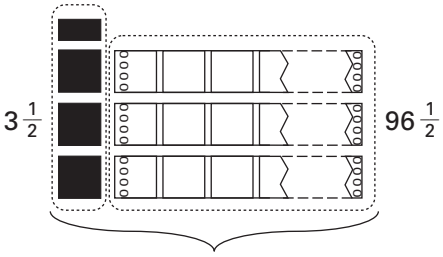
8 Copy the following table on the overhead or board and ask the students to find the missing values for x, assuming (x,y) is on the graph from Action 7:

	x	y
a)		8
b)		–9
c)		–14.2
d)		100

Discuss the students' methods of determining the values for x. Then ask them to relate each ordered pair (x,y) from the table to the sequence of arrangements for which $v(x) = 7/2 - 3x$. Discuss.



$v(x) = -9 = -18/2$



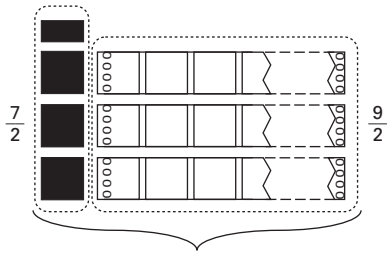
$v(x) = 100$

COMMENTS

8 The table shown in Action 8 differs from those given for the Connector activity in that this is simply an "organizer" for recording selected ordered pairs from a graph rather than an ordered listing that represents all ordered pairs for a graph.

a) The students might see from their graphs that there is exactly 1 point (x,y) for which $y = 8$. For that point, one sees (if the graph is constructed carefully enough) that $x = -1.5$.

Alternatively, the students may use Algebra Pieces to determine x. For example, if the value of the xth arrangement shown here is 8, then the $-x$-frames must have a total value of $8 - 7/2$ or $9/2$. Since there are 3 of them, each $-x$-frame has a value of $3/2$. Hence, the value of x is $-3/2$.



$v(x) = 8 = 16/2$

b) When $y = -9$, it is difficult to determine the exact value of x from the graph. Referring to a sketch of the xth arrangement, as shown at the left, one sees that if its value is –9, the $-x$-frames have a total value of $-9 - 7/2$ or $-(25/2)$. Since there are 3 of them, each $-x$-frame has a value of $-(25/6)$. Hence, $x = 25/6$.

It is important to acknowledge and discuss the students' preferences for methods of solving equations (e.g., symbols, Algebra Pieces, sketches, and hand-made graphs). A goal is for students to develop a variety of techniques for solving equations and for thinking critically about which technique is most appropriate for a given situation. It is also important to make connections between Algebra Piece, graphical, and symbolic representations. Note that in Lesson 14 students will also explore ways to use the graphing calculator to solve such problems.

c) Proceeding as in b), one finds $x = 5.9$.

d) From a picture of the xth arrangement, as shown at the left, one sees that if the value of the xth arrangement is 100, the total value of the $-x$-frames is $96\frac{1}{2}$. Since $96 \div 3 = 32$ and $\frac{1}{2} \div 3 = \frac{1}{6}$, each $-x$-frame has value $32\frac{1}{6}$. Hence, $x = -32\frac{1}{6}$.

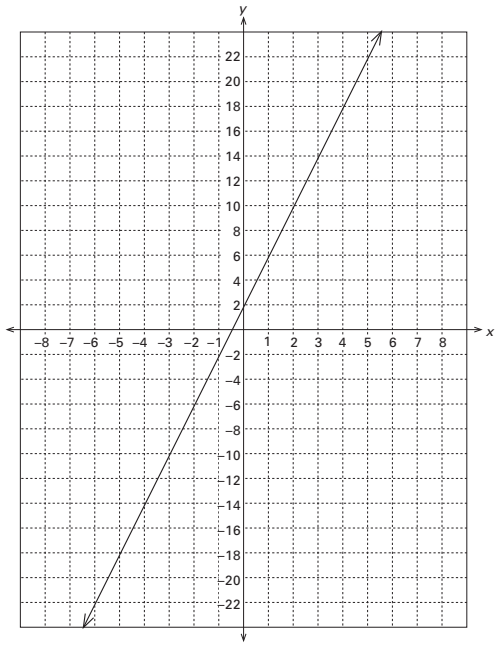
Focus Teacher Activity (cont.)

ACTIONS

9 Place a transparency of Focus Master A on the overhead and give a copy to each group. Tell the students this is a graph of $y = v(x)$ for a certain continuum of arrangements. Ask them to write a formula for $v(x)$ in the space provided on the bottom of Focus Master A and to sketch an Algebra Piece representation of the xth arrangement. Discuss.

Continuous Graphs Lesson **12**

Focus Master A



$v(x)$ = _____

Sketch the xth arrangement:

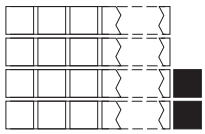
10 Discuss the students' ideas about ways the numbers, 4 and 2, in the formula for $v(x)$ relate to its graph. Discuss the terms *coefficient*, *slope,* and *intercept*.

COMMENTS

9 One line of reasoning is that the graph shows that $v(x)$ increases by 4 as x increases by 1. Thus, $v(1) = 4 + v(0)$, $v(2) = 8 + v(0)$, $v(3) = 12 + v(0)$ and so forth. Since $v(0) = 2$, this shows that the formula is $v(x) = 4x + 2$ which can be verified for other points on the graph. If the expression $v(x)$ is represented by y, the formula might be written $y = 4x + 2$.

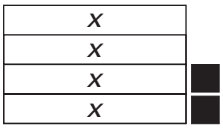
Before sketching the xth arrangement, some students may wish to form the arrangement with Algebra Pieces. An Algebra Piece arrangement is shown on the left below. A sketch of the xth arrangement is shown on the right.

pieces:



$v(x) = 4x + 2$

sketch:



$v(x) = 4x + 2$

Some students may generate a set of ordered pairs from the graph and find the equation for the graph by determining an arrangement of Algebra Pieces that represents all the ordered pairs. For example, one student described this process as follows:

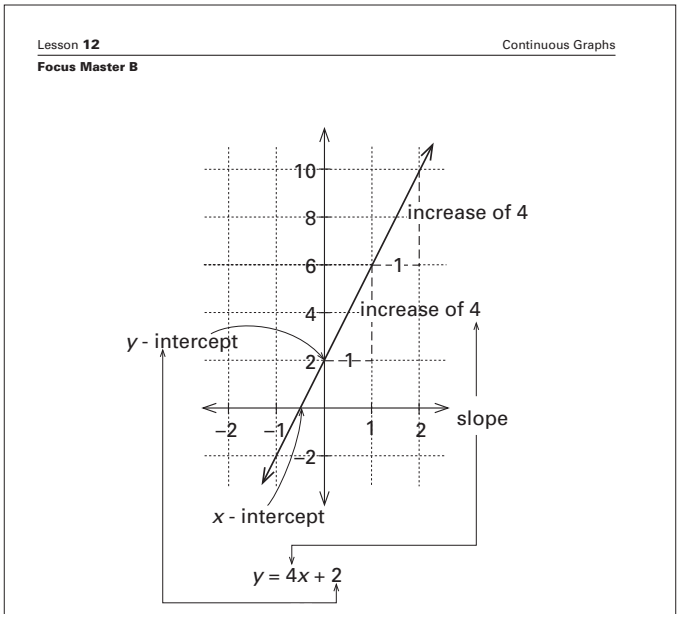
I started with the ordered pair (5,22). When x is 5, an x-strip contains 5 units. I determined that 22 units could be arranged in 4 strips of 5 units with 2 units left over, or 4x + 2 [no (5 × 5) x^2-mats could be used]. Then, I checked to be sure this arrangement could be used to represent all other ordered pairs.

10 You might have groups discuss their ideas before whole class sharing takes place.

Focus Master B, shown on the next page, may be useful for discussion. The number 4 in the product $4x$ is called the *coefficient of x*. It tells how much y values change as x values increase by 1 (for example, as x changes from 0 to 1, y changes from 2 to 6, an increase of 4). This rate of change, the change in y for each unit increase in x, is called the *slope* of the line. (Note that if y had decreased as x increased, the change, and hence the slope, would have been negative.)

Focus Teacher Activity (cont.)

ACTIONS



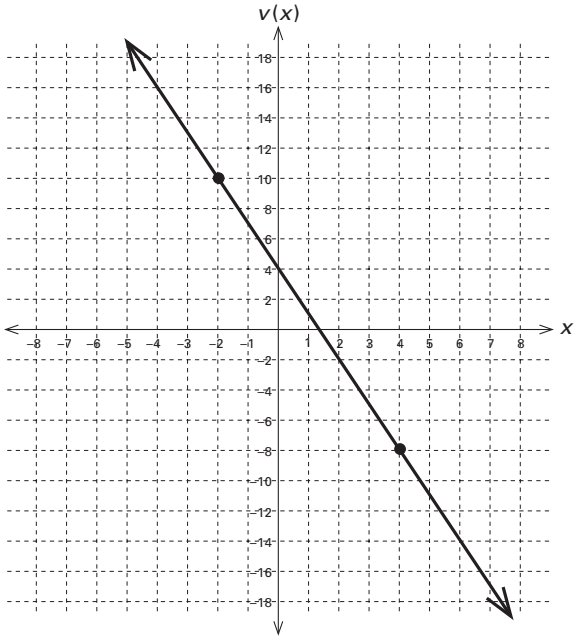
11 Give each student a sheet of coordinate grid paper. Tell them that the graph of $y = v(x)$ for a certain continuum of counting piece arrangements is a straight line which passes through the points (–2,10) and (4,–8). Ask them to graph and find a formula for $v(x)$. Then have them construct an Algebra Piece representation or draw a sketch of the xth arrangement. Discuss. Use their formulas as a context for discussing the *slope-intercep*t form of the equation of a line.

COMMENTS

The *constant* 2 is the value of y when x is 0 or, to put it another way, the y-coordinate of the point where the line crosses the y-axis. This value is called the *y-intercept* of the line. Students may comment that adding 2 to $y = 4x$ (the line with slope 4 that passes through the origin) shifts every point on the line $y = 4x$ "up 2 units."

A line may also cross the x-axis. If it does, the value of y is zero at the point of intersection with the x–axis and the value of x at that point is called the *x-intercept*. For $y = 4x + 2$, the x-intercept is $-\frac{1}{2}$.

11 Students will need to locate axes and then scale them. In the graph shown below, the x-axis is scaled so that each subdivision represents 1 unit and the y-axis is scaled so that each subdivision represents 2 units. The graph will appear differently for other scales, but will still be a straight line.

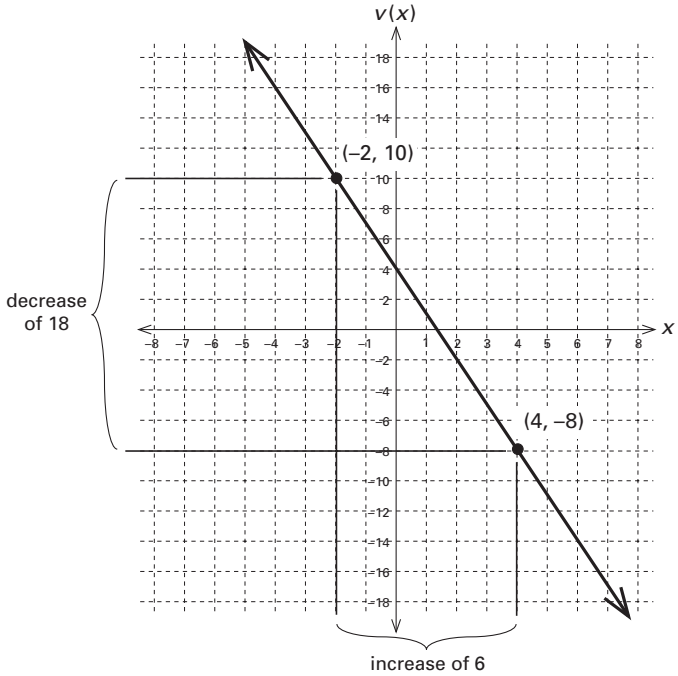


If the points (–2,10) and (4,–8) are located on a graph and a straightedge is used to draw a line through them, one sees that the y-intercept of the line is 4.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



slope = $^{-18}/_{6}$ = –3

COMMENTS

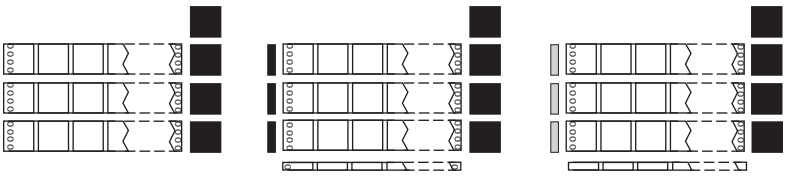
11 (continued.)
The slope of the line is –3 since y values decrease by 3 as x values increase by 1. One way to determine this is to note the y values decrease 18 units (from 10 to –8) as the x values increase 6 units (from –2 to 4), which is equivalent to a y-decrease of 3 units for every 1 unit x-increase.

Since the line has slope –3 and y-intercept 4, $y = v(x) = -3x + 4$. The students may use other methods to find a formula for $v(x)$.

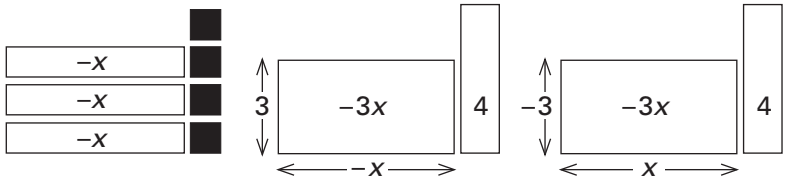
An equation that is written in the form $y = mx + b$, such as $y = -3x + 4$, is said to be in *slope-intercept* form; m is the slope of the line and b is the y-intercept.

Shown below are various sketches and Algebra Piece representations of $y = -3x + 4$, some of which include edge pieces. In the sketches, a numeral alongside the edge of a rectangle denotes the value of the edge. Note that the *value* of an edge may differ from its *length*—the length of an edge is always positive or zero, while the value of an edge can be positive, negative, or zero. If the value of an edge is positive, the value of the edge and its length are the same. If the value of an edge is negative, the value of the edge and its length are opposites. For example, if the value of an edge is –3, its length is 3.

$y = {}^{-}3x + 4$
Pieces:



$y = {}^{-}3x + 4$
Sketches:



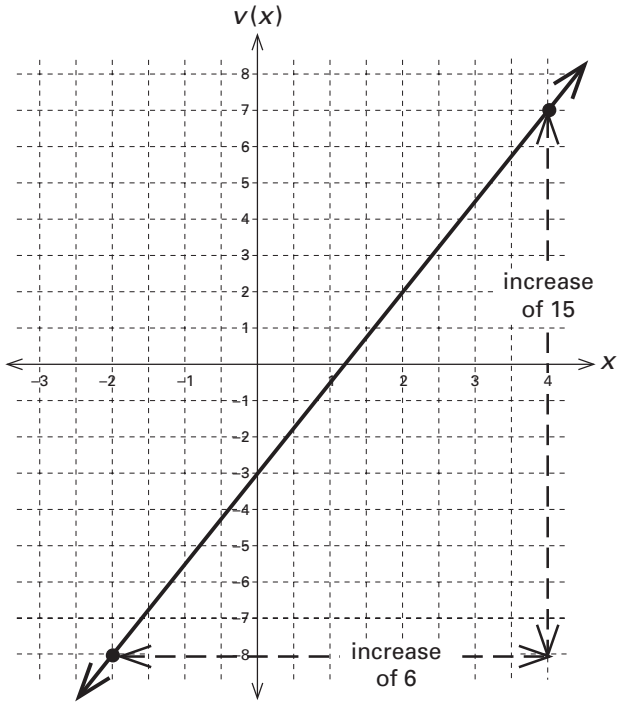
12 Repeat Action 11 for the points (–2,–8) and (4,7).

12 In the graph shown on the next page, the x-axis is scaled so that each subdivision represents ½ unit and the y-axis is scaled so that each subdivision represents 1 unit. The graph's appearance will differ for other scales, but will still be a straight line.

Using a straightedge to draw a line connecting the 2 given points on a graph with x- and y-axes (see diagram on the next page), one sees that the y-intercept is –3. Also, y values increase by 15 as x values increase by 6. This is equivalent to a y-increase of 2½ for each

Focus Teacher Activity (cont.)

ACTIONS



13 Give each student 4 sheets of coordinate grid paper and ask them to construct or sketch an Algebra Piece representation of the xth arrangement of a continuum of arrangements for which the graph of $v(x)$ is a straight line that satisfies the conditions in a) below. Invite several volunteers to sketch their xth arrangements at the overhead and then have the groups graph $v(x)$ and verify that the conditions in a) are met. Discuss relationships among the Algebra Piece representations and the equations and graphs that represent them. Repeat for b)-h).

a) y-intercept is 4

b) slope is 3

c) slope is –2 and y-intercept is 3

d) slope is 0 and y-intercept is 7

e) slope is undefined

f) slope is $\frac{3}{4}$

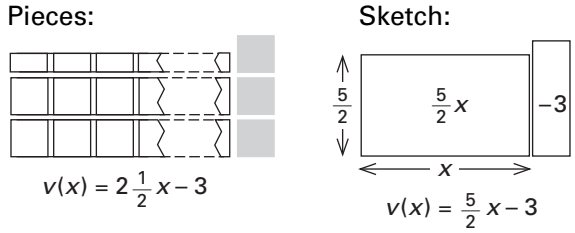
g) slope is $-\frac{4}{6}$ and y-intercept is –3

h) y intercept is 3 and slope is the same as the slope of $y = \frac{2}{5}x$

COMMENTS

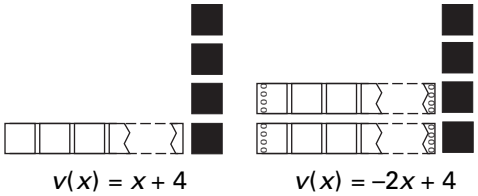
x-increase of 1. Thus, the line has slope $\frac{5}{2}$. Hence, $y = v(x) = (\frac{5}{2})x - 3$. The students can verify this formula by showing that it provides the correct values for $v(-2)$ and $v(4)$, namely –8 and 7.

On the left below is an Algebra Piece representation in which an x-frame has been cut in half. On the right is a sketch in which the values of edges and regions are shown.



13 The equations that students generate for a) form a *family of lines* with y-intercept 4. Similarly, the sets of equations students generate for each of b), e), and f) form families of lines whose relationships are defined by the given conditions. This idea is investigated further in Lesson 14. The conditions given in c), d), g), and h) each produce a unique line.

a) If the graph of $v(x)$ has y-intercept 4, then $v(0) = 4$. This will be the case if the "constant" part of the arrangements has value 4. Shown below are 2 possibilities.



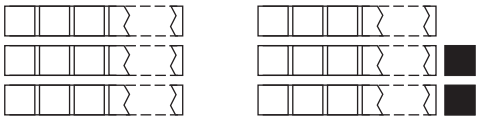
The graphs of the above equations have y-intercept 4 and slopes 1 and –2, respectively.

b) If the graph of $v(x)$ is a straight line whose slope is 3, then the values of the arrangements must increase by 3 as x increases by 1. One possibility is that $v(0) = 0$, $v(1) = 3$, $v(2) = 6$, and so forth. This will be the case if $v(x) = 3x$. Other possibilities can be obtained by adding a constant

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

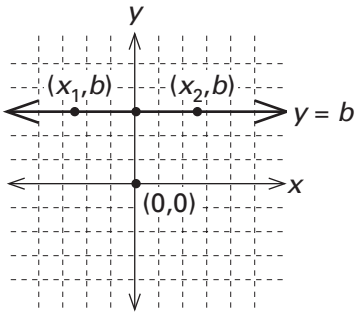


$v(x) = 3x$ $v(x) = 3x + 2$

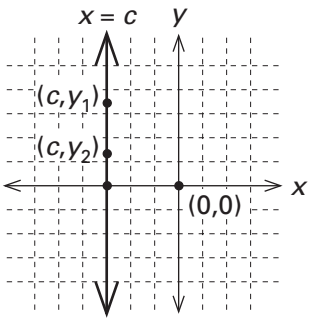
COMMENTS

13 (continued.)
to this expression, e.g., $v(x) = 3x + 2$ (see left). The difference in the graphs of these 2 expressions is that the y-intercept of the 1st is 0 and that of the 2nd is 2.

c) If the graph of $v(x)$ is a straight line and both the slope and y-intercept are given, there is only one possibility. In this case, $v(x) = -2x + 3$, or an equivalent expression.



d) The line with 0 slope and y-intercept (0,7) has equation $y = 7$. The slope of any horizontal line $y = b$, for b a real number, is 0. This is because, for any 2 distinct points (x_1,b) and (x_2,b) on the line $y = b$, the slope of the line is $(b - b)/(x_2 - x_1) = 0/(x_2 - x_1) = 0$ (see diagram at the left).



e) For any vertical line $x = c$, where c is a real number, passing through the points (c,y_1) and (c,y_2), the slope of the line is $(y_2 - y_1)/(c - c)$, or $(y_2 - y_2)/0$. Since division by 0 is undefined (see Lesson 6 of this course), the slope of a vertical line is undefined. Notice in the diagram at the left that an equation of the form $x = c$, where c is a real number, does not represent a function, since there are an infinite number of y values in the range associated with the one value $x = c$ in the domain.

f) Since the slope of this line is $3/4$, then the line must "rise" (i.e., change upwards vertically) $3/4$ unit for every 1 unit of "run" (i.e., left to right horizontal change). Or, in other words, it must rise 3 units for every run of 4 units. Hence, any line whose equation can be written in the form $y = (3/4)x + b$, where b is a real number, has slope $3/4$.

g) The equation for this line is $y = (-4/6)x - 3$. Some students may point out this could also be written as $y = (-2/3)x - 3$. In fact, any fraction equivalent to $-4/6$ can be substituted as the coefficient of x and the equation will have the same graph as $y = (-4/6)x - 3$.

Note also that the graph of the equation $y = (-4/6)x - 3$ is identical to the graph of $-4x + -6y = 18$. The latter equation is commonly referred to as the standard form of the equation. In general, an equation $ax + by = c$, where a, b,

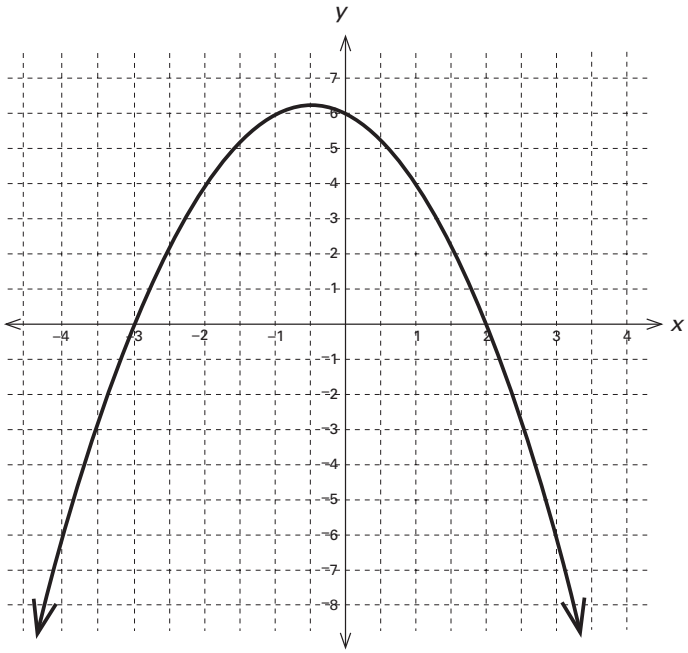
Focus Teacher Activity (cont.)

ACTIONS

14 Give each student another sheet of coordinate grid paper.

a) Ask the students to build or sketch an Algebra Piece representation of the xth arrangement of a continuum of arrangements for which $v_1(x) = (2 - x)(3 + x)$.

b) Ask the students to predict what the graph of $v_1(x)$ looks like. Discuss their predictions and then ask them to draw the graph, starting with integer values for x. Ask the students for their observations.

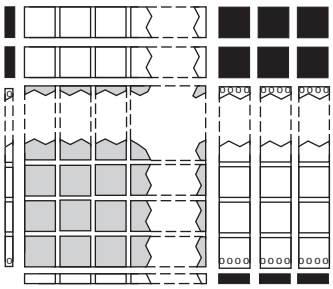


$y = v_1(x) = (2 - x)(3 + x)$

COMMENTS

and c are integers, is in *standard form*. Notice the variables are on one side of the equation and the constant is on the other. It is not as easy to "see" the slope and intercept of the graph of a linear equation in standard form as compared to an equation in slope-intercept form. Note that in *Math Alive! Course IV*, students will examine in more depth different forms of equations of lines.

14 a) Shown below is an Algebra Piece representation, with edge pieces.



$v_1(x) = (2 - x)(3 + x)$

b) Since $v_1(0) = 6$, the y-intercept is 6. The students may observe that the x-intercepts are 2 and –3 since $v_1(x) = 0$ for those values of x. If x is in the interval between the x-intercepts, both factors of $v_1(x)$ are positive and $v_1(x)$ is positive (i.e., for $-3 < x < 2$, $v_1(x) > 0$). Outside this interval, one factor of $v_1(x)$ is positive and the other is negative, so $v_1(x)$ is negative (i.e., for $x < -3$ and $x > 2$, $v_1(x) < 0$).

An alternate form for $v_1(x)$ is $-x^2 - x + 6$, as can be seen from the Algebra Piece representation shown in Comment a).

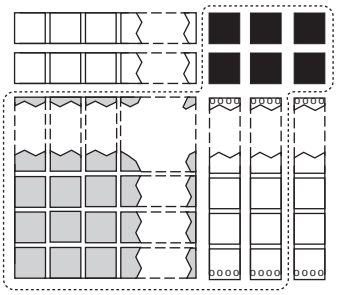
In the graph of $y = v_1(x)$ shown at the left, every subdivision of the x-axis represents ½ unit and every subdivision of the y-axis represents 1 unit. The shape of the graph will vary slightly for other scalings of the axes; however, regardless of the scaling, the graph is a *parabola* that is symmetric about the vertical line $x = -½$ and opens downward.

Some students may find a few points on the graph and connect these points with straight line segments. If so, you might have them find more points on the graph to help show that it is rounded rather than angular.

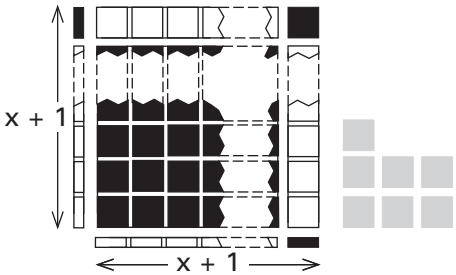
Focus Teacher Activity (cont.)

ACTIONS

15 Ask the students to graph $v_2(x) = x$ on the same coordinate axes as their sketch of $v_1(x) = (2 - x)(3 + x)$. Have them find the exact and approximate coordinates of the points where the graphs intersect. Discuss the students' ideas about how these intersection points relate to the sequences of arrangements represented by the graphs, and discuss their observations, in general, about the graph.

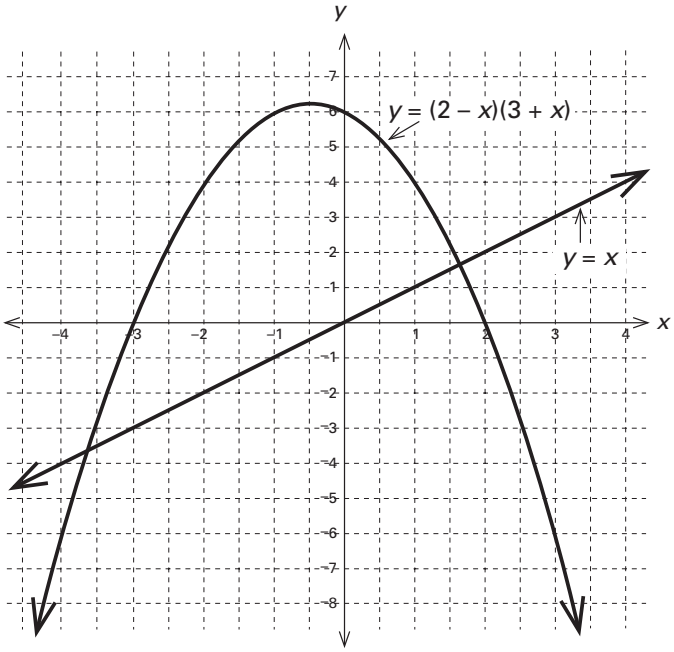


value of uncircled portion is x
value of circled portion is 0

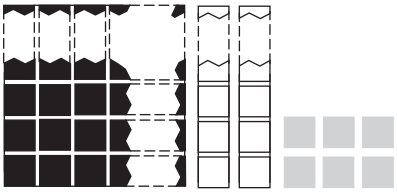


COMMENTS

15 If $y = x$ and $y = (2 - x)(3 + x)$ are graphed on the same coordinate system, it appears that the two graphs intersect when x is about 1.7 and –3.7 (see diagram below). The x-coordinates of these points are the numbers of the arrangements for which the 2 sequences have the same value.



The exact values of x where the 2 graphs intersect can be found by determining when $v_1(x) = v_2(x)$, i.e., when $(2 - x)(3 + x) = x$. This is the case if the value of the circled portion of the Algebra Piece representation for $v(x)$, shown at the left, is 0. If the value of the circled collection is 0, the value of its opposite collection, shown below, is also 0.



If a red counting piece and a black counting piece are added to the above collection, its value is unchanged. The resulting collection can be arranged into a square with edge $(x + 1)$, and with 7 red counting pieces left over, as shown at the left. Since the total collection has value 0, the square must have value 7. Thus, $x + 1$ equals $\sqrt{7}$ or $-\sqrt{7}$, so $x = -1 + \sqrt{7}$ or $x = -1 - \sqrt{7}$. Since $\sqrt{7} \approx 2.65$, the two graphs intersect when $x \approx 1.65$ or $x \approx -3.65$.

Focus Teacher Activity (cont.)

ACTIONS

16 If the following did not come up during Actions 14 and 15, pose them for discussion. Invite volunteers to mark the graph to indicate points that satisfy the given conditions. As needed, discuss symbolic ways of denoting inequality relationships.

a) For what values of x, if any, is the graph of $y = x$ above the graph of $y = (2 - x)(3 + x)$? Relate this information to the sequences of counting piece arrangements represented by the graphs.

b) For what values of x, if any, is the graph of $y = x$ below the graph of $y = (2 - x)(3 + x)$? Relate this information to the sequences represented by the graphs.

c) For what values of x, if any, is $(2 - x)(3 + x) = 0$? Relate this information to the sequences represented by the graphs.

d) When $x = 0$, what is the value of $(2 - x)(3 + x)$? Relate to the sequences.

e) Will the graph of $y = (2 - x)(3 + x)$ ever intersect the line $y = x$ again? Why or why not? Relate to the sequences.

17 Give each group a copy of coordinate grid paper and a copy of Focus Master C (see next page) and ask them to carry out the instructions. Discuss their results and their ideas about the usefulness of graphing as a tool for solving equations and inequalities.

COMMENTS

16 The intent here is to informally discuss some ideas involving inequality relationships and to continue making connections between graphical and Algebra Piece representations of sequences.

a) Notice, the graph of $y = x$ is above the graph of $y = (2 - x)(3 + x)$, when $x < -1 - \sqrt{7}$ and when $x > -1 + \sqrt{7}$. In terms of the sequence, when the arrangement number is less than $-1 - \sqrt{7}$, or ≈ -3.65, and when the arrangement number is greater than $-1 + \sqrt{7}$, or ≈ 1.65, then the values of the arrangements for the sequence $y = x$ are greater than the values of the arrangements for $y = (2 - x)(3 + x)$. Another way of writing this is $x > (2 - x)(3 + x)$ when $x < -1 - \sqrt{7}$ and when $x > -1 + \sqrt{7}$.

b) The graph of $y = x$ is below the graph of $y = (2 - x)(3 + x)$, i.e., $x < (2 - x)(3 + x)$, when $-1 - \sqrt{7} < x < -1 + \sqrt{7}$. Hence, the values of the arrangements for $y = x$ are less than the values of the arrangements for $y = (2 - x)(3 + x)$ when x is between ≈ -3.65 and ≈ 1.65.

c) $x = 2$ and $x = -3$. These are the x intercepts of the graph of $y = (2 - x)(3 + x)$ and they are the numbers of the arrangements whose values are zero.

d) When $x = 0$, the graph of $y = (2 - x)(3 + x)$ crosses the y-axis. Since $v(0) = (2 - 0)(3 + 0) = 6$, the y–intercept is (0,6). In terms of the sequence of arrangements for whose values are $v(x) = (2 - x)(3 + x)$, 6 is the value of the 0th arrangement.

e) Since the value of $y = (2 - x)(3 + x)$ decreases for $x > -1 + \sqrt{7}$ and the value of $y = x$ increases over that interval, it is not possible they will ever intersect again for any $x > -1 + \sqrt{7}$. Further, since $y = (2 - x)(3 + x)$ decreases at a faster rate than $y = x$ for all $x < -1 - \sqrt{7}$, it is not possible that they intersect for any $x < -1 - \sqrt{7}$. In terms of the sequences, the only arrangement numbers for which the sequences have the same value are ≈ -3.65 and ≈ 1.65.

17 You may find it helpful to have the students complete and discuss each problem before proceeding to the next.

a) b) In the graph shown below, both the x- and y-axes are scaled so that 1 subdivision is 1 unit.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Continuous Graphs Lesson **12**

Focus Master C

a) Suppose that the value of the xth arrangement of a certain continuous sequence of arrangements is, $v_1(x) = \frac{1}{2}x^2 - x - 8$. Graph v_1 and determine the value of x if $v_1(x) = 4$; if $v_1(x) = -2.5$.

b) Suppose the xth arrangement of a 2nd continuous sequence has value $v_2(x) = 4$, and a 3rd has value $v_3(x) = -2.5$. Sketch the graphs of v_2 and v_3 on the same coordinate axes as v_1.

c) Circle and label the coordinates of the points where v_2 and v_3 intersect v_1. Explain how these points relate to the sequences of arrangements represented by the graphs.

d) Find all the values of x for which $\frac{1}{2}x^2 - x - 8 > 4$. Explain the relationship between these values and the sequences of arrangements for v_1 and v_2.

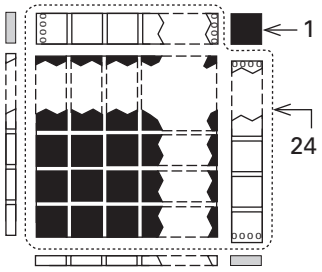
e) Find all the values of x for which $\frac{1}{2}x^2 - x - 8 < -2.5$. Relate these values to the sequences of arrangements for v_1 and v_3.

f) Draw a box around and label the coordinates of all x-intercepts of the graphs of v_1, v_2, and v_3. Relate your results to the sequences of arrangements.

g) Label the coordinates of the y-intercepts, if any, of the graphs of v_1, v_2, and v_3 and place a black dot at each y-intercept. Relate these points to the sequences of arrangements.

h) Place a V where you think v_1 stops decreasing and begins to increase. Explain how this turning point relates to the sequence of arrangements for v_1.

i) Challenge: give the coordinates of the turning point of v_1. Explain your methods.

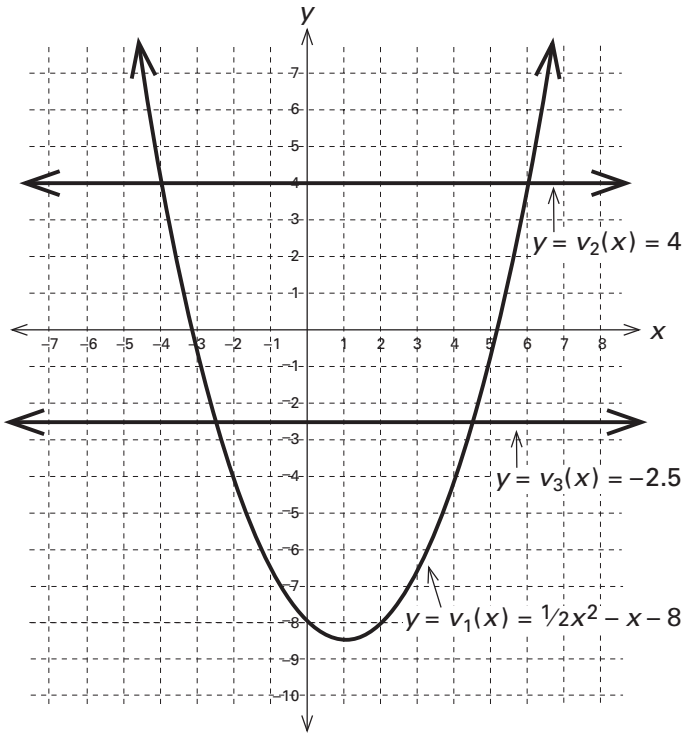


$x^2 - 2x = 24$

So, $x^2 - 2x + 1 = (x - 1)^2 = 25$

COMMENTS

17 (continued.)



c) If the students have drawn their graphs carefully, they can determine from them that $v_1(x) = v_2(x)$ when x is –4 or 6. This conclusion can also be reached by using Algebra Pieces, as illustrated at the left. Because of the $(\frac{1}{2})x^2$ in the expression for $v_1(x)$, it is easier to work with $2v_1(x)$. Thus, to determine when $v_1(x)$ is 4, one can determine when $2v_1(x)$ is 8, that is, when $x^2 - 2x - 16 = 8$. This will be the case if $x^2 - 2x = 24$ or, adding one black unit to complete the square, if $(x - 1)^2 = 25$ (see diagram to the left). Thus, $x - 1 = 5$ or $x - 1 = -5$. If $x - 1 = 5$, then $x = 6$; if $x - 1 = -5$, then $x = -4$.

When $v_1(x) = v_3(x)$, it appears from the graph that x is somewhere near 4.5 or –2.5. More precise values can be found by using Algebra Piece methods to find when $2v_1(x) = -5$. In this case, $x^2 - 2x = 16 - 5 = 11$, and so $x^2 - 2x + 1 = 12$. Hence, the square shown in the diagram to the left has value 12. Therefore, $x - 1$ has value $\sqrt{12}$ or $-\sqrt{12}$. Thus, $x = \sqrt{12} + 1$ or $-\sqrt{12} + 1$. Using a calculator, one finds $\sqrt{12} + 1 \approx 4.46$ and $-\sqrt{12} + 1 \approx -2.46$.

Focus Teacher Activity (cont.)

ACTIONS

18 Give each student 2 sheets of coordinate grid paper and a copy of Focus Master D. Ask the students to carry out the instructions. Discuss their results and observations about the use of graphs, Algebra Pieces, and algebra symbols for solving equations.

Lesson **12** Continuous Graphs

Focus Master D

Each equation listed below describes the value of the xth arrangement of a continuous sequence of arrangements. For each pair of equations, complete a)-e). Record your responses to b)-e) next to each graph.

Pair 1: $v_1(x) = \frac{2}{3}x + 7$
$v_2(x) = 0$

Pair 2: $v_1(x) = 3x^2 - 9$
$v_2(x) = 0$

Pair 3: $v_1(x) = \frac{1}{2}x^2 + 3x - \frac{27}{2}$
$v_2(x) = -4x - 3$

Pair 4: $v_1(x) = (x - 4)(x - 2)$
$v_2(x) = -x^2 + 6x - 8$

a) Graph the pair on the same coordinate axes.

b) Sketch the xth arrangement of each sequence.

c) Find the value(s) of x, if any, for which $v_1(x) = v_2(x)$. Explain your methods and tell how you checked to be sure you are correct. Are the values exact or approximate?

d) Find the x-intercepts and y-intercepts, if any, of v_1 and v_2. Explain your methods and tell how you checked to be sure you are correct. Are your answers exact or approximate?

e) Write two inequality statements that describe relationships between the two graphs.

f) List your observations or conjectures.

19 Write the equation shown below on the overhead:

$$v(x) = \begin{cases} x, \text{ for nonnegative } x \\ -x, \text{ for negative } x \end{cases}$$

Ask the students to sketch the –3rd through 3rd and the –2½th, 3¼th, –1⅓rd, and ⅕th arrangements, assuming $v(x)$ is the value of the xth arrangement of a continuum of arrangements. Then give each student a sheet of coordinate grid paper and ask them to graph $v(x)$. Discuss their observations about the graph and introduce the term *absolute value* and the notation $|x|$.

COMMENTS

18 This activity could be completed in class or as homework. You might encourage students to verify their results by using more than one strategy for answering c) and d). That is, students could use graphs, Algebra Pieces, or symbolic representations of Algebra Piece actions.

Pair 1:
c) $v_1(x) = v_2(x)$ for $x = -10\frac{1}{2}$.

d) $v_1(x)$: x-intercept, $-21\frac{1}{2}$; y-intercept, 7.
$v_2(x)$: x-intercepts, all real numbers; y-intercept, 0.

e) $v_1(x) < v_2(x)$ for $x < -10\frac{1}{2}$; $v_1(x) > v_2(x)$ for $x > -10\frac{1}{2}$.

Pair 2:
c) $v_1(x) = v_2(x)$ for $x = \sqrt{3}$ and $x = -\sqrt{3}$.

d) $v_1(x)$: x intercepts, $\sqrt{3}$ and $-\sqrt{3}$; y-intercept, –9.
$v_2(x)$: x-intercepts, all real numbers; y-intercept, 0.

Pair 3:
c) $v_1(x) = v_2(x)$ for $x = -7 + \sqrt{70}$ and $x = -7 - \sqrt{70}$.

d) $v_1(x)$: x-intercepts, 3 and –9; y-intercept, $-13\frac{1}{2}$.
$v_2(x)$: x-intercept, $-\frac{3}{4}$; y-intercept, –3.

Pair 4:
c) $v_1(x) = v_2(x)$ for $x = 4$ and $x = 2$.

d) $v_1(x)$: x-intercepts, 2 and 4; y-intercept, 8.
$v_2(x)$: x-intercepts, 2 and 4; y-intercept, –8.

19 The 2-part equation given in this action could be replaced by $v(x) = |x|$, where $|x|$ is read "*absolute value* of x." For example, if $x = 7$, $v(x) = |7| = 7$. If $x = -10$, then $v(x) = |-10| = -(-10) = 10$. The notation $|x|$ is a shorthand way of writing the given 2-part statement. To build comfort with the absolute value notation, you might ask the students to identify the coordinates of the points associated with the following values of x: 6, 9, –100, $17\frac{1}{2}$, 0, $-200\frac{1}{3}$. Here is a table of values showing the absolute value of the preceding numbers:

n	$v(n) = \lvert n \rvert$
6	$\lvert 6 \rvert = 6$
9	$\lvert 9 \rvert = 9$
–100	$\lvert -100 \rvert = -(-100) = 100$
$17\frac{1}{2}$	$\lvert 17\frac{1}{2} \rvert = 17\frac{1}{2}$
0	$\lvert 0 \rvert = 0$
$-200\frac{1}{3}$	$\lvert -200\frac{1}{3} \rvert = -(-200\frac{1}{3}) = 200\frac{1}{3}$

Focus Teacher Activity (cont.)

ACTIONS

20 Give each group 2 sheets of coordinate grid paper and a copy of Focus Master E and ask them to complete a) and b) for Situation 1. Invite volunteers to share their results at the overhead. Discuss the students' ideas about whether the graph is continuous, whether it represents a function, and what are the domain and range. Repeat for Situations 2-4.

Continuous Graphs Lesson **12**

Focus Master E

For each of the following situations, make (and label carefully) the indicated graph, and next to each graph write the following:

a) a mathematical formula that represents the relationships in the situation,

b) 2 mathematical conclusions that are based on information that you can "see" in your graph.

Situation 1 Jonathan earns $3 per hour baby-sitting. Make a graph that shows the amount Jonathan earns as related to the number of hours he works.

Situation 2 Erica's grandfather gives her $25 each year on her birthday. In addition, he gives her $1.50 for every year of her age. Make a graph that shows the amount of the birthday gift Erica receives from her grandfather as related to her age on her birthday.

Situation 3 The students in Ms. Cooper's math class are raising money for a field trip. The Whatsit Production Company has agreed to donate $5 for every hour a student works on roadside cleanup, plus an additional $10 if the student works more than 5 hours. Make a graph of the amount donated as related to the number of hours 1 student works.

Situation 4 When an object is dropped from an initial height of h_0 feet above the ground, it falls at a rate of 16 feet per second squared (Galileo observed this in 1604). Linden dropped a marble

(Continued on back.)

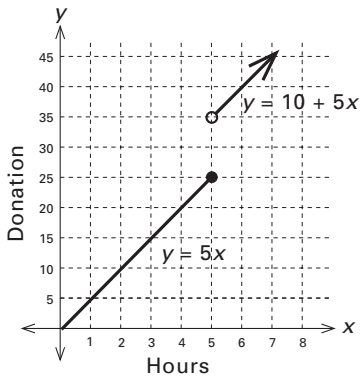
COMMENTS

20 This could also be assigned as homework. If so, each student will need a copy. To reduce sharing time at the overhead, as you circulate you can provide groups with blank transparencies and overhead pens for recording their results before coming to the overhead.

Situation 1: This can be represented by the graph of $y = 3x$, for $x \geq 0$ (points associated with values of $x < 0$ make no sense in this context).

Situation 2: The amount that Erica receives on each birthday depends on her age in years. Since she only receives a birthday gift on her birthday and not on the days in between, this graph is a set of discrete points whose domain is the positive integers and the path of the points follows the path of the line $y = 25 + 1.5x$, where x is her age in years and y is the amount she receives.

Situation 3: The situation can be represented by the following graph:



The graph for $x \leq 5$ is $v(x) = 5x$, and the graph for $x > 5$ is $v(x) = 10 + 5x$. You may need to point out the use of closed and open circles such as the ones shown above to indicate that (5,25) is included in the lower portion of the graph and the point (5,35) is not included in the upper portion of the graph (but all points whose x-coordinates are greater than 5 are included).

Notice that the relation:

$$v(x) = \begin{cases} 5x, \text{ for } 0 \leq x \leq 5 \\ 10 + 5x, \text{ for } x > 5 \end{cases}$$

is a function with domain the real numbers greater than or equal to 0. The range is the set of all real numbers y such that $0 \leq y \leq 25$ or $y > 35$. Notice the graph is discontinuous, i.e., when tracing the graph from left to right

Focus Teacher Activity (cont.)

ACTIONS

Lesson **12** Continuous Graphs

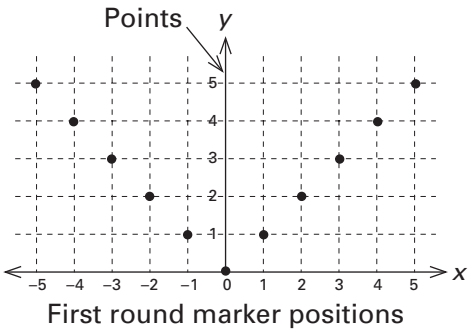
Focus Master E (cont.)

from the top of a 60-foot building into a pond at the base of the building. At the same time, Joel, who was standing next to Linden, dropped a brick. Make a graph that shows the height of the marble as related to the number of seconds after it is dropped. Do the same for the brick.

Situation 5 Katie and Malia designed the following game:

- The game board contains a number line. Both players start with a game marker at the origin of the number line (this is their first board position).
- Katie rolls a pair of standard dice (1 red die and 1 black) to determine where to move her marker. The number showing on the black die tells the number of spaces to move her game marker in the positive direction on the number line. The number showing on the red die tells the number of spaces to move the game marker in the negative direction.
- The number of points Katie earns is equal to the distance between the origin and her game marker after completing the moves indicated by the dice.
- Next Malia repeats the above procedures. Play continues until a player earns 25 points.

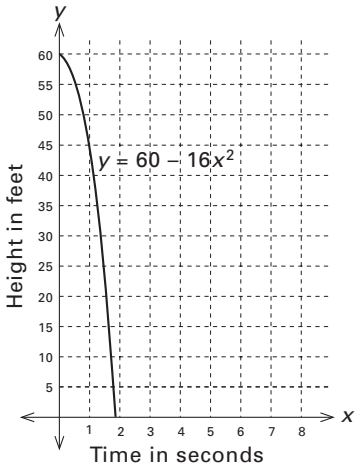
Graph all the possible ordered pairs (x,y) where x is a possible location of a player's marker after the first round of play and y is the number of points earned by landing in that position.



COMMENTS

with a pencil, it is necessary to pick up the pencil at the point (5,25).

Situation 4: Students may need clarification that 16 feet per second squared does not mean the 16 is squared, but rather that the time in seconds is squared. The following graph shows the height of the marble and brick as compared to the number of seconds after dropping them. Students may be surprised that the brick and marble will land at the same time, assuming there is no wind and the objects are dropped (not thrown or released with any initial "force").



Depending on the scaling that students use, they may think the graph looks like a straight line; however, the height, y, of an object at the end of each amount x of time is determined by $y = 60 - 16x^2$, a quadratic whose graph is a parabola. The part of the parabola that relates to this situation is in the first quadrant.

If it doesn't come up, you might ask the students to determine the number of seconds it takes for the marble or the brick to hit the ground. This requires finding the x-intercept, i.e., the value of x when $60 - 16x^2 = 0$. This equation is true when $x^2 = {}^{60}\!/_{16}$, i.e., when $x = \sqrt{{}^{60}\!/_{16}} \approx 1.94$ seconds. (Note: the negative solution, $x = -\sqrt{{}^{60}\!/_{16}}$, makes no sense in this context.)

Situation 5. Students may need to be reminded that distance is always a positive amount. Hence, if the marker lands on a negative number on the number line, the distance from the marker to the origin is the absolute value of the negative number. An equation for this graph is $y = |x|$, for x an integer between –5 and 5. The graph (see illustration at the left) is discontinuous,

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

21 Give each student a copy of Focus Student Activity 12.4 and explain your expectations regarding this activity.

Continuous Graphs Lesson **12**

Focus Student Activity 12.4

NAME _____ **DATE** _____

Although you will have many opportunities during this course to become familiar with your graphing calculator, it will be helpful if you are comfortable with the functions listed below as soon as possible. Please investigate each function on your calculator and, if needed, in your calculator manual. A way to test yourself to see if you can comfortably use and recall a calculator function is to demonstrate its use to someone else (a family member, a classmate, a neighbor, etc.). Try to check off all functions in Part 1 below by the following date _____.

1 I am comfortable using the following calculator functions:

____ ON/OFF
____ CLEAR the screen
____ show blank coordinate axes in the calculator viewing screen
____ move the cursor around a blank coordinate axes
____ change the viewing WINDOW size
____ FORMAT the axes
____ determine the "standard" WINDOW size on my calculator (on many it is $-10 \le x \le 10$ and $-10 \le y \le 10$)
____ enter an equation y =
____ GRAPH an equation y =
____ TRACE a graph (What shows on the screen when you do this?)
____ ZOOM in on a graph
____ ZOOM in again—and again
____ ZOOM out on a graph
____ ZOOM back to the standard window
____ TRACE the graph of a function to determine the approximate value of the function at $x = 0$, $x = 19.75$, and $x = -37.5$
____ TRACE the graph of a function to determine the value of x when $y = 75$, when $y = -75$
____ GRAPH 2 equations on the same coordinate axes.
____ TRACE to approximate the intersection of 2 graphs
____ ZOOM and TRACE to improve your approximation
____ DRAW a horizontal line on coordinate axes and slide the line up and down
____ DRAW a vertical line on coordinate axes and slide the line left and right

(Continued on back.)

COMMENTS

20 (continued.)
but rather a set of discrete points. (Absolute value is explored again in Lesson 14.)

Notice that the most points a player can earn on a single roll of the dice is 5 and the fewest is 0. The minimum number of rolls to win the game is 5, by earning 5 points on each roll.

21 We recommend that students complete Follow-up Student Activity 12.5 prior to receiving Focus Student Activity 12.4.

If students have their own graphing calculators or, if they can check out calculators for use at home, you could ask that work on Focus Student Activity 12.4 be completed outside of class. Then, throughout Lesson 13 you could devote brief periods to discussion of students' questions and discoveries about procedures for using the calculator (notice graphing calculators are not used until Lesson 14). If students do not have access to calculators at home, then you will need to set aside a class period or two for experimentation with the calculators. You might have manuals available for reference during class.

The purpose of Focus Student Activity 12.4 is to encourage experimentation by students, so they can gain familiarity with the capabilities of the graphing calculator without requiring extensive class time. Using this approach, when the calculator is required in Lesson 14, students will already be familiar with many of its capabilities.

We encourage our students to purchase their own graphing calculators. Students who cannot purchase a calculator may check one out for use in and out of class. They are charged for replacement if the calculator is damaged or lost. We recommend that our students all use the TI-83, because that is what is recommended by the high school they enter and because of its range of capabilities. However, if someone has another brand/model at home, that is fine. The critical graphing functions are common to most brands and models, although procedures for using them may vary.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

Focus Student Activity 12.4 (cont.)

____ view a TABLE of x- and y-coordinates of an equation
____ view a table of coordinates of 2 equations listed simultaneously
____ use a table to find when $0 = 5x + 1$
____ clear MEMory
____ reset defaults in MEMory
____ solve equations using the "solver" function from the MATH menu
____ use the "maximum" and minimum" functions from the CALC menu to find the turning point of a parabola
____ use the "intersect" function from the CALC menu to find the intersection of 2 graphs
____ use the "zero" function from the CALC menu to find the x-intercepts of a graph
____ use the "value" function from the CALC menu to find $v(x)$ for specific values of x
____ set the graphing style to shade the region above a graph; the region below a graph

2 Here are some other graphing calculator functions that I can use:

3 Here are some other functions I have tried but don't understand.

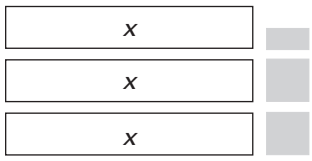
TEACHER NOTES:

Follow-up Student Activity 12.5

NAME _____ **DATE** _____

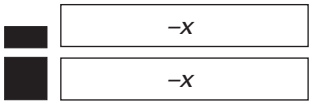
1 Using the given xth arrangement from a continuous sequence of arrangements, find the missing values in each table below. Explain how you determine the 3rd missing value in each table.

a)



x	$v(x)$
___	17
$-3\frac{1}{2}$	___
___	200
31	$90\frac{1}{2}$
x	___

b)



x	$v(x)$
55	___
___	–230
___	–121
___	178
x	___

2 For each of a)-d) below, sketch the graph of a straight line that satisfies the given conditions. Write an equation for the line.

a) slope of 3 and y-intercept of –2

b) passes through points (–2,–9) and (3,11)

c) passes through (2,1) and x-intercept is 3

d) slope is 0 and passes through (–3,–7)

3 Equations a) and b) below each represent the value of the xth arrangement of a continuous sequence of arrangements. Graph a) and b). Label the coordinates of the points for all x-intercepts and y-intercepts and the "turning points" of the graphs.

a) $v(x) = (x + 4)(x - 3)$

b) $v(x) = (2 - x)(5 + x)$

4 For each pair of equations given below: i) sketch the graphs of $v_1(x)$ and $v_2(x)$ on the same coordinate axes; ii) find a close approximation of the coordinates of the points where the graphs intersect; and iii) show the methods you use to approximate the intersection points.

a) $v_1(x) = {}^{-x}/_{2} + 8$ $\quad v_2(x) = 3x - 5$

b) $v_1(x) = (x + 4)(x - 3)$ $\quad v_2(x) = -x + 3$

c) $v_1(x) = x^2 - x - 12$ $\quad v_2(x) = -x^2 + x$

(Continued on back.)

Follow-up Student Activity (cont.)

5 For each pair of graphs in Problem 4 write an inequality statement that tells when these conditions are satisfied:

i) The graph of $v_1(x)$ is above the graph of $v_2(x)$;
ii) The graph of $v_1(x)$ is below the graph of $v_2(x)$.

6 Sketch the xth arrangement of a sequence of arrangements for which $v(x) = x^2 + 6x - 7$. Arrange the pieces to illustrate the factored form of the quadratic expression $x^2 + 6x - 7$. Then sketch the graph of $v(x)$ and do the following, if possible:

a) Label the x-intercepts and y-intercepts of $v(x)$.

b) Determine the value of x if $v(x) = 14\frac{1}{4}$.

c) Find all values of x for which $v(x) < 9$.

d) Find all values of x for which $v(x) > 9$.

e) Place an M where you think $v(x)$ stops decreasing and starts increasing.

f) Explain how each of your answers for a)-e) relates to the sequence of arrangements for which $v(x) = x^2 + 6x - 7$.

7 For each of the following: i) graph the situation; ii) write 2 thoughtful mathematical questions whose answers can be determined from the graphs; iii) write the answers to your questions.

a) Maria can burn 4 calories per minute by using a treadmill. Sketch a graph that shows the amount of calories burned as related to the number of minutes she works out on the treadmill.

b) Bob's neighbor agreed to pay Bob \$8 for adjusting his lawn mower plus \$2 for every hour it runs without breaking down. Make a graph of the amount of money Bob will receive as related to the number of hours the lawn mower runs.

c) A seagull flying 80 feet above the ground drops a clam shell. The height of the shell can be represented by the equation $v(x) = 80 - 16x^2$, where x is the number of seconds since the seagull dropped the shell. Sketch a graph that shows the height as related to the number of seconds after release.

Connector Student Activity 12.1

NAME _____ **DATE** _____

The 1st column of each table below lists, in order, the arrangement numbers of a sequence of counting piece arrangements, and the 2nd column lists the values of the arrangements. For each sequence:

a) Sketch 4 consecutive arrangements. Label the number and value of each arrangement.

b) Sketch the *n*th arrangement of the sequence.

c) On a coordinate grid, plot ordered pairs that represent several arrangements.

d) Fill in the blanks in the table.

e) Write a concise mathematical description of the set of all arrangement numbers in the sequence. Then describe mathematically the set of all numbers that are the values of the arrangements.

1

n	$v(n)$
1	4
2	9
3	14
4	19
5	24
⋮	⋮
13	__
14	__
⋮	⋮
__	154
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

2

n	$v(n)$
⋮	⋮
–3	18
–2	8
–1	2
0	0
1	2
2	8
⋮	⋮
__	450
⋮	⋮
43	__
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

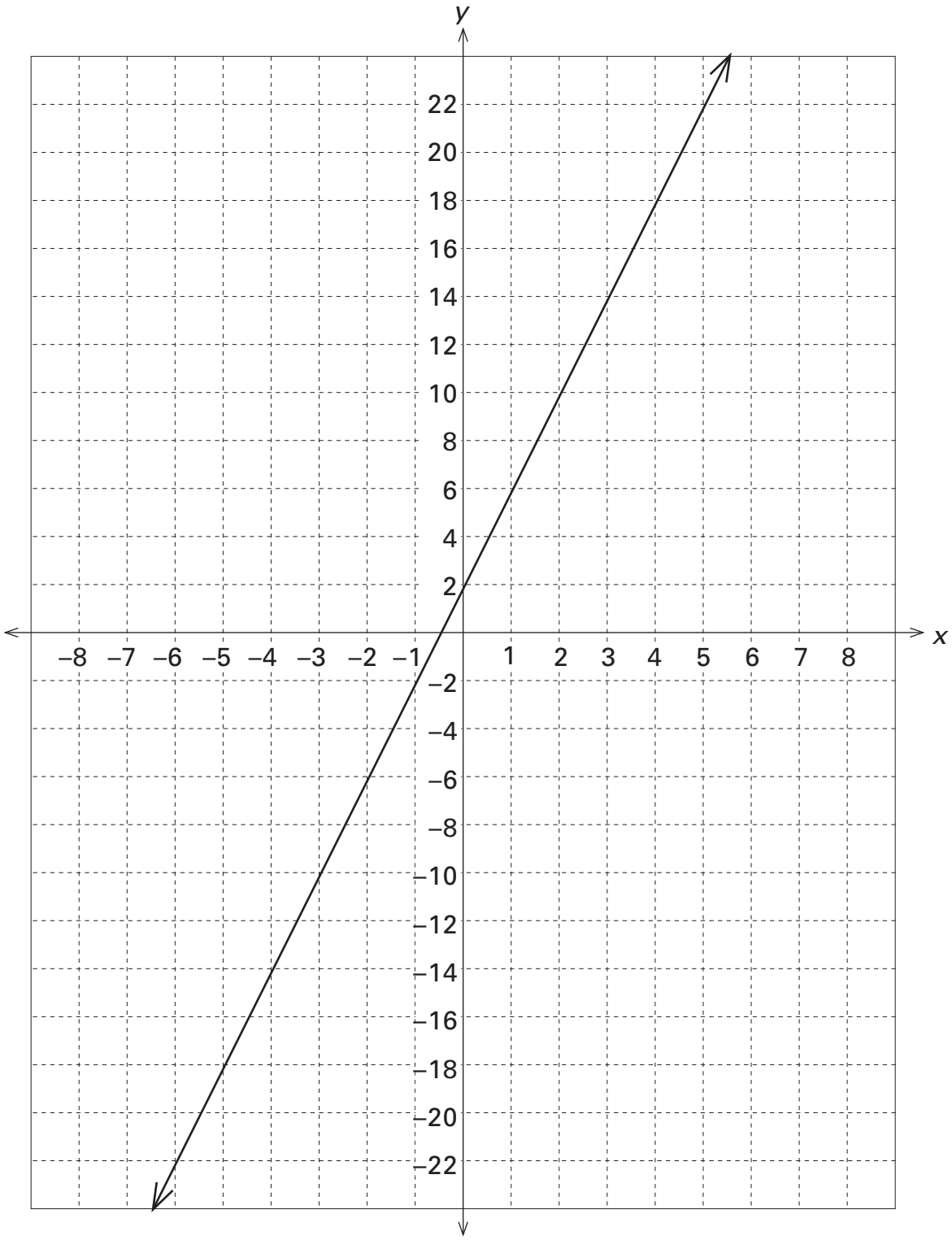
3

n	$v(n)$
–1	1
–2	2
–3	3
–4	4
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

4

n	$v(n)$
1	3
2	–4
3	9
4	–8
5	15
6	–12
⋮	⋮
n	$v(n)$ = _____
⋮	⋮

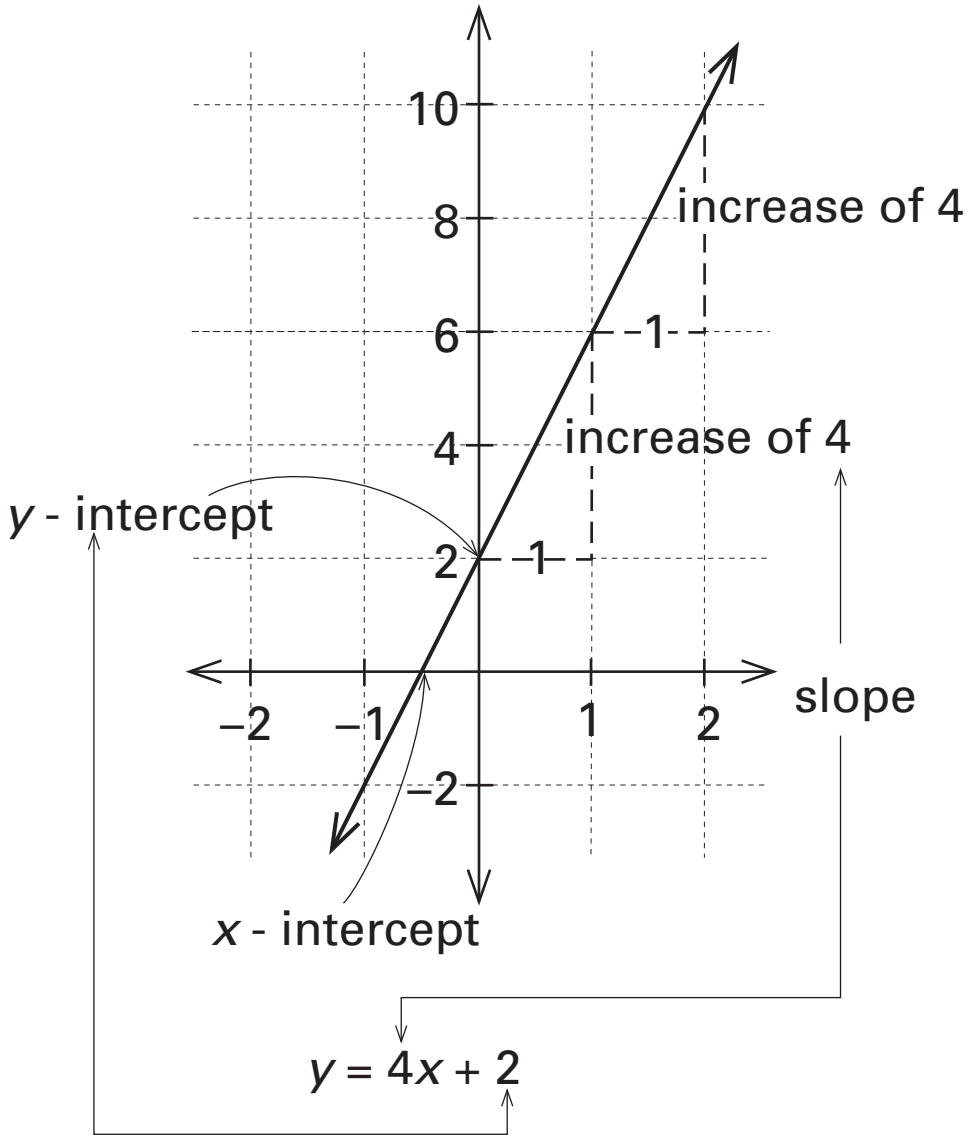
Focus Master A



$v(x)$ = _____

Sketch the xth arrangement:

Focus Master B



Focus Master C

a) Suppose that the value of the xth arrangement of a certain continuous sequence of arrangements is, $v_1(x) = \frac{1}{2}x^2 - x - 8$. Graph v_1 and determine the value of x if $v_1(x) = 4$; if $v_1(x) = -2.5$.

b) Suppose the xth arrangement of a 2nd continuous sequence has value $v_2(x) = 4$, and a 3rd has value $v_3(x) = -2.5$. Sketch the graphs of v_2 and v_3 on the same coordinate axes as v_1.

c) Circle and label the coordinates of the points where v_2 and v_3 intersect v_1. Explain how these points relate to the sequences of arrangements represented by the graphs.

d) Find all the values of x for which $\frac{1}{2}x^2 - x - 8 > 4$. Explain the relationship between these values and the sequences of arrangements for v_1 and v_2.

e) Find all the values of x for which $\frac{1}{2}x^2 - x - 8 < -2.5$. Relate these values to the sequences of arrangements for v_1 and v_3.

f) Draw a box around and label the coordinates of all x-intercepts of the graphs of v_1, v_2, and v_3. Relate your results to the sequences of arrangements.

g) Label the coordinates of the y-intercepts, if any, of the graphs of v_1, v_2, and v_3 and place a black dot at each y-intercept. Relate these points to the sequences of arrangements.

h) Place a V where you think v_1 stops decreasing and begins to increase. Explain how this turning point relates to the sequence of arrangements for v_1.

i) Challenge: give the coordinates of the turning point of v_1. Explain your methods.

Focus Master D

Each equation listed below describes the value of the xth arrangement of a continuous sequence of arrangements. For each pair of equations, complete a)-e). Record your responses to b)-e) next to each graph.

Pair 1: $v_1(x) = \frac{2}{3}x + 7$
$v_2(x) = 0$

Pair 2: $v_1(x) = 3x^2 - 9$
$v_2(x) = 0$

Pair 3: $v_1(x) = \frac{1}{2}x^2 + 3x - \frac{27}{2}$
$v_2(x) = -4x - 3$

Pair 4: $v_1(x) = (x - 4)(x - 2)$
$v_2(x) = -x^2 + 6x - 8$

a) Graph the pair on the same coordinate axes.

b) Sketch the xth arrangement of each sequence.

c) Find the value(s) of x, if any, for which $v_1(x) = v_2(x)$. Explain your methods and tell how you checked to be sure you are correct. Are the values exact or approximate?

d) Find the x-intercepts and y-intercepts, if any, of v_1 and v_2. Explain your methods and tell how you checked to be sure you are correct. Are your answers exact or approximate?

e) Write two inequality statements that describe relationships between the two graphs.

f) List your observations or conjectures.

Focus Master E

For each of the following situations, make (and label carefully) the indicated graph, and next to each graph write the following:

a) a mathematical formula that represents the relationships in the situation,

b) 2 mathematical conclusions that are based on information that you can "see" in your graph.

Situation 1 Jonathan earns $3 per hour baby-sitting. Make a graph that shows the amount Jonathan earns as related to the number of hours he works.

Situation 2 Erica's grandfather gives her $25 each year on her birthday. In addition, he gives her $1.50 for every year of her age. Make a graph that shows the amount of the birthday gift Erica receives from her grandfather as related to her age on her birthday.

Situation 3 The students in Ms. Cooper's math class are raising money for a field trip. The Whatsit Production Company has agreed to donate $5 for every hour a student works on roadside cleanup, plus an additional $10 if the student works more than 5 hours. Make a graph of the amount donated as related to the number of hours 1 student works.

Situation 4 When an object is dropped from an initial height of h_0 feet above the ground, it falls at a rate of 16 feet per second squared (Galileo observed this in 1604). Linden dropped a marble

(Continued on back.)

Focus Master E (cont.)

from the top of a 60-foot building into a pond at the base of the building. At the same time, Joel, who was standing next to Linden, dropped a brick. Make a graph that shows the height of the marble as related to the number of seconds after it is dropped. Do the same for the brick.

Situation 5 Katie and Malia designed the following game:

- The game board contains a number line. Both players start with a game marker at the origin of the number line (this is their first board position).

- Katie rolls a pair of standard dice (1 red die and 1 black) to determine where to move her marker. The number showing on the black die tells the number of spaces to move her game marker in the positive direction on the number line. The number showing on the red die tells the number of spaces to move the game marker in the negative direction.

- The number of points Katie earns is equal to the distance between the origin and her game marker after completing the moves indicated by the dice.

- Next Malia repeats the above procedures. Play continues until a player earns 25 points.

Graph all the possible ordered pairs (x,y) where x is a possible location of a player's marker after the first round of play and y is the number of points earned by landing in that position.

Focus Student Activity 12.2

NAME _____ **DATE** _____

A

B

C

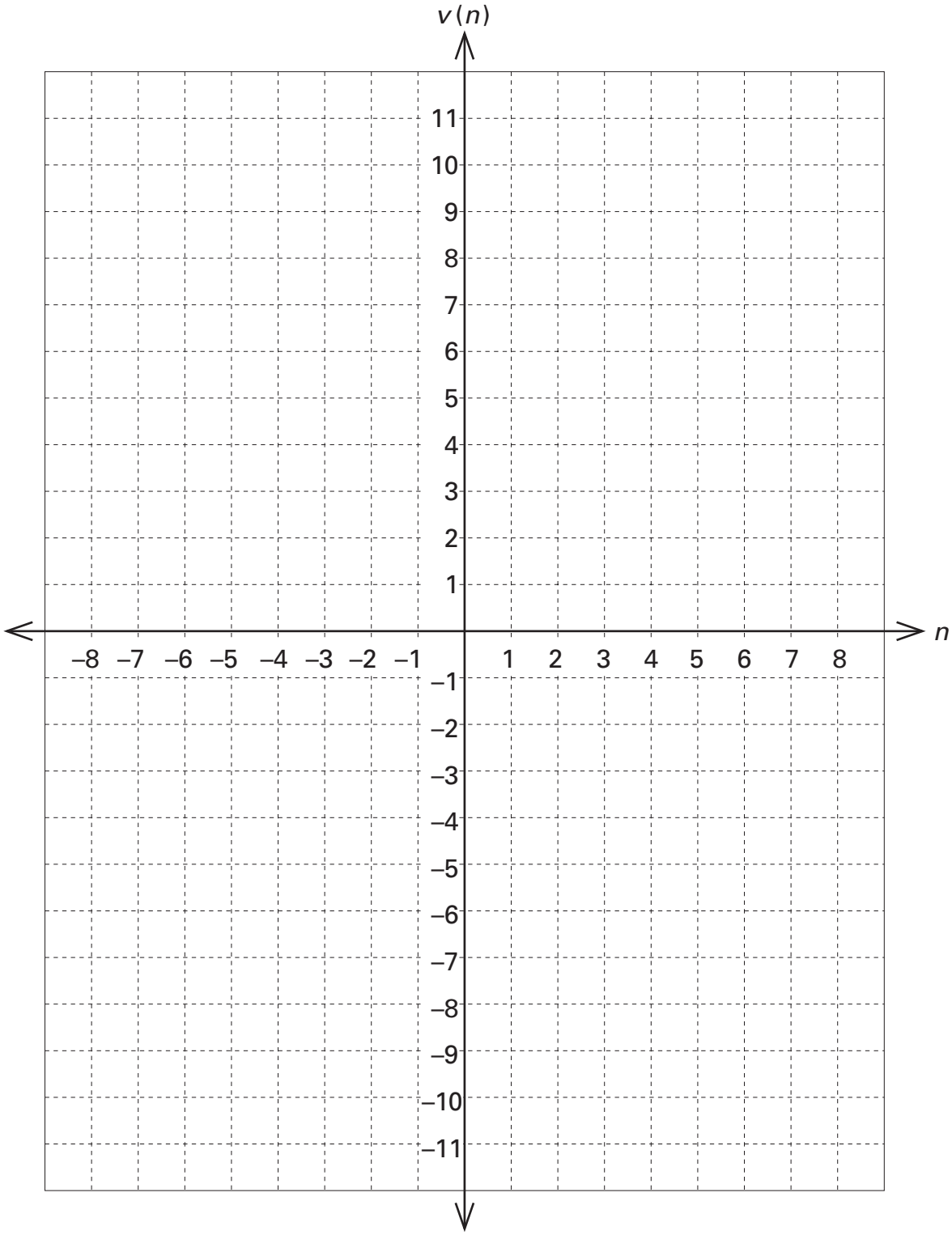
D

E

F

Focus Student Activity 12.3

NAME _____ **DATE** _____



$v(n) =$ _____

Focus Student Activity 12.4

NAME _____ **DATE** _____

Although you will have many opportunities during this course to become familiar with your graphing calculator, it will be helpful if you are comfortable with the functions listed below as soon as possible. Please investigate each function on your calculator and, if needed, in your calculator manual. A way to test yourself to see if you can comfortably use and recall a calculator function is to demonstrate its use to someone else (a family member, a classmate, a neighbor, etc.). Try to check off all functions in Part 1 below by the following date _____.

1 I am comfortable using the following calculator functions:

____ ON/OFF
____ CLEAR the screen
____ show blank coordinate axes in the calculator viewing screen
____ move the cursor around a blank coordinate axes
____ change the viewing WINDOW size
____ FORMAT the axes
____ determine the "standard" WINDOW size on my calculator (on many it is $-10 \le x \le 10$ and $-10 \le y \le 10$)
____ enter an equation y =
____ GRAPH an equation y =
____ TRACE a graph (What shows on the screen when you do this?)
____ ZOOM in on a graph
____ ZOOM in again—and again
____ ZOOM out on a graph
____ ZOOM back to the standard window
____ TRACE the graph of a function to determine the approximate value of the function at $x = 0$, $x = 19.75$, and $x = -37.5$
____ TRACE the graph of a function to determine the value of x when $y = 75$, when $y = -75$
____ GRAPH 2 equations on the same coordinate axes.
____ TRACE to approximate the intersection of 2 graphs
____ ZOOM and TRACE to improve your approximation
____ DRAW a horizontal line on coordinate axes and slide the line up and down
____ DRAW a vertical line on coordinate axes and slide the line left and right

(Continued on back.)

Focus Student Activity 12.4 (cont.)

____ view a TABLE of x- and y-coordinates of an equation
____ view a table of coordinates of 2 equations listed simultaneously
____ use a table to find when $0 = 5x + 1$
____ clear MEMory
____ reset defaults in MEMory
____ solve equations using the "solver" function from the MATH menu
____ use the "maximum" and minimum" functions from the CALC menu to find the turning point of a parabola
____ use the "intersect" function from the CALC menu to find the intersection of 2 graphs
____ use the "zero" function from the CALC menu to find the x-intercepts of a graph
____ use the "value" function from the CALC menu to find $v(x)$ for specific values of x
____ set the graphing style to shade the region above a graph; the region below a graph

2 Here are some other graphing calculator functions that I can use:

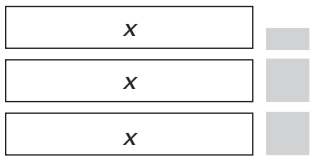
3 Here are some other functions I have tried but don't understand.

Follow-up Student Activity 12.5

NAME _____ **DATE** _____

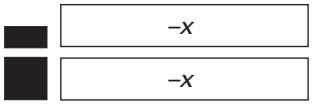
1 Using the given xth arrangement from a continuous sequence of arrangements, find the missing values in each table below. Explain how you determine the 3rd missing value in each table.

a)



x	$v(x)$
___	17
$-3\frac{1}{2}$	___
___	200
31	$90\frac{1}{2}$
x	___

b)



x	$v(x)$
55	___
___	–230
___	–121
___	178
x	___

2 For each of a)-d) below, sketch the graph of a straight line that satisfies the given conditions. Write an equation for the line.

a) slope of 3 and y-intercept of –2

b) passes through points (–2,–9) and (3,11)

c) passes through (2,1) and x-intercept is 3

d) slope is 0 and passes through (–3,–7)

3 Equations a) and b) below each represent the value of the xth arrangement of a continuous sequence of arrangements. Graph a) and b). Label the coordinates of the points for all x-intercepts and y-intercepts and the "turning points" of the graphs.

a) $v(x) = (x + 4)(x - 3)$ b) $v(x) = (2 - x)(5 + x)$

4 For each pair of equations given below: i) sketch the graphs of $v_1(x)$ and $v_2(x)$ on the same coordinate axes; ii) find a close approximation of the coordinates of the points where the graphs intersect; and iii) show the methods you use to approximate the intersection points.

a) $v_1(x) = {}^{-x}/_{2} + 8$ $v_2(x) = 3x - 5$

b) $v_1(x) = (x + 4)(x - 3)$ $v_2(x) = -x + 3$

c) $v_1(x) = x^2 - x - 12$ $v_2(x) = -x^2 + x$

(Continued on back.)

Follow-up Student Activity (cont.)

5 For each pair of graphs in Problem 4 write an inequality statement that tells when these conditions are satisfied:

i) The graph of $v_1(x)$ is above the graph of $v_2(x)$;
ii) The graph of $v_1(x)$ is below the graph of $v_2(x)$.

6 Sketch the xth arrangement of a sequence of arrangements for which $v(x) = x^2 + 6x - 7$. Arrange the pieces to illustrate the factored form of the quadratic expression $x^2 + 6x - 7$. Then sketch the graph of $v(x)$ and do the following, if possible:

a) Label the x-intercepts and y-intercepts of $v(x)$.

b) Determine the value of x if $v(x) = 14\frac{1}{4}$.

c) Find all values of x for which $v(x) < 9$.

d) Find all values of x for which $v(x) > 9$.

e) Place an M where you think $v(x)$ stops decreasing and starts increasing.

f) Explain how each of your answers for a)-e) relates to the sequence of arrangements for which $v(x) = x^2 + 6x - 7$.

7 For each of the following: i) graph the situation; ii) write 2 thoughtful mathematical questions whose answers can be determined from the graphs; iii) write the answers to your questions.

a) Maria can burn 4 calories per minute by using a treadmill. Sketch a graph that shows the amount of calories burned as related to the number of minutes she works out on the treadmill.

b) Bob's neighbor agreed to pay Bob \$8 for adjusting his lawn mower plus \$2 for every hour it runs without breaking down. Make a graph of the amount of money Bob will receive as related to the number of hours the lawn mower runs.

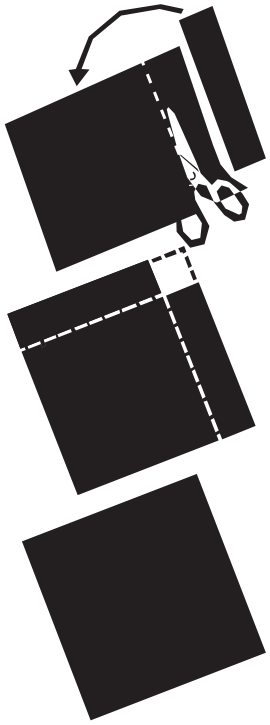
c) A seagull flying 80 feet above the ground drops a clam shell. The height of the shell can be represented by the equation $v(x) = 80 - 16x^2$, where x is the number of seconds since the seagull dropped the shell. Sketch a graph that shows the height as related to the number of seconds after release.

Modeling Situations

Lesson 13

THE BIG IDEA

The use of diagrams and sketches and concrete models to represent the mathematical relationships in a situation is a powerful problem-solving strategy that fosters insights about the concept of a variable, reinforces the notion that symbols are representations of "real" relationships and actions, and provides important conceptual links between algebra and geometry.



CONNECTOR

OVERVIEW

Students investigate L-shapes and use those shapes as a basis for generalizing about squares and their differences.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 transparency.
- ✔ Miscellaneous materials (e.g., scissors, grid paper, Algebra Pieces, geoboards or geoboard paper, and tile), available for each pair of students.

FOCUS

OVERVIEW

Students use sketches and diagrams to model the mathematical relationships in various everyday and mathematical situations. Students reason from the sketches to solve "puzzle problems" involving the situations, and in some cases, invent symbolic ways of representing their reasoning and methods.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Masters A, D, and E, 1 transparency of each.
- ✔ Focus Masters B, C, and F, 1 copy of each per pair of students and 1 transparency of each.

FOLLOW-UP

OVERVIEW

Students investigate and report their findings about rectangles that can be dissected into congruent staircases. They use diagrams and sketches to represent and solve a variety of puzzle problems.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 13.1, 1 copy per student.

LESSON IDEAS

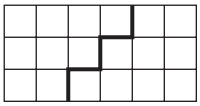
ASSESSMENT
Problem 1 from the Follow-up could be used as a student portfolio entry. If used for this purpose, you might provide (or have students create) an expanded assessment guide (see *Starting Points*).

The following questions may be useful for encouraging detailed responses to Problem 1: Can some rectangles be cut into 2 congruent staircases in more than one way, and if so, which rectangles and in how many ways? Can all "odd by even" rectangles be cut into 2 congruent staircases? What numbers can be written as the sum of consecutive integers and how does this relate to the problem? What happens if the steps are 2 or more grid squares high and wide? Etc.

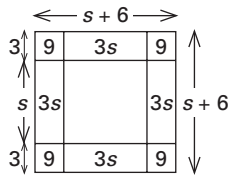
QUOTE
Different representations of problems serve as different lenses through which students interpret the problems and the solutions. If students are to become mathematically powerful, they must be flexible enough to approach situations in a variety of ways and recognize the relationships among different points of view.
NCTM Standards

SELECTED ANSWERS

1. The sides of the rectangle must both be greater than 1; the length of 1 dimension must be an odd number and the other an even number, e.g., see 3×6 rectangle at the right. See Lesson Ideas above for questions to prompt other ideas.



2-3. a) Inner Square X has sides of length *s*.

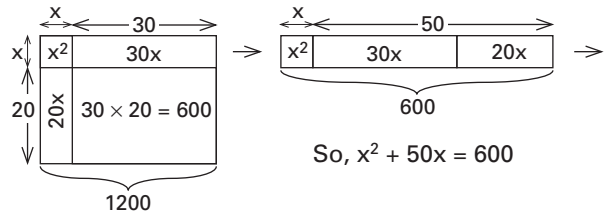


$$4(9) + 4(3s) = 63$$
$$36 + 12s = 63$$
$$12s = 27$$
$$s = 2\tfrac{1}{4}$$

So, perimeter of Square X is 9.

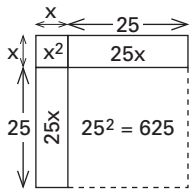
b) Average speeds are 40 and 60 miles per hour.

c)



So, $x^2 + 50x = 600$

$x^2 + 30x + 20x + 600 = 1200$

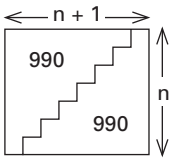


Thus, $x^2 + 50x + 625 = 600 + 625$
or, $(x + 25)^2 = 1225$
So, $x + 25 = \sqrt{1225}$
$x = 35 - 25 = 10$

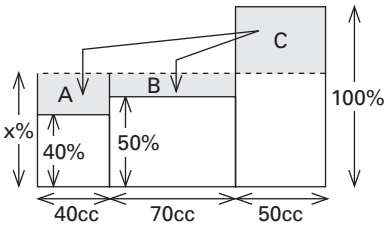
So, strips will have widths of 10 meters.

d) 2 and 9
e) 12 feet by 15 feet

f) Two copies of Kay's staircase have 1980 tile. $44 \times 45 = 1980$, so $n = 44$.



g) $\sqrt{\frac{18}{5}}$
h) 69, 71, and 73
i) $10 + 10\sqrt{2}$
j) 120
k) The pure acid in Region C is used to level off the acid in Regions A and B at a height *x*.



$$A + B = C$$
$$40(x - .40) + 70(x - .50) = 50(1.00 - x)$$
$$40x - 16 + 70x - 35 = 50 - 50x$$
$$160x = 101$$
$$x \approx .63$$

So, a 63% acid solution results.

l) 50cc
m) shilling = \$.14 and franc = \$.204
n) There are 52 black balls and 32 white balls. The probability of selecting a white ball is $32/84 = 8/21 \approx 38\%$.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students investigate L-shapes and use those shapes as a basis for generalizing about squares and their differences.

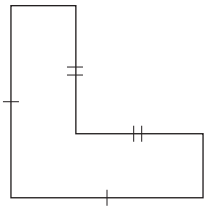
MATERIALS

✔ Connector Master A, 1 transparency.

✔ Miscellaneous materials (e.g., scissors, grid paper, Algebra Pieces, geoboards or geoboard paper, and tile), available for each pair of students.

ACTIONS

1 Arrange the students in pairs and sketch the following L-shape on the overhead.



L-Shape

Ask the students to list some possible questions for investigation about L-shapes. Record these questions.

2 Select questions posed by the students during Action 1 and/or place a transparency of Connector Master A on the overhead and ask the pairs to investigate ways to use models of L-shapes as the basis for answering these questions. Call attention to materials (e.g., scissors, grid paper, Algebra Pieces, geoboards or geoboard paper, and tile) that are available for student use.

Modeling Situations Lesson **13**

Connector Master A

For each of the following, suppose that whole square tile are used to form L-shapes.

a) Can 24 square tile be arranged in an L-shape? If so, in how many ways? What about 36 tile? 45 tile?

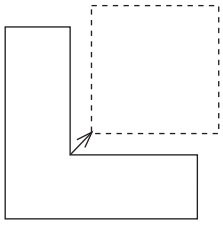
b) Determine ways to dissect an L-shape, using only straight cuts along edges of whole tile, so the pieces can be reassembled to form a rectangle. What is the minimum number of cuts necessary for any L-shape?

c) What rectangles can be dissected with exactly 1 straight cut (along edges of whole tile) and reassembled to form an L-shape?

d) What counting numbers can be written as the difference of 2 perfect squares? of 2 consecutive perfect squares? Why?

COMMENTS

1 Notice that an L-shape is formed by removing a square from one corner of a larger square:

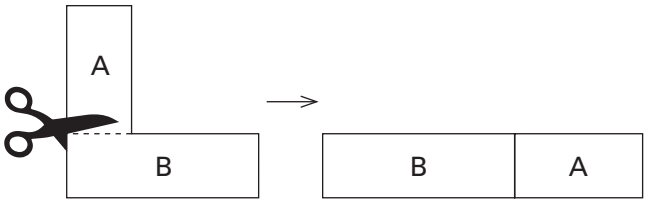


Students questions may include those listed on Connector Master A (see Comment 2), and they may pose others.

2 The intent here is to emphasize the mathematical information and relationships that can be accessed and proven by working with models of L-shapes. Following are observations that may come up in response to the questions on Connector Master A.

a) There are 2 ways that 24 square tile can be arranged into L-shapes (as a 7 by 7 square with a 5 by 5 square cut from the upper right corner, and as a 5 by 5 square with a 1 by 1 square cut from the upper right corner); 1 way for 36 tile; and 3 ways for 45 tile. Some students may investigate L-shapes for other numbers of tile and conjecture about L-shapes made from *n* square tile. Note that investigating questions b)-d) may provide insights about the general case.

b) The following single cut on any L-shape produces 2 pieces that contain only whole tiles and can be reassembled to form a rectangle:



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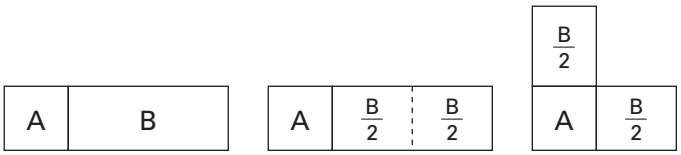
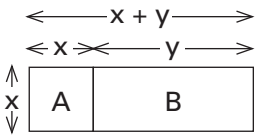
Connector Teacher Activity (cont.)

ACTIONS

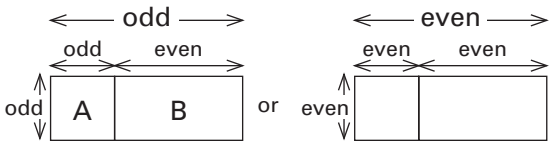




COMMENTS

2 (continued.)
c) Not all rectangular arrangements of tile can be dissected with a single straight cut and reassembled to form an L-shape. If a square (e.g., see region A in the diagram at the left) is marked off the end of a rectangle, the remaining rectangle (see region B) must be cut into 2 equal parts in order to make an L-shape. Hence, assuming that only whole tile are allowed, the length of region B must be an even number.

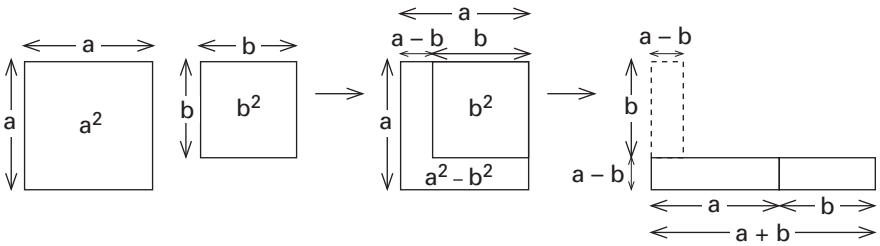
Notice that, since the sum of 2 even numbers is an even number and the sum of an even number and an odd number is an odd number (see *Math Alive! Course II,* Lesson 4), whenever the length x in the diagram at left is even and the length $(x + y)$ is even, then the length y must be even; similarly, if x is even and $(x + y)$ is odd, then y must be odd. Therefore, only rectangles whose dimensions are both even or both odd can be dissected with a single straight cut and reassembled to form an L-shape containing only whole tile. That is, the dimensions must be of the same *parity*, as illustrated below:



d) Students responses may vary; as they share their ideas, you might ask them to determine whether their various conclusions are equivalent.

Based on the results of c) above, students may notice that a given number can be written as the difference of 2 perfect squares if an L-shape can be formed with area equal to the given number. This is possible if the number can be expressed as the product of 2 even numbers or 2 odd numbers.

Further, for any 2 squares, the following diagram shows why $a^2 - b^2 = (a - b)(a + b)$:



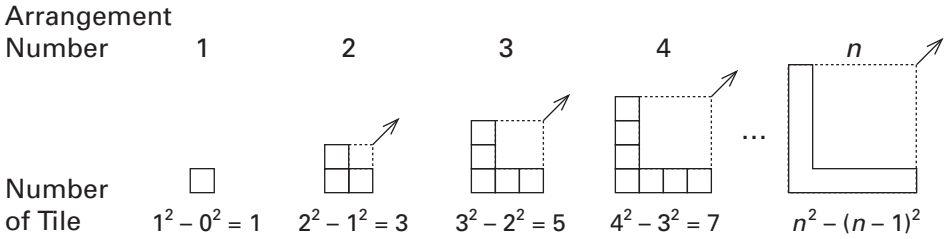
Students may notice that the difference between *consecutive* perfect squares, n^2 and $(n - 1)^2$ is always an L-shape

Connector Teacher Activity (cont.)

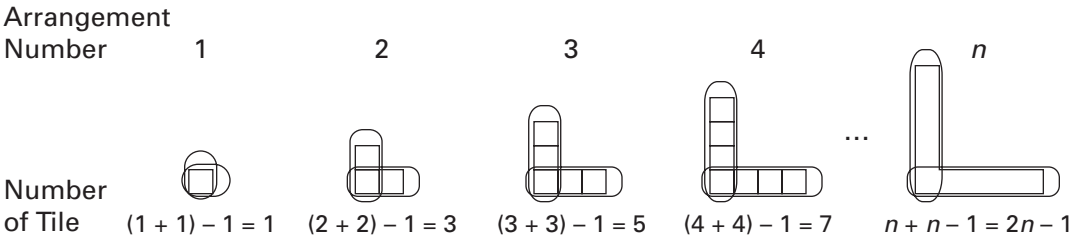
ACTIONS

COMMENTS

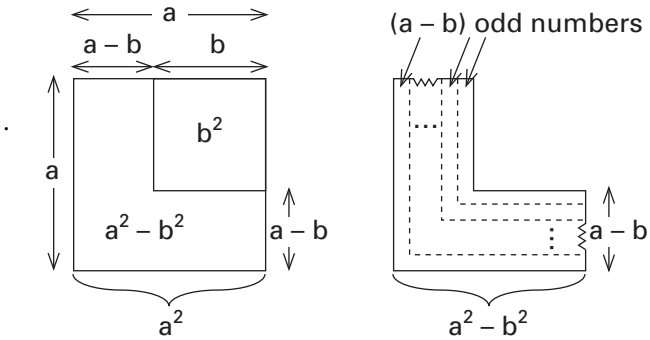
that is only 1 tile wide, and the number of tile in such an L-shape is the *n*th odd number, as illustrated by the following sequence of tile arrangements.



Some students may comment that the *n*th odd number can be written in the form $2n - 1$, for all whole numbers *n*, based on prior experiences with the following sequence of tile arrangements (see diagram below and see *Math Alive! Course II*, Lesson 4). Thus, $n^2 - (n - 1)^2 = 2n - 1$.



Students may make other observations that relate differences of squares to odd and even numbers. For example, the following diagram shows why the number of tile in any L-shape that is more than 1 tile wide is the sum of 2 or more consecutive odd numbers:



Notice also that an L-shape formed by the difference of 2 perfect squares, a^2 and b^2, contains an odd number of tile if $a - b$ is odd, since the sum of an odd number of odd numbers is an odd number. Further, the L-shape contains an even number of tile if $a - b$ is even, since the sum of an even number of odd numbers is an even number. Some students may point out that if an L-shape contains an even number of tile, that number must be divisible by 4.

Connector Teacher Activity (cont.)

ACTIONS

3 Sketch an L-shape (see diagram for Action 1) on the overhead once again. Ask for volunteers to offer several conjectures and generalizations about L-shapes together with any new questions that may be interesting to investigate.

COMMENTS

3 Students' statements and questions may vary depending on the questions investigated in Action 2. Based on the needs and interest of your students, you could suggest for investigation some ideas from Comment 2 that didn't come up.

Focus Teacher Activity

OVERVIEW & PURPOSE

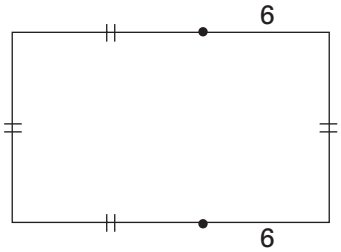
Students use sketches and diagrams to model the mathematical relationships in various everyday and mathematical situations. Students reason from the sketches to solve "puzzle problems" involving the situations, and in some cases, invent symbolic ways of representing their reasoning and methods.

MATERIALS

- ✔ Focus Masters A, D, and E, 1 transparency of each.
- ✔ Focus Masters B, C, and F, 1 copy of each per pair of students and 1 transparency of each.

ACTIONS

1 Arrange the students in pairs and sketch a copy of the following rectangle on the overhead.



Ask the students to describe what they see. After they have had an opportunity to respond, ask them what more they can say about the rectangle if its perimeter is 56 units. Discuss the students' observations.

COMMENTS

1 Following are examples of students' observations:

The quadrilateral is a rectangle.

Opposite sides of the figure are equal and parallel.

One dimension is 6 units longer than the other.

Once they know the perimeter, students may conclude the following:

Half the perimeter is 28, so 2 of the unknown segments total 28 – 6 = 22 linear units.

If the unknown lengths are d, then an equation that represents the perimeter is 4d + 12 = 56 linear units.

Some students may use "symbolic shorthand" to record their thought processes, and give verbal explanations based on the diagram. For example:

Explanation of thinking:	Recording:
The perimeter consists of 2 segments of length 6, and 4 other segments all of equal length—let's call this length d—and since the perimeter is 56, these lengths total 56.	$12 + 4d = 56$
The 4 segments have a total length of 56 – 12 or 44.	$4d = 56 - 12 = 44$
Thus, the length of each segment is 44 ÷ 4 or 11.	$d = 44 \div 4 = 11$
So, the dimensions of the rectangle are 11 and 11 + 6.	width = d = 11 length = $d + 6$ = 17

Notice, in the above case, the algebraic equations reflect a chain of thought based on the student's knowledge and insight.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

2 Ask the students to draw a sketch, using as few words and symbols as possible, that portrays a rectangle of unknown dimensions whose length is 4 units longer than 3 times its width. Have several students replicate their drawings on the chalkboard. Discuss whether the drawings adequately convey the information given about the rectangle and whether the words and symbols used are essential.

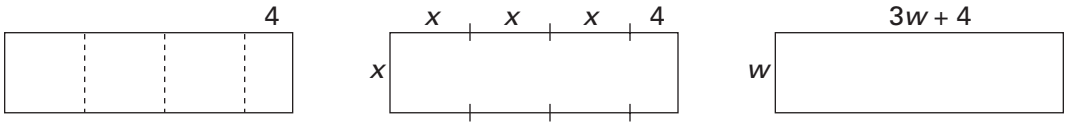
3 Ask the students to suppose the perimeter of the rectangle they drew in Action 2 is 48 inches. Then ask them to determine the dimensions of the rectangle. Ask for volunteers to describe their thinking.

COMMENTS

1 (continued.)
Note: Some students may question the assumption that the sketch is a rectangle since no right angles are marked. It is often the case that *sketches* do not include such markings; however, the class may prefer to set a policy that right angles be marked. Students may find the following related question interesting to investigate: If opposite sides of a quadrilateral are congruent, does one need to be given that 0, 1, 2, 3, or all 4 angles are right angles in order to prove the quadrilateral is a rectangle? (Since opposite sides are congruent, the quadrilateral is a parallelogram; since opposite angles of a parallelogram are congruent and consecutive angles are supplementary, if 1 angle of a parallelogram is a right angle, then it is possible to show that all angles must be right angles.)

2 Having the students draw sketches of a situation before a problem is posed focuses their attention on creating a sketch that portrays the essential features of the situation.

Below are some possible sketches. Notice that, in the last sketch shown, the essential information is carried in the symbols and not the sketch—that is, if the symbolic phrase "$3w + 4$" is erased, the distinguishing feature of the rectangle is lost.



3 The students will use various methods to arrive at the dimensions. One way is to note that the perimeter of 48 inches consists of 2 segments of length 4 and 8 other segments of equal length. Hence, the lengths of the 8 segments total 40 inches, so each is 5. Thus, the dimensions of the rectangle are 5 inches and $3 \times 5 + 4 = 19$ inches.

Focus Teacher Activity (cont.)

ACTIONS

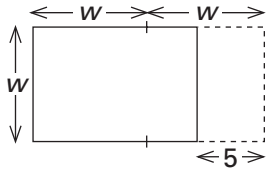
4 Repeat Action 2 for a rectangle whose length is 5 inches less than twice its width. Then ask the students to determine the dimensions of the rectangle if its perimeter is 32 inches. Have several students show their sketches and describe their thinking in determining the dimensions of the rectangle.

5 Ask the students to sketch a square. Then have them sketch an equilateral triangle whose sides are 2 units longer than the sides of the square. Ask the students to reason from their diagrams to determine the length of the side of the square if the square and the triangle have equal perimeters. Ask for volunteers to show their sketches and describe their thinking.

6 Ask the students to draw diagrams or sketches which represent a number and that number increased by 6. Show the various ways in which students have done this. Then ask the students to use one of the sketches to determine what the numbers are if their sum is 40.

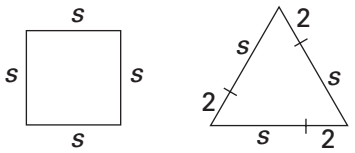
COMMENTS

4 To emphasize the mathematical relationships in the rectangle, it is helpful to discuss the students' sketches before telling them the perimeter of the rectangle. Here is one sketch:

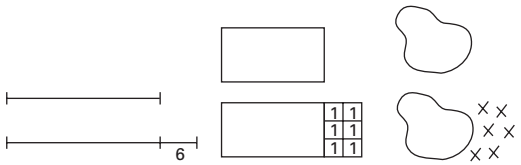


The extended rectangle shown above has a perimeter of 42 inches—10 inches longer than the original rectangle. These 42 inches are composed of 6 equal lengths. So each of these lengths is 7 inches. The width of the original rectangle is one of these lengths, or 7 inches; the length of the rectangle is 5 inches less than 2 of these lengths, or 9 inches.

5 The perimeter of the square, in the following drawing, contains 4 segments of length *s*; that of the equilateral triangle contains 3 segments of length *s* and 3 of length 2. Thus, the 3 segments of length 2 must sum to *s*. So *s* is 6.



6 Since numbers have no particular shape, the students must invent a way of portraying number. They might do this in a variety of ways, e.g., as a length or as an area or as a "blob."



Three sketches of a number and that number plus 6.

Looking at the sketch on the left above, the sum of the lengths of the segments portraying the numbers is 40. The small segment has length 6. Hence, the sum of lengths of the other 2 segments is 34. Since these 2 segments are congruent, the length of each is 34 ÷ 2, or 17. Hence the 2 numbers are 17 and 17 + 6, or 23.

Focus Teacher Activity (cont.)

ACTIONS

7 Ask the students to draw sketches that represent 2 numbers such that 4 times the smaller number is 1 less than the larger. Then ask them to reason from their sketches to determine the numbers if their sum is 36. Ask for volunteers to show their sketches and explain their reasoning.

8 Read aloud Part 1 shown below and ask the students to make a sketch that illustrates the mathematical relationships in the given situation. Then read aloud Part 2 and ask the students to add to their sketches to show this new information. Invite volunteers to make mathematical observations by reasoning from their sketches. Invite volunteers to share their observations and reasoning. If no one suggests it, ask the students to show how to reason from a diagram to determine how much money and what coins Larry and Mike each have.

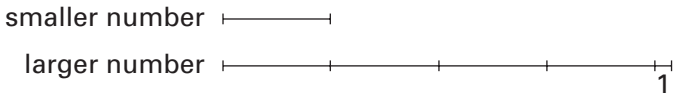
Part 1: **Mike has 3 times as many nickels as Larry has dimes.**

Part 2: **Mike has 45¢ more than Larry.**

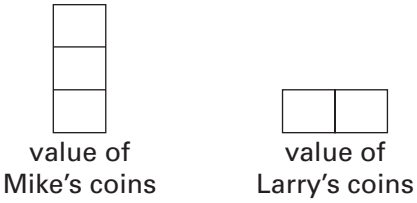
9 Place a transparency of Focus Master A on the overhead revealing *only the first sentence* of Situation A. Ask the students to make a quick sketch to represent the given information. Then reveal the second sentence and ask the students to adapt their sketches to include this information. Repeat for the 3rd sentence. Then ask the students to *reason from their sketches* to make mathematical observations and conclusions about the situation. Discuss, inviting volunteers to share their sketches, conclusions, and reasoning at the overhead. If no one suggests it, ask the students to reason from their diagrams to determine the number of people in each group.

COMMENTS

7 If the sum of the 2 numbers is 36, in the sketch shown below, the sum of the lengths of the 5 congruent segments (1 in the smaller number and 4 in the larger number) is 36 – 1, or 35. Hence, the length of each is 35 ÷ 5, or 7. Thus the numbers are 7 and 4(7) + 1, or 29.

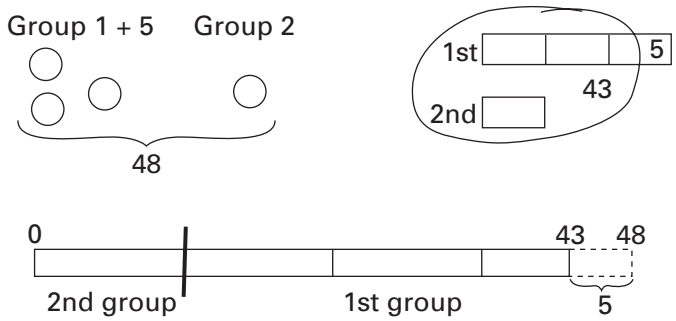


8 In the following sketch, the value of Mike's and Larry's coins are represented by boxes, all of which have the same value. Since Mike has 3 times as many coins as Larry, his stack of boxes is 3 times as high as Larry's. Larry's stack is twice the width of Mike's since each of Larry's coins is worth twice as much as each of Mike's.



Mike's stack contains 1 more box then Larry's. Since Mike has 45¢ more than Larry, this box is worth 45¢. Thus, Mike has 3 × 45¢ = \$1.35 in 3 × 9 = 27 nickels, while Larry has 2 × 45¢ = 90¢ in 9 dimes.

9 As examples, shown below are 3 different students' sketches of Situation a).



One can reason from each of the diagrams above to determine there are 31 people in the 1st group and 12 in the 2nd. If students have difficulty, you might copy these 3 sketches on the overhead and ask the students to speculate about ways students may have reasoned from each diagram to determine the number of people in each group.

Focus Teacher Activity (cont.)

ACTIONS

Modeling Situations Lesson **13**
Focus Master A

Situations

a) The people at a meeting are separated into 2 groups.
The 1st group has 5 less people than 3 times the number in the 2nd group.
There are 43 people at the meeting.

10 Slowly reveal the 4 sentences in Situation b) on Focus Master A, pausing between sentences so that students can create and adjust sketches as information is revealed. Then ask the students to *reason from their diagrams* to determine the 3 numbers. Invite volunteers to show their sketches at the overhead, without giving explanations, so that other students can speculate about the reasoning used. Then discuss the actual reasoning used by the volunteers. Where appropriate, encourage students to discuss ways to represent their visual solutions and thought processes with algebraic symbols and equations.

Modeling Situations Lesson **13**
Focus Master A

Situations

a) The people at a meeting are separated into 2 groups.
The 1st group has 5 less people than 3 times the number in the 2nd group.
There are 43 people at the meeting.

b) There are 3 numbers.
The 1st number is twice the 2nd number.
The 3rd is twice the 1st.
The sum of the 3 numbers is 112.

c) The sum of 2 numbers is 40.
Their difference is 14.

d) The sides of square A are 2 inches longer than the sides of square B.
The area of square A is 48 square inches greater than the area of square B.

e) Melody has $2.75 in dimes and quarters.
She has 14 coins altogether.

f) Three particular integers are consecutive.
The product of the 1st and 2nd integers is 40 less than the square of the 3rd integer.

g) Karen is 4 times as old as Lucille.
In 6 years, Karen will be 3 times as old as Lucille.

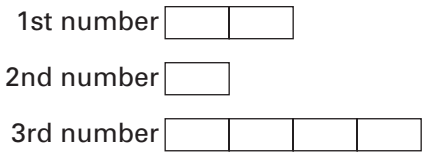
COMMENTS

It is intended here that emphasis be on modeling relationships and reasoning from a diagram rather than on getting answers. Revealing the information in parts and having the students make observations promotes such emphasis. Some students may point out that "guessing and checking" is one way to determine the number in each group; while this is true, such methods do not typically lead to generalizations or strong connections to algebra.

10 Some students may use Algebra Pieces or sketches of Algebra Pieces to represent some of these situations. This is appropriate.

It may be helpful to point out that reasoning from a diagram does not preclude using computation; rather the diagram suggests what to compute. To facilitate the sharing process, you might have several overhead pens and half-sheets of blank transparencies available so they can prepare their diagrams prior to presenting them to the class.

b) Here is one student's sketch and reasoning.



Each box represents 112 ÷ 7. or 16. Therefore, the numbers are 32, 16, and 64.

Following is an algebraic representation of the visual reasoning used above:

If x = 2nd number,
$2x$ = 1st number,
$2(2x) = 4x$ = 3rd number

So, $2x + x + 4x = 112$
$7x = 112$
$x = {}^{112}/_{7} = 16.$

Therefore, the 3 numbers are 16, 32, and 64.

Focus Teacher Activity (cont.)

ACTIONS

11 Repeat Action 10 for the remaining situations on Focus Master A, inviting students to pose questions for the class to answer about each situation.

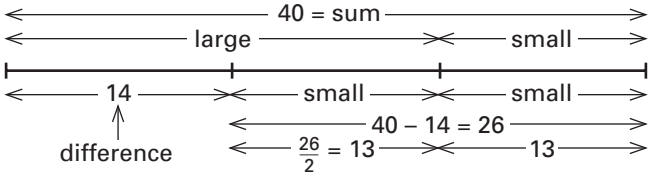
COMMENTS

11 It may be helpful to suggest that students view the questions they pose as "puzzle problems" to solve geometrically. Asking for geometric solutions encourages students to rely on their understanding of mathematical concepts and leads to connections between geometry and algebra. If students use algebraic symbols, encourage them to draw diagrams that illustrate what the symbols represent and why the algebraic solutions work. And, for some problems, you might ask the students to represent their geometric solutions and reasoning using algebraic symbols.

Following are sample questions and diagrams posed by students, together with their reasoning:

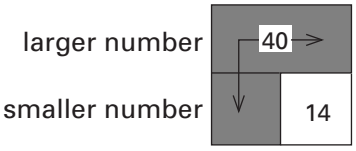
c) What are the numbers?

Solution 1



The smaller number is 26 ÷ 2, or 13.
The larger number is 40 – 13, or 27.

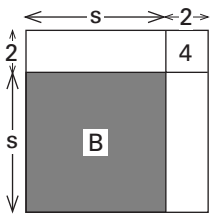
Solution 2



The area of the shaded region is the sum of the 2 numbers; the area of the unshaded region is the difference. The combined area of the shaded and unshaded regions is twice the larger number. Hence, the larger number is (40 + 14) ÷ 2, or 27. The smaller number is 27 – 14, or 13.

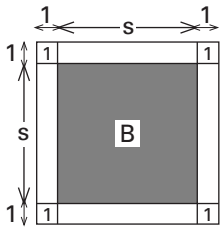
d) What is the length of the small square? In each of the following, *s* is the side of the smaller square.

Solution 1



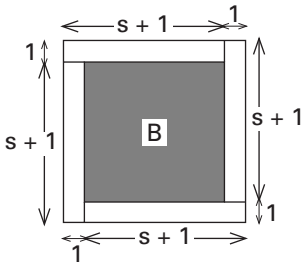
The area of the unshaded border is 48. Hence, the area of each of the two 2 × *s* rectangles is (48 – 4) ÷ 2 or 22. Thus, *s* is 11.

Solution 2



The area of each of the four 1 × *s* rectangles is (48 – 4) ÷ 4, or 11. Thus, *s* is 11.

Solution 3



The area of the unshaded border is 48. Hence, the area of each of the four 1 × (s + 1) rectangles is 48 ÷ 4, or 12, and *s* is 11.

Focus Teacher Activity (cont.)

ACTIONS

Modeling Situations Lesson **13**

Focus Master A

Situations

a) The people at a meeting are separated into 2 groups.
The 1st group has 5 less people than 3 times the number in the 2nd group.
There are 43 people at the meeting.

b) There are 3 numbers.
The 1st number is twice the 2nd number.
The 3rd is twice the 1st.
The sum of the 3 numbers is 112.

c) The sum of 2 numbers is 40.
Their difference is 14.

d) The sides of square A are 2 inches longer than the sides of square B.
The area of square A is 48 square inches greater than the area of square B.

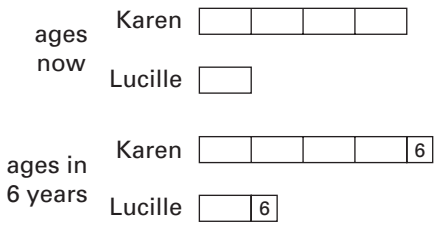
e) Melody has $2.75 in dimes and quarters.
She has 14 coins altogether.

f) Three particular integers are consecutive.
The product of the 1st and 2nd integers is 40 less than the square of the 3rd integer.

g) Karen is 4 times as old as Lucille.
In 6 years, Karen will be 3 times as old as Lucille.

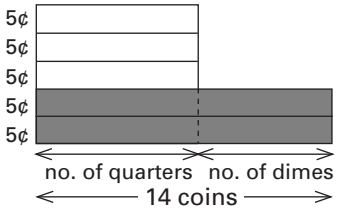
$$
\begin{aligned}
x &= \text{1st integer} \\
x + 1 &= \text{2nd integer} \\
x + 2 &= \text{3rd integer} \\
(x + 2)^2 - x(x + 1) &= 40 \\
x^2 + 4x + 4 - x^2 &= 40 \\
3x + 4 &= 40 \\
3x &= 40 - 4 = 36 \\
x &= {}^{36}\!/_{3} = 12
\end{aligned}
$$

So, the numbers are 12, 13, and14.



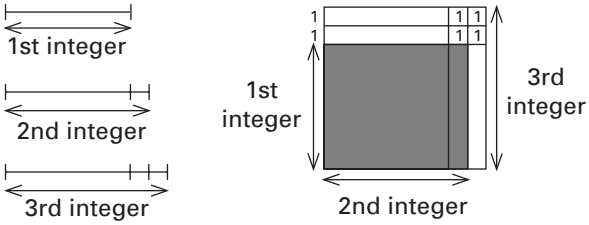
COMMENTS

e) How many dimes and how many quarters does Melanie have?



The value of each shaded bar is 5×14, or 70¢. Hence, the value of each unshaded bar is $(275 - 140) \div 3$, or 45¢. So, there are 9 quarters and 5 dimes.

f) What are the 3 integers? Note: Some students may represent this situation with Algebra Pieces, using edge pieces for lengths and area pieces for products. Or, they use diagrams such as the following:

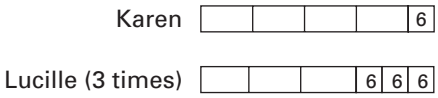


The area of the shaded rectangle is the product of the first 2 integers. The area of the unshaded region is the difference between that product and the square of the 3rd integer which is given to be 40. So, each of the 3 unshaded rectangles has area $(40 - 4) \div 3$, or 12. Thus, the 3 numbers are 12, 13, and 14.

Shown at the left is an algebraic representation of the above visual soluton.

g) How old is Lucille?

Comparison of Karen's age in 6 years with 3 times Lucille's age in 6 years:



These have the same value if each box represents two 6's, or 12. So, Karen is now 48 and Lucille is 12.

Focus Teacher Activity (cont.)

ACTIONS

12 Write the following on the overhead:

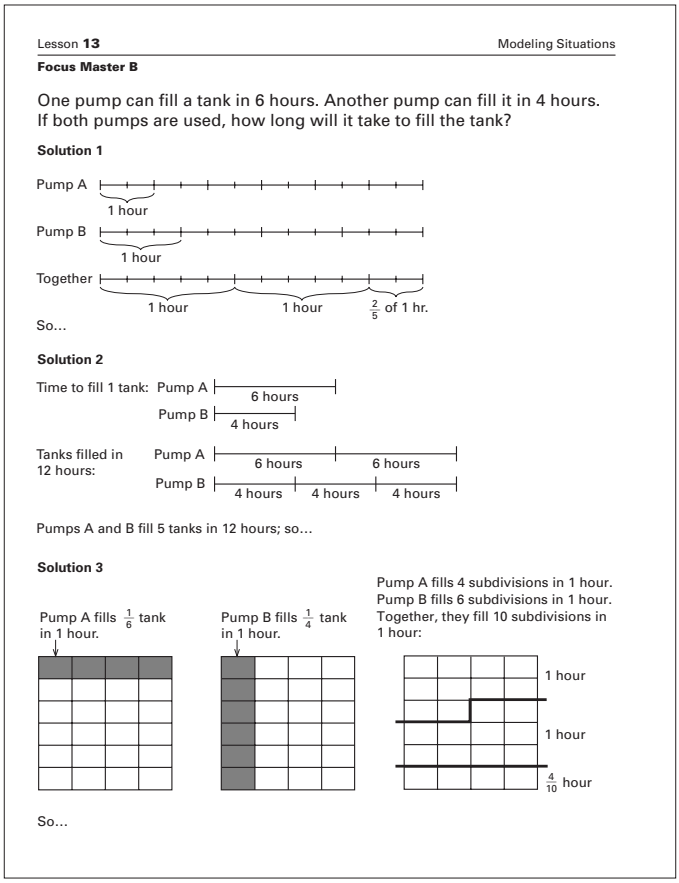
> **One pump can fill a tank in 6 hours.**
> **Another pump can fill it in 4 hours.**
> **Both pumps are turned on together to fill the tank.**

Ask the pairs of students to create sketches that illustrate the mathematical relationships in the given information, and to reason from their sketches to determine how long it will take to fill the tank using both pumps together. After the pairs have worked for 10-15 minutes, distribute a copy of Focus Master B to each pair, pointing out that these are sketches made by 3 students from another class. Ask the pairs to discuss their ideas and questions about the thinking behind the 3 solutions, to complete the "so…" statements, and to compare their methods with those shown on Focus Master B. Discuss as a large group.

13 Give each pair of students a copy of Focus Master C. Select (or have the students select) situations from Focus Master C and ask the students to make diagrams or sketches that illustrate the mathematical relationships in those situations. Then ask the students to pose (or you could pose) mathematical questions about the situations and to answer the questions by reasoning from their diagrams and sketches. Discuss.

COMMENTS

12 The intent here is to engage the students in thinking about the problem and then to encourage students to reflect on other students' work. Some students may not reach a solution before you distribute Focus Master B; you might give them the option not to examine Focus Master B until they are ready.


Lesson **13** Modeling Situations

Focus Master B

One pump can fill a tank in 6 hours. Another pump can fill it in 4 hours.
If both pumps are used, how long will it take to fill the tank?

Solution 1

Pump A — 1 hour
Pump B — 1 hour
Together — 1 hour, 1 hour, $\frac{2}{5}$ of 1 hr.

So…

Solution 2

Time to fill 1 tank: Pump A — 6 hours; Pump B — 4 hours

Tanks filled in 12 hours: Pump A — 6 hours, 6 hours; Pump B — 4 hours, 4 hours, 4 hours

Pumps A and B fill 5 tanks in 12 hours; so…

Solution 3

Pump A fills $\frac{1}{6}$ tank in 1 hour.

Pump B fills $\frac{1}{4}$ tank in 1 hour.

Pump A fills 4 subdivisions in 1 hour.
Pump B fills 6 subdivisions in 1 hour.
Together, they fill 10 subdivisions in 1 hour:

1 hour
1 hour
$\frac{4}{10}$ hour

So…

13 Select situations according to the students' levels of comfort and interest. You might encourage the students to view questions about these situations as "visual puzzles." For selected situations, after students have completed their visual solutions, ask them use to algebra symbols to represent their solutions and thought processes.

On the following 4 pages are sample questions and student solutions for situations a)-l).

Focus Teacher Activity (cont.)

ACTIONS

Modeling Situations Lesson **13**

Focus Master C

More Situations

a) A tank has 2 drains of different sizes.
If both drains are used, it takes 3 hours to empty the tank.
If only the first drain is used, it takes 7 hours to empty the tank.
On Tuesday only the 2nd drain is used to empty the tank.

b) Yesterday Maria and Lisa together had 20 library books.
Today Maria and Lisa visited the library; Lisa checked out new books and now has double the number of books that she had yesterday; Maria returned 3 of her books.
Now Maria and Lisa together have 30 books.

c) Of the students in Ms. Quan's class, $\frac{3}{5}$ are girls.
Ms. Nelson's class joined Ms. Quan's for a project; this doubled the number of boys and increased the number of girls by 6.
There are an equal number of boys and girls in the combined class.

d) On Moe's walk home from school, after 1 mile he stopped for a drink of water.
Next, Moe walked $\frac{1}{2}$ the remaining distance and stopped to rest at the park bench.
When Moe reached the park bench, he still needed to walk 1 mile more than $\frac{1}{3}$ the total distance from school to his home.

e) Jill has a gallon of paint that contains 20% red paint and 80% blue paint.
Jill adds more red paint until she has 50% red paint.

f) Standard quality coffee sells for $18.00 per kg.
Prime quality coffee sells for $24.00 per kg.
Every Saturday morning Moonman's Coffee Shop grinds a 40kg batch of a standard/prime blend to sell for $22.50/kg.

(Continued on back.)

Lesson **13** Modeling Situations

Focus Master C (cont.)

g) Alex's collection of nickels, dimes, and quarters has 3 fewer nickels than dimes and 3 more quarters than dimes.
Alex's collection is worth $4.20.

h) For a school play, Kyle sold 6 adult tickets and 15 student tickets.
Kyle collected $48 for his ticket sales.
Matt sold 8 adult tickets and 7 student tickets for the same school play.
Matt collected $38 for his ticket sales.

i) The doctor mixed a 1200 ml of an 85% sugar solution (i.e., the container is 85% sugar and the rest is water).
The nurse added enough of a 40% sugar solution to create a 60% solution.

j) On Wednesday, Steve drove from Gillette to Spearfish in 1 hr. and 30 min.
On Thursday, driving 8 miles per hour faster, Steve made the return trip in 1 hr. and 20 min.

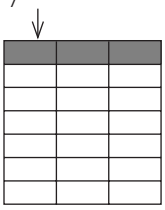
k) Michael averaged 78 points on 3 history tests.
His score on the 1st test was 86 points.
His average for the 1st 2 tests was 3 points more than his score on the 3rd test.

l) Traveling by train and then by bus, a 1200 mile trip took Wally 17 hours.
The train averaged 75 mph and the bus averaged 60 mph.

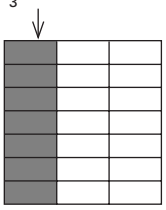
COMMENTS

a) How long does it take to empty the tank using only the 2nd drain?

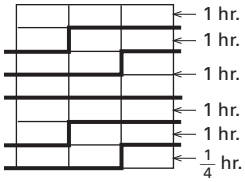
1st drain empties $\frac{1}{7}$ tank in 1 hour.



Both drains empty $\frac{1}{3}$ tank in 1 hour.



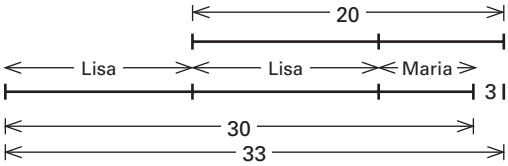
So, 2nd drain must empty $\frac{4}{21}$ tank in 1 hour.



Working together, both drains empty 7 subdivisions in 1 hour. The 1st drain empties 3, so the 2nd drain empties 4.

It takes the 2nd drain 5¼ hours to empty the tank.

b) How many books does each girl have now?



The difference in length of the top and bottom arrows is the number of books Lisa has. Hence, she has 33 – 20, or 13, books. So, Maria has 20 – 13, or 7, books.

c) How many students are in Ms. Quan's class?

Each of the boxes below contains the same number of students; 3 of the boxes contain girls and 2 contain boys:

G
G
G
B
B

Ms. Quan's class

Doubling the boys gives 4 boxes of boys. Adding 6 to the girls (each X is a girl), gives 6 more than 3 boxes of girls:

B	G
B	G
B	G
B	X X X X X X

If the number of boys and girls are equal, the last box of boys must contain 6 boys. Thus, all boxes contain 6 students and, to begin with there are 18 girls and 12 boys Ms. Quan's class.

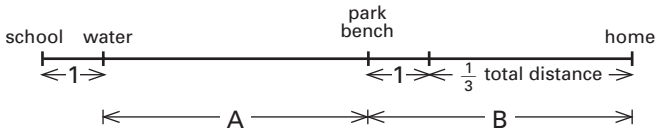
(Continued next page.)

Focus Teacher Activity (cont.)

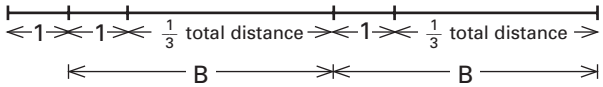
COMMENTS

13 (continued.)
d) How far is it from school to Moe's home?

Distance from school to home:



Segments A and B are equal. Thus, replacing A with B:



The 3 segments of length 1 comprise the other third of the distance. Hence, the distance from school to home is 9 miles.

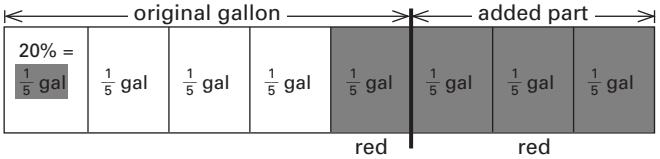
An algebraic representation of the above visual solution could be: Let x = the total distance in miles. Then,

$$\begin{aligned} (1/3)x + 1 &= 1/2(x - 1). \\ \text{So,}\quad 1 + (x/3 + 1) + (x/3 + 1) &= x \\ 3 + 2/3x &= x \\ 3 &= 1/3x \\ 9 &= x. \end{aligned}$$

Therefore, the distance from school to home is 9 miles.

e) How much pure red paint does Jill add?

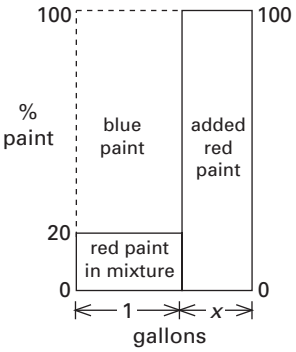
Solution 1



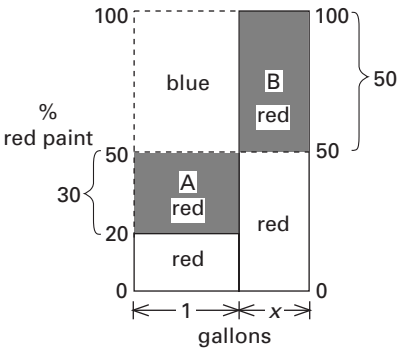
The areas of the rectangles shown at the left in Sketch I represent the amount of red paint in 1 gallon of the mixture and x gallons of added red paint. If the resulting mixture is to be 50% red paint, the 2 rectangles should be "leveled off" at 50. This will be the case if, in Sketch II, area A = area B. Since area A is 30 and area B is $50x$, the areas are equal if $30 = 50x$, that is, if $x = 3/5$. Hence, $3/5$ gallon of red paint must be added.

Solution 2

Sketch I:



Sketch II:

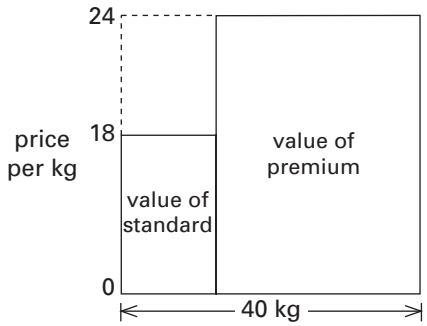


Focus Teacher Activity (cont.)

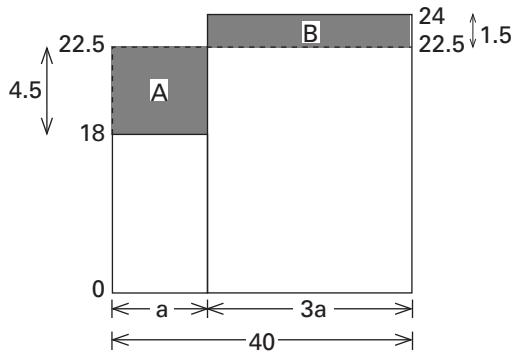
COMMENTS

f) How much prime coffee and how much standard coffee are needed to produce 40 kg of blend?

Sketch I:

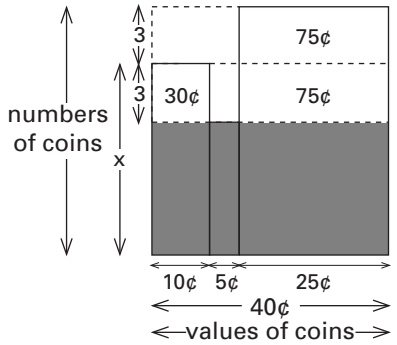


Sketch II:



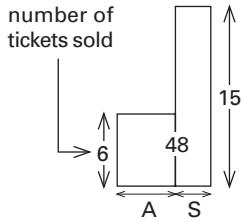
The areas of the rectangles in Sketch I at the left represent the values of the coffees in the blend. If the blend is to sell for $22.50, the 2 rectangles should "level off" at 22.5. This will be the case if, in Sketch II, area A = area B. Since the height of B is ⅓ the height of A (see diagrams at the left), for the areas to be equal, the base of B must be 3 times the base of A. So, if the base of A is *a*, the base of B is 3*a*. Thus, 4*a* = 40 and *a* = 10. Hence, there should be 10 kg of standard coffee and 30 kg of premium coffee.

g) How many of each coin does Alex have?

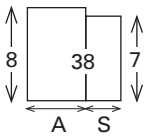


The heights of the rectangles shown at the left represent the number of coins and their bases the values, so the sum of the areas of the rectangles is the total value of the collection. The value of the unshaded portion is $1.80. Hence, the value of the shaded rectangle is $4.20 – $1.80, or $2.40. Since the value of its base is 40¢, its height is 2.40 ÷ .40 = 6. Thus, there are 6 nickels, 9 dimes, and 12 quarters.

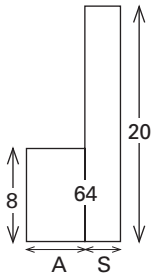
h) What is the cost of student and adult tickets?



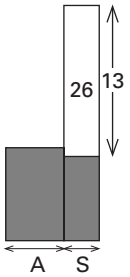
I. Kyle's sales



II. Matt's sales



III. Kyle's sales increased by a factor of ⅓



IV. Removing II from III

A is the cost of an adult ticket; S is the cost of a student ticket. Vertical dimensions represent the number of tickets sold.

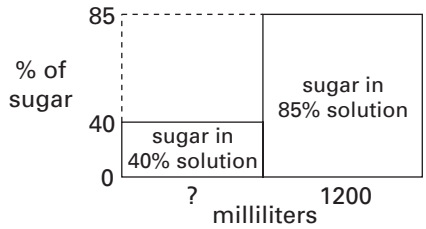
Increasing Kyle's sales by a factor of ⅓ and removing Matt's sales from the result, as shown in sketch IV, shows that 13 student tickets cost $26, so each cost $2. Thus, in sketch I, the 15 student tickets cost $30, so the 6 adult tickets cost $18, and each ticket costs $3.

(Continued next page.)

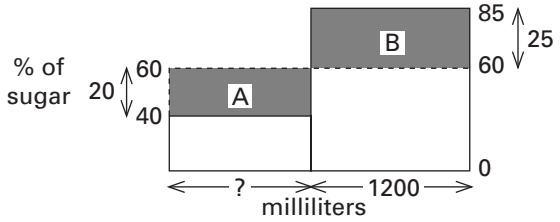
Focus Teacher Activity (cont.)

COMMENTS

Sketch I:



Sketch II:

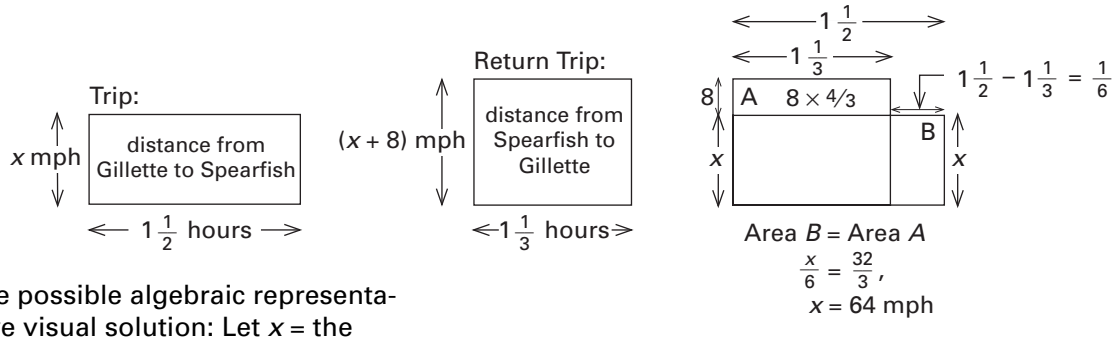


13 (continued.)

i) How much of the 40% solution did the nurse add?

The areas of the rectangles in Sketch I at the left represent the amount of sugar in the solutions. If the resulting solution is to be 60% sugar, the 2 rectangles should "level off" at 60. This will be the case if, in Sketch II, area A = area B. Since the area of B is 1200×25 and the height of A is 20, for the areas to be equal, the width of A must be $(1200 \times 25) \div 20$, or 1500. Hence, 1500 ml of the 40% solution should be added.

j) How far is it from Gillette to Spearfish?



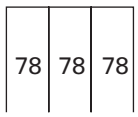
Following is one possible algebraic representation of the above visual solution: Let x = the speed.

$$
\begin{aligned}
(1\tfrac{1}{2})x &= (1\tfrac{1}{3})(x+8)\\
(\tfrac{3}{2})x &= (\tfrac{4}{3})x + (\tfrac{4}{3})8\\
(\tfrac{3}{2})x - (\tfrac{4}{3})x &= \tfrac{32}{3}\\
\tfrac{x}{6} &= \tfrac{32}{3}\\
x &= 64\text{mph}
\end{aligned}
$$

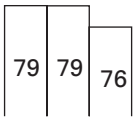
So, $64 \times 1\tfrac{1}{2} = 96$ miles is the distance from Gillette to Spearfish.

The areas of the above rectangles represent distances traveled. Since the distances are the same, the areas are equal. Thus, if one rectangle is superimposed on the other as shown above, the areas of A and B are equal. So, the distance between Gillette and Spearfish is $64 \times (1\tfrac{1}{2}) = 96$ miles.

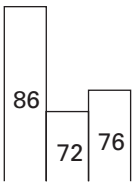
k) What were Michael's 3 test scores?



Average score is 78:



Moving 1 point from last score to each of 1st 2 scores, so average of 1st 2 scores is 3 greater than 3rd score:



Moving 7 points from 2nd score to 1st score, makes 1st score 86, and the 2nd and 3rd scores are 72 and 76, respectively.

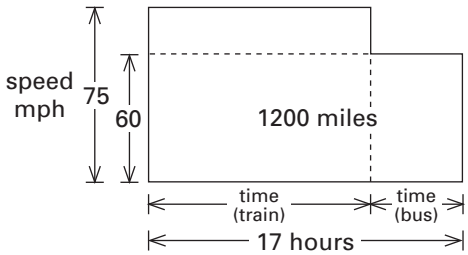
Focus Teacher Activity (cont.)

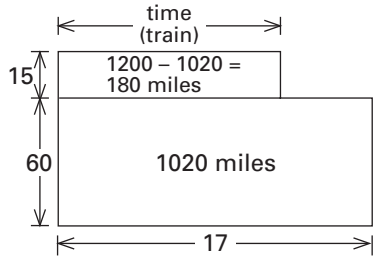
ACTIONS

COMMENTS

l) How far did the train travel?

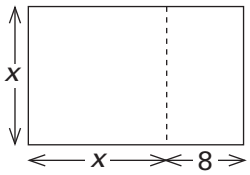
The distance traveled is represented by the area of the region in the 1st sketch below. This region can be divided into the 2 rectangles shown in the 2nd sketch. The area of the lower rectangle is 1020 miles. Hence, that of the upper is 180 miles, so its length is 180 ÷ 15, or 12 hours. Thus, 12 hours of the trip were by train, and the distance traveled by train was 75 × 12, or 900, miles.





14 Ask the students to sketch a rectangle whose length is 8 units greater than its width. Then tell them the area of the rectangle is 1428 and ask them to find its dimensions. Discuss the equations that have been solved.

14 Here is one sketch of the rectangle:



One way to determine the dimensions of the rectangle is to find 2 numbers which differ by 8 and whose product is 1428. If the rectangle were a square, its dimension would be $\sqrt{1428}$ which is about 38. Since its not a square, one dimension should be somewhat larger than this and one dimension somewhat smaller. If one guesses the dimensions are 34 and 42, a check will verify that this is correct. Making an educated guess and then checking to see if it is correct would be more difficult if the dimensions were not integers.

Another way to proceed is by "completing the square," as shown in the sketches on the left. If the strip of width 8 in the above sketch is split in two and half of it is moved to an adjacent side, as shown in Figure 1, the result is a square with a 4 × 4 corner missing. Adding this corner produces a square of area 1428 + 16, or 1444, and edge $x + 4$, as shown in Figure 2. Hence, $x + 4$ is $\sqrt{1444}$, or 38, and x is 34. So the dimensions of the original rectangle are 34 and 34 + 8, or 42. Thus, the equation $x(x + 8) = 1428$ has been solved without the use of guessing.

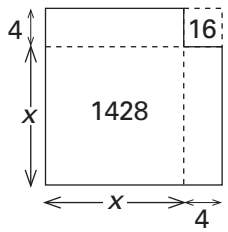


Figure 1

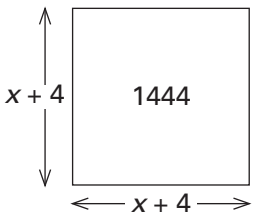


Figure 2

Focus Teacher Activity (cont.)

ACTIONS

15 Ask the students to draw sketches to solve the following equations:

a) $x^2 - 4x + 6 = 5$

b) $x^2 + 9x = 400$

c) $x(3x - 4) = 4$

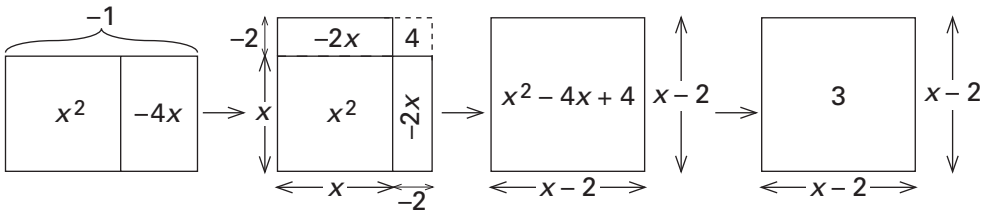
d) $2x(3 - x) = 3$

COMMENTS

15 All of these equations can be solved by completing the square. Note that, in *Math Alive! Course IV*, students develop the quadratic formula and investigate quadratics with solutions that are complex numbers (i.e., not real numbers).

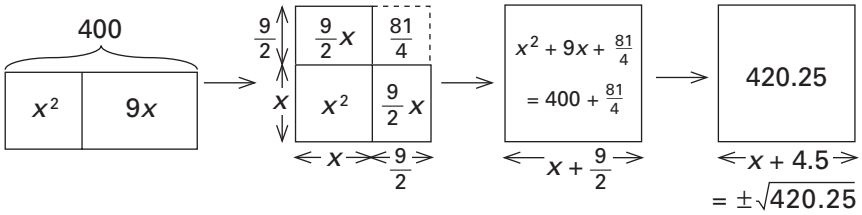
In the sketches that follow, differences are treated as sums, e.g., $x - 2$ is thought of as $x + (-2)$ and is portrayed by a line segment of value x augmented by a segment of value -2.

a) One can complete the square as shown in the following sequence of sketches. Notice that, since $x^2 - 4x + 6 = 5$, then $x^2 - 4x = -1$ and $x^2 - 4x + 4 = 3$.



If a square region has value 3, its edges have value $\sqrt{3}$ or $-\sqrt{3}$. Hence, $x - 2 = \sqrt{3}$ or $x - 2 = -\sqrt{3}$, and $x = 2 + \sqrt{3}$ or $x = 2 - \sqrt{3}$.

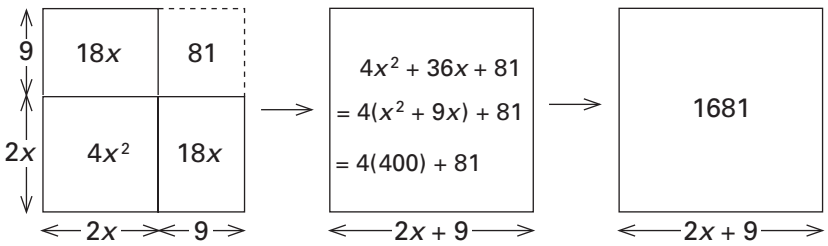
b) Completing the square gives the sequence of sketches shown below.



With the help of a calculator, one finds $\sqrt{420.25} = 20.5$. Thus, $x + 4.5 = \pm 20.5$ and $x = 16$ or $x = -25$.

Fractions can be avoided by doubling dimensions as shown in the sketches below.

Since $\sqrt{1681} = 41$, $2x + 9 = \pm 41$ and the result follows.

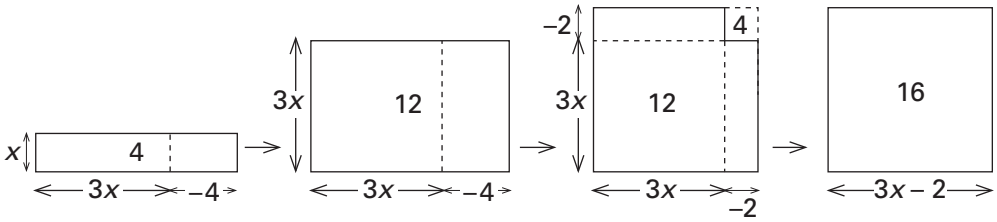


Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

c) In the following sequence, the second rectangular region is obtained from the first by increasing its height by a factor of 3.



$3x - 2 = \pm 4$
$3x = 6$ or $3x = -2$
$x = 2$ or $x = -\frac{2}{3}$

d) $x = \frac{3}{2} \pm \frac{\sqrt{3}}{2}$

Many sequences of sketches shown in the solutions above, and elsewhere in this lesson, contain more figures than may be in the sketches the students draw. In a number of instances, several figures shown in a sequence of sketches could be combined into a single figure, especially if an oral presentation is being made concurrently, or if solutions are being developed for private use and not for the benefit of a reader.

16 Place a transparency of Focus Master D on the overhead, revealing a) only. Ask the students to use sketches to represent the mathematical relationships in Situation a), and to reason from the diagram to make mathematical observations. Invite volunteers to share their mathematical observations and reasoning. Discuss equations that are solved by students' sketches. Repeat for b)-f).

Modeling Situations Lesson **13**
Focus Master D

a) The difference between 2 numbers is 6.
The sum of their squares is 1476.

b) The length of a rectangle is 6 units less than twice its width.
Its area is 836 square units.

c) The product of 2 consecutive even numbers is 2808.

d) The perimeter of a certain rectangle is 92 linear units.
The area of the rectangle is 493 square units.

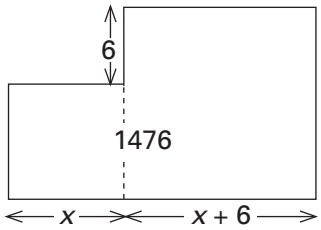
e) The sum of 2 numbers is 32.
The sum of the squares of the numbers is 520.

f) A 40 foot by 60 foot rectangular garden is bordered by a sidewalk of uniform width.
The area of the sidewalk is 864 square feet.

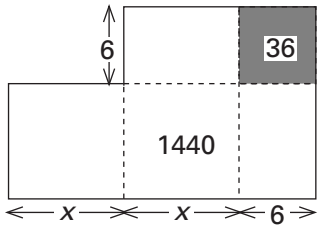
16 If students' observations are limited, you could pose a question for students to answer by reasoning from their diagrams. One possible problem related to each of a)-f) is shown below together with a visual solution to each problem.

a) What are the 2 numbers?

The difference of 2 numbers is 6 and the sum of their squares is 1476:



The shaded area below is 36; so the unshaded area is 1476 – 36, or 1440:



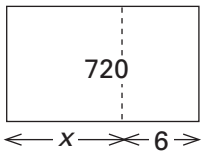
(Continued next page.)

Focus Teacher Activity (cont.)

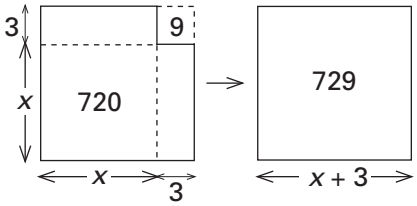
ACTIONS

COMMENTS

16 (continued.)
Half the unshaded area is 720:



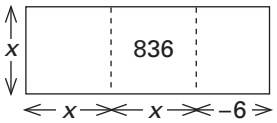
Completing the square:



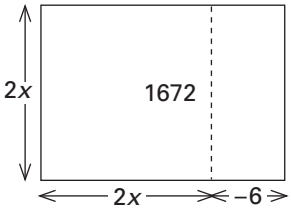
Since $\sqrt{729} = 27$, $x + 3 = \pm 27$. Thus, x is either 24 or –30 and $x + 6$ is, respectively, 30 or –24. Hence, the 2 numbers are 24 and 30 or –24 and –30.

b) What are the dimensions of the rectangle?

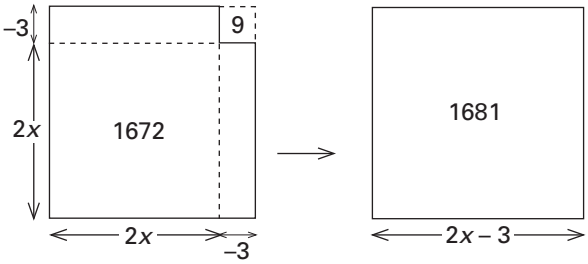
A rectangular region whose value is 836 with 1 edge whose value is 6 less than twice the value of the other edge.



Doubling this region:



Completing the square:

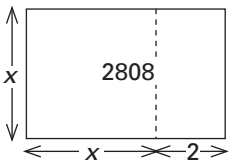


$2x - 3 = \sqrt{1681} = 41$, $x = 22$. So, the dimensions are 22 and 2(22) – 6, or 38.

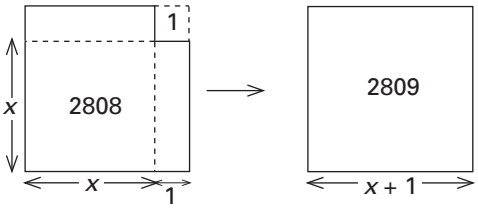
Note: $2x - 3 = -\sqrt{1681}$ is not considered above since dimensions (i.e., lengths) are always positive.

c) What are the numbers?

Two consecutive even numers whose product is 2808:



Completing the square:

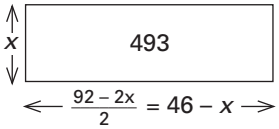


$x + 1 = \pm\sqrt{2809} = \pm 53$, $x = 52$ or $x = -54$. So, the 2 numbers are 52 and 54 or –54 and –52.

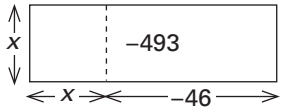
Focus Teacher Activity (cont.)

ACTIONS

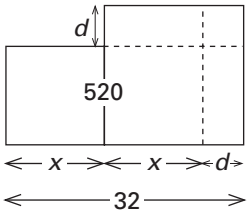
A rectangle whose perimeter is 92 and area is 493:



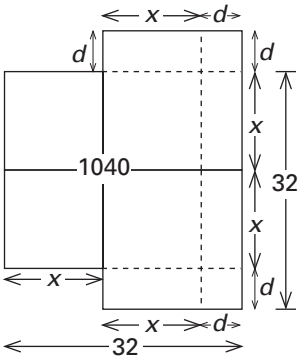
Changing the value of the base and, hence, the value of the region by a factor of –1:



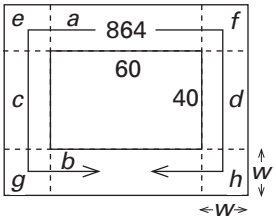
Two numbers, x and $x + d$, whose sum is 32 and whose squares add to 520:



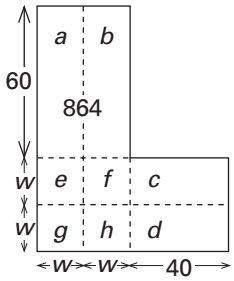
Double the figure:



A 40×60 rectangular garden with an 864 square foot border of uniform width, w:



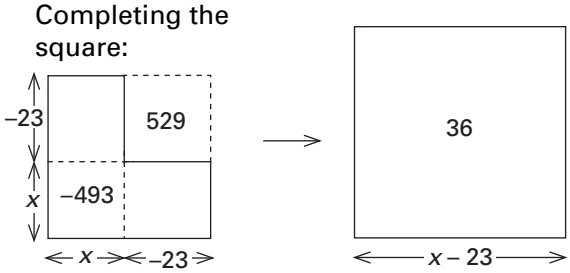
Rearranging the border:



COMMENTS

d) What are the dimensions of a rectangle whose perimeter is 92 and area is 493?

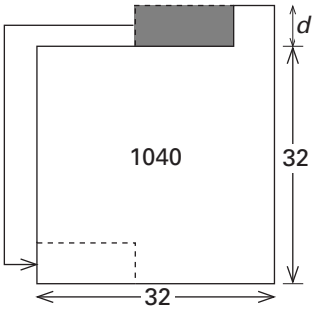
Completing the square:



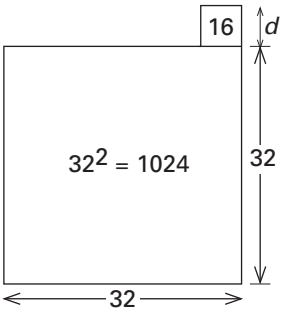
$x - 23 = \sqrt{36} = 6$, $x = 29$. So, the dimensions are 29 and 46 – 29, or 17.

e) What are the numbers?

Move the shaded region from the top to the bottom of the figure:



The result is 2 squares, one 32×32 and one $d \times d$, which sum to 1040:

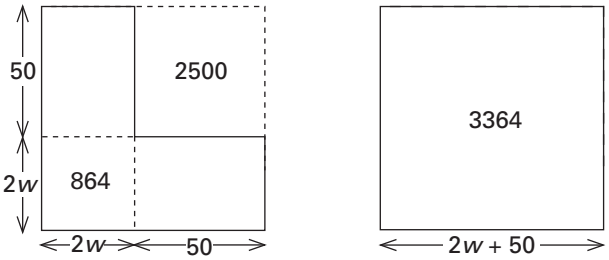


$d^2 = 1040 - 1024 = 16$, so, $d = 4$.

Since $d = 4$, then in the 1st sketch, $2x + 4 = 32$. Thus, $x = 14$ and the 2 numbers are 14 and 18.

f) What is the width of the sidewalk?

Moving 10 feet from the end of rectangles a and b to the end of rectangles c and d, and then completing the square:



$2w + 50 = \sqrt{3364} = 58$, so, $w = 4$ (i.e., the width of the border is 4 feet).

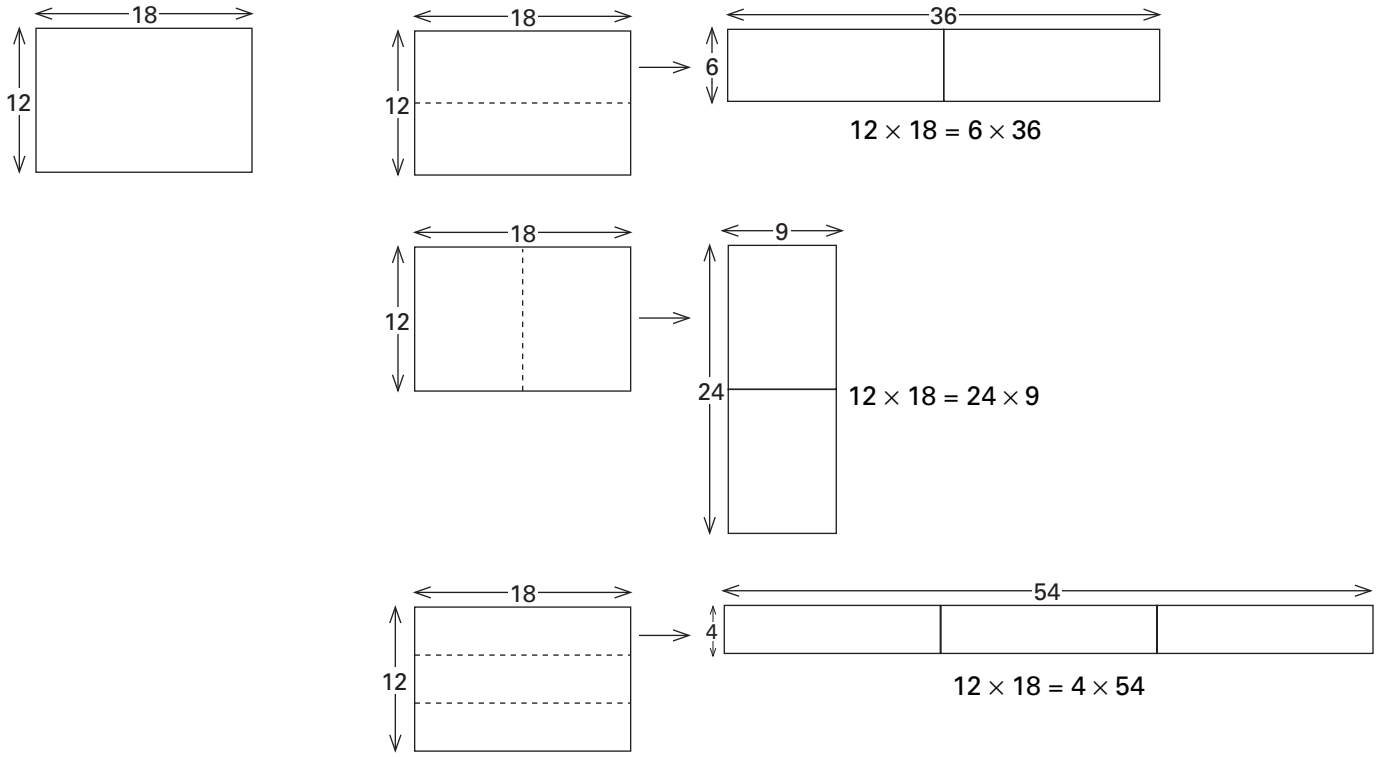
Focus Teacher Activity (cont.)

ACTIONS

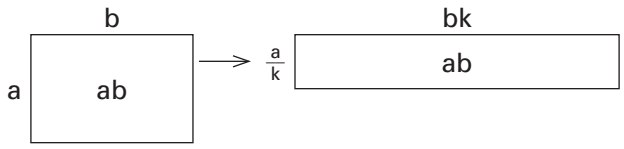
17 Ask the students to each sketch a rectangle with edges 12 and 18. Then have them use "rectangle maneuvers" to form noncongruent rectangles whose values are equal to 12×18. Invite volunteers to show examples. Use this as a context for introducing the term *inverse variation.*

COMMENTS

17 Based on rectangle maneuvers for forming equal products, as explored in Lesson 7 of this course and Lesson 20 of *Math Alive! Course II*, here are methods of forming 3 noncongruent rectangles each with value 12×18:



The above examples illustrate the general relationship that multiplying one edge of a rectangle by a factor k, and the other edge by the inverse, $1/k$, does not change the value of the rectangle. This is an example of *inverse variation*. That is, the edges of the rectangle *vary inversely* with each other and, although the values of the edges change, the product is constant. Inverse variation is also referred to as *indirect variation*.



Focus Teacher Activity (cont.)

ACTIONS

18 Place a transparency of Focus Master E on the overhead, revealing Situation A only. Ask the groups to make a sketch that models the mathematical conditions in the given situation. Invite volunteers to show their sketches and make mathematical observations. Then, if the students haven't brought it up, ask them to show how to reason from a sketch to determine how long the rations will last the 40 people in Lifeboat A.

Lesson **13** Modeling Situations
Focus Master E

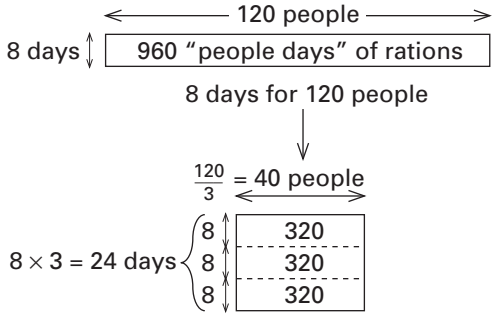
Situation A
There are several lifeboats on the USS Mathstar, and each lifeboat has enough rations to last 120 people for 8 days. The number of days that rations will last varies inversely with the number of people in a lifeboat. In a storm, the USS Mathstar sinks; 40 people climb aboard Lifeboat A.

19 Ask the students to again sketch a rectangle with edge values 12 and 18. Then ask them to form several new rectangles by multiplying one edge of the 12×18 rectangle by a whole number or fraction and without changing the other edge. Invite volunteers to show examples, labeling the value of each rectangle and the values of its edges. Discuss their observations. Use this as a context for introducing the term *direct variation.*

COMMENTS

18 Following is one possible diagram and reasoning.

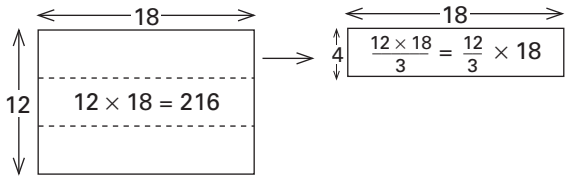
The amount of rations in each lifeboat is fixed. One edge of the rectangle represents the number of people on board and the other edge, represents the number of days the rations will last.



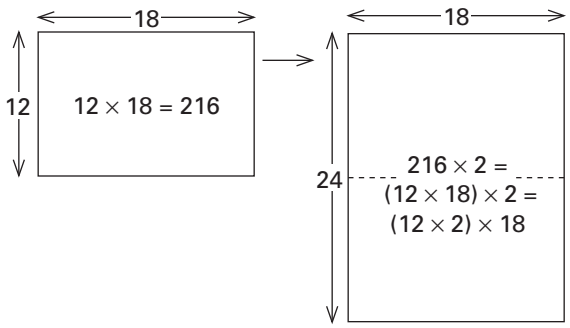
So, 8×3 days for 40 people.

19 Based on the equal quotients strategy explored in Lesson 7 of this course and Lesson 20 of *Math Alive! Course II*, multiplying the value of one edge of a rectangle by a factor multiplies the value of the rectangle by the same factor but does not change the value of the other edge. That is, the value of the rectangle *varies directly* as the value of one edge varies and vice versa. This is an example of *direct variation.* Here are 2 examples that illustrate this property:

Multiplying an edge and the area by ⅓:



Multiplying an edge and the area by 2:



Focus Teacher Activity (cont.)

ACTIONS

20 Place a transparency of Focus Master E on the overhead again, revealing Situation B. Ask the groups to make a sketch that illustrates the type of variation in the given situation. Invite volunteers to share their sketches and observations at the overhead. Then, if no one has suggested it, ask the students to show how to use a sketch to determine Ollie's wages for 44 hours of work.

Lesson **13** Modeling Situations

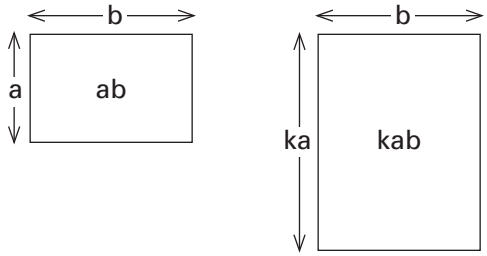
Focus Master E

Situation A
There are several lifeboats on the USS Mathstar, and each lifeboat has enough rations to last 120 people for 8 days. The number of days that rations will last varies inversely with the number of people in a lifeboat. In a storm, the USS Mathstar sinks; 40 people climb aboard Lifeboat A.

Situation B
Ollie's wages vary directly with the time he works. Last week his wages for 20 hours of work were $350. This week he will work 44 hours.

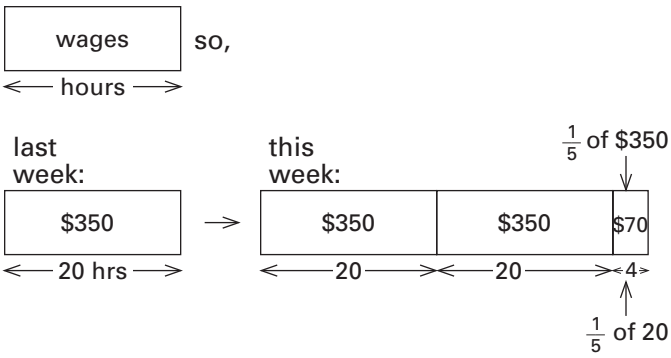
COMMENTS

In general,



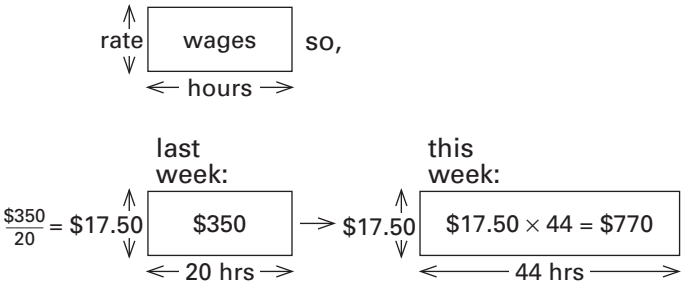
Notice in the examples given above, that the ratio of the edge that is changed to the area remains constant. Another way to describe a situation in which one quantity varies directly as another is to say the 2 quantities are *directly proportional*. That is, when 2 quantities are directly proportional, if one changes, the other changes proportionally. Note: 2 quantities that vary inversely with each other (see Actions 16 and 17) are called *inversely proportional*.

20 Here is one possibility:



Since Ollie's wages vary directly with the hours he works, and since 20 hours × 2⅕ = 44 hours, this week Ollie will earn $350 × 2⅕ = $770.

Some students may solve this problem by computing Ollie's hourly rate of pay, which is $350 ÷ 20 = $17.50, and then multiplying by 44 hours:



Focus Teacher Activity (cont.)

ACTIONS

21 Place a transparency of Focus Master F on the overhead, revealing Puzzle Problem a) only for the groups to complete. Invite volunteers to show their visual solutions. Then give each pair of students a copy of Focus Master F and have them complete selected problems from b)-i). Discuss. Finally, assign Problem 2 and discuss the students' results.

Modeling Situations Lesson **13**

Focus Master F

Puzzle Problems

1. For each of the following problems, make sketches that illustrate the type of variation given. Then show how to reason from your sketches to solve the problem. Mark your sketches and write equations to show your thinking and calculations.

a) The amount of profit for burger sales at Al's Burger Bar varies directly with the number of burgers sold. During the lunch rush yesterday Al sold 85 burgers and made $93.50 profit. During lunch today, Al sold 50 burgers. What was his profit today?

b) Each year the Math Club receives the same grant for students to attend the state math contest. The amount each student receives varies inversely with the number of Math Club students who attend the contest. Last year 12 students each received $25. If 16 students attend this year, what amount will each receive?

c) The amount raised during the school magazine subscription sale varies directly with the number of sales. This week the students raised $630 from sales of 450 subscriptions. If they sell 550 subscriptions this week, how much will they earn?

d) The distance an object falls in a given time varies directly as the square of the time. A certain object falls 64 feet in 2 seconds and hits the ground in 5 seconds. From what height did the object fall?

e) The current in a simple electrical circuit varies inversely with the resistance. If the current is 20 amps when the resistance is 5 ohms, what is the current if the resistance is 8 ohms?

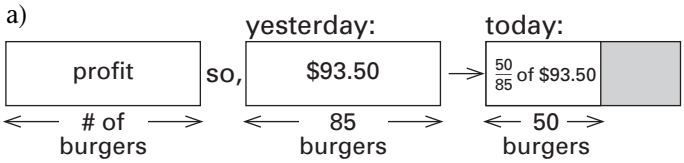
(Continued on back.)

COMMENTS

21 Notice that Problem 1 contains a mix of problems that involve direct and inverse variation. Select problems according to your students' comfort, needs, and interest. These problems could also be completed as homework (if so, each student will need a copy of Focus Master F).

The topics of inverse and direct variation are explored further in *Math Alive! Course IV*.

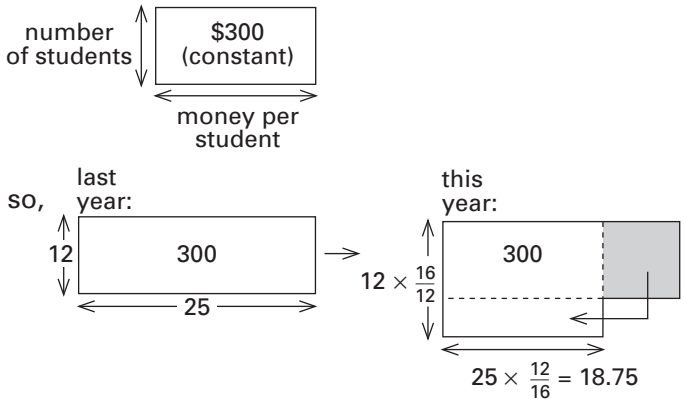
a)



Since Al sold $^{50}/_{85}$ of 85, or 50, burgers today, his profit today was $^{50}/_{85}$ of $93.50 = $55.

Another approach is to determine that the amount of profit per burger is $93.50 ÷ 85 = $1.10, and then multiply to determine that the total profit for 50 burgers is 50 × $1.10 = $55.

b)

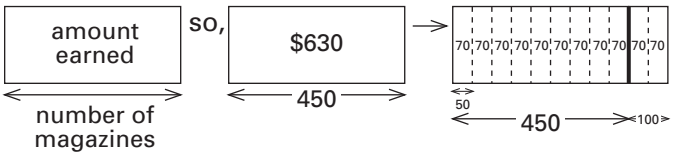


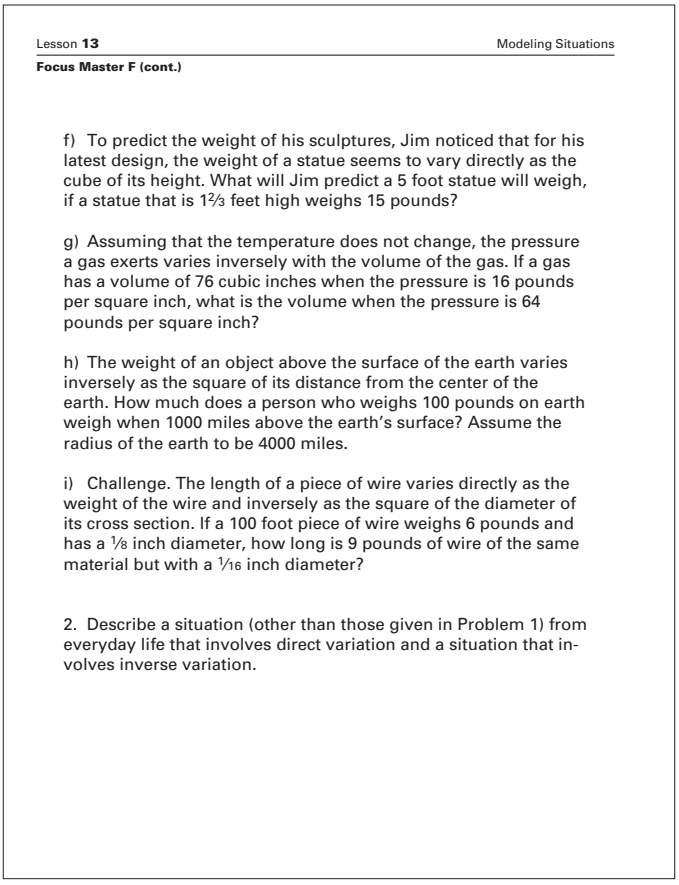
Since the number of students this year is the number last year multiplied by $^{16}/_{12}$ (i.e., 12 × $^{16}/_{12}$ = 16) and since the grant amount does not change, then the amount per student is multiplied by $\frac{1}{16/12}$ = $^{12}/_{16}$. Hence, each student receives $25 × $^{12}/_{16}$ = $18.75. Note: some students may work with $^{12}/_{16}$ = $^{3}/_{4}$ and $^{16}/_{12}$ = $^{4}/_{3}$.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS





Lesson **13** Modeling Situations

Focus Master F (cont.)

f) To predict the weight of his sculptures, Jim noticed that for his latest design, the weight of a statue seems to vary directly as the cube of its height. What will Jim predict a 5 foot statue will weigh, if a statue that is $1\frac{2}{3}$ feet high weighs 15 pounds?

g) Assuming that the temperature does not change, the pressure a gas exerts varies inversely with the volume of the gas. If a gas has a volume of 76 cubic inches when the pressure is 16 pounds per square inch, what is the volume when the pressure is 64 pounds per square inch?

h) The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. How much does a person who weighs 100 pounds on earth weigh when 1000 miles above the earth's surface? Assume the radius of the earth to be 4000 miles.

i) Challenge. The length of a piece of wire varies directly as the weight of the wire and inversely as the square of the diameter of its cross section. If a 100 foot piece of wire weighs 6 pounds and has a $\frac{1}{8}$ inch diameter, how long is 9 pounds of wire of the same material but with a $\frac{1}{16}$ inch diameter?

2. Describe a situation (other than those given in Problem 1) from everyday life that involves direct variation and a situation that involves inverse variation.

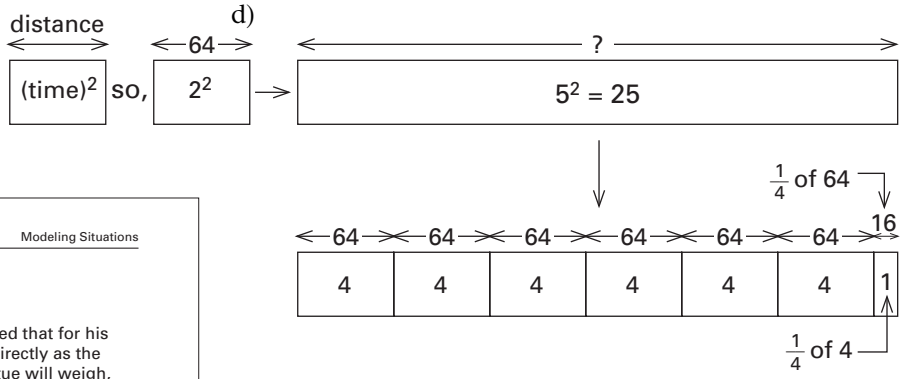
COMMENTS

21 (continued.)

c)

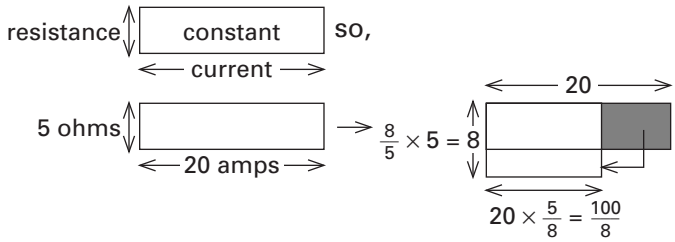
For every 50 magazines sold, the students earn $70, so an increase of 100 magazines earns an additional $140. Hence, for 550 magazines students earn $770. That is, the ratio $\frac{\$630 \text{ earned}}{450 \text{ magazines}}$ is equal to the ratio $\frac{\$770 \text{ earned}}{550 \text{ magazines}}$.

d)



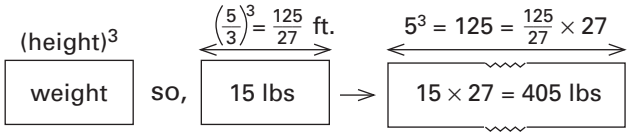
Since the square of the time (i.e., $2^2 = 4$) is multiplied by $6\frac{1}{4}$ to get 25, then the distance (i.e., 64 feet) is multiplied by $6\frac{1}{4}$ to get 400 feet.

e)



Since current and resistance vary inversely and since the resistance, 5, is multiplied by $\frac{8}{5}$ (i.e., $\frac{8}{5} \times 5 = 8$), then the current is multiplied by $\frac{5}{8}$, the reciprocal of $\frac{8}{5}$. Hence, for 8 ohms of resistance the current is $20 \times \frac{5}{8} = \frac{100}{8} = 12.5$ amps.

f)



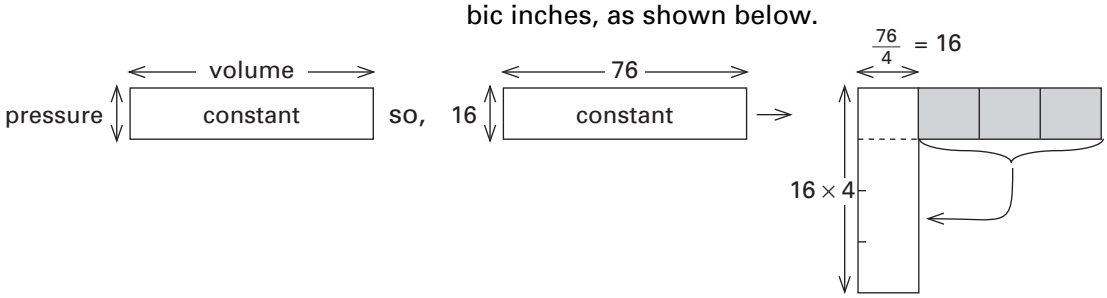
Since the height cubed of a 15 pound statue is multiplied by 27 to get the height cubed of a 5 foot statue (i.e., $\frac{125}{27} \times 27 = 125 = 5^3$), then the weight of a 15 pound statue is multiplied by 27 to get the weight of a 5 foot statue, $15 \times 27 = 405$ pounds.

Focus Teacher Activity (cont.)

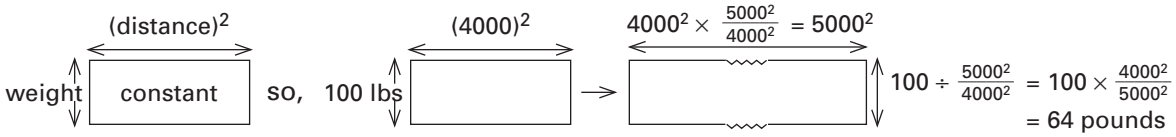
ACTIONS

COMMENTS

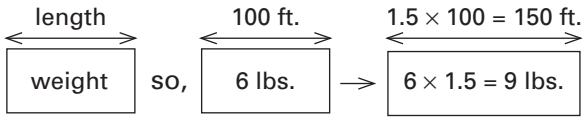
g) Since pressure and volume vary inversely and since the pressure is multiplied by 4, then the volume is multiplied by ¼. So, when pressure is $16 \times 4 = 64$ psi, the volume is $76 \times \frac{1}{4} = 19$ cubic inches, as shown below.



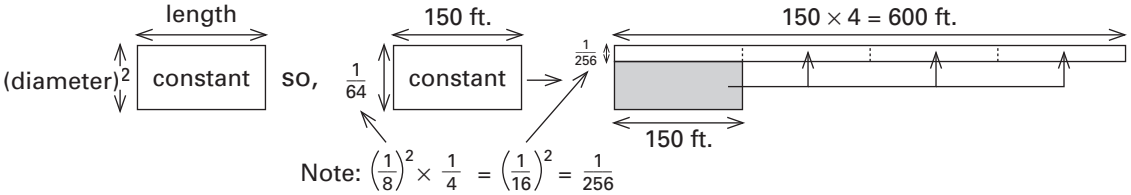
h) Since weight varies inversely with distance squared, and since $5000^2 = 4000^2 \times \frac{5000^2}{4000^2}$, a 100 pound person who is 4000 miles from the earth's center weighs $100 \times \frac{4000^2}{5000^2} = 64$ pounds when 5000 miles from the earth's center:



i) Since length and weight vary directly, multiplying the weight, 6 pounds, of ⅛" wire by 1.5 multiplies the length, 100, by 1.5. Hence, a 150 foot length of ⅛" wire weighs 9 lbs, as shown here:



Since the diameter squared for a ⅛" diameter wire is multiplied by ¼ to get the diameter squared for a 1⁄16" wire, i.e., $(\frac{1}{8})^2 \times \frac{1}{4} = (\frac{1}{16})^2$, and since length varies inversely with diameter squared, then the length of the ⅛" wire is multiplied by 4 (the reciprocal of ¼) to get the length of the 1⁄16" wire. Hence, a 1⁄16" wire that weighs 9 pounds is $150 \times 4 = 600$ feet long, as shown here:



TEACHER NOTES:

Follow-up Student Activity 13.1

NAME _____ **DATE** _____

1 Investigate the following situation and write a detailed summary of your investigation, including: what you do and how you do it (don't forget to tell what doesn't work, as well as what does), what conclusions you make and how you know they are true, what you conjecture and why, what you wonder about when you finish, and how long you spend on the investigation.

If rectangles are formed on grids so the edges of the rectangles lie along grid lines, some rectangles can be cut along grid lines to form 2 congruent staircases and others cannot. Note: assume 1 stairstep is a 1×1 square of the grid.

2 For each of the following puzzle problems, make a sketch that illustrates the mathematical conditions of the problem. Then reason from your sketch to solve the problem. Mark each sketch to show your thinking and reasoning. If needed to fully communicate your thought processes and calculations, add brief comments next to each diagram.

a) If each side of Square X increases by 3 feet, the area increases by 63 square feet. What is the perimeter of Square X?

b) Two cars start from points 400 miles apart and travel toward each other. They meet after 4 hours. Find the average speed of each car if one travels 20 miles per hour faster than the other.

c) An ice-skating rink is 30 meters by 20 meters. Plans are made to double the rink's area by first adding a rectangular strip along one end of the rink, and then adding a strip of the same width along one side retaining a rectangular shape for the rink. What will be the width of these strips?

d) Four times the larger of 2 numbers exceeds their sum by 25; four times the smaller number exceeds their difference by 1. What are the numbers?

e) The length of a room is 3 feet more than its width. If the length increases by 3 feet and the width decreases by 2 feet, the area of the floor does not change. What are the dimensions of the room?

(Continued on back.)

Follow-up Student Activity (cont.)

f) Kay added a set of consecutive integers, $1 + 2 + 3 + \ldots + n$, to get the total 990. What is n?

g) If the length of each side of a square is decreased by 20%, the area is decreased by 72 square inches. What is the length of an edge of the original square.

h) What are 3 consecutive odd numbers whose sum is 213?

i) What is the length of the side of a square whose diagonal is 10 inches longer than the side?

j) At Henry High School, 1 less than $\frac{1}{5}$ of the students are seniors, 3 less than $\frac{1}{4}$ are juniors, $\frac{7}{20}$ are freshmen, and the remaining 28 students are sophomores. How many students attend Henry High?

k) If 40 cc of a 40% acid solution, 70 cc of a 50% acid solution, and 50 cc of pure acid are combined, what % acid solution results?

l) How many cubic centimeters of pure sulfuric acid must be added to 100 cc of a 40% solution to obtain a 60% solution?

m) If 8 shillings and 5 francs are worth \$2.14, and 9 shillings and 70 francs are worth \$15.54, what is the value of a shilling and the value of a franc?

n) A bag contains only white balls and black balls. Ten more than $\frac{1}{2}$ the total number of balls are black, and 6 more than $\frac{1}{2}$ the number of black balls are white. If 1 ball is randomly selected at random from the bag, what is the probability it will be white?

3 Select 3 or more of the puzzle problems from Problem 2 and write algebraic equations to represent important parts of your sketches and the steps of your thought processes. Be sure to tell what each variable and equation represents.

4 Use diagrams and brief explanations to show what each of the following means: a) direct variation and b) inverse variation. For each of a) and b), describe an everyday situation involving that type of variation and make a diagram that illustrates the mathematical relationships in the situation.

Connector Master A

For each of the following, suppose that whole square tile are used to form L-shapes.

a) Can 24 square tile be arranged in an L-shape? If so, in how many ways? What about 36 tile? 45 tile?

b) Determine ways to dissect an L-shape, using only straight cuts along edges of whole tile, so the pieces can be reassembled to form a rectangle. What is the minimum number of cuts necessary for any L-shape?

c) What rectangles can be dissected with exactly 1 straight cut (along edges of whole tile) and reassembled to form an L-shape?

d) What counting numbers can be written as the difference of 2 perfect squares? of 2 consecutive perfect squares? Why?

Focus Master A

Situations

a) The people at a meeting are separated into 2 groups.
The 1st group has 5 less people than 3 times the number in the 2nd group.
There are 43 people at the meeting.

b) There are 3 numbers.
The 1st number is twice the 2nd number.
The 3rd is twice the 1st.
The sum of the 3 numbers is 112.

c) The sum of 2 numbers is 40.
Their difference is 14.

d) The sides of square A are 2 inches longer than the sides of square B.
The area of square A is 48 square inches greater than the area of square B.

e) Melody has $2.75 in dimes and quarters.
She has 14 coins altogether.

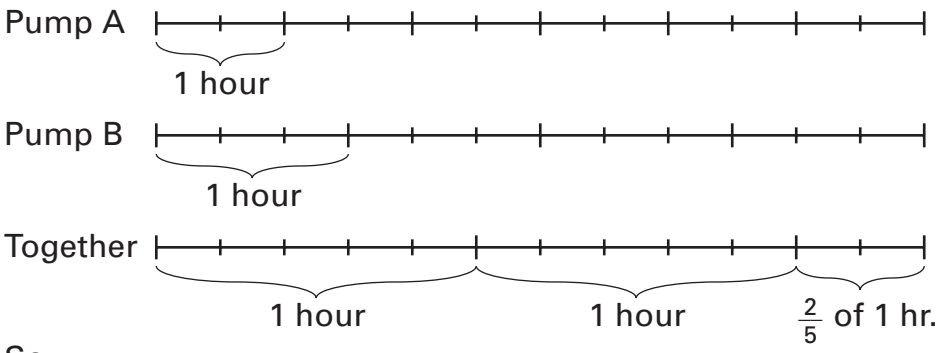
f) Three particular integers are consecutive.
The product of the 1st and 2nd integers is 40 less than the square of the 3rd integer.

g) Karen is 4 times as old as Lucille.
In 6 years, Karen will be 3 times as old as Lucille.

Focus Master B

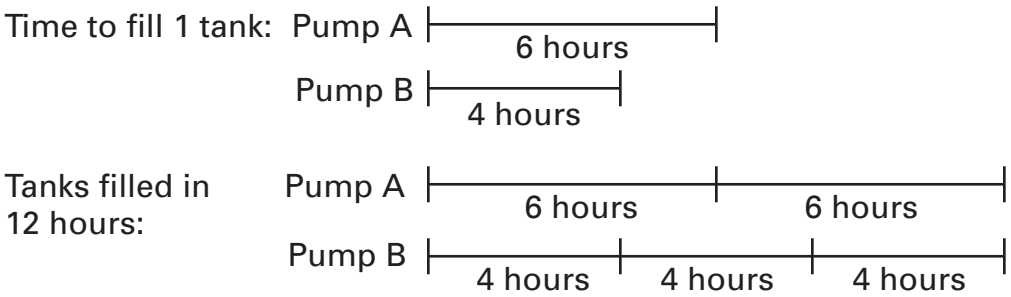
One pump can fill a tank in 6 hours. Another pump can fill it in 4 hours. If both pumps are used, how long will it take to fill the tank?

Solution 1



So…

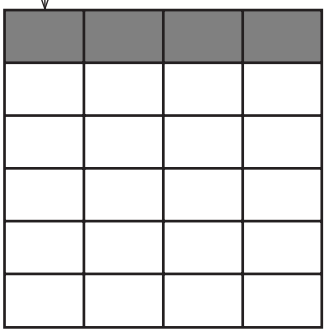
Solution 2



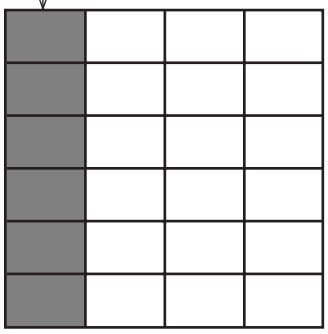
Pumps A and B fill 5 tanks in 12 hours; so…

Solution 3

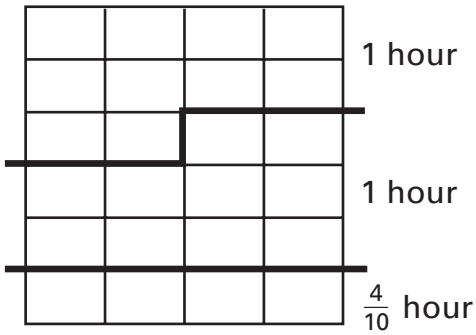
Pump A fills $\frac{1}{6}$ tank in 1 hour.



Pump B fills $\frac{1}{4}$ tank in 1 hour.



Pump A fills 4 subdivisions in 1 hour.
Pump B fills 6 subdivisions in 1 hour.
Together, they fill 10 subdivisions in 1 hour:



So…

Focus Master C

More Situations

a) A tank has 2 drains of different sizes.
If both drains are used, it takes 3 hours to empty the tank.
If only the first drain is used, it takes 7 hours to empty the tank.
On Tuesday only the 2nd drain is used to empty the tank.

b) Yesterday Maria and Lisa together had 20 library books.
Today Maria and Lisa visited the library; Lisa checked out new books and now has double the number of books that she had yesterday; Maria returned 3 of her books.
Now Maria and Lisa together have 30 books.

c) Of the students in Ms. Quan's class, $\frac{3}{5}$ are girls.
Ms. Nelson's class joined Ms. Quan's for a project; this doubled the number of boys and increased the number of girls by 6.
There are an equal number of boys and girls in the combined class.

d) On Moe's walk home from school, after 1 mile he stopped for a drink of water.
Next, Moe walked $\frac{1}{2}$ the remaining distance and stopped to rest at the park bench.
When Moe reached the park bench, he still needed to walk 1 mile more than $\frac{1}{3}$ the total distance from school to his home.

e) Jill has a gallon of paint that contains 20% red paint and 80% blue paint.
Jill adds more red paint until she has 50% red paint.

f) Standard quality coffee sells for $18.00 per kg.
Prime quality coffee sells for $24.00 per kg.
Every Saturday morning Moonman's Coffee Shop grinds a 40kg batch of a standard/prime blend to sell for $22.50/kg.

(Continued on back.)

Focus Master C (cont.)

g) Alex's collection of nickels, dimes, and quarters has 3 fewer nickels than dimes and 3 more quarters than dimes.
Alex's collection is worth $4.20.

h) For a school play, Kyle sold 6 adult tickets and 15 student tickets.
Kyle collected $48 for his ticket sales.
Matt sold 8 adult tickets and 7 student tickets for the same school play.
Matt collected $38 for his ticket sales.

i) The doctor mixed a 1200 ml of an 85% sugar solution (i.e., the container is 85% sugar and the rest is water).
The nurse added enough of a 40% sugar solution to create a 60% solution.

j) On Wednesday, Steve drove from Gillette to Spearfish in 1 hr. and 30 min.
On Thursday, driving 8 miles per hour faster, Steve made the return trip in 1 hr. and 20 min.

k) Michael averaged 78 points on 3 history tests.
His score on the 1st test was 86 points.
His average for the 1st 2 tests was 3 points more than his score on the 3rd test.

l) Traveling by train and then by bus, a 1200 mile trip took Wally 17 hours.
The train averaged 75 mph and the bus averaged 60 mph.

Focus Master D

a) The difference between 2 numbers is 6.
The sum of their squares is 1476.

b) The length of a rectangle is 6 units less than twice its width.
Its area is 836 square units.

c) The product of 2 consecutive even numbers is 2808.

d) The perimeter of a certain rectangle is 92 linear units.
The area of the rectangle is 493 square units.

e) The sum of 2 numbers is 32.
The sum of the squares of the numbers is 520.

f) A 40 foot by 60 foot rectangular garden is bordered by a sidewalk of uniform width.
The area of the sidewalk is 864 square feet.

Focus Master E

Situation A

There are several lifeboats on the USS Mathstar, and each lifeboat has enough rations to last 120 people for 8 days. The number of days that rations will last varies inversely with the number of people in a lifeboat. In a storm, the USS Mathstar sinks; 40 people climb aboard Lifeboat A.

Situation B

Ollie's wages vary directly with the time he works. Last week his wages for 20 hours of work were $350. This week he will work 44 hours.

Focus Master F

Puzzle Problems

1. For each of the following problems, make sketches that illustrate the type of variation given. Then show how to reason from your sketches to solve the problem. Mark your sketches and write equations to show your thinking and calculations.

a) The amount of profit for burger sales at Al's Burger Bar varies directly with the number of burgers sold. During the lunch rush yesterday Al sold 85 burgers and made $93.50 profit. During lunch today, Al sold 50 burgers. What was his profit today?

b) Each year the Math Club receives the same grant for students to attend the state math contest. The amount each student receives varies inversely with the number of Math Club students who attend the contest. Last year 12 students each received $25. If 16 students attend this year, what amount will each receive?

c) The amount raised during the school magazine subscription sale varies directly with the number of sales. This week the students raised $630 from sales of 450 subscriptions. If they sell 550 subscriptions this week, how much will they earn?

d) The distance an object falls in a given time varies directly as the square of the time. A certain object falls 64 feet in 2 seconds and hits the ground in 5 seconds. From what height did the object fall?

e) The current in a simple electrical circuit varies inversely with the resistance. If the current is 20 amps when the resistance is 5 ohms, what is the current if the resistance is 8 ohms?

(Continued on back.)

Focus Master F (cont.)

f) To predict the weight of his sculptures, Jim noticed that for his latest design, the weight of a statue seems to vary directly as the cube of its height. What will Jim predict a 5 foot statue will weigh, if a statue that is $1\frac{2}{3}$ feet high weighs 15 pounds?

g) Assuming that the temperature does not change, the pressure a gas exerts varies inversely with the volume of the gas. If a gas has a volume of 76 cubic inches when the pressure is 16 pounds per square inch, what is the volume when the pressure is 64 pounds per square inch?

h) The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. How much does a person who weighs 100 pounds on earth weigh when 1000 miles above the earth's surface? Assume the radius of the earth to be 4000 miles.

i) Challenge. The length of a piece of wire varies directly as the weight of the wire and inversely as the square of the diameter of its cross section. If a 100 foot piece of wire weighs 6 pounds and has a $\frac{1}{8}$ inch diameter, how long is 9 pounds of wire of the same material but with a $\frac{1}{16}$ inch diameter?

2. Describe a situation (other than those given in Problem 1) from everyday life that involves direct variation and a situation that involves inverse variation.

Follow-up Student Activity 13.1

NAME _____ **DATE** _____

1 Investigate the following situation and write a detailed summary of your investigation, including: what you do and how you do it (don't forget to tell what doesn't work, as well as what does), what conclusions you make and how you know they are true, what you conjecture and why, what you wonder about when you finish, and how long you spend on the investigation.

If rectangles are formed on grids so the edges of the rectangles lie along grid lines, some rectangles can be cut along grid lines to form 2 congruent staircases and others cannot. Note: assume 1 stairstep is a 1×1 square of the grid.

2 For each of the following puzzle problems, make a sketch that illustrates the mathematical conditions of the problem. Then reason from your sketch to solve the problem. Mark each sketch to show your thinking and reasoning. If needed to fully communicate your thought processes and calculations, add brief comments next to each diagram.

a) If each side of Square X increases by 3 feet, the area increases by 63 square feet. What is the perimeter of Square X?

b) Two cars start from points 400 miles apart and travel toward each other. They meet after 4 hours. Find the average speed of each car if one travels 20 miles per hour faster than the other.

c) An ice-skating rink is 30 meters by 20 meters. Plans are made to double the rink's area by first adding a rectangular strip along one end of the rink, and then adding a strip of the same width along one side retaining a rectangular shape for the rink. What will be the width of these strips?

d) Four times the larger of 2 numbers exceeds their sum by 25; four times the smaller number exceeds their difference by 1. What are the numbers?

e) The length of a room is 3 feet more than its width. If the length increases by 3 feet and the width decreases by 2 feet, the area of the floor does not change. What are the dimensions of the room?

(Continued on back.)

Follow-up Student Activity (cont.)

f) Kay added a set of consecutive integers, $1 + 2 + 3 + \ldots + n$, to get the total 990. What is n?

g) If the length of each side of a square is decreased by 20%, the area is decreased by 72 square inches. What is the length of an edge of the original square.

h) What are 3 consecutive odd numbers whose sum is 213?

i) What is the length of the side of a square whose diagonal is 10 inches longer than the side?

j) At Henry High School, 1 less than $\frac{1}{5}$ of the students are seniors, 3 less than $\frac{1}{4}$ are juniors, $\frac{7}{20}$ are freshmen, and the remaining 28 students are sophomores. How many students attend Henry High?

k) If 40 cc of a 40% acid solution, 70 cc of a 50% acid solution, and 50 cc of pure acid are combined, what % acid solution results?

l) How many cubic centimeters of pure sulfuric acid must be added to 100 cc of a 40% solution to obtain a 60% solution?

m) If 8 shillings and 5 francs are worth \$2.14, and 9 shillings and 70 francs are worth \$15.54, what is the value of a shilling and the value of a franc?

n) A bag contains only white balls and black balls. Ten more than $\frac{1}{2}$ the total number of balls are black, and 6 more than $\frac{1}{2}$ the number of black balls are white. If 1 ball is randomly selected at random from the bag, what is the probability it will be white?

3 Select 3 or more of the puzzle problems from Problem 2 and write algebraic equations to represent important parts of your sketches and the steps of your thought processes. Be sure to tell what each variable and equation represents.

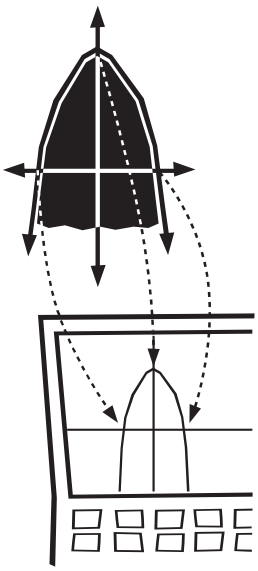
4 Use diagrams and brief explanations to show what each of the following means: a) direct variation and b) inverse variation. For each of a) and b), describe an everyday situation involving that type of variation and make a diagram that illustrates the mathematical relationships in the situation.

Analyzing Graphs

Lesson 14

THE BIG IDEA

Analyzing the graphs and Algebra Piece representations of families of linear and quadratic equations prompts intuitions and conjectures about the general characteristics of linear and quadratic functions. Exploring a variety of equation solving options—"by-hand" graphs, graphing calculators, Algebra Pieces, algebra symbols, and mental methods—provides students a powerful "tool kit" of problem-solving strategies.



CONNECTOR

OVERVIEW

Students discuss their successes and challenges with the graphing calculator exploration started in the last action of Lesson 12.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activity 12.4 (see Lesson 12), each student needs their completed copy.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).
- ✔ Butcher paper, 1 sheet per class.

FOCUS

OVERVIEW

Students use graphing calculators to graph, solve, and evaluate linear and quadratic equations and inequalities. Special functions of graphing calculators provide information about graphs of everyday situations. Students compare the advantages and disadvantages of the graphing calculator as a tool for solving and graphing equations to by-hand graphing, Algebra Piece and symbolic methods, and mental strategies.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Master A, 1 transparency.
- ✔ Focus Masters B-C, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master D, 1 copy per student and 1 transparency.
- ✔ Focus Student Activities 14.1-14.2, 1 copy of each per student and 1 transparency.
- ✔ Algebra Pieces for each student.
- ✔ Algebra Pieces for the overhead.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).

FOLLOW-UP

OVERVIEW

Students use graphs to represent and solve equations and system of equations. They write math expressions that represent graphs of equations and inequalities. They use graphs to solve problems regarding ice cream sales.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 14.3, 1 copy per student.
- ✔ Coordinate grid paper (see *Blackline Masters*), 6 sheets per student.

LESSON IDEAS

QUOTE

Calculators and computers with appropriate software transform the mathematics classroom into a laboratory much like the environment in many science classes, where students use technology to investigate, conjecture, and verify their findings. In this setting, the teacher encourages experimentation and provides opportunities for students to summarize ideas and establish connections with previously studied topics.

NCTM Standards

SELECTED ANSWERS

2. Coordinates of points of intersection to nearest tenth:
   a) (–1.2,4.1)
   b) no points of intersection
   c) (1,5)
   d) (–2.3,6.7) and (.6,2.9)
   e) (3.8,1.1) and (–1.6,5.2)
   f) (0,–2)

5. a) $y = (-\frac{3}{4})x + 3$; function; domain and range are all real numbers.

   b) $y = -|x|$ or $y = x$ for $x \leq 0$, $-x$ for $x > 0$. This is a function with domain all real numbers and range all nonpositive real numbers.

   c) $x > 3$, i.e., the region to the right of the vertical line $x = 3$. Not a function.

   d) $y \geq x^2 + 3$; not a function.

   e) $y < 2x + 1$; not a function.

   f) Parabola $y = -x^2 - 1$; function; domain all real numbers and range all real numbers less than or equal to –1.

6. a) $y_1 = 200 + .25x$
   b) $y_2 = 1.5x$
   d) 161 (for 160, expenses = income)
   e) $y_2 - y_1$ for $x = \$300$,
   or 1.5(\$300) – [\$200 + .25(\$300)] = \$175

TEACHER NOTES:

Connector Teacher Activity

OVERVIEW & PURPOSE

Students discuss their successes and challenges with the graphing calculator exploration started in the last action of Lesson 12.

MATERIALS

- ✔ Focus Student Activity 12.4 (see Lesson 12), each student needs their completed copy.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).
- ✔ Butcher paper, 1 sheet per class.

ACTIONS

1 Arrange the students in groups and ask them to get out their completed copies of Focus Student Activity 12.4 (the graphing calculator checklists distributed in Lesson 12). Have the groups do a round robin share in which group members describe the calculator functions they find difficult to use and those they feel most comfortable using. Allow a few minutes for groups to discuss these and to determine 2 or 3 questions about the use of the calculator to pose to the class. Post one sheet of butcher paper on the wall. As the groups report their lists of questions, record the questions on the butcher paper. Invite individuals to add any other questions they feel are especially important. Discuss 2 or 3 of the listed questions.

COMMENTS

1 It isn't necessary to immediately address every question that students post. Rather, you might discuss a few questions now and leave the list posted for reference throughout this lesson. Periodically, you might check to see if there are questions that need attention, ones that can be crossed off, or others that need to be added.

An overhead graphing calculator and/or a transparency of a calculator keyboard are convenient for demonstrations and discussion. Calculator manuals should be referenced for specific procedures.

Continuous Graphs Lesson **12**

Focus Student Activity 12.4

NAME _____ **DATE** _____

Although you will have many opportunities during this course to become familiar with your graphing calculator, it will be helpful if you are comfortable with the functions listed below as soon as possible. Please investigate each function on your calculator and, if needed, in your calculator manual. A way to test yourself to see if you can comfortably use and recall a calculator function is to demonstrate its use to someone else (a family member, a classmate, a neighbor, etc.). Try to check off all functions in Part 1 below by the following date _____.

1 I am comfortable using the following calculator functions:

____ ON/OFF
____ CLEAR the screen
____ show blank coordinate axes in the calculator viewing screen
____ move the cursor around a blank coordinate axes
____ change the viewing WINDOW size
____ FORMAT the axes
____ determine the "standard" WINDOW size on my calculator (on many it is $-10 \le x \le 10$ and $-10 \le y \le 10$)
____ enter an equation y =
____ GRAPH an equation y =
____ TRACE a graph (What shows on the screen when you do this?)
____ ZOOM in on a graph
____ ZOOM in again—and again
____ ZOOM out on a graph
____ ZOOM back to the standard window
____ TRACE the graph of a function to determine the approximate value of the function at $x = 0$, $x = 19.75$, and $x = -37.5$
____ TRACE the graph of a function to determine the value of x when $y = 75$, when $y = -75$
____ GRAPH 2 equations on the same coordinate axes.
____ TRACE to approximate the intersection of 2 graphs
____ ZOOM and TRACE to improve your approximation
____ DRAW a horizontal line on coordinate axes and slide the line up and down
____ DRAW a vertical line on coordinate axes and slide the line left and right

(Continued on back.)

Lesson **12** Continuous Graphs

Focus Student Activity 12.4 (cont.)

____ view a TABLE of x- and y-coordinates of an equation
____ view a table of coordinates of 2 equations listed simultaneously
____ use a table to find when $0 = 5x + 1$
____ clear MEMory
____ reset defaults in MEMory
____ solve equations using the "solver" function from the MATH menu
____ use the "maximum" and minimum" functions from the CALC menu to find the turning point of a parabola
____ use the "intersect" function from the CALC menu to find the intersection of 2 graphs
____ use the "zero" function from the CALC menu to find the x-intercepts of a graph
____ use the "value" function from the CALC menu to find $v(x)$ for specific values of x
____ set the graphing style to shade the region above a graph; the region below a graph

2 Here are some other graphing calculator functions that I can use:

3 Here are some other functions I have tried but don't understand.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students use graphing calculators to graph, solve, and evaluate linear and quadratic equations and inequalities. Special functions of graphing calculators provide information about graphs of everyday situations. Students compare the advantages and disadvantages of the graphing calculator as a tool for solving and graphing equations to by-hand graphing, Algebra Piece and symbolic methods, and mental strategies.

MATERIALS

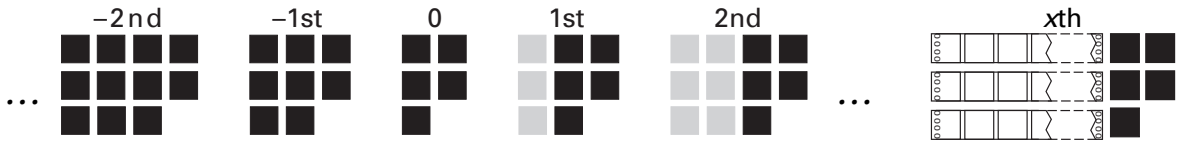
- ✔ Focus Master A, 1 transparency.
- ✔ Focus Masters B-C, 1 copy of each per group and 1 transparency of each.
- ✔ Focus Master D, 1 copy per student and 1 transparency.
- ✔ Focus Student Activities 14.1-14.2, 1 copy of each per student and 1 transparency.
- ✔ Algebra Pieces for each student.
- ✔ Algebra Pieces for the overhead.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).

ACTIONS

1 Arrange the students in groups and distribute Algebra Pieces to each student. Write the formula $v(x) = -3x + 5$ on the overhead and tell the students that this formula represents the xth arrangement of a continuous sequence of counting piece arrangements. Ask the groups to form the –3rd, –2nd, –1st, 0th, 1st, 2nd, 3rd, and xth arrangements of this sequence. Discuss.

COMMENTS

1 Students investigated continuous sequences of counting piece arrangements in Lesson 12 of this course. Several Algebra Piece arrangements for the given sequence are shown below.



2 Ask the students to leave the arrangements formed in Action 1 on their desks/tables and to *imagine* the graph of $y = -3x + 5$ in enough detail that they can "see" important features of the graph. Ask for volunteers to sketch and explain their ideas at the overhead. Use this as a context for recalling the terms slope, x-intercept, and y-intercept, and how to determine the value of each.

If it hasn't come up previously, point out to students that an x-intercept of a graph is also referred to as a *zero* of the equation, since it is a point where the value of y is zero.

2 In the above sequence, each time the arrangement number increases by 1, the value of the arrangement decreases by 3. Hence, the graph of $y = -3x + 5$ is a line that falls from left to right at the rate of 3 vertical units for every 1 horizontal unit, i.e., its slope is –3. The line passes through the y-axis at the point (0,5), the y-intercept.

Some students may predict the x-intercept as "a point on the x-axis between $x = 1$ and $x = 2$, and closer to 2." Others may mentally solve the equation $-3x + 5 = 0$ to determine the x-intercept is $x = 5/3$. And, some may use Algebra Piece representations of $y = -3x + 5$ or use algebra symbols to solve for x when $y = 0$.

Still others may note that, since the slope is –3, the line drops 3 units vertically for every 1 unit of horizontal change (to the right), or down 1 unit for every $1/3$ unit to the right. Hence, since the y–intercept is at (0,5), one can drop down 3 units and move to the right 1 unit to locate the point (1,2) and from there move down 2 units and over $2/3$ unit to locate the x-intercept.

Focus Teacher Activity (cont.)

ACTIONS

3 If it didn't come up in Actions 1 or 2, ask the groups to determine how the slope, y-intercept, and x-intercept of the graph relate to the arrangements formed in Action 1. Discuss.

4 Distribute graphing calculators (if students don't have them). Ask the students to graph $v(x) = -3x + 5$ on their calculators, and to determine various methods of using a graphing calculator to determine a) below. Discuss their ideas regarding the advantages and disadvantages of the graphing calculator methods when compared to: hand graphing, mental strategies, and algebraic procedures (either with Algebra Pieces or with symbols representing the pieces). Are some methods more reasonable than others for this problem? Why? Repeat for b) and c).

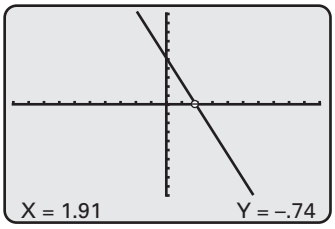
a) the x-intercept

b) the y-intercept

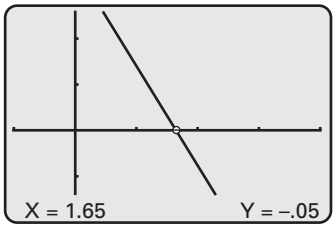
c) the point where $x = 49$

Using ZOOM and TRACE to locate the x-intercept:

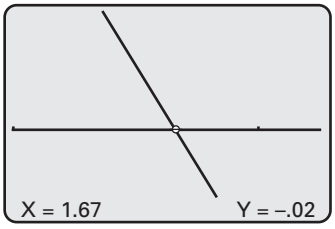
1st trace:



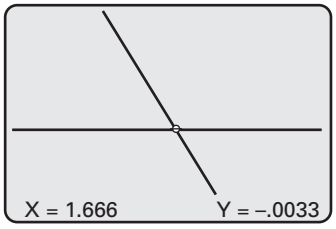
Trace after 1st zoom:



Trace after 2nd zoom:



Trace after 3rd zoom:



COMMENTS

3 The slope of a line is the ratio of the difference in the values of 2 arrangements to the difference in the corresponding arrangement numbers. The value of the y-intercept is the value of the 0th arrangement. The x-intercept is the number of the arrangement whose value is 0.

4 If the calculators were used by other classes, students may need to clear or turn off graphs that were stored in the calculator.

Throughout this lesson students have opportunities to use the calculator functions that were listed on Focus Student Activity 12.4, and they are introduced to other functions as needed or appropriate for use in the activity. The names of functions and menus that are referenced in this lesson may vary among calculator brands, and some brands may not have some of the functions. Hence, you may need to adapt some actions according to the calculators used by your students. All of the examples described in this lesson are based on the TI-83.

a) The students should notice that the TRACE function can give a very close "approximation" for the x-intercept, but not necessarily an exact value. A series of traces and zooms for $y = -3x + 5$ is shown below at the left. Each ZOOM obtains a closer approximation of the x-intercept.

Many students may suggest that mentally calculating the x-intercept (by mentally determining when $0 = -3x + 5$) is simple and therefore using the calculator is not needed to compute the x-intercept. And others may note that symbolic procedures are quick and exact for this equation. Two important purposes of this lesson are for students to: 1) develop a "tool kit" of options for graphing and solving equations and 2) develop a sense for the appropriate uses of the available options.

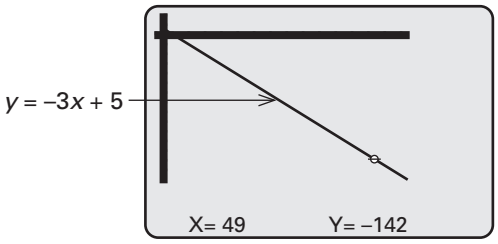
On the TI-83, for example, the "zero" and "intersect" functions from the CALC menu, are both appropriate for locating the x-intercept. The "solver" function from the MATH menu can also be used to determine the value of x for which $0 = -3x + 5$. Or, students may use the table function.

b) Students may feel that mentally evaluating the equation at $x = 0$, by substituting 0 for x to get $y = (-3)(0) + 5 = 5$ is the most "reasonable" approach for finding the y-intercept of this equation. However, on the TI-83, for example, they could also use the "value" function from the CALC menu by entering the value 0 for x. Or, they may ZOOM and TRACE the graph.

Focus Teacher Activity (cont.)

ACTIONS

Use WINDOW to set minimum and maximums:
x min 0 x max 50
y min –150 y max 20

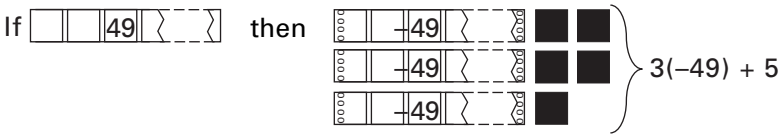


The y-value at $x = 49$.

COMMENTS

c) Students may graph $y = -3x + 5$ in a standard window and attempt tracing the graph to determine the y value for $x = 49$. However, this is not possible if $x = 49$ is outside the viewing window. (Note: in graphing calculators there is a standard default viewing window, such as [–10,10] for y, and [–10,10] for x.) Hence, use the WINDOW function to resize the window so that $x = 49$ is included and so the corresponding y-value also appears (see example at the left). This requires making a mental estimate of the y-value when $x = 49$. For example, one could note that $y = -3(49) + 5$ should be a little more than $-3(50)$, and therefore, set the minimum y at –150.

Once $y = -3x + 5$ is entered in the calculator and the window is set so that $x = 49$ is included, one can also use the "value" function from the TI-83 CALC menu. Or, some students may "see" $v(x)$ using Algebra Pieces:



5 Ask the students to determine various methods of identifying the points on $y = -3x + 5$ where the following are true. Discuss.

a) $y = -75$

b) $y = 10\frac{2}{3}$

c) $x = -28.75$

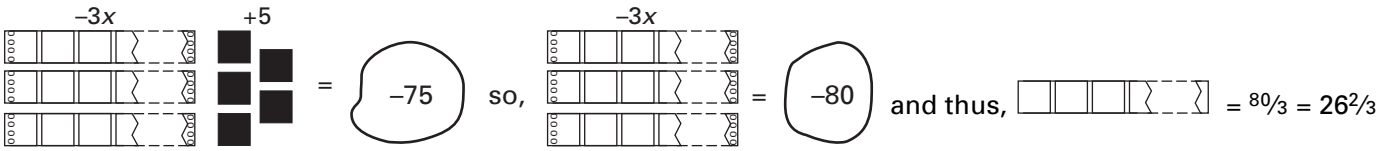
d) $x = 10$

e) $x = 257$

5 As students discuss the advantages and disadvantages of various techniques, you might encourage them to make connections among the representations they use for these techniques. For example, a particular counting piece arrangement corresponds to a point on the graph; a point on the graph can be described by a pair of values, x and y; and the relationship between the values x and y can be described by a general formula (in this case, $y = -3x + 5$). Understanding connections among these mathematical representations empowers students as algebraic thinkers.

The intent here is to have students continue exploring various techniques for solving and evaluating equations while developing comfort with the techniques and a sense about their appropriate uses. Following are some methods that students may suggest. Others are possible (e.g., although not discussed in this lesson if computers are available, you might have the students explore the use of one or more computer graphing utilities).

a) One way to solve $-3x + 5 = -75$ is to use Algebra Pieces, as shown below:



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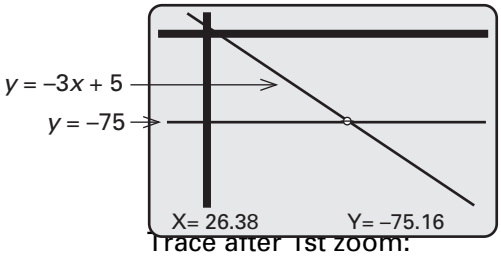
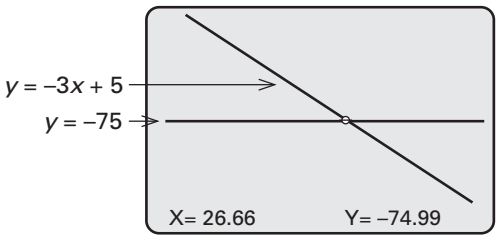
Focus Teacher Activity (cont.)

ACTIONS

1st trace:



Trace after 1st zoom:



COMMENTS

5 (continued.)
Instead of Algebra Pieces, one can use symbols representing Algebra Piece actions, as shown here:

$$\begin{aligned} -3x + 5 &= -75 \\ -3x + 5 - 5 &= -75 - 5 \\ -3x &= -80 \\ {}^{-3x}/_{3} &= {}^{-80}/_{3} \\ -x &= {}^{-80}/_{3} \\ x &= {}^{80}/_{3} \end{aligned}$$

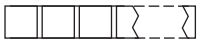
Another possibility is to adjust the window, then graph both $y = -3x + 5$ and $y = -75$ simultaneously on a graphing calculator, and finally zoom and trace to approximate the x-value where these graphs intersect. (Note: to determine the window size, one must approximate the value of x when $y = -75$). The line $y = -75$ can be graphed by using the Y = function and the GRAPH function on the calculator, or by using the "horizontal" function from the DRAW menu to form a horizontal line and then slide the horizontal line vertically until it intersects $y = -3x + 5$ at $y = -75$. In the graphs shown at the left the horizontal axis was set for $-10 \le x \le 50$, and the vertical axis was set for $-150 \le y \le 20$.

Students could also adjust the window so $y = -75$ is included, then graph the lines $y = -3x + 5$ and $y = -75$ simultaneously on the calculator, and finally select the function "intersect" from the TI-83 CALC menu to determine the point of intersection.

Yet another method of solving $-3x + 5 = -75$ is to use the TI-83 "solver" function from the MATH menu. Since this function requires that equations be entered in the form "0 = …," and since the difference between the values of the two equations $y_1 = -75$ and $y_2 = -3x + 5$ is zero at the point of intersection of the lines $y_1 = -75$ and $y_2 = -3x + 5$, one can enter the equation $0 = (-3x + 5) - (-75)$, or the equivalent $0 = -3x + 80$, for the calculator to solve.

Still another method is to use the TABLE function from the graphing calculator to view a table of values for $y = -3x + 5$. Scroll to the entry closest to $y = -75$. Increments in x may need adjustment in order to locate an x-value that produces a y-value closer to $y = -75$.

b) $x = -1\,{}^{8}/_{9}$

c) If  $= -28.75$, then $-3x + 5$
$= -3(-28.75) + 5$
$= 86.25 + 5$
$= 91.25$

Focus Teacher Activity (cont.)

ACTIONS

6 Place a transparency of Focus Master A on the overhead, revealing Part a) only. Discuss the groups' ideas. Then reveal and discuss Parts b)-d).

Analyzing Graphs Lesson **14**

Focus Master A

I $y = -3x + 5$
II $y = -x + 5$
III $y = x + 5$
IV $y = 3x + 5$

a) Imagine the graph of each of equations I-IV. What similarities and differences does your group *predict* about the graphs?

b) Now graph the 4 equations simultaneously on your graphing calculators. Do the results agree with your predictions? What else do you notice?

c) Equations I-IV are a "family" of equations. What characteristic(s) do you think make these equations a family? What are two other equations that could belong to this family?

d) What are similarities and differences among Algebra Piece representations of the xth arrangements of the sequences represented by equations I-IV?

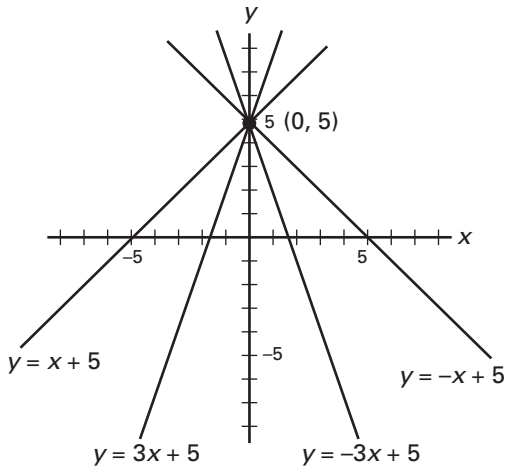
COMMENTS

One could also set the window of the graphing calculator to include $x = -28.75$ and $y = v(-28.75)$, and then trace and zoom to approximate $v(-28.75)$ or use the "value" function from the CALC menu. To determine a window that will include $v(-28.75)$, one could set the upper bound for y at $v(-30)$, which is easy to compute mentally. Another approach is to locate the y-value that corresponds to $x = -28.75$ in a table for $y = -3x + 5$.

d)-e) Methods similar to those used for c) are appropriate for d) and e).

6 a) You may need to remind the students to *imagine and predict* the graphs at this point, not to draw or use their calculators. Students may mention differences in the slope, both its steepness and rise/fall. They may predict that I and IV are mirror images of each other across the y-axis, as are lines II and III; and many may point out they all have y-intercept 5.

b) Here are graphs of the 4 equations. You might suggest that students be sure to determine the equation associated with each graph on their calculators.



c) Some students may describe these as a "family of lines with a common y-intercept." Hence, other family members could be equations of any lines whose y-intercept is 5. Other students may suggest that the lines must have slopes of +1, –1, +3, or –3. Still others may say that, for every line in the family, if the line has slope m, then another family member must have slope $-m$, and others may suggest any lines whose slopes are integers and whose y-intercepts are 5 belong in the family.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

7 Give each student a copy of Focus Student Activity 14.1 and repeat Action 6 for 5 or more of the equation families listed. Encourage conjectures and generalizations about relationships between the graph of a line or parabola and the constant, coefficients, and variables in the equation for the line or parabola. Encourage discussion about the information revealed by different forms of an equation (e.g., factored or expanded forms of a quadratic).

Analyzing Graphs Lesson **14**

Focus Student Activity 14.1

NAME _____ DATE _____

For each equation family below, record the following on separate paper:

a) your predictions about the graphs of the 4 equations,

b) your observations about calculator graphs of the equations,

c) the characteristic(s) that you think make the equations a family,

d) two additional equations that would fit in the family,

e) similarities and differences among Algebra Piece representations of the 4 equations.

1 I $y = -3x + 5$
II $y = -3x - 5$
III $y = -3x + 2$
IV $y = -3x - 2$

2 I $y = x^2 - 6$
II $y = x^2 + 6$
III $y = -x^2 - 6$
IV $y = -(x^2 - 6)$

3 I $y = 4x^2$
II $y = (\frac{1}{4})x^2$
III $y = -3x^2 - 6$
IV $y = (-\frac{3}{4})x^2$

4 I $y = x(x - 3)$
II $y = x^2 - 2x$
III $y = x^2 + 2x$
IV $y = x(x + 3)$

5 I $y = (x - 3)(x - 4)$
II $y = x^2 - 7x + 12$
III $y = (x + 2)(x + 3)$
IV $y = (x + 1)(x - 2)$

6 I $y = (x - 2)(x - 5)$
II $y = 2(x - 2)(x - 5)$
III $y = -(x - 2)(x - 5)$
IV $y = -2(x^2 - 7x + 10)$

7 I $28x + 8y = 0$
II $7x + 2y = 6$
III $14x + 4y = 4$
IV $21x + 6y = -12$

8 I $y = -5x + \frac{2}{3}$
II $3y = -15x + 2$
III $0 = -5x - y + \frac{2}{3}$
IV $-2 = -15x - 3y$

COMMENTS

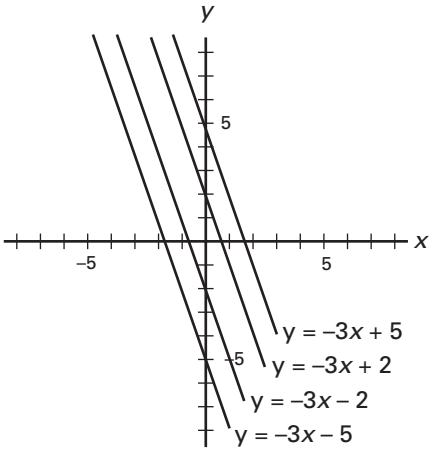
6 (continued.)
d) For example, the *x*th arrangements of the sequences represented by these equations each contain only *x*-frames and 5 units or –*x*-frames and 5 units.

7 Students could explore these in groups or individually as homework. And you might create other families for students to examine, based on mathematical ideas or relationships you feel students need to discuss further or based on prior conjectures that students have made.

The intent here is for students to continue their search for insights about relationships among equations, their graphs, and their Algebra Piece representations. This search will also extend throughout *Math Alive! Course IV*.

Conjectures that surface may shift discussion in a number of directions; the direction to pursue can be based on students' interest and needs.

1) Changes in the *y*-intercept generate a family of parallel lines, each with slope –3 in this case.

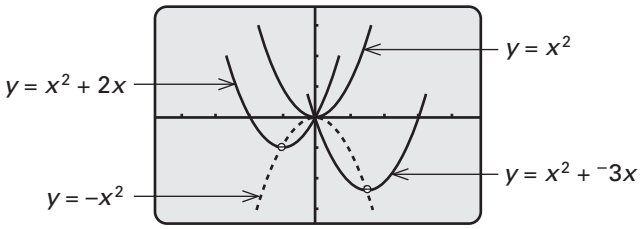


2) Notice that graphs of equations I and II are identical to the parabola $y = x^2$ after translating it 6 units down (I) or up (II) the *y*-axis. Similarly, graphs of III and IV are translations of the parabola, $y = -x^2$, 6 units down (III) or up (IV) the *y*-axis. Graphs of II and III are reflections of each other across the *x*-axis, as are graphs of I and IV. Graphs of I and III are reflections of each other across the line $y = -6$, while graphs of II and IV are reflections of each other across the line $y = 6$.

3) Some may refer to this as a family of quadratic functions whose graphs are parabolas with vertices are on the *y*-axis. This is also true for the quadratics in 2) above.

Focus Teacher Activity (cont.)

ACTIONS



COMMENTS

4) These are all quadratic functions and, when written in standard form, the coefficients of the x^2 and x terms are integers and the constant is 0. When expressed in factored form, all have x as one factor.

Students may be interested in pursuing the effects of adding various x-terms to the equation $y = x^2$. One interesting conjecture that may come up is that the vertices of all the parabolas of the form $y = x^2 + bx$, where b is a real number, lie on the parabola $y = -x^2$ (see left); and vertices of all parabolas of the form $y = -x^2 + bx$ lie on the parabola $y = x^2$.

5) Some students may call this a family of parabolas that can be written in the form $y = x^2 + bx + c$, where b and c are real numbers not equal to zero. Some may notice that the x-intercepts of a parabola are easy to identify when the equation is in factored form (assuming it factors). For example, if the equation is of the form $y = (x - r)(x - s)$, where r and s are real numbers, the x-intercepts are $x = r$ and $x = s$.

Notice that when the two factors in equation I are multiplied, the product is equation II. Hence, equations I and II have identical graphs, with x-intercepts 3 and 4.

6) Students may describe this as a family of parabolas whose vertices lie on the vertical line midway between $x = 2$ and $x = 5$, i.e., parabolas whose vertices lie on the vertical line $x = 3.5$. For each family member, its reflection across the x-axis is also in the family. Students may make conjectures about equations for lines whose graphs are reflections of each other across the x-axis. Encourage this. To prompt thinking you might pose questions, such as, "How can an equation be altered to create a graph that is a reflection across the y-axis? across the line $y = x$?" Note: replacing x with $-x$ in an equation creates a reflection across the y-axis, and exchanging x and y in an equation creates a reflection across the line $y = x$; however, students may not reach these conclusions now.

7) These equations are all written in standard linear form ($ax + by = c$). Rewriting each equation in slope-intercept form shows that all 4 lines have the same slope but different y-intercepts. Hence, this is a family of parallel lines with slope $^{-}7/2$. Notice, the standard form of a linear equation does not give away as many explicit "clues" about its graph as does the slope-intercept form.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

8 Give each student a copy of Focus Student Activity 14.2 and ask the groups to complete i)-vi) for the pair of equations given in a). Discuss. Then repeat for b)-g). Encourage students to make conjectures about relationships between equations, their graphs, and their Algebra Piece representations.

Lesson **14** Analyzing Graphs

Focus Student Activity 14.2

NAME _____ **DATE** _____

For each pair of equations given in a)-g):

i) Sketch and label counting piece arrangements to represent y_1 for $x = -3, -2, -1, 0, 1, 2,$ and 3, and sketch an Algebra Piece representation of the xth arrangement of y_1. Repeat for y_2.

ii) Make a table that shows the corresponding values of x, y_1, and y_2 for the arrangements formed in i).

iii) Predict how you think the graphs of y_1 and y_2 will look in comparison to the graph of $y = |x|$. Then, on the same coordinate axes, sketch a graph of y_1, y_2, and $y = |x|$ over the domain of real numbers such that $-10 \le x \le 10$. Label the coordinates of the points from the table in ii).

iv) On the same coordinate axes of the graphing calculator, graph y_1, y_2, and $y = |x|$ over the domain given in iii).

v) Write 3 additional equations that form a family with y_1, and y_2, and explain the relationship that makes the 5 equations a family.

vi) Record conjectures and generalizations based on your observations from i)-v).

a) $y_1 = |x| - 3$ $\quad y_2 = |x| + 4$

b) $y_1 = |x - 5|$ $\quad y_2 = |x + \frac{1}{2}|$

c) $y_1 = |2x|$ $\quad y_2 = |-2x|$

d) $y_1 = 5|x|$ $\quad y_2 = -5|x|$

e) $y_1 = |x - 2x|$ $\quad y_2 = |3x + 2|$

f) $y_1 = |x| - |2x|$ $\quad y_2 = |3x| + |2|$

g) $y_1 = \frac{1}{x}$ $\quad y_2 = \frac{3}{x}$

COMMENTS

7 (continued.)
8) The graphs of these equations are all identical; hence the equations are all equivalent. Students may add other equations equivalent to these, or they may add a set of equivalent equations that represent another line.

8 Students could also work independently on Focus Student Activity 14.2 as homework and then share their ideas during group discussions or during class discussions.

a) The absolute value function was introduced in Lesson 12 of this course. On the TI-83, for example, the absolute value function is "abs" in the "NUM" menu of the MATH menu.

Many students will probably create a family of equations of the form $y = |x| + b$, where b is a real number. Adding b to $|x|$ in $y = |x|$ shifts the graph of $y = |x|$ up or down the *y-axis*, depending on whether b is a positive or negative number.

b) Students may need help interpreting the notation here. For example, $|x - 5|$ implies "subtract 5 from x and then find the absolute value of the result," whereas $|x| - 5$, implies "compute the absolute value of x and then subtract 5."

Students may create a family of equations of the form $y = |x + b|$, whose graphs are horizontal translations of the graph $y = |x|$.

c) The graphs of these 2 equations are identical to each other and rise/fall at double the rate of $y = |x|$. The graph of an equation of the form $y = |bx|$ is V-shaped, opens up, and the vertex of the V is at (0,0). The sign of b has no affect on the graph, while $|b|$ determines the steepness with which the sides of the V rise or fall.

d) Since the coefficients, 5 and –5, are outside the absolute value notation, the graphs of y_1 and y_2 are reflections of each other across the horizontal axis. Graphs of equations of the form $y = b|x|$ are V-shaped, and the vertex of the V is at (0,0). The V opens up if $b > 0$, opens down if $b < 0$, and $|b|$ determines the steepness with which the sides of the V rise of fall.

e)-f) Notice that the absolute value of a sum or difference of two expressions is not necessarily equal to the sum or difference of the absolute values of the expressions.

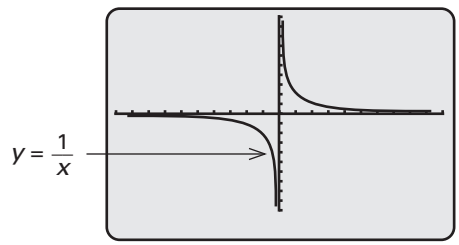
Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

g) The intent here is only to introduce a new "type" of graph that students may find of interest.

The graph of $y = 1/x$ (see diagram below) is called a *hyperbola*. Notice that, as the value of x gets closer to 0 from the right, the value of $1/x$ increases. For example: if $x = 1/2$, then $1/x = 2$; if $x = 1/4$, then $1/x = 4$; if $x = 1/100$, then $1/x = 100$, etc. Thus, as x gets closer and closer to 0, y gets larger and larger. Similarly, as x gets closer to 0 from the left, the value of $y = 1/x$ gets increasingly large in the negative direction. Hence, one can say that, as x approaches 0 from the right, y approaches ∞ and as x approaches zero from the left, y approaches $-\infty$. Further, the value of $y = 1/x$ is undefined for $x = 0$, since $1/0$ is undefined. Hence, x can never take on the value 0. Note: The *infinity symbol*, ∞, is used to indicate that a quantity gets larger and larger without bounds.



Notice also that, as the value of x increases in either the positive or negative direction, the value of $1/x$ gets closer to 0. For example, when $x = 100$, $y = 1/100$ and when $x = -1000$, $y = 1/-1000$. In the graph, $y = 1/x$, the x- and y-axis are called *asymptotes* of the graph because they are lines that the graph gets increasingly closer to, but never touches or crosses. Such ideas are examined in depth in calculus courses.

The graph of $y = 3/x$ is also a hyperbola with branches in the 1st and 3rd quadrants, and with the x- and y-axes as asymptotes. The graphs of all equations of the form $y = k/x$ are hyperbolas with asymptotes the x- and y-axes; the branches of such hyperbolas are in the 1st and 3rd quadrants if $k > 0$ and the branches are in the 2nd and 4th quadrants if $k < 0$, while $|k|$ determines how close the hyperbola comes to the origin.

Notice that the equation $y = k/x$ is equivalent to the equation $xy = k$. Such equations and their graphs represent situations in which x and y vary inversely. That is, since the product of xy is equal to a constant k, then x and y are inversely proportional (see Lesson 13 of this course, and see *Math Alive! Course IV*).

Focus Teacher Activity (cont.)

ACTIONS

9 Write the equations $y_1 = 2x + 1$ and $y_2 = x^2 - 2$ on the overhead or board. Point out that these are referred to as *a system of 2 equations in 2 variables*. Ask the groups to *solve this system of equations*, using each of the methods listed below. Discuss.

a) the TRACE and ZOOM functions on the graphing calculator

b) Algebra Pieces (or sketches of the pieces)

c) algebraic symbols

d) the calculator "solver" function

e) the calculator "intersect" function

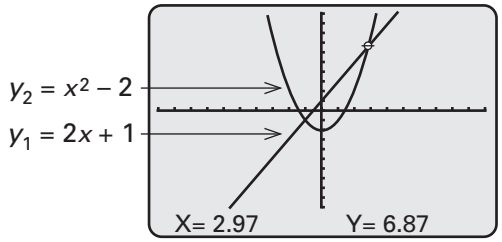
f) the calculator TABLE function

COMMENTS

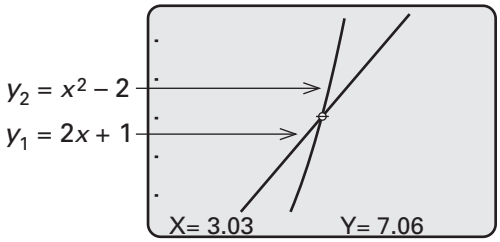
9 *Solving this system of equations* means to find the value of x for which $2x + 1 = x^2 - 2$, i.e., to *solve the equations simultaneously*. In terms of the graph, solving the system means finding the points of intersection of the two graphs. In terms of sequences of counting piece arrangements, solving the system is equivalent to finding the values of x for which the 2 different xth arrangements of the sequences represented by the equations have the same value.

a) The diagrams below show a trace and zoom to locate $x = 2.97$ and $x = 3.03$ as approximations for one solution. Additional zooms improve the approximation, suggesting the graphs intersect at $x = 3$.

Trace:



Zoom and then trace again:

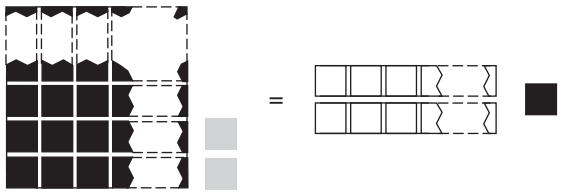


A series of zooms and traces of the other intersection point suggests $x = -1$ as a solution. One can verify that $x = 3$ and -1 are solutions of the system by testing those points in equations for y_1 and y_2. Since $v_1(3) = 2(3) + 1 = 7 = 3^2 - 2 = v_2(3)$, and $v_1(-1) = 2(-1) + 1 = -1 = (-1)^2 - 2 = v_2(-1)$, the points (3,7) and (–1,–1) are intersection points of these graphs.

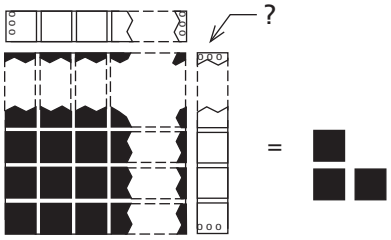
Focus Teacher Activity (cont.)

ACTIONS	COMMENTS

b) Illustrated below is an Algebra Piece solution.

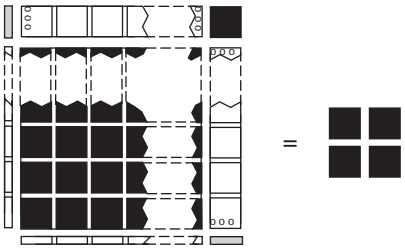


$x^2 - 2 = 2x + 1$



$x^2 - 2x = 3$

Adding $-2x + 2$ to both collections produces these 2 collections.



$x^2 - 2x + 1$ or $(x - 1)^2 = 4$

"Completing the squares" by adding 1 unit to the upper right hand corner of each collection produces this diagram. Since the squares are equal in value, their edges must be equal. Hence, $x - 1 = 2$ or $x - 1 = -2$, so $x = 3$ or $x = -1$. Therefore, $x = 3$ and $x = -1$ are the x-coordinates of the intersection points of the 2 graphs.

c) One can also *imagine* the Algebra Piece actions and record symbolic procedures that represent those actions. For example:

$x^2 - 2 = 2x + 1$		(form the 2 equal collections)
$(x^2 - 2) + (-2x + 2) = (2x + 1) + (-2x + 2)$		(add $-2x + 2$ to both collections)
$x^2 - 2x = 3$		(simplify)
$x^2 - 2x + 1 = 3 + 1$		(add 1 to both collections)
$(x - 1)^2 = 2^2$ or $(-2)^2$		(form squares of each collection)
$x - 1 = 2$ or -2		(take the square root of the value of
so, $x = 3$ or -1		each square)

d) One can use the TI-83 "solver" function to determine when the difference $(2x + 1) - (x^2 - 2) = 0$.

e) If students' calculators do not have this function, the students may have other methods to suggest.

f) This requires generating side-by-side tables for the 2 equations, and then scrolling to locate the values of x for which the y-values from the 2 tables are equal.

Focus Teacher Activity (cont.)

ACTIONS

10 Write 3 or more of the following systems of equations on the overhead and ask the students to determine the solution(s), if any, to each system, using the approach of their choice. Ask that students verify each solution using a second method, and so that one of their methods utilizes the graphing calculator and one does not. Discuss as needed.

a) $y = 4 + 2x$; $y = x + 3$

b) $y = 7 - x^2$; $y = -7 + x^2$

c) $y = 3x - 2$; $y = 3x + 1$

d) $y = 2x + 7$; $y = 4x^2 - 3x + 2$

e) $2x + 3y = 4$; $x - y = 7$

f) $y - 2x = 5$; $x + 3y = 6$

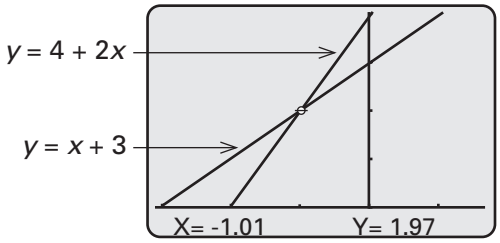
g) $y = x^2 + 7$; $y = 0$

h) $y = -2x^2 - 1$; $y = 0$

COMMENTS

10 Allow plenty of time for students to explore and discuss their approaches and results with their groupmates before discussing as a class. Students may use a hand graph, a calculator graph, Algebra Piece procedures, symbolic procedures, mental strategies, the solver or intersect functions on the calculator, tables generated by hand or by a calculator, or combinations of these; or, they may invent other strategies.

a) In the following example, zooming and tracing yields the estimate (–1.01,1.97). This suggests that the lines intersect at, or very near, $x = -1$. Since, $y = 4 + 2x$ and $y = x + 3$ both equal 2 when $x = -1$, the point (–1,2) is the exact point of intersection of the 2 graphs.



An algebraic solution based on what students imagine as Algebra Piece actions might look like the following:

$$\begin{aligned} 4 + 2x &= x + 3 && \text{(form the 2 equal collections)} \\ 4 + x &= 3 && \text{(remove } x \text{ from both collections)} \\ 1 + x &= 0 && \text{(remove 3 units from both collections)} \\ x &= -1 && \text{(add } -1 \text{ to both collections)} \end{aligned}$$

b) Here is a solution using algebra symbols to represent Algebra Pieces:

$$\begin{aligned} 7 - x^2 &= -7 + x^2 \\ 14 &= 2x^2 \\ 7 &= x^2 \\ \sqrt{7} &= x \text{ or } -\sqrt{7} = x \end{aligned}$$

Since $7 - (\pm\sqrt{7})^2 = 0$, and $-7 + (\pm\sqrt{7})^2 = 0$, the graphs intersect at the points $(-\sqrt{7},0)$ and $(\sqrt{7},0)$. Note that these are exact points of intersection; calculator methods give decimal approximations for the x-coordinates.

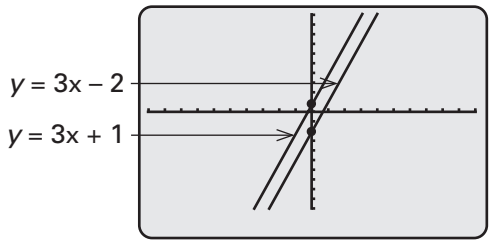
c) There is no value of x for which these 2 expressions are equal. Some students may reason that it is not possible to add 1 to a number and produce the same result as subtracting 2 from the number. Or, students may reason that since these 2 graphs are different straight lines with the same slope, they are parallel and hence,

Focus Teacher Activity (cont.)

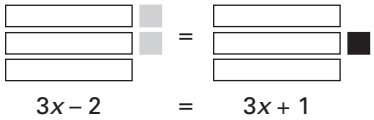
ACTIONS

COMMENTS

never intersect. Zooming out on the calculator graph (see diagram below) can help verify this; however, it is important to note that, when 2 lines are not parallel, it is possible to miss an intersection point by not zooming out far enough.



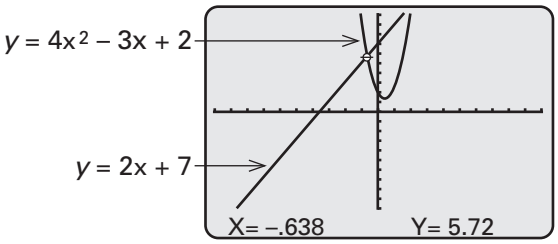
Here is a representation using sketches of Algebra Pieces:



Adding –3*x* to each collection above leaves: 

But it is not possible that –2 = 1, so there are no solutions to the system.

d) As shown in the calculator display below, one approximation of an intersection point is (–.638,5.72). The other intersection point is outside the window. Changing the window ranges for *x* and *y* enables one to approximate the other intersection point.



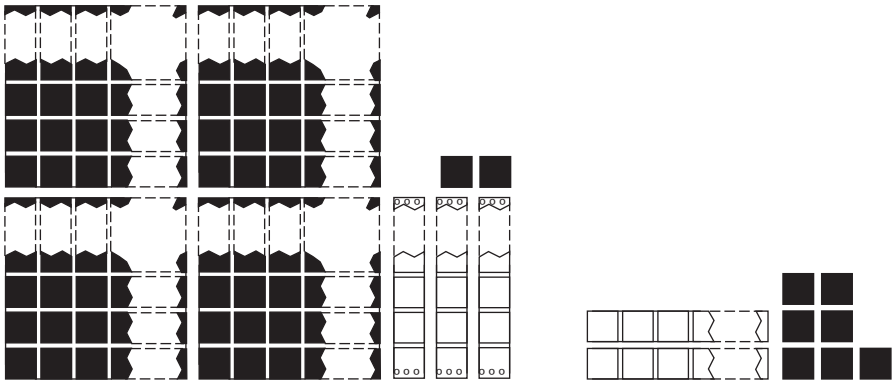
Using Algebra Pieces (see next page) to complete the square for this quadratic equation illustrates that these graphs do not intersect at a point whose coordinates are whole or rational numbers. Rather, the coordinates of the points of intersection are irrational numbers.

(Continued next page.)

Focus Teacher Activity (cont.)

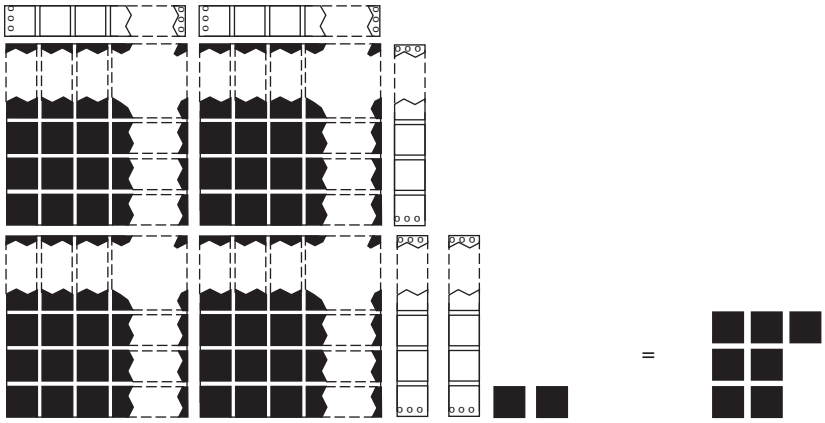
COMMENTS

10 (continued.)



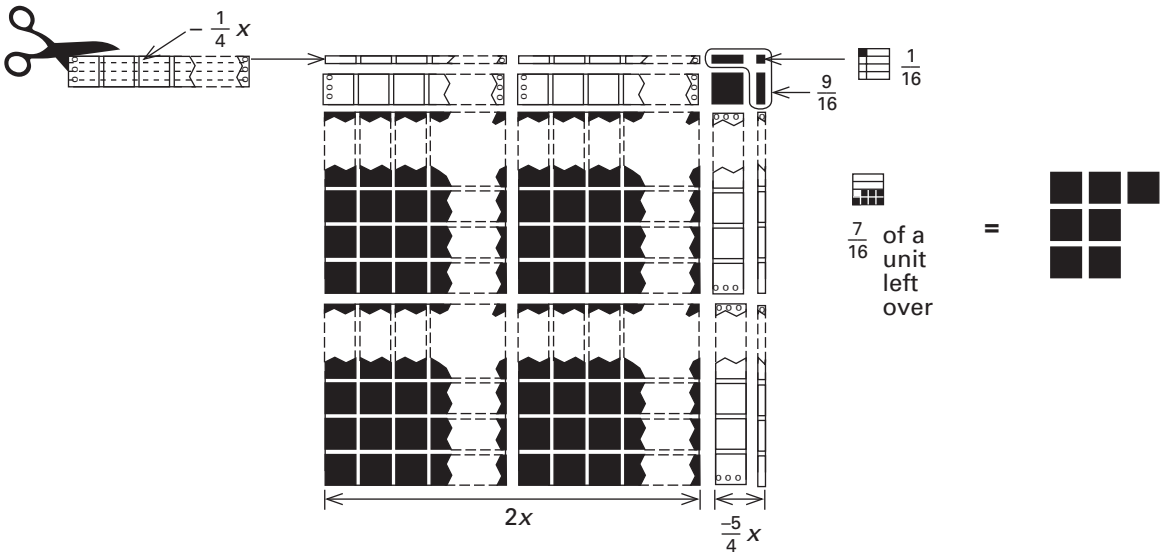
$4x^2 - 3x + 2 \quad = \quad 2x + 7$

Adding $-2x$ to both collections in the above diagram produces the following collections:



$4x^2 - 5x + 2 \quad = \quad 7$

Cutting and rearranging the pieces in the above collection produce the following (note that the pieces on the upper and right edges of the square are not edge pieces; rather, they are quartered $-x$-frames):



$(2x - \frac{5}{4})^2 + \frac{7}{16} \quad = \quad 7$

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

Subtracting $7/16$ from both collections above produces $(2x - 5/4)^2 = 69/16 = 105/16$, and so $2x - 5/4 = \pm\sqrt{105/16}$. Thus, $x = \frac{\left(5/4 \pm \sqrt{105/16}\right)}{2}$ and the graphs cross at approximately $x \approx 1.906$ and $x \approx -.656$. When $x = -.656$, $2x + 7 = 4x^2 - 3x + 2 \approx 5.69$. When $x \approx 1.906$, $2x + 7 = 4x^2 - 3x + 2 \approx 10.8$. Thus, the 2 points of intersection for these graphs occur at approximately (–.656,5.69) and (1.906,10.8). These are rational approximations to irrational coordinates.

This example illustrates the convenience of the graphing calculator for quickly finding approximate solutions to equations. Recall that the TRACE function obtained the approximation $x = -.638$ (see page 383), which is close to the algebraic approximation of $x = -.656$. Repeated zooms and traces would improve the calculator approximation.

e) In order to graph these equations on the calculator, one must rewrite them in slope intercept form as $y = (-2/3)x + 4/3$ and $y = x - 7$. Then one can graph and trace to find the approximate intersections, or use the "intersect" function from the CALC menu.

Or, one could use the "solver" function from the MATH menu to determine when $0 = [(-2/3)x + 4/3] - (x - 7)$.

One way to solve this system of equations symbolically is to solve $(-2/3)x + 4/3 = x - 7$ for x.

Another symbolic method is to solve for y in one equation, substitute that value in the other equation, and then solve for x. For example, solving the equation $x - y = 7$ for y, one gets $y = x - 7$. Then substituting $x - 7$ for y in the equation $2x + 3y = 4$ produces the new equation $2x + 3(x - 7) = 4$. Hence, $2x + 3x - 21 = 4$, so $5x = 25$, and thus, $x = 5$. This is an example of the method called *solving by substitution*. Students will investigate this method further in *Math Alive! Course IV*.

f) Students will need to rewrite these equations in slope intercept form before graphing them on the calculator, using the "solver" function on the calculator, or solving them using Algebra Pieces or symbols. The method of substitution described in e) could also be used here.

g)-h) There are no solutions to either of these systems since neither the parabola $y = x^2 + 7$ nor the parabola $y = -2x^2 - 1$ intersects the x-axis (i.e., the line $y = 0$). Notice that completing the square to solve $x^2 + 7 = 0$

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

11 Give each group a copy of Focus Master B and ask the groups to:

i) Write a mathematical statement involving equalities and/or inequalities to describe each graph.

ii) Indicate whether each graph represents a function, and if so, identify the domain and range of the function.

iii) Recreate each graph on a graphing calculator.

Discuss their results. As needed, clarify graphing calculator procedures, the use of inequality symbols, and graphing conventions such as the use of dotted lines and open/closed circles.

COMMENTS

10 (continued.)
produces the solutions $x = \frac{\pm\sqrt{-28}}{2}$, which are not defined for any real number values of x, since there is no real number whose square is –28. Similarly, there are no real numbers x for which $-2x^2 - 1 = 0$. Note in *Math Alive! Course IV* students are introduced to the *complex number system* which includes solutions of equations involving square roots of negative numbers.

11 It may be helpful to have the students complete a) and then discuss before continuing with the others. Encourage students to write mathematical statements that use as few words as possible, but so that someone reading their statements could recreate the graph exactly.

a) One statement describing this graph is $y = 2$ for $x \geq 2$. This is a function whose domain is the real numbers ≥ 2, and 2 is the only number in the range. Note: Students may also point out that this graph is a ray whose end point is (2,2).

Note that the filled in dot at the point (2,2) implies that the point (2,2) is part of the graph. If the dot were not filled in then the domain would be $x > 2$ and the point (2,2) would not be considered on the graph. Students may not be familiar with this notation, but will probably speculate and you can clarify as needed.

b) An equation for this function is $y = |x| + 1$. The domain is all real numbers and the range is the reals ≥ 1.

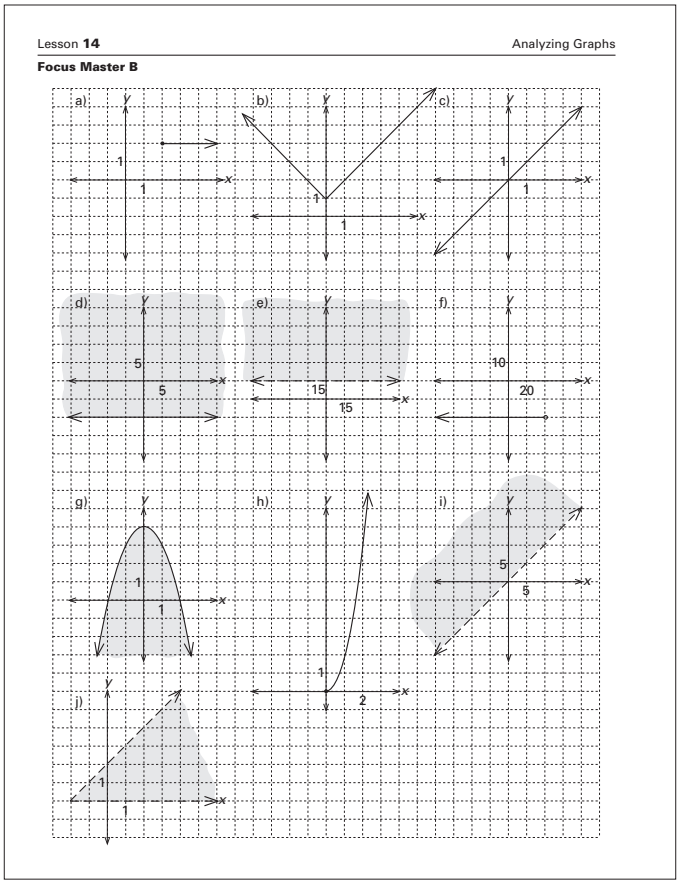
c) The domain and range of this function, $y = x$, are the real numbers.

d) You may need to tell students that all points in the shaded region are part of the graph to be described. This is the graph of an inequality and since for any point on the graph, each x-value has an infinite number of y-values, it is not a function. This graph includes the set of all ordered pairs, (x,y) for which x and y are real numbers and $y \geq -10$. The fact that the line $y = -10$ is a solid line implies the points on the line $y = -10$ are included in the graph. To graph inequalities on the TI-83, for example, it is necessary to set the graphing style to shade above or below a graph (i.e., adjust the icon in the first column for the "Y =" function, see manual).

e) The dotted line implies the points on the line are not included in the graph, but all of the shaded region is part of the graph. Hence, this is a graph of all the ordered pairs (x,y) for which x and y are real numbers and

Focus Teacher Activity (cont.)

ACTIONS



COMMENTS

$y > 15$. This is not a function. Using dotted lines and open circles may not be possible on students' calculators.

f) This is a graph of the function $y = -20$ for $x < 40$. The domain is all real numbers $x < 40$, and the range contains only the number –20.

g) This graph does not represent a function. It is a graph of the inequality $y \leq -x^2 + 4$ for all real numbers x. The y-values are all real numbers less than or equal to 4.

h) This is a graph of the function $y = 2x^2$ for $x \geq 0$. The domain and range are the real numbers ≥ 0.

i) This is not the graph of a function, but rather a graph of the inequality $y > x$ for all real numbers x.

j) This inequality may be a challenge for students to describe. It is the set of all ordered pairs for which $y < x + 2$ and, at the same time, $y > 0$. The values of x are all real numbers greater than –2. To graph j) on the graphing calculator, one must simultaneously graph $y < x + 2$ and $y > 0$. The shaded region shown in j) is the portion of the graph in which the shaded regions for the 2 graphs overlap. This overlapping shaded region is the *solution to the system of inequalities* $y < x + 2$ and $y > 0$, and is all the points for which $y < x + 2$ and $y > 0$. Note: keep in mind it may not be possible for students to distinguish between dotted and solid lines and curves on calculator graphs.

To give students additional experience solving systems of inequalities and to prompt further conjectures, you could replace the equal signs in systems a)-f) from Action 10 with various inequality symbols and have the students solve these new systems of inequalities. Note that, if there is no overlapping shaded region for a system of inequalities, then there are no solutions to the system. Students will explore inequalities further in *Math Alive! Course IV*.

Focus Teacher Activity (cont.)

ACTIONS

12 Give each group a copy of Focus Master C and have them carry out the instructions.

Analyzing Graphs Lesson **14**

Focus Master C

Franko and his son, Marcus, plan to race one another on a track.

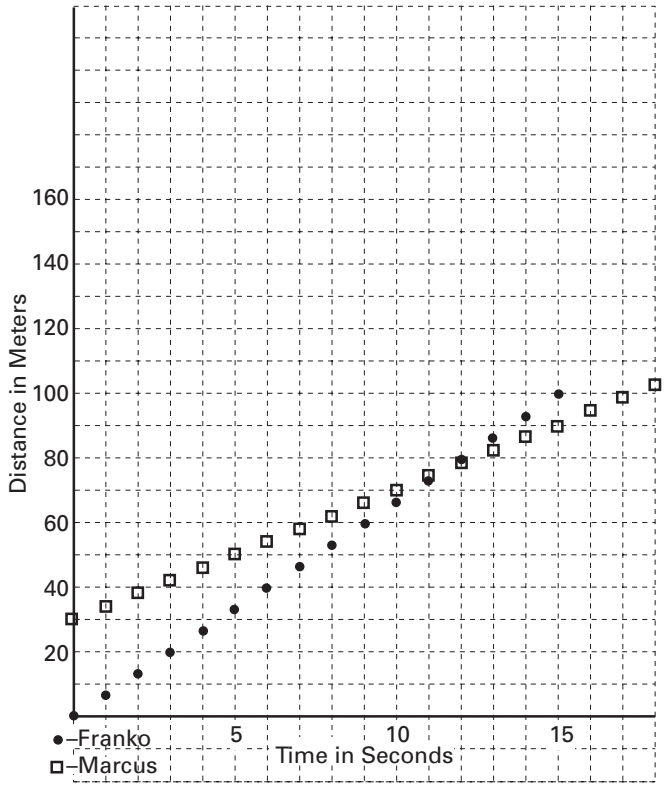
Marcus can run 20 meters in 5 seconds.

Franko can run 20 meters in 3 seconds.

They have agreed that Marcus will start 30 meters ahead of Franko.

Write several "We wonder…" statements about this situation.

13 Pick one or more of the groups' "We wonder…" statements to explore. Discuss their results.



COMMENTS

12 Students may wonder: Is this a fair head start for the son? Who would win a 100 meter race? When would the man catch up with the child? What length race would be most fair? etc. What are equations and graphs that could represent this situation?

13 If students have difficulty deciding what to explore, you might suggest something, basing your choice on its mathematical potential. Or you might suggest they investigate a question that they pose and a question that you pose. Have 1-cm grid paper available as needed.

As an example, the following discussion explores the question "Who would be favored to win a 100 meter race?"

There are several approaches that students could use to investigate the above question, such as to make a chart or table of values, and/or to graph the times and distances traveled by both runners. A graph is illustrated at the left. A table can be produced by hand or by using the table function on the graphing calculator. Notice that since Marcus runs 20 meters in 5 seconds, then he runs 4 meters in 1 second. Similarly, since Franko runs 20 meters in 3 seconds, he runs $20/3$ meters in 1 second.

Marcus	
time in seconds	distance in meters
0	30
1	34
2	38
3	42
⋮	⋮
10	70
11	74
12	78
13	82
14	86
15	90
16	94
17	98
18	102

Franko	
time in seconds	distance in meters
0	0
1	$6\frac{2}{3}$
2	$13\frac{1}{3}$
3	20
⋮	⋮
9	60
10	$66\frac{2}{3}$
11	$73\frac{1}{3}$
12	80
13	$86\frac{2}{3}$
14	$93\frac{1}{3}$
15	100

Focus Teacher Activity (cont.)

ACTIONS

14 If it hasn't already been suggested, ask the students to write an equation for y_1, the distance of Marcus from the starting line x seconds after the start of the race, and an equation for y_2 the distance of Franko from the starting line x seconds after the start of the race. Ask for volunteers to share their equations and have the class determine whether the equations accurately represent the distances. Then ask the groups to use calculator graphs of the equations to determine a fair length for the race (or to verify their results from Action 13).

COMMENTS

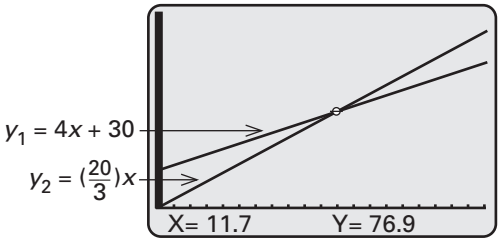
The table and graph show that Franko catches up with Marcus between 11 and 12 seconds. Also, at 12 seconds, Franko is at the 80 meter mark, and Marcus is at the 78 meter mark, so Franko is favored to win a 100 meter race.

14 Students may verify their equations by testing various numbers of seconds to see if the meters traveled match those in their tables from Action 13. For example, their equation for y_1 should yield 50 meters at 5 seconds (the initial 30 plus 20 more), 70 meters at 10 seconds, and so forth.

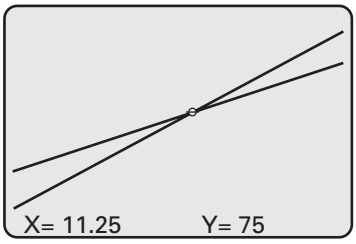
One possible pair of equations is $y_1 = 4x + 30$ and $y_2 = (20/3)x$.

In order to find the race length that makes the 30 meter head start fair, students must determine where Franko catches Marcus (i.e., where the graphs cross, or where the distances run are equal). To do this, they can graph $y_1 = 4x + 30$ and $y_2 = (20/3)x$ and use trace to determine the point of intersection (see diagram below).

1st trace:
x min 0 x max 20
y min 0 y max 150



Trace after several zooms:



Many students will probably suggest a "fair" race is about 75 meters. An extension question could be, "How much of a head start does Marcus need for 100 meters to be the length of a fair race?"

Focus Teacher Activity (cont.)

ACTIONS

15 Give each student a copy of Focus Master D and ask them to complete Situation 1. Discuss their results. Then repeat for Situations 2 and 3.

Lesson **14** Analyzing Graphs

Focus Master D

For each of the 3 Situations shown below, please do the following:

a) Make a diagram or sketch that illustrates the important mathematical relationships in the situation.

b) Write 3 or more worthwhile mathematical questions that a person might investigate about the situation.

c) Investigate one or more of your mathematical questions. While you investigate each question, keep a running account of your thought processes. Make note of your discoveries, stuck points, AHA's, important mathematical moments, changes in direction, etc.

d) After you complete your investigation of each question, write a summary that includes a restatement of the question, a clear and concise explanation of a solution process (this might be a refined version of your method or a different method you discovered during your work), your answer to the question, and verification that your answer works.

Situation 1
The Rent-A-Wreck and the We Hardly Try car rental companies charge the following prices:

We Hardly Try charges an initial fee of $10 and then charges $.10 per mile. Rent-A-Wreck does not charge an initial fee, but charges $.15 per mile.

Situation 2
The Saucey Pizza Company charges $7 for a pizza. The ingredients and labor for each pizza cost $2.50. The overhead costs (lights, water, heat, rent, etc.) are $100 per day.

Situation 3
Michael, the golf pro at U-Drive-It Golf Range, claims that when he hits the ball from the lower level tee, the height h of the ball after t seconds is: $h = 80t - 16t^2$.

Michael also claims that when he hits the ball from the elevated tee, the ball reaches the following height in t seconds: $h = 20 + 80t - 16t^2$.

COMMENTS

15 These situations could be explored in class, and/or as a homework activity. Either way, it is helpful to encourage students to discuss and compare ideas with classmates.

Situation 1. Examples of questions students may pose include: How much does it cost to drive each vehicle 10 miles? 20 miles? etc. Which company has the better deal? Is there a number of miles for which the cost will be the same for both companies? As an example, the following discussion addresses the question: Which company has the better deal?

Students could make a table, write and solve a system of equations describing each company's cost, or graph the equations by hand or on a graphing calculator and look for the points where the graphs intersect.

If y_1 is the cost at We Hardly Try, y_2 is the cost at Rent-A-Wreck, and x is the number of miles driven, equations for the total rental cost from each company could be represented as follows:

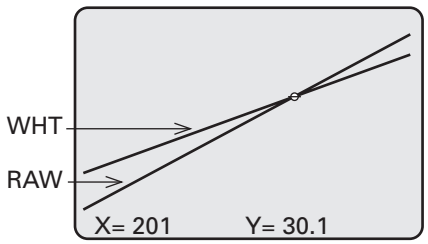
$y_1 = \$10 + .10x$
$y_2 = .15x$

A symbolic solution to this system could look like the following:

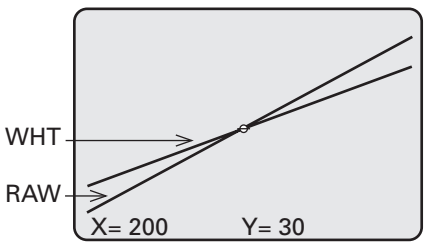
$$\begin{aligned} 10 + .10x &= .15x \\ 10 + .10x - .10x &= .15x - .10x \\ 10 &= .05x \\ 10\,(1/.05) &= .05x\,(1/.05) \\ 200 &= x \end{aligned}$$

When $x = 200$, $y_1 = 30 = y_2$. Hence, at 200 miles the cost of driving a car from either company is the same. For less than 200 miles, Rent-A-Wreck is a better deal, while We Hardly Try is a better deal for more than 200 miles. Traces on a graphing calculator illustrate this:

1st Trace:
x min 0 x max 300 (miles)
y min 0 y max 50 (cost)

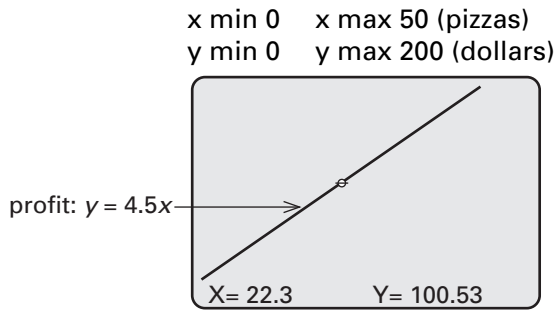


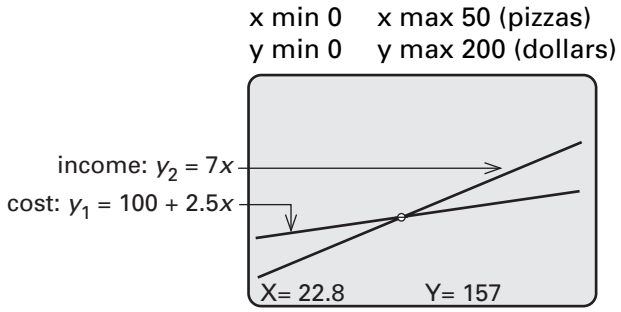
Trace, after several zooms:



Focus Teacher Activity (cont.)

ACTIONS





seconds x	lower level tee height in feet $y_1 = 80x - 16x^2$	elevated tee height in feet $y_2 = 80x - 16x^2 + 20$
0	0	20
0.5	36	56
1.0	64	84
1.5	84	104
2.0	96	116
2.5	100	120
3.0	96	116
3.5	84	104
4.0	64	84
4.5	36	56
5.0	0	20
5.5	–44	–24

COMMENTS

Situation 2. Examples of questions that may come up include: If Saucey's sells 100 pizzas, how much money will they make or lose that day? How many pizzas do they need to sell in a day to make a profit? Should they change their pricing? Following is one way of answering the question: How many pizzas do they need to sell in a day to make a profit?

Since a pizza is sold for \$7 and the ingredients and labor cost \$2.50, the profit on each pizza is \$7 – \$2.50 = \$4.50. If Saucey's sells x pizzas in a day, then the amount of daily profit, y, is $y = \$4.50x$. The minimum number of pizzas that must be sold to have a total income greater than the daily overhead cost of \$100 can be found by tracing a graph of $y = 4.5x$ to determine the value of x when y exceeds \$100, as shown at the left. Since y exceeds 100 between $x = 22$ and $x = 23$, Saucey must sell at least 23 pizzas per day or lose money.

Another approach to answering this question is to simultaneously graph the equation representing the daily cost, $y_1 = 100 + 2.50x$ (where x is the number of pizzas sold), and the equation for daily income, $y_2 = 7x$. The intersection of the graphs is the point at which Saucey's breaks even (see graph at the left). This also indicates Saucey must sell 23 pizzas in order for daily income to exceed daily cost.

Situation 3. Here are some questions students may pose: At what time does the golf ball reach its highest point? How long does it take before the golf ball hits the ground? How high is the elevated tee? Where did those formulas come from?

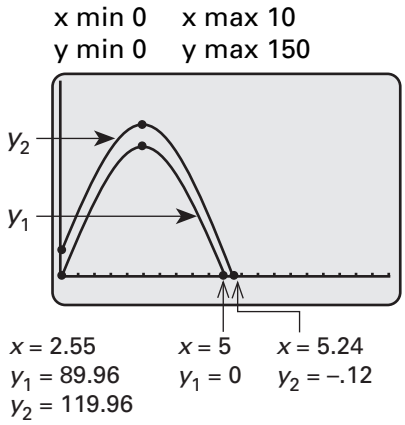
An example of a table generated by the calculator is shown at the left, where y_1 describes the height of the ball when hit from the lower tee, and y_2 describes the height of the ball from the elevated tee. The table suggests the ball reaches a high point after 2.5 seconds, and then starts back down again, hitting the ground after 5 seconds (a table with smaller increments can be used to see if the high point is slightly more or less than 2.5). Notice the table lists negative heights, but since the ball stops descending at ground level, those do not make sense and are irrelevant.

Graphing and tracing both equations on a graphing calculator, one can find the high point of the golf ball and the time when the ball hits the ground (see diagram on the next page).

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS



$$y_1 = 80x - 16x^2$$
$$y_2 = 80x - 16x^2 + 20$$

A ball hit from the elevated tee stays 20 feet higher, and hits the ground about a quarter second later. Note: the graphs of the equations are *not* the paths of the balls, but rather give the heights of the ball for various times.

COMMENTS

15 (continued.)
Notice, in the diagram at the left, the 2 graphs are the same shape, the graph of y_2 is a vertical translation 20 units above the graph of y_1, and it takes longer for the ball hit from the elevated tee to hit the ground.

The coefficients of the variables have physical significance. For example, the "80" means the golf ball rises at a rate of 80 feet/sec when it is first hit. On the other hand, the "–16" is the effect of the pull of the earth's gravity on the golf ball (the force of gravity is a downward force of 16 feet per second squared (i.e., per x^2, where x is the number of seconds). Gravity slows the height gain of the ball over time and eventually the ball starts to descend. The 20 in the elevated tee equation is the height of the ball at time 0, so the elevated tee is 20 feet above the ground level.

16 Write the topics from a)-h) below on the overhead or board, editing according to the functions available on your students' graphing calculators. Ask the groups to each create a list of advantages and disadvantages of using these methods to solve equations or systems of equations, and to evaluate equations. Discuss the groups' lists.

a) a hand sketched graph

b) the graph, zoom, and trace functions on the graphing calculator

c) Algebra Pieces

d) symbolic manipulations

e) mental strategies

f) the "solver" function from the graphing calculator

g) the "intersect," "zero," and "value" functions from the graphing calculator

h) tables of values (hand or calculator)

16 The intent here is to encourage the students to think critically about the appropriate uses of technology, and to keep in mind that the graphing calculator is one of several options for graphing and solving equations. One way to carry out this action is to give each group several strips of butcher paper and some marking pens and have them record their comments on the strips. Then, a class list of recommendations could be formed by sorting and consolidating the groups' lists.

It is important that students understand that not everyone will choose the same option for a particular situation; what seems most reasonable or easy to one person is not necessarily most reasonable or easy for another; and not every option is a reasonable choice for every situation. Here are examples of students' comments:

Making tables or graphs by hand may only give us a rough idea about solutions; a more complete graph may be needed for a better approximation to the answer. This was true when we solved the problem with the two runners, Franko and Marcus.

The graphing calculator can often make tables and graphs more quickly than we can by hand.

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
	Sometimes it is difficult to know what values to set for the WINDOW *ranges in order to see important parts of the graph in the calculator window (e.g., the golf problem).*
	Symbolic strategies are very efficient for solving equations, unless the equations are complicated or involve really "messy" numbers.
	Sometimes using calculator functions like table, solver, intersect, and zero is more time-consuming because mental strategies or simple symbolic procedures are all that are needed.
	To use mental strategies to graph equations, we need to understand the relationship between the numbers in the equation and the shape and important points of the graph.
	The Algebra Pieces help us make sense out of what the points on a graph represent and they help us "see" solutions.
	An Algebra Piece solution can be complicated for equations with coefficients and constants that are large numbers or fractions.
	When we solve an equation with Algebra Pieces or symbols that represent them, we find the exact solution. Graphs (made either by hand or on the calculators) often only give us approximations.
	The trace and zoom functions on the graphing calculator may only approximate a solution. We can test the result in the equation to see if it is exact or approximate.
	The equation solver, intersect, and zero functions on the calculator give exact answers quickly, but they don't show where the answers comes from.
	If all we need to know are things like the shape of the graph, whether the values of y continue to increase as x gets larger, the value of the y-intercept, whether a line rises or falls, or whether a quadratic equation has a high or low point, we can often tell just by looking at the equation and imagining the graph.
	If we have to evaluate an equation for a certain value for x, we could trace on the graphing calculator to get an approximation, we could compute it by hand or mentally if the numbers are reasonable, or we could use the table or value function on the calculator. How messy or large the numbers are helps us decide which method to use.

TEACHER NOTES:

Follow-up Student Activity 14.3

NAME _____ **DATE** _____

1 For each of the following families of 3 equations, graph the equations on 1 coordinate axis, and list the characteristics that make the equations a family. Then create and graph 2 or more additional equations that have those characteristics. Label each graph with its equation.

a) $y = 4x - 1$
$y = 4x + 2$
$y = 4x - 5$

b) $y = 3x - 2$
$y = \frac{x}{5} - 2$
$y = -6x - 2$

c) $y = x^2 + 3$
$y = -2x^2 + 3$
$y = \frac{x^2}{4} + 3$

d) $y = |x| + 2$
$y = -3|x|$
$y + 2|x| - 3$

e) $y = \frac{1}{x} + 5$
$y = \frac{1}{x} - 4$
$y = \frac{1}{x} + 3$

f) $y = 4(x - 2)(x + 5)$
$y = -3(x - 2)(x + 5)$
$y = (x - 2)(x + 5)$

2 For each of the following systems of equations: i) show or describe how you solve the system; ii) sketch a graph of the system on coordinate grid paper; and iii) label the coordinates to the nearest tenth of all points of intersection.

a) $6x + 3y = 5$
$2y - 3x = 12$

b) $y = x^2 - 8x + 18$
$2x + y = 7$

c) $y = \frac{2}{x} + 3$
$y = x^2 + 4$

d) $y = \frac{x^2}{2} - \frac{x}{2} + 3$
$y = -x^2 - 3x + 5$

e) $y = x^2 - 3x - 2$
$x^2 + 3y - 18 = 0$

f) $y = -x^2 - x - 2$
$y = (x + 1)(x - 2)$

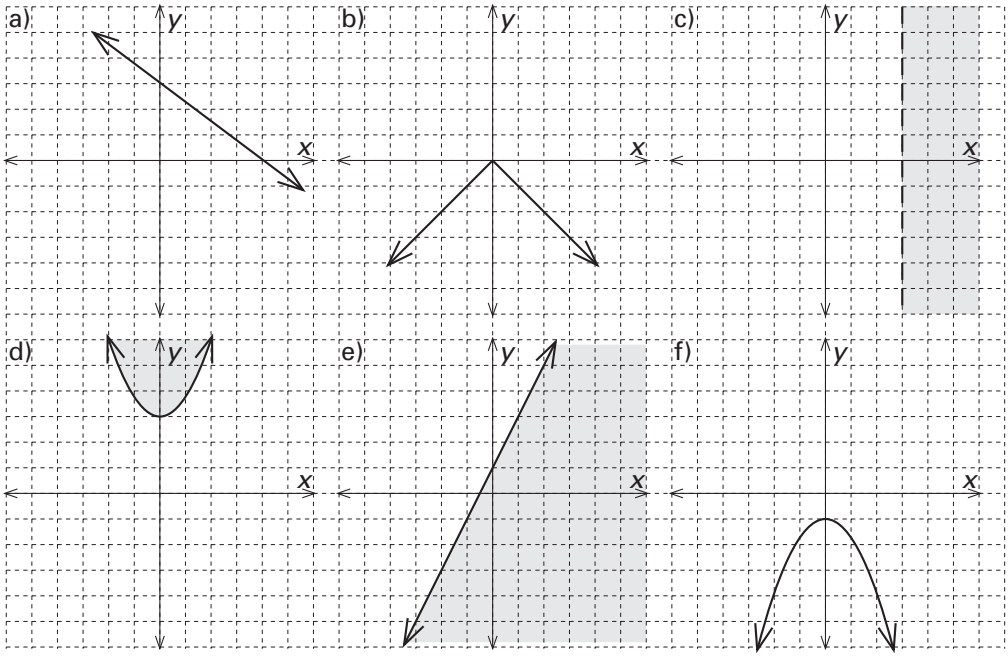
3 Verify your solutions to Problems 2a) and 2b) by solving each using another method. Show your thinking and reasoning.

4 Discuss the advantages and disadvantages of using the graphing calculator to solve equations. Give examples to illustrate your ideas.

(Continued on back.)

Follow-up Student Activity (cont.)

5 Write a mathematical statement involving equalities or inequalities to describe each graph. Tell whether the graph represents a function; if it does, tell the domain and range of the function.



6 During the summer Patty rents an ice cream truck and sells ice cream; she pays $200 per month to rent the truck; she sells ice-cream bars for $1.50 each; and she pays 25 cents for each bar.

a) Write an equation for y_1, Patty's expenses per month, if she sells x ice cream bars per month.

b) Write an equation for y_2, Patty's monthly income (before she pays her expenses) for selling x ice cream bars per month.

c) Graph the two equations in a) and b) and label the coordinates of the intersection of the graphs. Explain how you determine these coordinates and how they relate to the given situation.

d) What is the minimum number of ice cream bars Patty must sell in one month in order for her income to be greater than her expenses. Explain your reasoning.

e) If Patty sells 300 ice cream bars in one month, how much profit will she make? Explain your reasoning.

Focus Master A

I $y = -3x + 5$
II $y = -x + 5$
III $y = x + 5$
IV $y = 3x + 5$

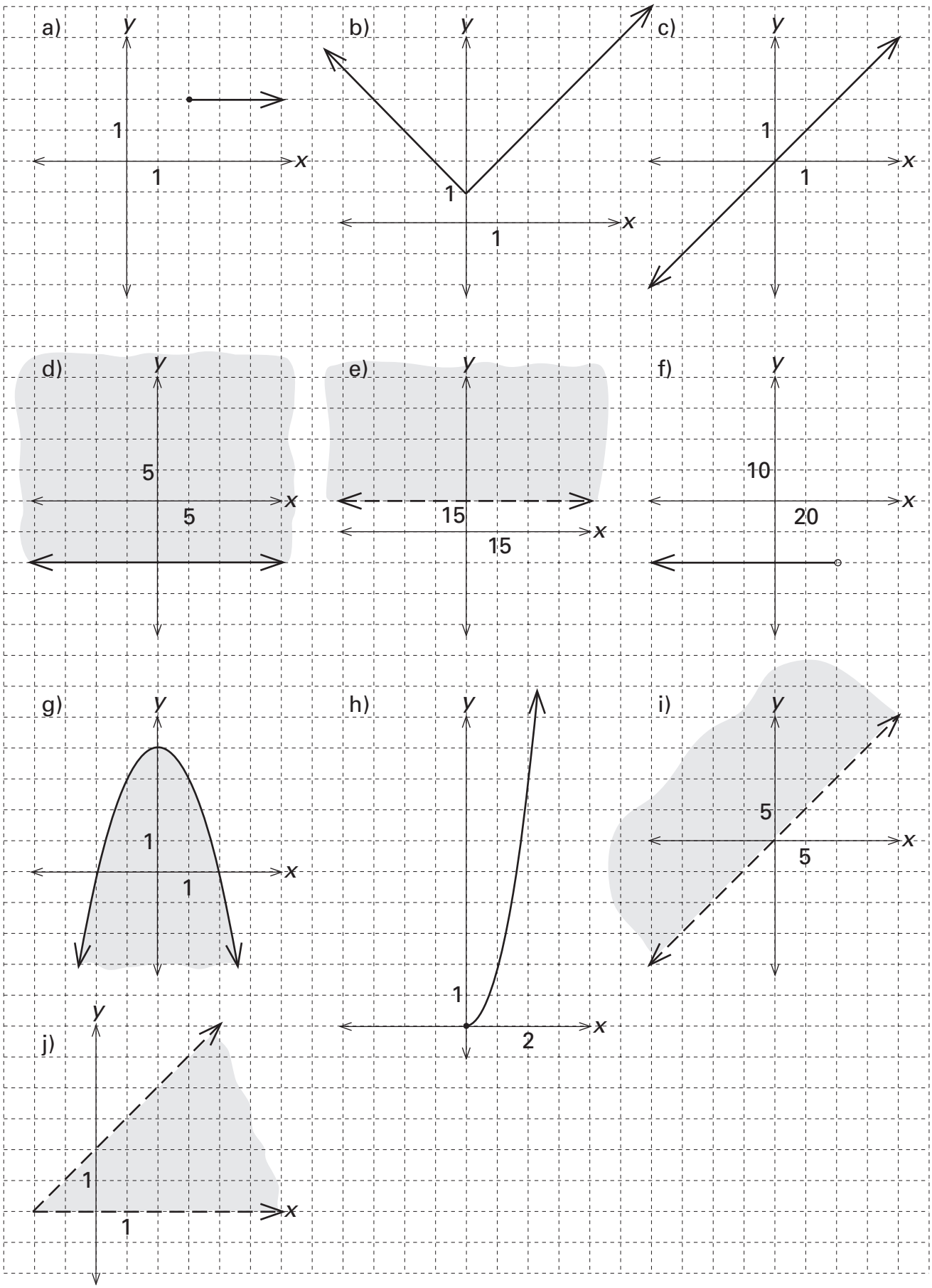
a) Imagine the graph of each of equations I-IV. What similarities and differences does your group *predict* about the graphs?

b) Now graph the 4 equations simultaneously on your graphing calculators. Do the results agree with your predictions? What else do you notice?

c) Equations I-IV are a "family" of equations. What characteristic(s) do you think make these equations a family? What are two other equations that could belong to this family?

d) What are similarities and differences among Algebra Piece representations of the xth arrangements of the sequences represented by equations I-IV?

Focus Master B



Focus Master C

Franko and his son, Marcus, plan to race one another on a track.

Marcus can run 20 meters in 5 seconds.

Franko can run 20 meters in 3 seconds.

They have agreed that Marcus will start 30 meters ahead of Franko.

Write several “We wonder…” statements about this situation.

Focus Master D

For each of the 3 Situations shown below, please do the following:

a) Make a diagram or sketch that illustrates the important mathematical relationships in the situation.

b) Write 3 or more worthwhile mathematical questions that a person might investigate about the situation.

c) Investigate one or more of your mathematical questions. While you investigate each question, keep a running account of your thought processes. Make note of your discoveries, stuck points, AHA's, important mathematical moments, changes in direction, etc.

d) After you complete your investigation of each question, write a summary that includes a restatement of the question, a clear and concise explanation of a solution process (this might be a refined version of your method or a different method you discovered during your work), your answer to the question, and verification that your answer works.

Situation 1
The Rent-A-Wreck and the We Hardly Try car rental companies charge the following prices:

We Hardly Try charges an initial fee of $10 and then charges $.10 per mile. Rent-A-Wreck does not charge an initial fee, but charges $.15 per mile.

Situation 2
The Saucey Pizza Company charges $7 for a pizza. The ingredients and labor for each pizza cost $2.50. The overhead costs (lights, water, heat, rent, etc.) are $100 per day.

Situation 3
Michael, the golf pro at U-Drive-It Golf Range, claims that when he hits the ball from the lower level tee, the height h of the ball after t seconds is:
$h = 80t - 16t^2$.

Michael also claims that when he hits the ball from the elevated tee, the ball reaches the following height in t seconds: $h = 20 + 80t - 16t^2$.

Focus Student Activity 14.1

NAME _____ **DATE** _____

For each equation family below, record the following on separate paper:

a) your predictions about the graphs of the 4 equations,

b) your observations about calculator graphs of the equations,

c) the characteristic(s) that you think make the equations a family,

d) two additional equations that would fit in the family,

e) similarities and differences among Algebra Piece representations of the 4 equations.

1
I $y = -3x + 5$
II $y = -3x - 5$
III $y = -3x + 2$
IV $y = -3x - 2$

2
I $y = x^2 - 6$
II $y = x^2 + 6$
III $y = -x^2 - 6$
IV $y = -(x^2 - 6)$

3
I $y = 4x^2$
II $y = (1/4)x^2$
III $y = -3x^2 - 6$
IV $y = (-3/4)x^2$

4
I $y = x(x - 3)$
II $y = x^2 - 2x$
III $y = x^2 + 2x$
IV $y = x(x + 3)$

5
I $y = (x - 3)(x - 4)$
II $y = x^2 - 7x + 12$
III $y = (x + 2)(x + 3)$
IV $y = (x + 1)(x - 2)$

6
I $y = (x - 2)(x - 5)$
II $y = 2(x - 2)(x - 5)$
III $y = -(x - 2)(x - 5)$
IV $y = -2(x^2 - 7x + 10)$

7
I $28x + 8y = 0$
II $7x + 2y = 6$
III $14x + 4y = 4$
IV $21x + 6y = -12$

8
I $y = -5x + 2/3$
II $3y = -15x + 2$
III $0 = -5x - y + 2/3$
IV $-2 = -15x - 3y$

Focus Student Activity 14.2

NAME _____ **DATE** _____

For each pair of equations given in a)-g):

i) Sketch and label counting piece arrangements to represent y_1 for $x = -3, -2, -1, 0, 1, 2$, and 3, and sketch an Algebra Piece representation of the xth arrangement of y_1. Repeat for y_2.

ii) Make a table that shows the corresponding values of x, y_1, and y_2 for the arrangements formed in i).

iii) Predict how you think the graphs of y_1 and y_2 will look in comparison to the graph of $y = |x|$. Then, on the same coordinate axes, sketch a graph of y_1, y_2, and $y = |x|$ over the domain of real numbers such that $-10 \le x \le 10$. Label the coordinates of the points from the table in ii).

iv) On the same coordinate axes of the graphing calculator, graph y_1, y_2, and $y = |x|$ over the domain given in iii).

v) Write 3 additional equations that form a family with y_1, and y_2, and explain the relationship that makes the 5 equations a family.

vi) Record conjectures and generalizations based on your observations from i)-v).

a) $y_1 = |x| - 3$ $\quad$ $y_2 = |x| + 4$

b) $y_1 = |x - 5|$ $\quad$ $y_2 = |x + \frac{1}{2}|$

c) $y_1 = |2x|$ $\quad$ $y_2 = |-2x|$

d) $y_1 = 5|x|$ $\quad$ $y_2 = -5|x|$

e) $y_1 = |x - 2x|$ $\quad$ $y_2 = |3x + 2|$

f) $y_1 = |x| - |2x|$ $\quad$ $y_2 = |3x| + |2|$

g) $y_1 = \frac{1}{x}$ $\quad$ $y_2 = \frac{3}{x}$

Follow-up Student Activity 14.3

NAME _____ **DATE** _____

1 For each of the following families of 3 equations, graph the equations on 1 coordinate axis, and list the characteristics that make the equations a family. Then create and graph 2 or more additional equations that have those characteristics. Label each graph with its equation.

a) $y = 4x - 1$
$y = 4x + 2$
$y = 4x - 5$

b) $y = 3x - 2$
$y = \frac{x}{5} - 2$
$y = -6x - 2$

c) $y = x^2 + 3$
$y = -2x^2 + 3$
$y = \frac{x^2}{4} + 3$

d) $y = |x| + 2$
$y = -3|x|$
$y + 2|x| - 3$

e) $y = \frac{1}{x} + 5$
$y = \frac{1}{x} - 4$
$y = \frac{1}{x} + 3$

f) $y = 4(x - 2)(x + 5)$
$y = -3(x - 2)(x + 5)$
$y = (x - 2)(x + 5)$

2 For each of the following systems of equations: i) show or describe how you solve the system; ii) sketch a graph of the system on coordinate grid paper; and iii) label the coordinates to the nearest tenth of all points of intersection.

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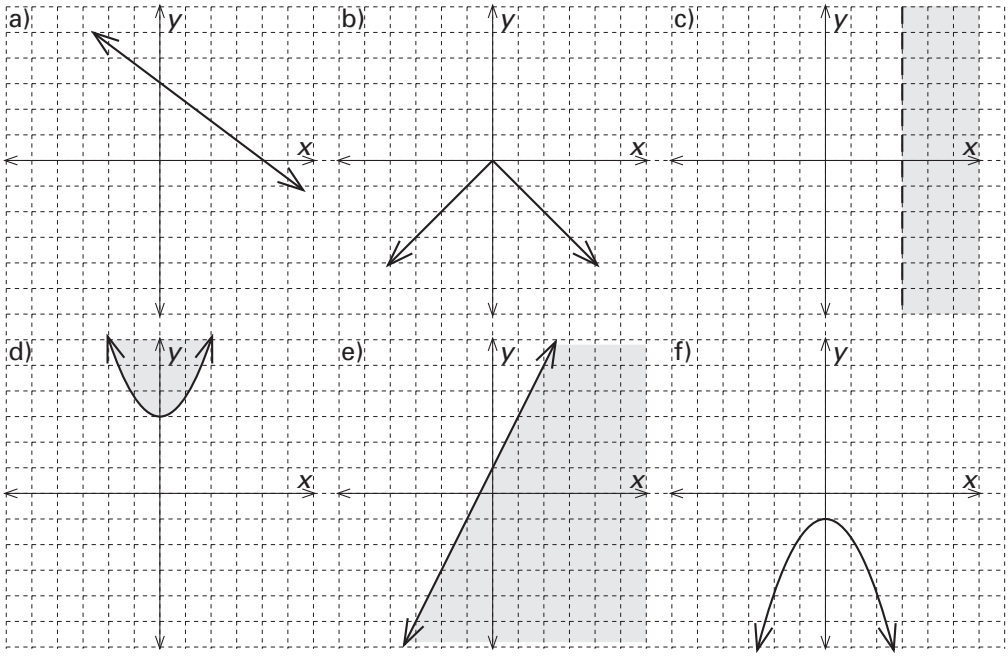
3 Verify your solutions to Problems 2a) and 2b) by solving each using another method. Show your thinking and reasoning.

4 Discuss the advantages and disadvantages of using the graphing calculator to solve equations. Give examples to illustrate your ideas.

(Continued on back.)

Follow-up Student Activity (cont.)

5 Write a mathematical statement involving equalities or inequalities to describe each graph. Tell whether the graph represents a function; if it does, tell the domain and range of the function.



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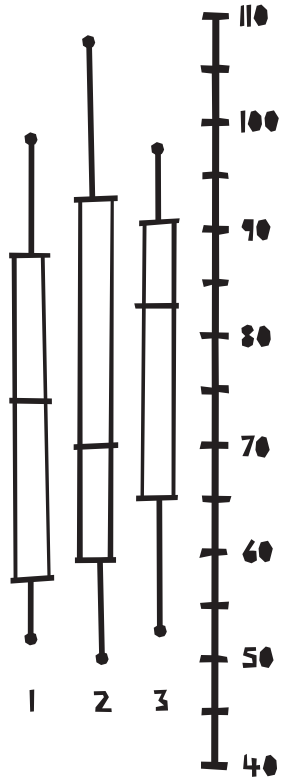
e) If Patty sells 300 ice cream bars in one month, how much profit will she make? Explain your reasoning.

Data–Variability & Spread

Lesson 15

THE BIG IDEA

Histograms, stem and leaf plots, line plots, and box plots are visual representations of one-variable data, while scatter plots are representation of two-variable data. These plots reveal information about the distribution, variability, and spread of data and can be used for making predictions.



CONNECTOR

OVERVIEW

Students form stem and leaf plots and line plots of the intervals of time between eruptions of Old Faithful, and they compute the mean, median, mode, and range of these times. They make predictions from these graphs and data.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Student Activity 15.1, 1 copy per student and 1 transparency.

FOCUS

OVERVIEW

Students form stem and leaf plots, histograms, line plots, box and whisker plots, and scatter plots of the time intervals between eruptions and durations of eruptions of Old Faithful. Students compare these graphs and use them as a basis for making predictions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Focus Student Activity 15.2, 1 copy per group and 1 transparency.
- ✔ Focus Student Activities 15.3-15.7, 1 copy per student and 1 transparency of each.
- ✔ Focus Masters A and B, 1 transparency of each.
- ✔ Focus Masters C and D, 1 copy of each per student and 1 transparency of each.
- ✔ 1-cm grid paper, 1 sheet per student and 1 transparency.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).
- ✔ Straightedges, 1 per student.

FOLLOW-UP

OVERVIEW

Students examine data from eruptions of Kilauea. They form line plots, stem and leaf plots, histograms, box and whisker plots, and scatter plots to compare changes in lengths of eruptions and intervals between eruptions over a 60-year period.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 15.8, 1 copy per student.
- ✔ ¼" grid paper, 1 sheet per student.

LESSON IDEAS

QUOTE

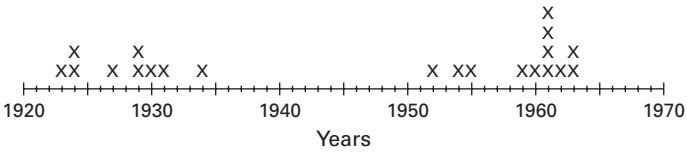
Collecting, organizing, describing, displaying, and interpreting data, as well as making decisions and predictions on the basis of that information, are skills that are increasingly important in a society based on technology and communication. These processes are particularly appropriate for young children because they can be used to solve problems that often are inherently interesting, represent significant applications of mathematics to practical questions, and offer rich opportunities for mathematical inquiry. The study of statistics and probability highlights the importance of questioning, conjecturing, and searching for relationships when formulating and solving real-world problems.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

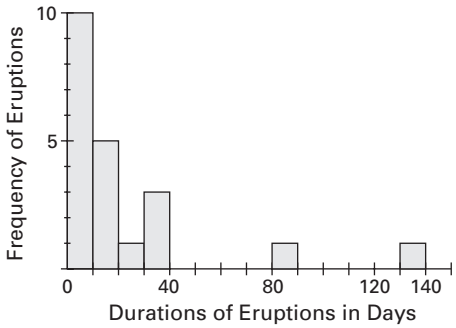
1. Line plot of 21 Eruptions of Kilauea from 1923 to 1963.



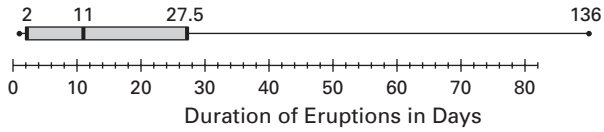
Stem & Leaf Plot of Durations in Days

13	6
12	
11	
10	
9	
8	8
7	
6	
5	
4	
3	3,6,6
2	2
1	1,3,4,7,9
0	1,1,1,2,2,2,3,3,4,7

Histogram



Box & Whisker Plot



Possible observations: Large gap from 1930's to 1950's shows a quiet period for Kilauea; cluster on line plot in early 1960's shows frequent outbreaks; histogram and stem and leaf plot shows that durations of 1 to 10 days were the most common durations; the box plot shows that approximately 25% of the durations were 2 or less days, 50% were 11 days or less, 25% lasted more than 27.5 days. Since the duration of 136 days is more than 1.5 times the interquartile range above the upper quartile ($136 > 1.5 \times 25.5 + 27.5$), this duration is an outlier. By the same test, the duration of 88 days is also an outlier.

3. A median-fit line for the scatter plot will show the length of time before the next eruption to be between 1.4 and 1.8 years.

4. a) The histogram for the 1964-82 period shows there were more durations of less than 10 days and four large durations of more than 200 days. The box plot for the 1964-82 data shows that 50% of the durations were less than 5 days, as compared to 11 days for the box plot in 1.

   b) There were 21 eruptions of Kilauea during the 40-year period from 1923 to 1963, and the next 21 eruptions occurred in less than 20 years. The eruptions are occurring more frequently. The largest gap is 3 years as compared to a gap of 17 years for the 1923-63 period.

   c) The eruptions are occurring more frequently and there is wider variation in the durations of eruptions.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students form stem and leaf plots and line plots of the intervals of time between eruptions of Old Faithful, and they compute the mean, median, mode, and range of these times. They make predictions from these graphs and data.

MATERIALS

✔ Connector Student Activity 15.1, 1 copy per student and 1 transparency.

ACTIONS

1 Arrange the students in groups. Invite them to tell what they know about the geyser, Old Faithful. Discuss. Then give each student a copy of Connector Student Activity 15.1 and point out that the data listed was gathered, in the order listed, during the working hours of a 3-day period at Old Faithful. Ask the students to record several *I notice…* and several *I wonder…* statements about the data. Then ask them to do a round robin share in their groups and to pick the group's 3 favorite *I notice…* and 3 favorite *I wonder…* statements to share with the class. Discuss.

Data—Variability and Spread Lesson **15**

Connector Student Activity 15.1

NAME _____ **DATE** _____

Period 1—Length of Time in Minutes Between Eruptions of Old Faithful

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

Regarding the above data,

I notice…

2 Ask the groups to create a *stem and leaf plot* of the Old Faithful data listed on Connector Student Activity 15.1. Invite volunteers to demonstrate their methods at the overhead. Clarify as needed, and ask the students to make observations regarding information revealed by the plot.

COMMENTS

1 Old Faithful is a geyser in Yellowstone National Park in the state of Wyoming. Throughout this lesson, students will examine data about Old Faithful and use that data to make predictions about the geyser's eruptions. Some students may enjoy exploring the Internet and other resources for information and photographs of Old Faithful.

Students may observe variability in the data—that short and long time intervals seem to alternate. They may calculate some statistics such as the mean, median, or the range. They may wonder: Why do the lengths of the time intervals alternate? Is this typical?

Note: throughout this lesson, the students will use all activity sheets and masters distributed during the Connector and Focus activities. It may be helpful to provide groups or individual students each a file folder for storing these sheets.

2 *Stem and leaf plots* were introduced in Lesson 17 of *Math Alive! Course II*. For this set of data, the digits from the tens place of each number comprise the stem, and the units digits are the leaves. For example, the stem of 5 and the leaves of 1, 2, 7, 7, and 9 represent the numbers 51, 52, 57, 57, and 59.

Stem and Leaf Plot
Old Faithful Period 1 Data

Stem	Leaves
5	1,2,7,7,9
6	2
7	2,3
8	7,7,8,8
9	4,4,8

(Continued next page.)

Connector Teacher Activity (cont.)

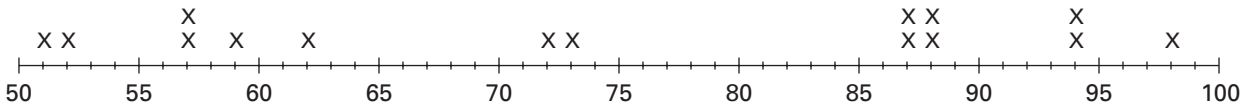
ACTIONS

COMMENTS

2 (continued.)
It is helpful to order the data before or while making a stem and leaf plot, although one could first make a plot with an ordered stem and unordered leaves, and then make another plot with the leaves ordered. A stem and leaf plot is particularly useful for organizing a large number of data entries; the shape and spread (i.e., the bumps, gaps, clusters, etc.) of the plot can reveal information about the data. However, note that when data is ordered numerically, any information related to the original order is lost. For example, once the Old Faithful data is ordered, one can no longer "see" that the lengths of time intervals seem to alternate between long and short.

3 Ask the students to create a *line plot* of the Old Faithful period 1 data. Invite volunteers to show their methods and results. Discuss the students' observations about information revealed by the line plot.

3 Students were introduced to the use of *line plots* in *Math Alive! Course II*, Lesson 17. You might refer to that lesson for additional discussion ideas. Here is a line plot of the Old Faithful period 1 data :



In a line plot the horizontal baseline shows the values of the data and the number of X's in a column indicate the *frequency* (i.e., the actual count) of each value. Notice in this data set there is little repetition of exact time intervals, and the line plot shows that intervals tend to be either "short" or "long." Some students may comment that the stem and leaf plot is more convenient to construct than the line plot because of the wide spread of this data.

4 Ask the groups to reach agreement on the meanings of the terms *mean*, *median*, *mode*, *maximum*, *minimum*, and *range*. Discuss. Then ask the groups to compute each of these statistics for the Old Faithful period 1 data and to relate those values to their stem and leaf plots and line plots. Discuss, clarifying as needed.

4 One way to determine the *mean* of a set of numbers is to "level off" the numbers by subtracting from some numbers and adding the amounts subtracted to others. Students may relate this to prior experiences (see *Math Alive! Course II*, Lesson 17) in which they represented each number in a data set as the height of a single column of cubes, moving cubes from one column to another until all of the columns are equal in height. The leveled-off height of the columns is the mean of the numbers. Or, one could find the leveled-off height by combining all of the original columns into a single column and then dividing that column into as many equal columns as the original number of columns. This is equivalent to adding the numbers together and dividing by the number of numbers.

Connector Teacher Activity (cont.)

ACTIONS

Ordered Old Faithful Period 1 Data:

51 52 57 57 59 62 72 73 87 87 88 88 94 94 98

↑

Median

Stem & Leaf Plot
Old Faithful Period 1 Data

5	1,2,7,7,9
6	2
7	2,3 Median
8	7,7,8,8
9	4,4,8

number of data entries = 15
modes = 57, 87, 88, and 94
median = 73
mean = 74.6

COMMENTS

Both of the above methods could be used to find the leveled-off, or mean, interval between eruptions of Old Faithful. Or, another method is to note that all of the data entries are greater than 40. Hence, one could find the mean of the differences between 40 and each data entry, and then the mean of the data is 40 plus the mean of these differences. Using each of these methods, the mean interval between eruptions of Old Faithful is 74.6 minutes.

Note: the *mean* of a set of numbers is often referred to as the *average* of the numbers; it is important to note that the median and mode are also types of averages. An average (mean, median, or mode) of a set of numerical data is also referred to as a *center* of the data.

The *mode* is the most frequently occurring data entry in a data set. If all of the data entries occur only once, then there is no mode. If two or more different numbers each repeat the same number of times and no other numbers repeat more often, then those numbers are each called a mode. For example, in the Old Faithful period 1 data, 57, 87, 88, and 94 each repeat twice and no other numbers repeat as many or more times; hence, 57, 87, 88, and 94 are all modes of the data set. That is, the modal intervals between eruptions are 57 minutes, 87 minutes, 88 minutes, and 94 minutes.

The *median* is the middle of a set of numerical data that has been ordered from smallest to largest. The median of the Old Faithful period 1 data is 73 minutes as indicated in the ordered listing of the data shown at the left.

The median can be easily located in a line or stem and leaf plot by counting simultaneously from both ends of the data (after it has been arranged in numerical order) until the middle of the data is reached. If there is an odd number of data entries, the median is the center entry. If there is an even number of data entries, the median is the mean of the center 2 entries. For example, if the data for Old Faithful were 60, 63, 70, 75, 81, and 86, the median of these 6 data entries would be the mean of the 3rd and 4th pieces of data in the ordered list, or $\frac{(70+75)}{2}$ = 72.5. Another method of locating the median is to count the total number of data entries in the data set and then find the middle of that number of entries (e.g., for the Old Faithful period 1 data, the middle entry is the 8th in a set of 15 entries). This is an efficient way of finding the median when there is a large number of entries in a data set.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

4 (continued.)
The *range* of a set of numbers is the difference between the maximum and minimum. For the Old Faithful period 1 data, the range is 98 – 51 = 47 minutes. The minimum and maximum are the first and last data entries, respectively, after the data has been ordered numerically from least to greatest. The maximum, minimum, distribution, and range of a set of numerical data reveal the "spread" of the data, and can prompt predictions about general patterns.

The statistical terms discussed here were all explored in Lesson 17 of *Math Alive! Course II*. Refer to that lesson for other discussion ideas.

5 Ask the groups to use the Old Faithful period 1 data as a basis for predicting Old Faithful data for the working hours of another day, i.e., Is there anything about Old Faithful that seems "faithful?" Discuss their predictions, reasoning, and confidence levels.

5 To promote thought you might ask the students to make predictions such as: What are the shortest and the longest intervals between eruptions? What percentage of the time does Old Faithful take between 60 and 70 minutes to erupt? What is the next time interval after the 51 minute interval? Notice, for example, each short time interval was followed by a considerably longer time interval.

Some students may point out that it is difficult to predict because this data sample is so small. This is a valid point; however, it is also the case that students can begin to make hypotheses based on patterns they see in the data and then test those hypotheses as they get additional data.

Focus Teacher Activity

OVERVIEW & PURPOSE

Students form stem and leaf plots, histograms, line plots, box and whisker plots, and scatter plots of the time intervals between eruptions and durations of eruptions of Old Faithful. Students compare these graphs and use them as a basis for making predictions.

MATERIALS

- ✔ Focus Student Activity 15.2, 1 copy per group and 1 transparency.
- ✔ Focus Student Activities 15.3-15.7, 1 copy per student and 1 transparency of each.
- ✔ Focus Masters A and B, 1 transparency of each.
- ✔ Focus Masters C and D, 1 copy of each per student and 1 transparency of each.
- ✔ 1-cm grid paper, 1 sheet per student and 1 transparency.
- ✔ Graphing calculators, 1 per student.
- ✔ Graphing calculator for the overhead (optional).
- ✔ Straightedges, 1 per student.

ACTIONS

1 Arrange the students in groups and give each group a copy of Focus Student Activity 15.2. Have the groups write the period 1 data in numerical order and then ask them to make observations and conjectures about how they think the *box and whisker plot* is constructed and what it reveals about the period 1 data. Discuss, clarifying as needed. Use students' observations as a context for introducing the terms *upper, lower,* and *middle quartile*.

Data—Variability and Spread Lesson **15**

Focus Student Activity 15.2

NAME _____ DATE _____

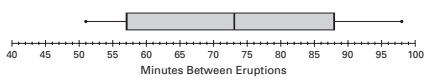
Period 1—Length of Time in Minutes Between Eruptions of Old Faithful.

Data in order collected:

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

Data in numerical order:

Box and Whisker Plot



Observations:

COMMENTS

1 The graph shown on Focus Student Activity 15.2 is a *box and whisker plot*, sometimes called a *box plot*, of the given Old Faithful period 1 data. This may be many students' first introduction to a box and whisker plot. Part of the detective work of data analysis is making sense out of graphs created by others, so allow plenty of time for the students to examine this graph before you reveal information about it.

The shaded rectangle is the *box*; this box encloses approximately the middle 50% of the data. The *whiskers* are the line segments extending from each end of the box. They extend across approximately the lower 25% and approximately the upper 25% of the data. Notice that the length of the plot, from the end of one whisker to the end of the other, shows the total range of the data, from 51 to 98. The vertical mark inside the box indicates the median of the data, and the two ends of the box, 57 and 88, show the ¼- and ¾-marks of the ordered data. The ¼- and ¾-marks are the medians of the lower and upper halves of the data, and are called the *lower quartile* and the *upper quartile*, respectively. Sometimes the median is called the *middle quartile*. If, during their examination of the plot, students don't notice that the ends of the boxes are at the ¼- and ¾-marks in the data, or that these are the medians of the upper and lower halves of the data, you might prompt such discoveries by asking the students to determine how they think the location of the endpoints of the box relates to the data and how the length of the box might be determined.

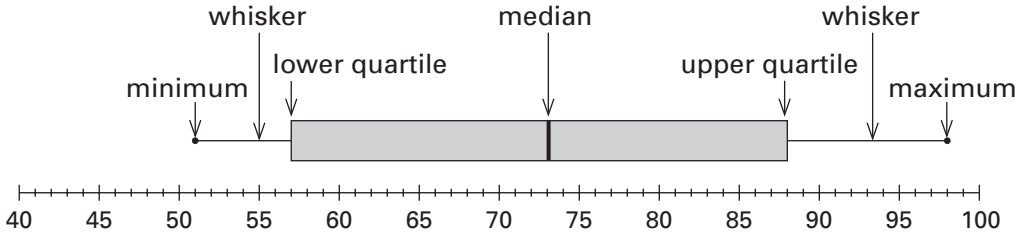
(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)
In this data set, the median is the 8th of 15 data entries. Thus, there are 7 data entries in each of the 2 halves above and below the median. The lower quartile is 57 (written LQ = 57), the median of the lower 7 data entries. The upper quartile (UQ) is 88, the median of the upper 7 data entries.



Note: this type of box plot is also called a *50% box plot* because approximately 50% of the data occurs on or between the ends of the box. Because both 57 and 88 repeat in the period 1 data set, there is more than 50% of the data contained by the box. In general, for a 50% box plot, the ends of the box are always at the upper and lower quartiles. However, there may be more than 50% of the data inside the box if 2 or more data values equal the upper and/or lower quartiles. In Lesson 17 students also explore 90% box plots.

2 Give each student a copy of Focus Student Activity 15.3. Ask them to make observations and conjectures about how the *histogram* was constructed and what it reveals about the period 1 data. Discuss. Then ask each group to discuss and compare the information revealed by the line plot, stem and leaf plot, box and whisker plot, and histogram, recording their observations at the bottom of Focus Student Activity 15.3. Discuss the groups' ideas and list these ideas on a poster.

2 While students may have had experience with a standard bar graph, they may be less familiar with the use of histograms. Hence, allow plenty of time for them to make and test conjectures about ways this graph is formed.

A *histogram* is a bar graph in which the heights of the bars represent the frequency, or count, of the data entries that fall within that interval. The width of each bar on the histogram of the Old Faithful data represents an interval of time in minutes, and the height of the bar tells the number of data entries within that time interval. For example, there are 2 entries for 50 through 54 minutes, and there are 3 entries for 55 through 59 minutes, 7 entries for 60 through 64 minutes, 0 entries for 65 through 69 minutes, 2 entries for 70 through 74 minutes, etc. It is common practice on a histogram for the interval represented by the width of a bar to include the number marking the left edge of the bar but not the number marking the right edge. For example, the time intervals of 50 through 54 minutes are counted in the first bar, and intervals of 55 through 59 minutes are counted in the second bar. A gap between bars indicates there is no data in that interval.

Focus Teacher Activity (cont.)

ACTIONS

Lesson **15** Data—Variability and Spread

Focus Student Activity 15.3

NAME _____ DATE _____

Period 1—Length of Time in Minutes Between Eruptions of Old Faithful.

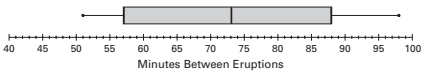
Data in order collected:

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

Data in numerical order:

51, 52, 57, 57, 59, 62, 72, 73, 87, 87, 88, 88, 94, 94, 98

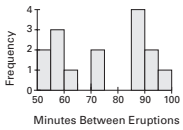
Box and Whisker Plot



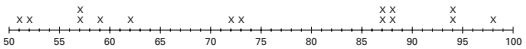
Stem and Leaf Plot

5	1,2,7,7,9
6	2
7	2,3
8	7,7,8,8
9	4,4,8

Histogram



Line Plot



Observations:

Average deviation(s):

Method(s):

COMMENTS

In making comparisons of the graphs, students should notice that each type of plot highlights different features of the data. Some examples of observations students have made include the following (it is not necessary that all of these observations surface now).

Gaps in the time intervals show up in the histogram better than on the stem and leaf plot. We think this is partly because the stems are groups of 10 minutes and the histogram has intervals of 5 minutes.

We think the data is too spread out and varies too much for a line plot to be very useful. The stem and leaf plot is more compact and works better than the line plot when data is spread out.

The line plot shows all gaps, small or large, because every piece of data is shown. The histogram lumps several pieces of data into one bar.

Both the stem plot and the histogram show two peaks in the data. The peaks don't show in the box plot.

We think making the stem and leaf plot is a fast way to order the numbers from smallest to largest. And we can see all of the actual data in this plot.

You can't see the numerical ordering of the data in the box and whisker plot or the histogram. You have to have the data ordered before you can make your box and whisker plot or histogram.

On the histogram and stem and leaf plot the shape of the data helps us see how the data is distributed.

We can see the maximum, minimum, range, and median in the box plot, line plot, and stem plot, but not the histogram. In the line and stem plots, we can see the mode and we could compute the mean from the data in the plots. We can approximate the range from the histogram.

The box plot seems more complicated to construct. We think it would help to make a stem and leaf or line plot of the data first and find the median, lower quartile, and upper quartile before forming the box plot.

Small gaps aren't as obvious on the stem plot as on the line plot.

The box plot and the histogram are summaries of the data and don't show as many details as the line and stem and leaf plots, but the box plot focuses attention on the median, maximum, minimum, and quartiles.

Focus Teacher Activity (cont.)

ACTIONS

3 Ask the groups to make some predictions regarding what might be a set of 15 data entries collected during another 3-day period, based on their examination of the plots on Focus Student Activity 15.3. Discuss.

4 Give each student a copy of Focus Student Activity 15.4 and a straightedge. Ask them to make and label, in the spaces provided, a line plot, stem and leaf plot, box and whisker plot, and histogram of the Old Faithful period 2 data. Discuss and compare their results, encouraging students to describe information revealed by one graph but not by others, and information that repeats from graph to graph.

Data—Variability and Spread Lesson **15**

Focus Student Activity 15.4

NAME _____ DATE _____

Period 2—Length of Time in Minutes Between Eruptions of Old Faithful.

Data in order collected:

93, 86, 70, 63, 91, 82, 58, 97, 59, 70, 58, 98, 55, 107, 61

Data in numerical order:

Box and Whisker Plot

Stem and Leaf Plot **Histogram**

Line Plot

Observations: Average deviation(s):

Method(s):

COMMENTS

3 Many may respond that they still don't have enough information to predict with confidence. However, based on the information revealed by the various plots, students may feel increased or decreased confidence in their earlier predictions. Some students may compute the average of the short-time intervals between eruptions and the average of the long-time intervals, and predict those averages as approximations for alternating short and long intervals during another period of 15 entries. You might remind the students to keep their copies of all Focus Student Activities used during this lesson.

4 Procedural questions regarding how to construct these graphs may surface. If so, you might suggest that groups pose questions to the class for clarification. Then, if additional clarification is needed, you could provide it. To save time at the overhead, you might ask volunteers to sketch their groups' graphs on a blank transparency prior to coming to the overhead.

Some groups may choose different scales for the graphs and, hence, have difficulty comparing them. Such a situation could prompt a meaningful class discussion.

Completed graphs of the period 2 data are shown below:

Data—Variability and Spread Lesson **15**

Focus Student Activity 15.4

NAME _____ DATE _____

Period 2 - Length of Time in Minutes Between Eruptions of Old Faithful.

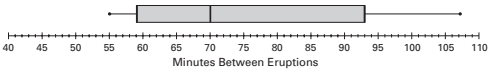
Data in order collected:

93, 86, 70, 63, 91, 82, 58, 97, 59, 70, 58, 98, 55, 107, 61

Data in numerical order:

55, 58, 58, 59, 61, 63, 70, 70, 82, 86, 91, 93, 97, 98, 107

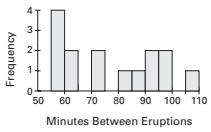
Box and Whisker Plot



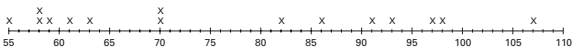
Stem and Leaf Plot

5	5,8,8,9
6	1,3
7	0,0
8	2,6
9	1,3,7,8
10	7

Histogram



Line Plot



Observations: Average deviation(s):

Method(s):

Focus Teacher Activity (cont.)

ACTIONS

5 Give each student a copy of Focus Student Activity 15.5 (see completed activity below) and repeat Action 4 for this activity. Discuss the students' results. Ask them to compare the graphs for periods 1, 2, and 3. What clues do the graphs give about the behavior of Old Faithful?

Lesson **15** Data—Variability and Spread

Focus Student Activity 15.5

NAME _____ **DATE** _____

Period 3—Length of Time in Minutes Between Eruptions of Old Faithful.

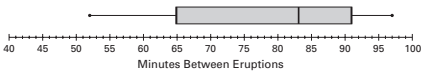
Data in order collected:

82, 91, 65, 97, 52, 94, 60, 94, 63, 91, 83, 84, 71, 83, 70

Data in numerical order:

52, 60, 63, 65, 70, 71, 82, 83, 83, 84, 91, 91, 94, 94, 97

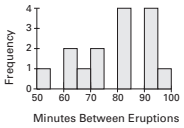
Box and Whisker Plot



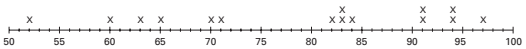
Stem and Leaf Plot

5	2
6	0,3,5
7	0,1
8	2,3,3,4
9	1,1,4,4,7

Histogram



Line Plot



Observations:

Average deviation(s):

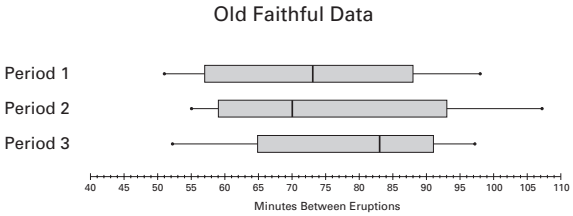
Method(s):

COMMENTS

5 You may find a transparency of Focus Master A, shown below, useful for discussing the 3 box plots simultaneously.

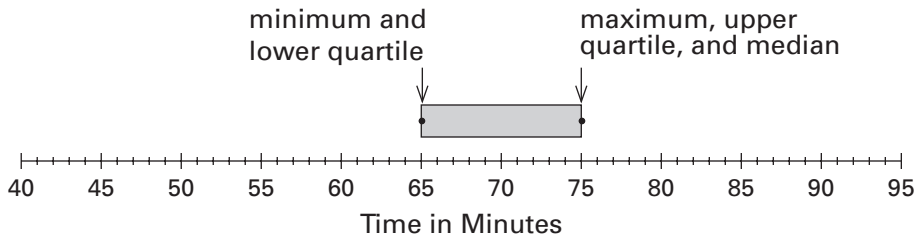
Data—Variability and Spread Lesson **15**

Focus Master A



Notice that the median for the period 3 data is greater than the medians for periods 1 and 2. This shows that, in general, many of the time intervals between eruptions for the upper 50% of period 3 are greater than the time intervals for the upper 50% of periods 1 and 2. Notice also that the lower quartile for the period 3 data is near the median for the period 2 data; this shows that almost 50% of the time intervals for period 2 are comparable to the lower 25% of the time intervals for period 3.

Notice also that box and whisker plots can have boxes that are relatively symmetric about the median, like period 1, or very asymmetric, like periods 2 and 3, depending on the data. The whiskers can be long in one direction and short in the other, such as periods 2 and 3, or they can be more balanced, such as period 1. There may also be situations for which there are no whiskers in a box plot. For example, if the data were 65, 65, 65, 65, 65, 65, 65, 65, 75, 75, 75, 75, 75, 75, 75, 75, 75 a 50% box plot would look like the one shown below. The lower quartile is 65, which is also the minimum. The median, upper quartile, and maximum all occur at 75.



(Continued next page.)

Focus Teacher Activity (cont.)

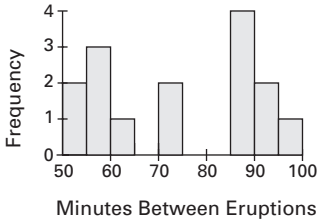
ACTIONS

COMMENTS

5 (continued.)
Some of the student observations listed in Comment 6 may come up here. Following are stem and leaf plots and histograms for the 3 periods.

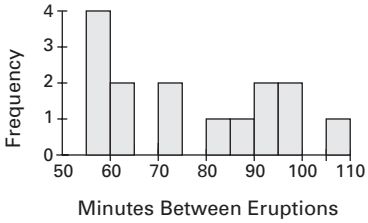
Period 1

5	1,2,7,7,9
6	2
7	2,3
8	7,7,8,8
9	4,4,8



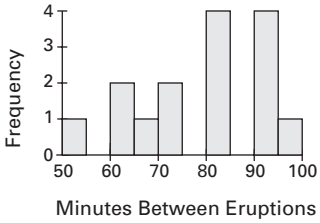
Period 2

5	5,8,8,9
6	1,3
7	0,0
8	2,6
9	1,3,7,8
10	7



Period 3

5	2
6	0,3,5
7	0,1
8	2,3,3,4
9	1,1,4,4,7



6 Ask the groups to discuss any new ideas they have regarding the advantages and disadvantages of the various graphing techniques explored so far. Add their ideas to the class poster started in Action 2.

6 Students may have changed their minds about some of their earlier statements, or they may wish to clarify some statements. Statements listed in Comment 2 that didn't surface at that time, may be suggested by students now. Here are some additional statements that students have given at this point:

Box and whisker plots show the median and the range of the data, and how stretched out the data is, in one or both directions. Quick visual comparisons across several data samples are convenient with box and whisker plots. However, except for the maximum and minimum, the individual data values are not shown.

A stem and leaf plot keeps a record of each actual data value, and it shows the shape of the data.

Histograms show the number of times that values in a certain interval occur. They show gaps in the data, and they are useful in comparing several sets of data. However, the histogram only summarizes the data, and the actual data values can't be seen.

A box and whisker plot could be used to compare data sets with unequal numbers of data entries. This would be much harder to do with the other plots (unless we changed the height of the histogram to show the percentage of the data entries in each bar rather than the number of entries).

Focus Teacher Activity (cont.)

ACTIONS

7 Point out the fact that the range of a data set (i.e., the distance between the endpoints of the whiskers) provides a measure of *variability* in the data. *Outliers*, values that are uncharacteristic of most values in the data set, provide additional information about variability within the data set.

Explain to the students the common practice of computing 1.5 times the *interquartile range* to locate outliers. Then ask them to use this method to determine whether there are any outliers in the Old Faithful data for period 1. Have them mark the location of any outliers on their box and whisker plots. Discuss. Repeat for periods 2 and 3.

8 Write the following statement on the overhead:

Another measure of the variability and spread of data is the distribution of the data around its center(s).

Allow time for the groups to speculate about and record ways this idea could apply to the Old Faithful data. Then, use their ideas as a context for introducing the use of *average deviation* from the center of a data set as a measure of how data is *dispersed*.

COMMENTS

7 To locate *outliers* on a box plot, determine the length of the box, also called the *interquartile range* (IQR); an outlier is any data value that is more than 1.5 interquartile ranges outside the box; i.e., (1.5 × IQR) greater than the upper quartile or (1.5 × IQR) less than the lower quartile).

For example, the maximum data value for period 2 is 107 minutes. The interquartile range for period 2 is 93 – 59 = 34, and 1.5 × 34 = 51. Since 107 is not greater than the upper quartile (93) plus 51, the time interval of 107 minutes is not an outlier. Students may point out that all of the whiskers on the box plots of the periods 1, 2, and 3 data are shorter than their corresponding boxes. Hence, since outliers are on the whiskers of a box and whisker plot are 1.5 or more box lengths from the end of the box, there are no outliers for the period 1, 2, and 3 data.

Note that since the upper and lower quartiles can also be located on a line plot and on a stem and leaf plot, it is also possible to use the method of calculating 1.5 times the inter-quartile range to determine outliers on those plots.

8 One way to examine the distribution of data about the *center* (i.e., mean, median, or mode) of a data set is to compute the difference between each data entry and the data center, and finally compute the mean difference. This average difference is called the *average deviation* from the center. Students may determine the center of the data as the mean, median, or mode—all are appropriate, although the mean and the median are more relevant to the Old Faithful data.

As an example, the average deviation from the median of the period 2 data is determined by first computing the difference between 70 (the period 2 median) and each data entry, e.g., 70 – 55 = 15, 70 – 58 = 12, 70 – 58 = 12, 70 – 59 = 11, etc. (The fact that some data entries are greater than the median, and some are less, is not important; when computing the differences, only positive values are used.) These differences, in numerical order, are: 0, 0, 7, 9, 11, 12, 12, 12, 15, 16, 21, 23, 27, 28, and 37 minutes.

Next, determine the average (mean) of the differences to be $15.\overline{3}$ minutes. Hence, for period 2 we say that $15.\overline{3}$ minutes is the average deviation from the center (median).

In general, if the average deviation from the center is small, then the spread or dispersion of the data is small; if the average deviation is great, the spread is also.

Focus Teacher Activity (cont.)

ACTIONS

9 Ask the students to determine average deviations for each of periods 1-3 and to record this information and their methods in the spaces provided on Focus Student Activities 15.3, 15.4, and 15.5. Discuss their results and the implications of this information regarding the "faithfulness" of Old Faithful.

10 Distribute a copy of Focus Student Activity 15.6 and a sheet of 1-cm grid paper to each student, and ask them to complete Problem 1. Discuss their observations.

COMMENTS

9 Students may compute average deviations from different centers (mean, median, or mode) and, hence, their results may vary. Encourage discussion of their rationale for using a particular center rather than another. Do they feel one gives a more representative picture of the dispersion of the Old Faithful data than another?

Period 1 differences between the median (73) and each data entry are: 0, 1, 11, 14, 14, 14, 15, 15, 16, 16, 21, 21, 21, 22, and 25 minutes. The mean of these differences, i.e., the average deviation for period 1, is 15.1 minutes to the nearest tenth.

As discussed in Action 8, the period 2 differences are: 0, 0, 7, 9, 11, 12, 12, 12, 15, 16, 21, 23, 27, 28, and 37 minutes and the mean of these differences is $15.\overline{3}$ minutes.

Period 3 differences between the median (83) and each data entry are: 0, 0, 1, 1, 8, 8, 11, 11, 12, 13, 14, 18, 20, 23, and 31 minutes, and the mean of these differences is 11.4 minutes.

Notice the average deviation from the median of the period 1 data and the average deviation from the median of the period 2 data (see Comment 8) are each approximately 15 minutes, while the average deviation for the period 3 data is 11.4 minutes. So the average deviation from the median is greater for the period 1 data and the period 2 data than for the period 3 data. This is one way to indicate that the data for periods 1 and 2 are more widely dispersed (i.e., spread out) than the data for period 3. However, in spite of the smaller deviation for the period 3 data, many students may feel the average deviations for the 3 sets of data are reasonably close and provide an indication of the "faithfulness" of Old Faithful.

10 Students were introduced to scatter plots in Lesson 18 of *Math Alive! Course II*. A completed scatter plot of the Problem 1 data from Focus Student Activity 15.6 is shown on Focus Master B. A transparency of this master may be useful during class discussion.

Focus Teacher Activity (cont.)

ACTIONS

Data—Variability and Spread Lesson **15**

Focus Student Activity 15.6

NAME _____ DATE _____

1 The following data about Old Faithful was recorded, in order, in April 1997. Form a scatter plot of this data by placing the duration of eruptions along the horizontal axis and the time interval before the next eruption along the vertical axis. Record your observations.

Duration of Eruption	Time Interval Before Next Eruption
4 min 24 sec	81 min
2 min 00 sec	60 min
4 min 43 sec	91 min
1 min 55 sec	51 min
4 min 14 sec	85 min
1 min 34 sec	55 min
1 min 34 sec	98 min
2 min 8 sec	49 min
4 min 30 sec	85 min
1 min 43 sec	65 min
4 min 27 sec	102 min
1 min 51 sec	56 min
4 min 35 sec	86 min
1 min 44 sec	62 min
4 min 35 sec	91 min

2 Following are the starting time and durations of 11 different eruptions randomly selected from a 2-day period in May 1997 (not listed in order). Use your scatter plot from Problem 1 to predict the time interval before the next eruption and the time of the next eruption. Explain your methods.

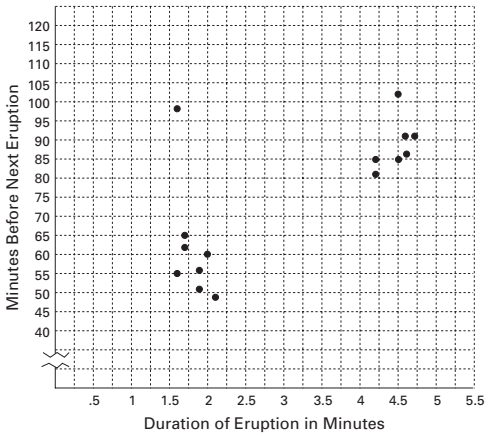
Starting Time	Duration	Predicted Interval Before Next Eruption	Predicted Time of Next Eruption
11:22 am	1 min 50 sec	_____	_____
7:16 am	4 min 19 sec	_____	_____
2:11 pm	1 min 50 sec	_____	_____
8:53 am	1 min 55 sec	_____	_____
12:20 pm	4 min 46 sec	_____	_____
8:30 am	4 min 26 sec	_____	_____
11:18 am	4 min 35 sec	_____	_____
12:40 pm	3 min 46 sec	_____	_____
9:52 am	4 min 35 sec	_____	_____
10:17 am	1 min 45 sec	_____	_____
7:35 am	1 min 55 sec	_____	_____

COMMENTS

Lesson **15** Data—Variability and Spread

Focus Master B

Eruptions of Old Faithful—April 1997



A scatter plot provides a visual description of data with two different types of measurements; such data is called *two-variable data*. Line plots, stem and leaf plots, histogram, and box and whisker plots provide descriptions of *one-variable data*. For example, the above scatter plot of Old Faithful data suggests that if the duration of the eruption is short, then the time interval before the next eruption is short; and if the duration of the eruption is long, then the time interval before the next eruption is long. The one point of the plot which is an exception to this has coordinates (1.6,98) and this point appears to be an outlier [note: in *Math Alive! Course IV* students will investigate statistical procedures for determining outliers on scatter plots; such procedures verify (1.6,98) is an outlier].

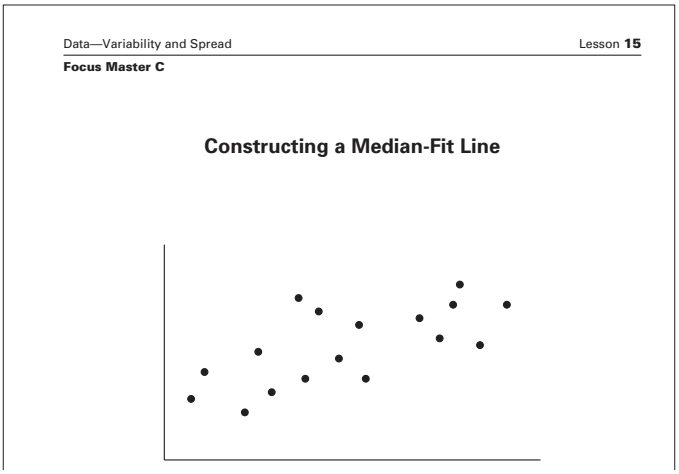
Some students may suggest drawing a "line of best fit" for making predictions. Notice that, in Action 12, students learn strategies for constructing such a line.

Focus Teacher Activity (cont.)

ACTIONS

11 Ask the students to complete Problem 2 from Focus Student Activity 15.6. Discuss their methods and reasoning.

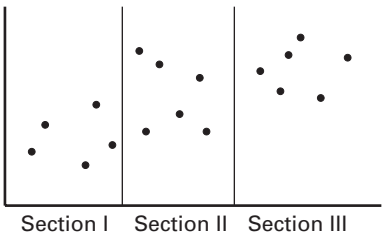
12 Give each student a copy of Focus Master C. Place a transparency of Focus Master C on the overhead and introduce the method of constructing a *median-fit line* by demonstrating the steps listed below.


Data—Variability and Spread Lesson **15**

Focus Master C

Constructing a Median-Fit Line

Step 1. Count the number of points in the plot and draw two vertical lines so that there are approximately the same number of points in each section of the plot.



COMMENTS

11 One method of predicting a time interval before the next eruption is to locate (on the horizontal axis of the scatter plot) the duration of eruption that is closest to the given duration in Problem 2; then locate the corresponding time interval before next eruption. If there are 2 or more time intervals on the plot for a given duration, their mean could be computed to predict the time interval. Students may invent a variety of other methods of predicting.

Another method is to use a *line of best fit*—also called a *median-fit line*, *fitted line, med-med line*, or *trend line*—which is introduced in the next action. Students may devise other strategies.

12 Note: the sample of data in this example is *not* related to the Old Faithful data. The purpose here is only to introduce students to the method of constructing a median-fit line. In Action 13, students will apply this method to the Old Faithful scatter plot.

Step 1. If possible, the first and last sections should have the same number of points. In the 3 sections shown at the left there are 5, 6, and 6 points; placing 6 points in the first section was inconvenient for this data because of the proximity of the 6th and 7th data points.

Steps 2-3. It helps to slide a straightedge horizontally and vertically to count and locate the median point for each section. Clear plastic rulers work well for this.

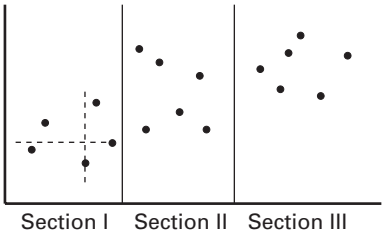
Notice that moving horizontally to find the middle point locates the median of the *x*-coordinates of the points in the section and moving vertically locates the median of the *y*-coordinates of the points.

Step 5. The line formed in this step is called the *median-fit line* for the given data. Emphasis here is on the standard procedure for constructing a median-fit line; the purpose in forming such a line is to aid in the analysis and interpretation of the data in a scatter plot. In the remaining actions of this activity students use median-fit lines (and other *lines of best fit*) to make predictions and identify relationships between the variables, duration of eruption and intervals between eruptions.

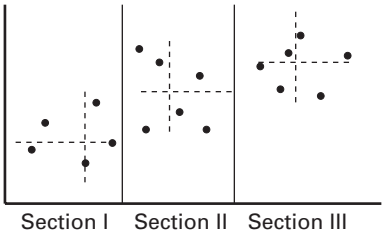
Focus Teacher Activity (cont.)

ACTIONS

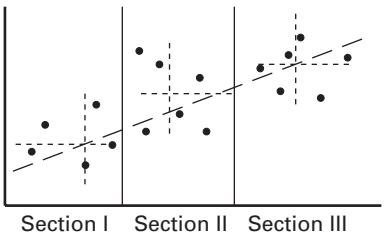
Step 2. Locate the "middle" of the points in Section I as follows: move horizontally from left to right to the horizontal middle of the 5 data points (in this case the 3rd point) and draw a dashed vertical line through it. Then move vertically from bottom to top to the 3rd point and draw a horizontal dashed line through it. The intersection of the dashed lines is the *median point* of Section I.



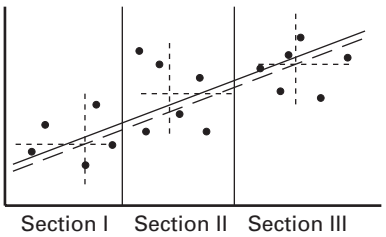
Step 3. Repeat Step 2 to locate the median points for Sections II and III. Since there are 6 points in each of these sections the dashed lines are half way between the 3rd and 4th points.



Step 4. Place a straightedge on the median points of Sections I and III (see diagonal line formed by long dashes).



Step 5. Slide the straightedge approximately ⅓ of the distance from the line formed in Step 4 toward the median point of Section II. The line drawn along this edge of the ruler produces a *median-fit* line (see solid line below).



COMMENTS

Focus Teacher Activity (cont.)

ACTIONS

13 Ask the students to use the methods from Action 12 to draw a median-fit line on their scatter plots from Action 10. Then:

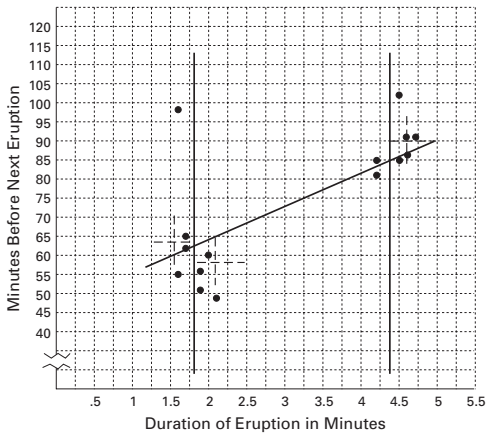
a) Have the students use the median-fit line as the basis for evaluating their predictions for Problem 2 on Focus Student Activity 15.6 (see Action 11) and/or for determining whether/how to adjust their predictions. Discuss the students' methods and reasoning.

b) Discuss the students' ideas regarding possibilities for other *lines of best fit* for their scatter plots, and any adjustments in the students' predictions prompted by these lines.

Lesson **15** Data—Variability and Spread

Focus Master B

Eruptions of Old Faithful—April 1998



COMMENTS

13 a) A median-fit line for this scatter plot is sketched on the copy of Focus Master B at the left. There will be slight variations in the median-fit lines which students form. This is due to variations in selecting the 3 sections for grouping the points from the scatter plot.

If students need help using their median-fit lines to make predictions, it may be helpful to ask specific questions for interpretation from their graphs, such as: For an eruption that lasts 2.5 minutes, what is the time interval before the next eruption? To use the median-fit line to answer this question, first locate the point on the median-fit line with the horizontal coordinate 2.5. The corresponding vertical coordinate, ≈ 68, is the median-fit line predicted number of minutes before the next eruption.

b) Following are examples of strategies students may suggest for constructing other lines of best fit: find the median point of each of the 2 data "clusters" and form the line that connects those 2 points; construct the mean-fit line; construct the mean-fit line without the point (1.6,98); etc. You might have the students draw such lines of best fit and the median-fit line on a transparency of Focus Master B, and then discuss the students' ideas regarding which line best fits the data and why. For example: For which line is more of the data closer to the line? Is there something important apparent about the data that shows in one line but not another?

You might also encourage discussion regarding why the students think the median-fit line is constructed as it is, and/or conditions for which it may be more appropriate to use a line constructed in a different way. In *Math Alive! Course IV* students examine line-fitting in more depth and, since there is not a line of best fit for all scatter plots, extend this idea to curve-fitting.

Focus Teacher Activity (cont.)

ACTIONS

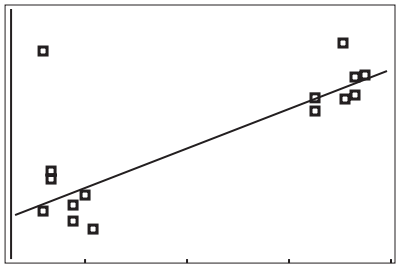
14 Ask the students to determine the slope, the *y*-intercept, and the equation of their median-fit lines from Action 13. Invite volunteers to show their methods. Then have them explore ways to *use their equation* to answer the following questions:

a) What is the interval in minutes before the next eruption if the duration of the current eruption is 4 minutes?

b) What is the duration of an eruption in minutes, if the interval following the eruption is 58 minutes?

c) (Optional) How do the results for a) and b) above compare to results that are based on equations for other lines of best fit formed in Action 13b)?

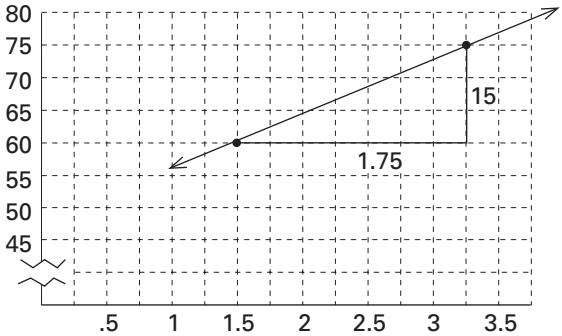
15 Distribute graphing calculators if the students do not have them. Introduce students to the use of graphing calculators to form scatter plots and median-fit lines. Have the students compare their predictions from Problem 2 of Focus Student Activity 15.6 (see Action 10) which were based on their handmade graphs to predictions based on the calculator graphs. Have them record adjusted predictions, if they feel adjustments are needed. Discuss.



COMMENTS

14 Due to slight variations in the locations of the students' median-fit lines, the slopes and intercepts and, hence, the equations for these lines will vary.

The following example refers to the median-fit line from Comment 13. Notice this line passes through points (1.5,60) and (3.25,75); hence, its slope is $\frac{(75-60)}{(3.25-1.5)}$, or $\frac{15}{1.75} \approx 8.6$.



The *y*-intercept of this median-fit line is ≈ 47. That is, if the line is extended it intersects the vertical axis at about (0,47). So, the equation of this line is $y = 8.6x + 47$. Based on this equation, the answers to a) and b) are: a) $8.6(4) + 47 = 81.4$ minutes; b) $8.6x + 47 = 58$ minutes, so $x \approx 1.3$ minutes. To answer c) students need to write equations for other lines of best fit constructed during Action 13b).

15 A computer printout from a TI-83 scatter plot and median-fit line is shown below at the left. (Note: on the TI-83, the median-fit line is called the *med-med line*.) The equation of this median-fit line is $11x + 38.7$ (slope and *y*-intercept are to the nearest tenth). Notice the difference in this equation and the one from Comment 14; variations are due to the way the graphing calculator is programmed and the way the points of the scatter plot are grouped in by-hand methods.

To obtain a scatter plot and median-fit line on a TI-83, press STAT ENTER to obtain the lists for entering data; enter the Duration of Eruption Times into one list and the Minutes Before Next Eruption into a second list; turn on Plot 1 from the STAT PLOT menu and select the scatter plot option; press GRAPH. To obtain an equation for a median-fit line, press STAT and choose the "med-med" function under the CALC menu (choosing "med-med" requires putting the cursor on "med-med" and pressing enter

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

15 (continued.)
twice); then press Y = and enter the equation; finally, adjust the WINDOW and press GRAPH to see the scatter plot and median-fit (med-med) line on the screen.

Starting Time	Duration	TI-83 Median-Fit Line Predicted Interval Before Next Eruption	TI-83 Median-Fit Line Predicted Time of Next Eruption
11:22 am	1 min 50 sec	59	12:21 pm
7:16 am	4 min 19 sec	86	8:42 am
2:11 pm	1 min 50 sec	59	3:10 pm
8:53 am	1 min 55 sec	60	9:53 am
12:20 pm	4 min 46 sec	91	1:51 pm
8:30 am	4 min 26 sec	87	9:57 am
11:18 am	4 min 35 sec	89	12:47 pm
12:40 pm	3 min 46 sec	80	2:00 pm
9:52 am	4 min 35 sec	89	11:21 am
10:17 am	1 min 45 sec	58	11:15 am
7:35 am	1 min 55 sec	60	8:35 am

The chart at the left shows predicted intervals for Problem 2 from Focus Student Activity 15.6 that were obtained by tracing the TI-83 median-fit line shown above. Students may wish to record calculator-generated predictions next to their predictions from Actions 10 and 13, or they may adjust their earlier predictions based on calculator results.

16 Distribute a copy of Focus Student Activity 15.7 to each student and ask them to complete Problem 1. Discuss their methods of comparing the actual starting times of next eruptions to the starting times predicted by the JASON Project.

Lesson **15** Data—Variability and Spread

Focus Student Activity 15.7

NAME _____ DATE _____

The predicted starting times of next eruptions of Old Faithful in column (4) below were obtained by the JASON Project in May 1997.

1 Compare the actual starting times of the next eruptions from column (2) in the table below to the JASON Project predicted starting times in column (4). How do you rate the JASON Project predictions? Support your rating with reasoning based on statistical evidence.

2 Complete column (3) of the table below by filling in each of your predicted starting times of next eruptions from Focus Student Activity 15.6 (notice that data in Column (1) is ordered according to its occurence—be sure to enter your data accordingly). Then use at least 2 different statistical methods to compare your predictions in column (3) to those by the JASON Project in column (4) and to the actual starting times of next eruptions in column (2). Use your comparisons as the basis for rating the quality of your predictions, and provide convincing statistical evidence to support your rating.

(1) Starting Time of Eruption	(2) Actual Starting Time of Next Eruption	(3) My Predicted Starting Time of Next Eruption (from Student Activity 15.6)	(4) JASON Project's Predicted Start of Next Eruption
7:16 am	8:53 am	_____	8:41 am
8:53 am	9:52 am	_____	9:48 am
9:52 am	11:22 am	_____	11:21 am
11:22 am	12:20 pm	_____	12:15 pm
12:20 pm	1:58 pm	_____	1:52 pm
7:35 am	8:30 am	_____	8:30 am
8:30 am	10:17 am	_____	9:59 am
10:17 am	11:18 am	_____	11:12 am
11:18 am	12:40 pm	_____	12:47 pm
12:40 pm	2:11 pm	_____	1:59 pm
2:11 pm	3:16 pm	_____	3:06 pm

16 The JASON Project is an educational project that, in 1997, produced research on geysers and disseminated data and information for use by teachers and students. See their website at http://www.jason.org.

Students may invent several ways to rate the JASON Project predictions. One method is to select a time interval that seems to be a reasonable margin of error, such as 10 minutes; next compute the difference between each actual time of eruption and the predicted time of eruption by the JASON Project; and then determine which JASON Project predictions are less than or equal to 10 minutes. Computing these differences shows that 8 of the JASON Project's 11 predictions satisfy this requirement. This can be used as the basis for assigning a rating to the predictions.

Another method is to compute the mean of the differences between the actual times of eruption and the predicted times of the JASON Project, and then consider a prediction to be acceptable if its difference is less than or equal to this mean. The differences in minutes between the actual times of the next eruptions and the times predicted by the JASON Project are 12, 4, 1, 5, 6, 0, 18, 6, 7, 12, and 10. The mean of these differences is 7.4 to the nearest tenth, and 7 of the JASON Project predictions satisfy this requirement, i.e., are less than this mean.

Focus Teacher Activity (cont.)

ACTIONS

17 Ask the students to complete Problem 2 on Focus Student Activity 15.7. Discuss the students' methods of comparing their predictions of the starting times of eruptions to the actual starting times and their methods of comparing their predictions to those of the JASON Project.

COMMENTS

17 Students may devise a variety of strategies for comparing and evaluating predictions. As an example (students will use their own predictions), Michael predicted the following starting times (in order) for column 3 from the chart on Focus Student Activity 15.7:

8:42 am; 9:53 am; 11:21 am; 12:21 pm; 1:51 pm; 8:35 am; 9:57 am; 11:15 am; 12:47 pm; 2:00 pm; 3:10 pm.

To compare his predicted starting times to the actual starting times of next eruptions, Michael computed the differences between his predictions and the actual starting times: 11, 1, 1, 1, 7, 5, 20, 3, 7,11, and 6 minutes. Then he computed the mean of these differences as 6.6 minutes, to the nearest tenth. Michael stated,

I think that predictions that are off by 7 or fewer minutes are "good" predictions. Since, on the average, my predicted starting times differ from the actual starting times by fewer than 7 minutes, and since 7 out of my 11 predicted starting times are off by 7 or fewer minutes , I rate my predictions as "good."

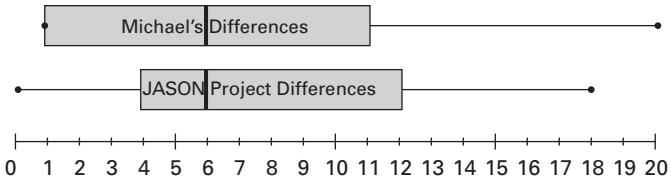
To compare his predictions to the JASON Project predictions, Michael computed the differences between the JASON Project predicted starting times and the actual starting times. These differences are: 12, 4, 1, 5, 6, 0, 18, 6, 7, 12, and 10 minutes, with a mean difference of 7.4 minutes to the nearest tenth. Based on the fact that his predictions differed from the actual times by an average of 6.6 minutes, and the JASON Project predictions differed from the actual times by an average of 7.4 minutes, Michael stated that his predictions were "better than the JASON Project predictions."

As a second method of comparing his predictions to the JASON Project predictions, Michael computed the differences between the JASON project differences (12, 4, 1, 5, 6, 0, 18, 6, 7, 12, and 10 minutes) and his differences (11, 1, 1, 1, 7, 5, 20, 3, 7, 11, and 6 minutes) to get 1, 3, 0, 4, 1, 5, 2, 3, 0, 1, and 4 minutes. The mean of these differences, to the nearest tenth, is 2.2 minutes. This lead Michael to conclude that his predictions were "close to the JASON Project predictions."

Focus Teacher Activity (cont.)

ACTIONS

18 One method of comparing predicted starting times of eruptions to actual starting times is to form a box plot of the differences between predicted and actual times (see differences in Comment 17). If it hasn't come up, ask the students to use this method to compare their differences to the JASON project differences. Discuss.



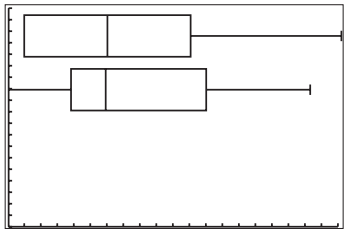
Difference between actual and predicted starting times.

19 If it didn't come up in Action 18, discuss the use of the graphing calculator to graph box plots. Ask the students to compare their by-hand box plots to calculator box plots. Discuss.

COMMENTS

18 The upper box plot at the left is for the set of differences between the actual starting times and Michael's predictions (i.e., for 11, 1, 1, 1, 7, 5, 20, 3, 7, 11, and 6 minutes; see example in Comment 17). The lower box plot is for the set of differences between the actual starting times and the JASON Project's predictions (i.e., for 12, 4, 1, 5, 6, 0, 18, 6, 7, 12, and 10 minutes). These sets of differences each have a median of 6, and the right hand whiskers show that the maximum difference between the predicted starting time and the actual starting time is greater for Michael's data than for the JASON Project's data. However, the middle 50% of both sets of data (the boxes) show that Michael's data has more lower values than the JASON Project, and this agrees with mean of the differences for Michael's data being 6.6 as compared to the mean of the differences for the JASON Project data of 7.4 (see Comment 17). Notice that there is no whisker on the lower end of the box for Michael's data because the lower 25% of the data each equal the minimum difference, 1.

19 As examples, the plots shown below are print-outs of a TI-83 box-plot of the differences between Michael's predicted starting times and the actual starting times (see example in Comment 17) and the differences between the JASON Project predicted times and the actual times.



Note: To obtain box-plots on the TI-83, students may: (1) enter the differences between their own predictions and the actual starting times in List 1, and the differences between the JASON Project predictions and the actual times in List 2; (2) turn on plots 1 and 2 from the STAT PLOT menu and select the "box plot" option; (3) press GRAPH; (4) TRACE the box plots to identify the minimums, maximums, medians, and quartiles for these plots.

Focus Teacher Activity (cont.)

ACTIONS

20 Place a transparency of Focus Master D on the overhead and give a copy to each student. Ask the groups to formulate a list of "We notice… We wonder…" statements about the given information. Invite volunteers to share a few of their group's statements with the class. Then ask each student, or each group of students, to:

a) Formulate a question or set of related questions about the data, noting that this question may change or evolve as students complete part b) and through input from classmates and you.

b) Investigate the given data and complete a statistical analysis of the data in order to answer the question or questions formed in a).

c) Report the details and reasoning behind their methods, conjectures and conclusions, supporting all conclusions with sound statistical evidence.

Lesson **15** Data—Variability and Spread

Focus Master D

U.S. Data*	Percentage of Population Not High School Graduates (1994, to nearest percent)	Average Amount of Money Spent for Education Per Student (1993, in thousands of dollars)	Percentage of Population Below Poverty Level (1993, to tenth of a percent)
Northeast, Mid-Atlantic			
Maine	21	6.1	15.4
New Hampshire	18	5.6	9.9
Vermont	19	6.7	10.0
Massachusetts	20	6.6	10.7
Rhode Island	28	6.6	11.2
Connecticut	21	8.2	8.5
New York	25	8.8	16.4
New Jersey	23	9.5	10.9
Pennsylvania	25	6.9	13.2
Delaware	23	6.4	10.2
Maryland	22	6.4	9.7
District of Columbia	27	8.0	26.4
West Virginia	34	5.7	22.2
Midwest			
Ohio	24	5.3	13.3
Indiana	24	5.4	12.2
Illinois	24	5.4	13.6
Michigan	23	6.4	15.4
Wisconsin	21	6.5	12.6
Minnesota	18	5.6	11.6
Iowa	20	5.2	10.3
Missouri	26	4.5	16.1
North Dakota	23	4.9	11.2
South Dakota	23	4.4	14.2
Nebraska	18	4.9	10.3
Kansas	19	5.5	13.1
South			
Virginia	25	5.3	9.7
North Carolina	30	4.9	14.4
South Carolina	32	4.7	18.7
Georgia	29	4.7	13.5
Florida	26	5.3	17.8
Kentucky	35	4.9	20.4
Tennessee	33	4.0	19.6
Alabama	33	3.8	17.4
Mississippi	36	3.4	24.7
Arkansas	34	3.8	20.0
Louisiana	32	4.3	26.4
Oklahoma	25	4.1	19.9
Texas	28	4.9	17.4
West			
Montana	19	5.5	14.9
Idaho	20	4.0	13.1
Wyoming	17	5.8	13.3
Colorado	16	5.1	9.9
New Mexico	25	4.6	17.4
Arizona	21	4.1	15.4
Utah	15	3.2	10.7
Nevada	21	4.9	9.8
Washington	16	5.5	12.1
Oregon	19	6.1	11.8
California	24	4.6	18.2
Alaska	13	9.3	9.1
Hawaii	20	5.8	8.0

*Statistical Abstract of the United States, 115th ed. (Washington: Bureau of the Census, 1995), pp.159, 168, 461, 482.

COMMENTS

20 Focus Master D contains 3 types of data for 4 regions of the United States. This action could be used to provide further opportunities for students to form plots and analyze data, and/or to assess students' understanding of statistical methods.

Prior to beginning this project you might provide, or have the class create, an assessment guide for the project (see *Starting Points* for ideas). You might require specific statistical graphs (e.g., histograms and line, box, stem and leaf, and scatter plots) and specific statistical summaries (e.g., mean, median, mode, average deviation, median-fit lines, quartiles, etc.). You could ask the students to relate the variability they see in the data to their conclusions. Reports should give evidence that the students know a wide range of statistical methods (both by hand and by calculator), how to use the methods appropriately, and how to communicate convincingly. You might also include specific requirements regarding the presentation of the students' reasoning and results.

Each of the 3 types of data on Focus Master D can be analyzed separately as one-variable data by constructing stem and leaf plots, histograms, and/or box plots of each type of data, or as two-variable by forming scatter plots and median-fit lines based on 2 types of data. For example, 4 box and whisker plots of the percentages of the population from each of the 4 U.S. regions that are not high school graduates could provide the basis for a one-variable analysis. Or, a scatter plot comparing 2 of the 3 types of data could provide the basis of a two-variable analysis.

Questions such as the following may help students develop their own questions about the data: How do the Midwest states compare with the Northeast Mid-Atlantic states in the amounts of money spent on education per student? If a scatter plot is formed of the two variables, Percentage of Population Not High School Graduates, and Average Amount of Money Spent for Education per Student, what predictions can be made from a median-fit line?

TEACHER NOTES:

Follow-up Student Activity 15.8

NAME _____ **DATE** _____

Construct all graphs by hand or by a graphing calculator. Show labeled sketches of all graphs you construct.

1 This table shows the starting dates of all eruptions of the Hawaiian volcano, Kilauea, from 1923 to 1963, the duration in days of each eruption, and the time in years before the next eruption. Form a line plot of the eruption dates by year, and form a stem and leaf plot, histogram, and box plot of the Duration in Days data. Next to each plot, write your observations, conjectures, and/or generalizations based on mathematical relationships in the plot. What can you see/conclude from one plot that another doesn't show?

Eruptions of Kilauea on Hawaii from 1923 to 1963

Starting Dates	Duration in Days	Time Interval Before Next Eruption in Years
August 25, 1923	1	.7
May 10, 1924	17	.2
July 19, 1924	11	3.0
July 7, 1927	13	1.6
February 20, 1929	2	.4
July 25, 1929	4	1.4
November 19, 1930	19	1.1
December 23, 1931	14	1.7
September 6, 1934	33	18.6
June 27, 1952	136	1.9
May 31, 1954	3	.7
February 28, 1955	88	4.7
November 14, 1959	36	.2
January 13, 1960	36	1.1
February 24, 1961	1	.2
March 3, 1961	22	.4
July 10, 1961	7	.2
September 22, 1961	3	1.2
December 7, 1962	2	.7
August 21, 1963	2	.1
October 5, 1963	1	1.4

(Continued on back.)

Follow-up Student Activity (cont.)

2 Form a line, stem and leaf, and box plot for the Time Interval Before Next Eruption data. State several observations, conjectures, and/or generalizations. Compare these plots with those from Problem 1. Describe the similarities and differences in the graphs and how these similarities and differences may relate to the data.

3 On ¼" grid paper, form a scatter plot of the eruptions of Kilauea with the Duration of Eruptions along the horizontal axis and the Time Interval Before Next Eruption along the vertical axis. Sketch a median-fit line for this plot. Suppose that in November of 1963 another eruption occurred and it lasted 30 days. Use your plot and median-fit line to predict the length of time before the next eruption of Kilauea.

4 The dates of outbreaks of eruptions at Kilauea from 1964 to 1982 and the duration in days of these eruptions are listed here: March 5, 1964, 10 days; December 14, 1965, < 1 day; November 5, 1967, 251 days; August 22, 1968, 5 days; October 7, 1968, 15 days; February 22, 1969, 6 days; May 24, 1969, 867 days; August 14, 1971, < 1 day; September 24, 1971, 5 days; February 4, 1972, 454 days; May 5, 1973, < 1 day; November 10, 1973, 30 days; December 12, 1973, 203 days; July 19, 1974, 3 days; September 19, 1974, < 1 day; December 31, 1974, < 1 day; November 29, 1975, < 1 day; September 13, 1977, 18 days; November 16, 1979, 1 day; April 31, 1982, < 1 day; September 5, 1982, < 1 day. Note: < stands for "less than."

a) Form a histogram and a box plot for the Eruption Duration in Days data for the years 1964 to 1982. Compare these to the histogram and box plot for Problem 1 and state a few observations.

b) Form a line plot of the dates of eruptions by year for 1964-82. Compare this to the line plot from Problem 1. What do these plots show about the differences in data for these 2 periods?

c) Since January 3, 1983, Kilauea has erupted continuously. This is the longest period of a volcanic eruption in Hawaii in historical times. Comparing the data and graphs from 1923-63 to the data from 1964-82, what clues can you find that might have indicated a drastic change for the years following 1983? Discuss your observations and reasoning.

Connector Student Activity 15.1

NAME _____ **DATE** _____

Period 1—Length of Time in Minutes Between Eruptions of Old Faithful

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

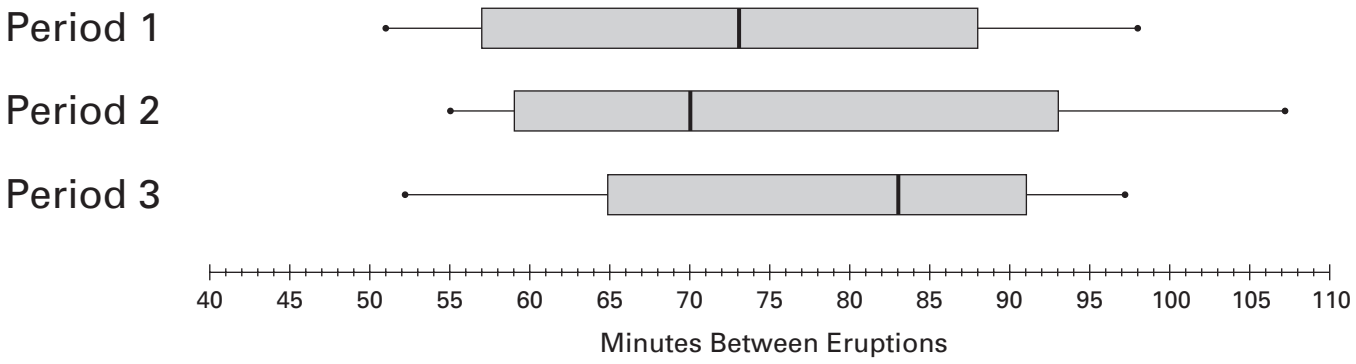
Regarding the above data,

I notice…

I wonder…

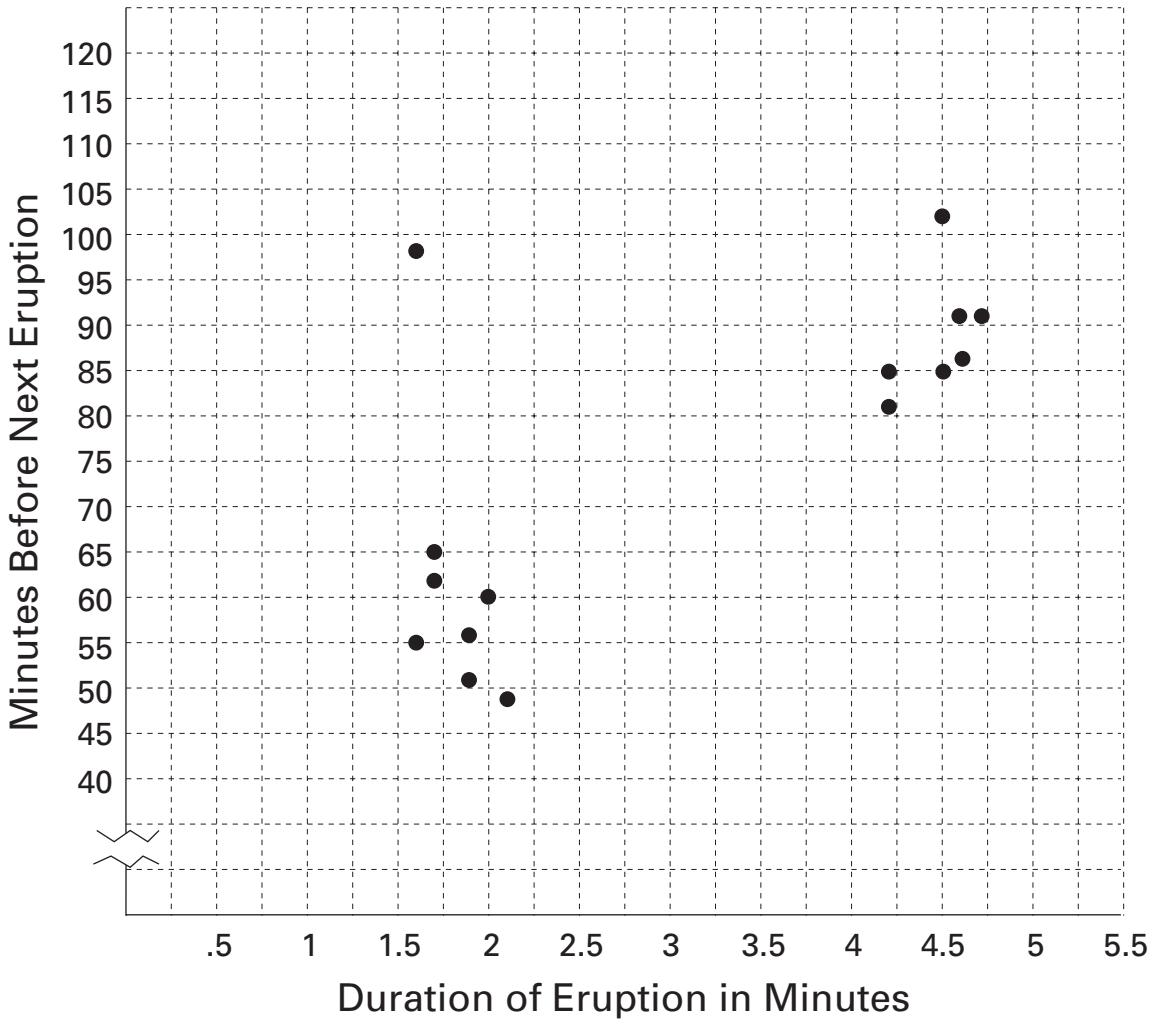
Focus Master A

Old Faithful Data



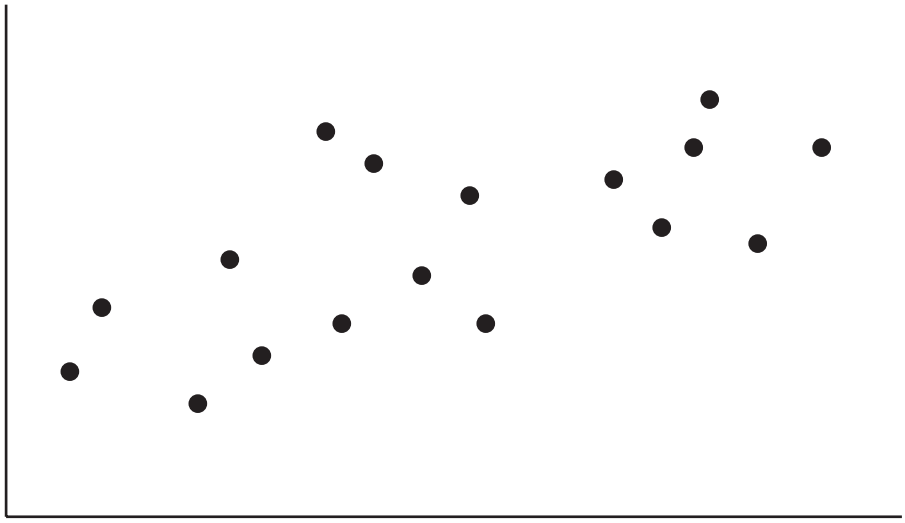
Focus Master B

Eruptions of Old Faithful—April 1997



Focus Master C

Constructing a Median-Fit Line



Focus Master D

U.S. Data*	Percentage of Population Not High School Graduates (1994, to nearest percent)	Average Amount of Money Spent for Education Per Student (1993, in thousands of dollars)	Percentage of Population Below Poverty Level (1993, to tenth of a percent)
Northeast, Mid-Atlantic			
Maine	21	6.1	15.4
New Hampshire	18	5.6	9.9
Vermont	19	6.7	10.0
Massachusetts	20	6.6	10.7
Rhode Island	28	6.6	11.2
Connecticut	21	8.2	8.5
New York	25	8.8	16.4
New Jersey	23	9.5	10.9
Pennsylvania	25	6.9	13.2
Delaware	23	6.4	10.2
Maryland	22	6.4	9.7
District of Columbia	27	8.0	26.4
West Virginia	34	5.7	22.2
Midwest			
Ohio	24	5.3	13.3
Indiana	24	5.4	12.2
Illinois	24	5.4	13.6
Michigan	23	6.4	15.4
Wisconsin	21	6.5	12.6
Minnesota	18	5.6	11.6
Iowa	20	5.2	10.3
Missouri	26	4.5	16.1
North Dakota	23	4.9	11.2
South Dakota	23	4.4	14.2
Nebraska	18	4.9	10.3
Kansas	19	5.5	13.1
South			
Virginia	25	5.3	9.7
North Carolina	30	4.9	14.4
South Carolina	32	4.7	18.7
Georgia	29	4.7	13.5
Florida	26	5.3	17.8
Kentucky	35	4.9	20.4
Tennessee	33	4.0	19.6
Alabama	33	3.8	17.4
Mississippi	36	3.4	24.7
Arkansas	34	3.8	20.0
Louisiana	32	4.3	26.4
Oklahoma	25	4.1	19.9
Texas	28	4.9	17.4
West			
Montana	19	5.5	14.9
Idaho	20	4.0	13.1
Wyoming	17	5.8	13.3
Colorado	16	5.1	9.9
New Mexico	25	4.6	17.4
Arizona	21	4.1	15.4
Utah	15	3.2	10.7
Nevada	21	4.9	9.8
Washington	16	5.5	12.1
Oregon	19	6.1	11.8
California	24	4.6	18.2
Alaska	13	9.3	9.1
Hawaii	20	5.8	8.0

*Statistical Abstract of the United States, 115th ed. (Washington: Bureau of the Census, 1995), pp.159, 168, 461, 482.

Focus Student Activity 15.2

NAME _____ **DATE** _____

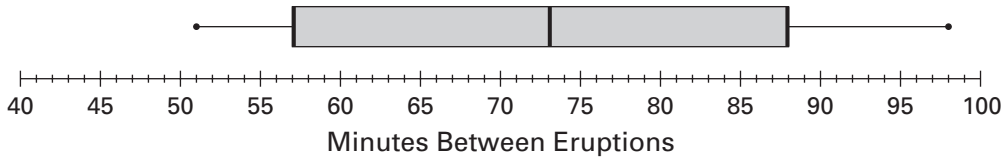
Period 1—Length of Time in Minutes Between Eruptions of Old Faithful

Data in order collected:

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

Data in numerical order:

Box and Whisker Plot



Observations:

Focus Student Activity 15.3

NAME _____ **DATE** _____

Period 1—Length of Time in Minutes Between Eruptions of Old Faithful

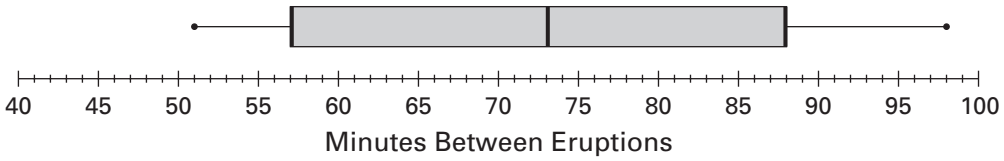
Data in order collected:

57, 87, 73, 94, 52, 88, 72, 88, 62, 87, 57, 94, 51, 98, 59

Data in numerical order:

51, 52, 57, 57, 59, 62, 72, 73, 87, 87, 88, 88, 94, 94, 98

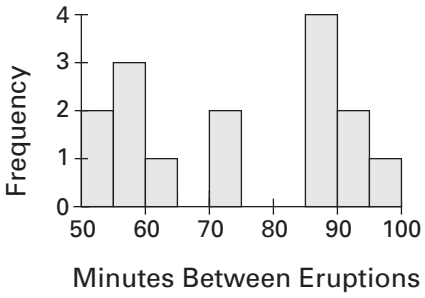
Box and Whisker Plot



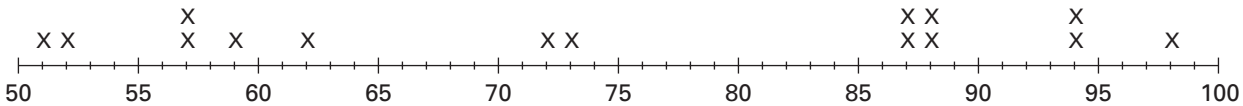
Stem and Leaf Plot

5	1,2,7,7,9
6	2
7	2,3
8	7,7,8,8
9	4,4,8

Histogram



Line Plot



Observations:

Average deviation(s):

Method(s):

Focus Student Activity 15.4

NAME _____ **DATE** _____

Period 2—Length of Time in Minutes Between Eruptions of Old Faithful

Data in order collected:

93, 86, 70, 63, 91, 82, 58, 97, 59, 70, 58, 98, 55, 107, 61

Data in numerical order:

Box and Whisker Plot

Stem and Leaf Plot

Histogram

Line Plot

Observations:

Average deviation(s):

Method(s):

Focus Student Activity 15.5

NAME _____ **DATE** _____

Period 3—Length of Time in Minutes Between Eruptions of Old Faithful

Data in order collected:

82, 91, 65, 97, 52, 94, 60, 94, 63, 91, 83, 84, 71, 83, 70

Data in numerical order:

Box and Whisker Plot

Stem and Leaf Plot

Histogram

Line Plot

Observations:

Average deviation(s):

Method(s):

Focus Student Activity 15.6

NAME _____ **DATE** _____

1 The following data about Old Faithful was recorded, in order, in April 1997. Form a scatter plot of this data by placing the duration of eruptions along the horizontal axis and the time interval before the next eruption along the vertical axis. Record your observations.

Duration of Eruption	Time Interval Before Next Eruption
4 min 24 sec	81 min
2 min 00 sec	60 min
4 min 43 sec	91 min
1 min 55 sec	51 min
4 min 14 sec	85 min
1 min 34 sec	55 min
1 min 34 sec	98 min
2 min 8 sec	49 min
4 min 30 sec	85 min
1 min 43 sec	65 min
4 min 27 sec	102 min
1 min 51 sec	56 min
4 min 35 sec	86 min
1 min 44 sec	62 min
4 min 35 sec	91 min

2 Following are the starting time and durations of 11 different eruptions randomly selected from a 2-day period in May 1997 (not listed in order). Use your scatter plot from Problem 1 to predict the time interval before the next eruption and the time of the next eruption. Explain your methods.

Starting Time	Duration	Predicted Interval Before Next Eruption	Predicted Time of Next Eruption
11:22 am	1 min 50 sec	_____	_____
7:16 am	4 min 19 sec	_____	_____
2:11 pm	1 min 50 sec	_____	_____
8:53 am	1 min 55 sec	_____	_____
12:20 pm	4 min 46 sec	_____	_____
8:30 am	4 min 26 sec	_____	_____
11:18 am	4 min 35 sec	_____	_____
12:40 pm	3 min 46 sec	_____	_____
9:52 am	4 min 35 sec	_____	_____
10:17 am	1 min 45 sec	_____	_____
7:35 am	1 min 55 sec	_____	_____

Focus Student Activity 15.7

NAME _____ **DATE** _____

The predicted starting times of next eruptions of Old Faithful in column (4) below were obtained by the JASON Project in May 1997.

1 Compare the actual starting times of the next eruptions from column (2) in the table below to the JASON Project predicted starting times in column (4). How do you rate the JASON Project predictions? Support your rating with reasoning based on statistical evidence.

2 Complete column (3) of the table below by filling in each of your predicted starting times of next eruptions from Focus Student Activity 15.6 (notice that data in Column (1) is ordered according to its occurence—be sure to enter your data accordingly). Then use at least 2 different statistical methods to compare your predictions in column (3) to those by the JASON Project in column (4) and to the actual starting times of next eruptions in column (2). Use your comparisons as the basis for rating the quality of your predictions, and provide convincing statistical evidence to support your rating.

(1) **Starting Time of Eruption**	(2) **Actual Starting Time of Next Eruption**	(3) **My Predicted Starting Time of Next Eruption** (from Student Activity 15.6)	(4) **JASON Project's Predicted Start of Next Eruption**
7:16 am	8:53 am	_____	8:41 am
8:53 am	9:52 am	_____	9:48 am
9:52 am	11:22 am	_____	11:21 am
11:22 am	12:20 pm	_____	12:15 pm
12:20 pm	1:58 pm	_____	1:52 pm
7:35 am	8:30 am	_____	8:30 am
8:30 am	10:17 am	_____	9:59 am
10:17 am	11:18 am	_____	11:12 am
11:18 am	12:40 pm	_____	12:47 pm
12:40 pm	2:11 pm	_____	1:59 pm
2:11 pm	3:16 pm	_____	3:06 pm

Follow-up Student Activity 15.8

NAME _____ **DATE** _____

Construct all graphs by hand or by a graphing calculator. Show labeled sketches of all graphs you construct.

1 This table shows the starting dates of all eruptions of the Hawaiian volcano, Kilauea, from 1923 to 1963, the duration in days of each eruption, and the time in years before the next eruption. Form a line plot of the eruption dates by year, and form a stem and leaf plot, histogram, and box plot of the Duration in Days data. Next to each plot, write your observations, conjectures, and/or generalizations based on mathematical relationships in the plot. What can you see/conclude from one plot that another doesn't show?

Eruptions of Kilauea on Hawaii from 1923 to 1963

Starting Dates	Duration in Days	Time Interval Before Next Eruption in Years
August 25, 1923	1	.7
May 10, 1924	17	.2
July 19, 1924	11	3.0
July 7, 1927	13	1.6
February 20, 1929	2	.4
July 25, 1929	4	1.4
November 19, 1930	19	1.1
December 23, 1931	14	1.7
September 6, 1934	33	18.6
June 27, 1952	136	1.9
May 31, 1954	3	.7
February 28, 1955	88	4.7
November 14, 1959	36	.2
January 13, 1960	36	1.1
February 24, 1961	1	.2
March 3, 1961	22	.4
July 10, 1961	7	.2
September 22, 1961	3	1.2
December 7, 1962	2	.7
August 21, 1963	2	.1
October 5, 1963	1	1.4

(Continued on back.)

Follow-up Student Activity (cont.)

2 Form a line, stem and leaf, and box plot for the Time Interval Before Next Eruption data. State several observations, conjectures, and/or generalizations. Compare these plots with those from Problem 1. Describe the similarities and differences in the graphs and how these similarities and differences may relate to the data.

3 On ¼" grid paper, form a scatter plot of the eruptions of Kilauea with the Duration of Eruptions along the horizontal axis and the Time Interval Before Next Eruption along the vertical axis. Sketch a median-fit line for this plot. Suppose that in November of 1963 another eruption occurred and it lasted 30 days. Use your plot and median-fit line to predict the length of time before the next eruption of Kilauea.

4 The dates of outbreaks of eruptions at Kilauea from 1964 to 1982 and the duration in days of these eruptions are listed here: March 5, 1964, 10 days; December 14, 1965, < 1 day; November 5, 1967, 251 days; August 22, 1968, 5 days; October 7, 1968, 15 days; February 22, 1969, 6 days; May 24, 1969, 867 days; August 14, 1971, < 1 day; September 24, 1971, 5 days; February 4, 1972, 454 days; May 5, 1973, < 1 day; November 10, 1973, 30 days; December 12, 1973, 203 days; July 19, 1974, 3 days; September 19, 1974, < 1 day; December 31, 1974, < 1 day; November 29, 1975, < 1 day; September 13, 1977, 18 days; November 16, 1979, 1 day; April 31, 1982, < 1 day; September 5, 1982, < 1 day. Note: < stands for "less than."

a) Form a histogram and a box plot for the Eruption Duration in Days data for the years 1964 to 1982. Compare these to the histogram and box plot for Problem 1 and state a few observations.

b) Form a line plot of the dates of eruptions by year for 1964-82. Compare this to the line plot from Problem 1. What do these plots show about the differences in data for these 2 periods?

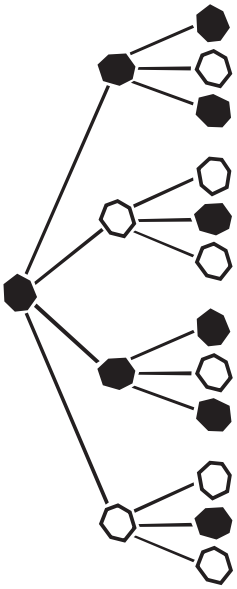
c) Since January 3, 1983, Kilauea has erupted continuously. This is the longest period of a volcanic eruption in Hawaii in historical times. Comparing the data and graphs from 1923-63 to the data from 1964-82, what clues can you find that might have indicated a drastic change for the years following 1983? Discuss your observations and reasoning.

Counting & Probability Diagrams

Lesson **16**

THE BIG IDEA

The fundamental counting principle is a powerful counting technique used in place of one-by-one counting. Investigations using concrete models, tree diagrams, and rectangle diagrams to organize and represent situations involving counting prompt insights about this principle. When such investigations emphasize combinatorial reasoning and probability, important conjectures about permutations, combinations, and the probabilities of certain outcomes also emerge.



CONNECTOR

OVERVIEW

Students use colored linking cubes and tree and rectangular diagrams to solve counting problems. These models and diagrams illustrate the fundamental counting principle.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Colored linking cubes (e.g., Hex-a-links), 40 or more per pair of students.
- ✔ Connector Masters A, B, and C, 1 transparency of each.
- ✔ Connector Student Activity 16.1, 1 copy per pair of students and 1 transparency.
- ✔ An overhead pen or masking tape for each pair of students.

FOCUS

OVERVIEW

Students develop strategies for solving counting problems involving permutations and combinations. They compute the probabilities of outcomes which require these strategies. Rectangles and probability trees are introduced as methods of determining sample spaces and for computing probabilities.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Colored linking cubes (e.g., Hex-a-links), 80 or more for each group of students.
- ✔ Focus Student Activities 16.2-16.4, 1 copy of each per student and 1 transparency of each.
- ✔ An overhead pen or masking tape for each group.

FOLLOW-UP

OVERVIEW

Students use diagrams and the fundamental counting principal to solve counting and probability problems, and provide explanations for their conclusions.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 16.5, 1 copy per student.

LESSON IDEAS

QUOTE

In grades K-8, counting typically involves matching the elements of a set with a finite subset of the natural numbers. But real-world problems that can be simplified to the form "How many different subsets of size k can be selected from the members of a set having n distinct members?" require an entirely different method of counting. To develop students' abilities to solve problems with this structure, instruction should emphasize combinatorial reasoning as opposed to the application of analytic formulas for permutations and combinations.

NCTM Standards

ASSESSMENT

You might provide class time for brainstorming content ideas and/or evaluation standards for the letters students produce for Problem 6 on the Follow-up. This letter could be used as a portfolio entry.

PURPOSE

The primary purpose of this lesson is to engage students in thinking and conjecturing about the concepts of counting rather than mastering formulas and definitions. Although many teacher comments include formulas and definitions, these are provided for teacher background and for comparison to generalizations that students may offer.

SELECTED ANSWERS

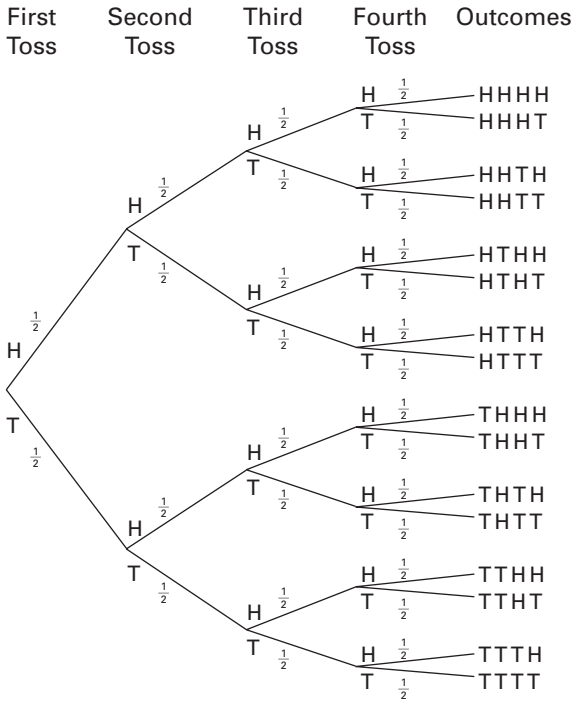
1. a) 20
   b) 120
   c) 800
   d) 324. The first house can be any one of 4 different colors and then each remaining house can be any one of 3 colors ($4 \times 3 \times 3 \times 3 \times 3$).
   e) 720. The 6 remaining positions can be filled in $6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways.
   f) 8, since $7 \times 7 \times 7$ is less than 365 and $8 \times 8 \times 8$ is greater than 365.
   g) 50,000, since the digit in the ones place must be even (0, 2, 4, 6, or 8) there are $10 \times 10 \times 10 \times 10 \times 5$ possible zip codes.
   h) 495

2. a) $\frac{1}{4} \times \frac{1}{8} = \frac{1}{32}$ b) $\frac{1}{8} \times \frac{1}{12} = \frac{1}{96}$
   c) $\frac{1}{4} \times \frac{1}{4} \times \frac{1}{6} = \frac{1}{96}$ d) $\frac{3}{8} \times \frac{1}{3} = \frac{1}{8}$

3. a) $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$ b) $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
   c) $\frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$ d) $\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

4. The probability of the 1st gumball being red is $\frac{2}{6}$, and if the 1st gumball is red, the probability of the 2nd being red is $\frac{1}{5}$. So, the probability that both are red is $\frac{2}{6} \times \frac{1}{5} = \frac{1}{15}$.

5. In the following probability tree, H represents heads and T represents tails. This tree shows that 8 of the 16 outcomes have 2 or more heads in a row, so the probability of this happening is $\frac{8}{16} = \frac{1}{2}$.



Connector Teacher Activity

OVERVIEW & PURPOSE

Students use colored linking cubes and tree and rectangular diagrams to solve counting problems. These models and diagrams illustrate the fundamental counting principle.

MATERIALS

- ✔ Colored linking cubes (e.g., Hex-a-links), 40 or more per pair of students.
- ✔ Connector Masters A, B, and C, 1 transparency of each.
- ✔ Connector Student Activity 16.1, 1 copy per pair of students and 1 transparency.
- ✔ An overhead pen or masking tape for each pair of students.

ACTIONS

1 Arrange the students in pairs and distribute colored linking cubes and an overhead pen or some masking tape to each pair. Place a transparency of Connector Master A on the overhead and ask the pairs to draw a diagram or build a model that shows Marcia's situation and helps determine the number of routes possible. Invite volunteers to show their diagrams or models at the overhead. If they aren't suggested by a student, introduce a *tree diagram* and a *rectangle diagram* as possibilities.

Counting and Probability Trees Lesson **16**

Connector Master A

Marcia's Routes to School

Each morning before school Marcia walks to the post office to mail letters for her parents, and then she continues on to school. Marcia has 4 different routes she can walk from her home to the post office: along the river; through the park; by the new housing development; or on the highway. When she leaves the post office, there are 3 different routes that she can walk to school; along the river; across the athletic fields; or by the mall. How many different routes can Marcia take on her trips to school?

COMMENTS

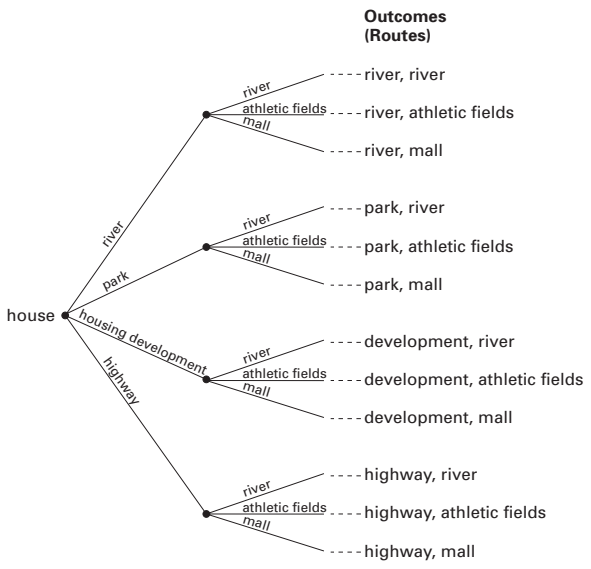
1 Some students may draw a "map" showing the 4 paths from Marcia's home to the post office and then draw the 3 paths from the post office to the school. Such a drawing may be labeled with details such as streets, river, home, post office, school, etc. The 12 possible different routes can be traced on the map.

Students' diagrams may resemble a *tree diagram*—a schematic diagram of the various paths, as illustrated on Connector Master B. A transparency of this master may be useful for introducing students to this method.

Lesson **16** Counting and Probability Diagrams

Connector Master B

Tree Diagram



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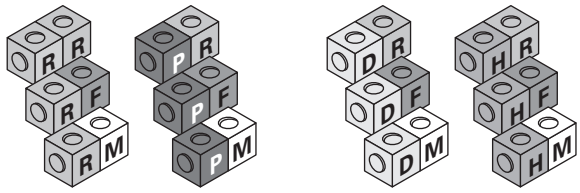
Connector Teacher Activity (cont.)

ACTIONS



Counting and Probability Diagrams Lesson **16**

Connector Master C

Routes from home to post office

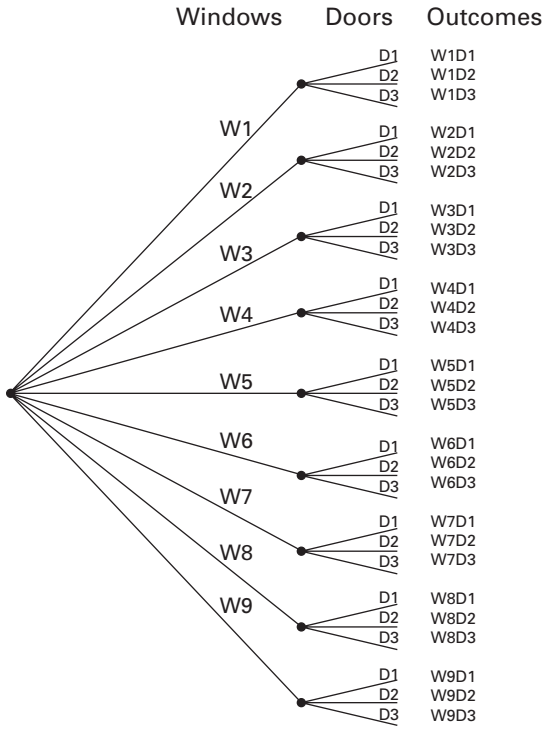
R	P	D	H

→

Routes from home via post office to school

RR	PR	DR	HR
RF	PF	DF	HF
RM	PM	DM	HM

2 Give each pair of students a copy of Connector Student Activity 16.1 and ask them to complete Problem 1. Discuss their reasoning and methods. If it isn't suggested by a student, ask the class to make a tree and/or rectangle diagram to illustrate the possibilities. Repeat for Problems 2-10.



COMMENTS

1 (continued.)
Students who use the linking cubes to form a model of this situation may use masking tape or an overhead pen to label the cubes. For example, they might label some cubes with R for river, P for park, D for housing development, and H for highway and then label other cubes with R for river, F for athletic fields, and M for mall. A way of placing the cubes to illustrate the 12 routes is shown at the left.

A *rectangle diagram* can also be used to determine the number of routes, as shown at the left on Connector Master C. The letters R, P, D, H, F, and M denote the routes as on the cubes above. The 4 parts of the first rectangle represent the 4 routes to the post office. Then each of these 4 parts is divided into 3 parts to represent the routes from the post office to the school; the second rectangle shows the 12 different routes from home to school. A transparency of Connector Master C may be useful for illustrating a rectangle diagram of Marcia's routes.

2 You may wish to have the pairs solve selected problems, and then assign the remaining problems as homework.

One possible diagram and/or explanation for each problem is provided below. The students may find it is *not* necessary to draw all the branches of a tree diagram or all the subdivisions of a rectangle or to form all the possible linking cube arrangements to reach a correct solution. That is, they may construct a partial diagram or model and "see" in it a pattern that can be generalized to solve a problem.

1) There are 27 ways, as shown by the tree diagram at the left, where W1 through W9 represent the 9 windows and D1, D2, and D3 represent the 3 doors.

Notice that, for every window the burglar enters, there are 3 doors he could exit. Hence, there are $9 \times 3 = 27$ routes for the burglar.

2) A tree diagram begins with 6 branches for the exterior colors, then each of the 6 branches has 5 branches for the interiors, and each of the 5 branches has 3 branches for the trims. Similarly, a rectangle can be divided into 6 parts to represent the 6 exterior colors; each of these can be divided into 5 parts to represent the 5 interior colors;

Connector Teacher Activity (cont.)

ACTIONS

Counting and Probability Diagrams Lesson **16**

Connector Student Activity 16.1

NAME _____ DATE _____

Solve each problem and show a diagram or write a brief explanation of your reasoning.

1 A house has 3 doors to the outside and 9 windows. In how many ways can a burglar enter the house through a window and leave through a door?

2 A car manufacturer provides 6 different exterior colors, 5 interior colors, and 3 different trims. How many different exterior color/interior color/trim schemes are available?

3 For breakfast Henry always chooses one of the following drinks: orange juice, tomato juice, or apple juice. For cereal he chooses either corn flakes or bran flakes. For pastry he has either a doughnut, a glazed muffin, or an apple tart. What are all the different possibilities for Henry's breakfasts if he always has cereal, a drink, and a pastry?

4 A teenager posed the following problem: How many different outfits can she wear if she has 2 skirts with different patterns, 3 different colored blouses and 2 different types of shoes?

5 Each student at Athey Creek Middle School has a 4-digit locker number. If the 1st digit of each 7th grader's locker number is a 7 and each of the other 3 digits is one of the digits from 0 through 9, how many different 4-digit numbers are available for the 7th graders? Note: digits can be repeated any number of times in a locker number.

6 A school cafeteria offers a selection of 2 types of meat, 3 types of fish, 4 different vegetables, and 3 desserts. In how many ways can a student select 1 vegetable, 1 dessert, and either 1 type of meat or 1 type of fish?

Lesson **16** Counting and Probability Diagrams

Connector Student Activity 16.1 (cont.)

7 A building developer may choose from 4 different roofing subcontractors, 5 different electrical subcontractors, 3 different plumbing subcontractors, 2 different carpenters, and 6 different painters. In how many ways can he select 1 of each?

8 The Tigers and the Wildcats are to play a 3-game series in soccer. The first team to win 2 games wins the series. Find all the possible outcomes for the 3 games.

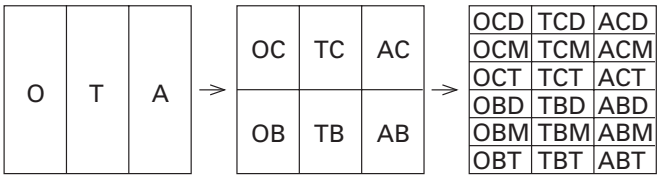
9 The annual Fourth of July raft race in Centerville has 7 entries. If a different prize is offered for each of the following categories; fastest raft, slowest raft, and most original raft, in how many ways is it possible to award the prizes?

10 A store carries 5 styles of backpacks in 4 different sizes. The customer also has a choice of 2 different kinds of material for 3 of the styles, and only 1 material for the other styles. How many different types of backpacks are there to choose from? If Larry is only interested in the 2 largest backpacks, how many different backpacks does he have to choose from?

COMMENTS

and each of these into 3 parts to represent the trim colors. Thus, there are $6 \times 5 \times 3 = 90$ different color-trim schemes.

3) The rectangle diagram below begins with 3 parts, 1 for each type of juice. Then each part is divided into 2 parts, 1 for each type of cereal. Finally, each of the 6 parts is divided into 3 parts, 1 for each type of pastry. The 18 parts in the last rectangle represent the 18 different breakfast possibilities. The letters in these rectangles are the first letters of each type of juice, cereal, and pastry.



Note that some students may form 18 arrangements of linking cubes labeled as shown in the 3rd rectangle above.

4) A tree diagram first has 2 branches, then each of these has 3 branches, and then each of these has 2 branches. So, there are $2 \times 3 \times 2 = 12$ different outfits.

5) Since the 1st digit must be a 7, there is only 1 choice for the 1st digit, but each of the remaining 3 digits can be any one of the digits from 0 through 9. A tree diagram begins with 1 branch for the 1st digit; this branch has 10 branches (0-9). Each of these 10 branches has 10 branches (0-9), and each of these 10 branches has 10 branches (0-9). Hence, there are $1 \times 10 \times 10 \times 10 = 1000$ different 4-digit numbers.

6) Since either 1 of the 2 meats or 1 of the 3 types of fish can be chosen, a tree diagram begins with 5 branches. Then each of these branches has 4 branches for the different types of vegetables, and each of the branches for the vegetables has 3 branches for desserts. So, there are $5 \times 4 \times 3 = 60$ ways.

7) A tree diagram begins with 4 branches; each of the 4 branches has 5 branches; each of the 5 branches has 3 branches; each of the 3 branches has 2 branches; and each of the 2 branches has 6 branches. So, there are $4 \times 5 \times 3 \times 2 \times 6 = 720$ different ways the building developer can select one of each.

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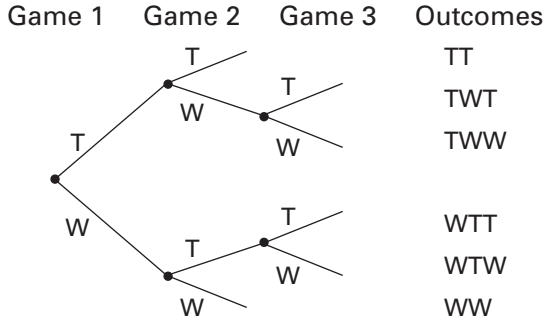
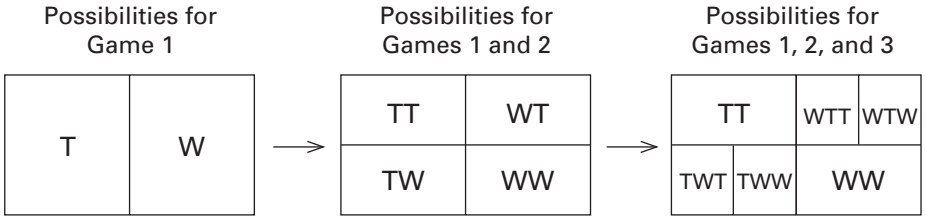
Connector Teacher Activity (cont.)

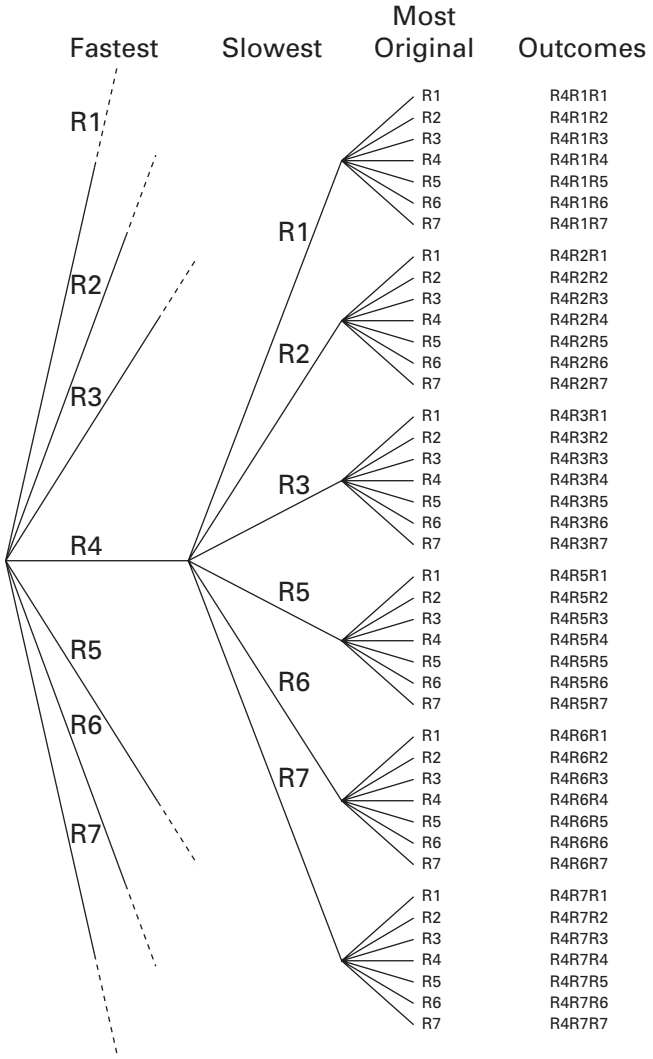
ACTIONS



COMMENTS

2 (continued.)
8) In contrast to the previous problems, not all outcomes in the tree diagram for this problem have the same number of branches. In the diagram at the left, T represents Tiger and W represents Wildcat. The diagram shows there are 6 possible outcomes: TT, TWT, TWW, WTT, WTW, and WW. In 2 outcomes the series ends in 2 games and in 4 outcomes the series requires 3 games. A similar result is shown below in the rectangle diagram for this problem.

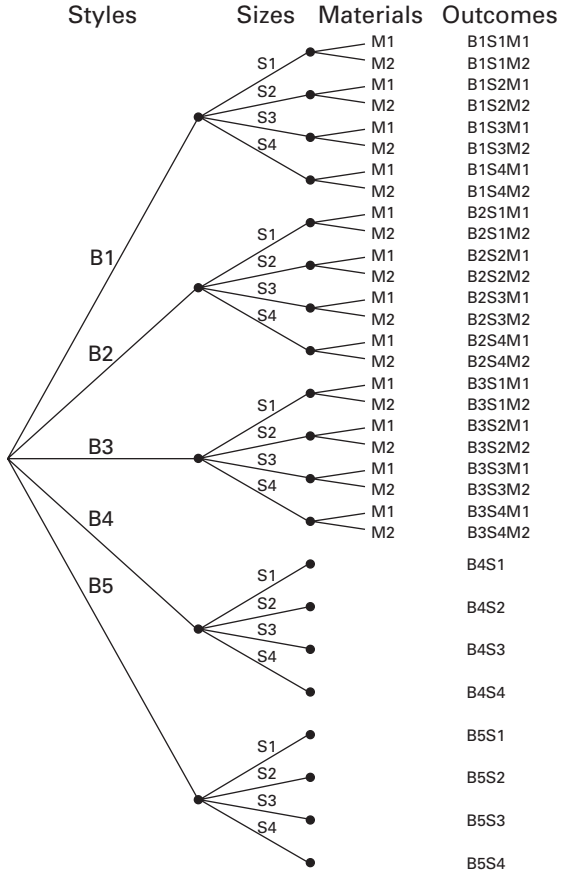




9) In the tree diagram on the left, R1 through R7 designate the 7 rafts. The diagram is incomplete—it only shows all of the branches when R4 has the fastest time. Note that if R4 has the fastest time, it cannot have the slowest time. However, any of the rafts, including the fastest and the slowest, can be judged the most original. If R4 is the fastest, there are $6 \times 7 = 42$ ways in which the prizes can be awarded. Similarly, if any of the other rafts is fastest, there are 42 ways the prizes can be awarded. Hence, there are $7 \times 42 = 294$ different ways in which the prizes can be awarded.

Connector Teacher Activity (cont.)

ACTIONS



COMMENTS

10) In the tree diagram shown at the left, B1 through B5 represent the 5 styles of backpacks, S1 through S4 represent the 4 sizes, and M1 and M2 represent the 2 types of material. Two types of material are available for styles B1, B2, and B3. The end points of the diagram show that there are $(3 \times 8) + (2 \times 4) = 32$ different types of backpacks to choose from. Regardless which 2 of the 4 sizes, S1, S2, S3, or S4 are chosen for the 2 largest sizes, Larry will have a choice of 16 different types of backpacks.

3 Ask the students to discuss with their groupmates their ideas about similarities and differences among the problems on Connector Student Activity 16.1. Ask them to list their observations, conjectures, questions, and generalizations. Discuss as a large group. Use this as a context for discussing use of the *Fundamental Counting Principal* to determine the outcomes of a *compound event*.

3 Notice that in each problem there are 2 or more types of activities that can be carried out in several ways. For example, in Problem 1) there are 2 types of activities: entering through windows and leaving through doors, and these are represented by 9 branches and 3 branches (or 9×3 parts of a rectangle), respectively. In Problem 2) there are 3 types of activities: choosing exterior colors; choosing interior colors, and choosing trim. These are represented by 6, 5, and 3 branches (or $6 \times 5 \times 3$ parts of a rectangle), respectively. An activity that is composed of 2 or more activities is called a *compound event*. A *simple event* involves only 1 activity (e.g., toss 1 coin, or choose 1 of 3 colors, or spin a spinner once).

In general, the total number of ways in which 1 activity can be followed by another is the product of the number of ways each activity can be carried out. That is, if 1 activity can be done in m ways, and another can be done in n ways, then there are $m \times n$ ways of doing both

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
	3 (continued.) activities. This is called the *Fundamental Counting Principle*. Each problem on Connector Student Activity 16.1 involves a compound event with several possible outcomes. Notice, for example, that in Problem 1) the number of routes a burglar can take is 9×3, and in Problem 2), the number of color/trim combinations is $6 \times 5 \times 3$.

Focus Teacher Activity

OVERVIEW & PURPOSE

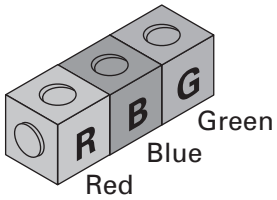
Students develop strategies for solving counting problems involving permutations and combinations. They compute the probabilities of outcomes which require these strategies. Rectangles and probability trees are introduced as methods of determining sample spaces and for computing probabilities.

MATERIALS

- ✔ Colored linking cubes (e.g., Hex-a-links), 80 or more for each group of students.
- ✔ Focus Student Activities 16.2-16.4, 1 copy of each per student and 1 transparency of each.
- ✔ An overhead pen or masking tape for each group.

ACTIONS

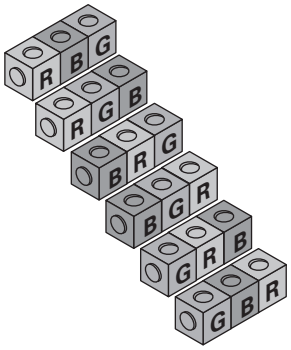
1 Arrange the students in groups and give each group a set of colored linking cubes and an overhead pen or some masking tape (for labeling cubes). Place a red cube, a blue cube, and a green cube in a row on the overhead or a table and ask the students to determine all the possible ways in which these 3 cubes can be arranged in a row. Ask them to sketch a diagram that illustrates their results and reasoning.



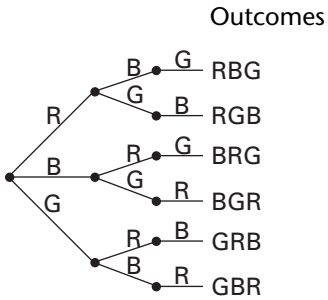
COMMENTS

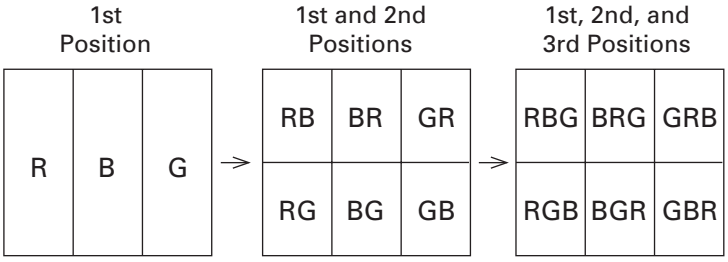
1 If all of the students' diagrams are similar, you might show other possibilities; doing so may prompt ideas for subsequent actions.

Shown below are the 6 different ways the cubes can be arranged in a row. Notice that if a red cube is placed first, for example, there are 2 ways to place the remaining blue and green cubes; if blue is placed first, there are 2 ways to place the remaining red and green; and if green is placed first, there are 2 ways to place the blue and red. So, there are 3×2 rows.



Students may use a tree diagram or rectangle diagram, such as shown here:

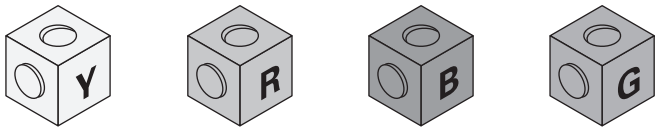




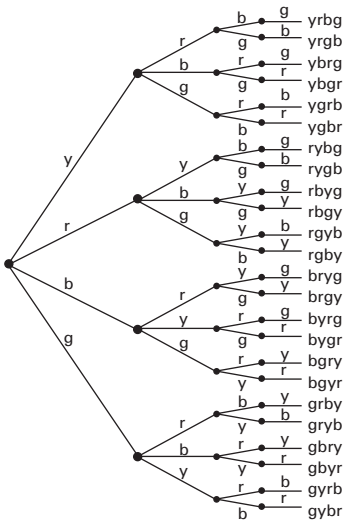
Focus Teacher Activity (cont.)

ACTIONS

2 Repeat Action 1, but this time with 4 different colored cubes; then for 5 different colored cubes; and finally for 6 different colored cubes. Introduce the term *permutations* to indicate the different ordered arrangements of the items in a set.



Post a blank "We conjecture… We Wonder…" poster and distribute butcher paper strips and marking pens. Have the groups add ideas to the poster as they come up now and throughout the lesson. Encourage any attempts to generalize and suggest that students will have opportunities to test and refine their conjectures during subsequent actions.



COMMENTS

2 There are 24 ways of arranging 4 different colored cubes in a row (i.e., 24 permutations of 4 different colors) as denoted by the following sequences of letters, where y stands for yellow, b for blue, r for red, and g for green:

yrbg	rybg	bryg	grby
yrgb	rygb	brgy	gryb
ybrg	rbyg	byrg	gbry
ybgr	rbgy	bygr	gbyr
ygrb	rgyb	bgry	gyrb
ygbr	rgby	bgyr	gybr

Creating models or diagrams that show all the permutations of more than 3 different colored cubes can be tedious. Hence, many students may generalize from partial diagrams or models, or they may reason based on their observations about 3 colored cubes. For example, students might reason, using the results of Action 1, that for 4 colors, whatever color is in the 1st position, there are 6 possible arrangements of the remaining 3 colors. Since any 1 of 4 colors can occupy the 1st position, there are 4×6 or $4 \times 3 \times 2$ permutations of 4 different colored cubes. A tree diagram for the 4 colors is shown at the left. One way of computing the number of ways 5 colors can be arranged in a row is to multiply 5 times the number of ways for 4 colors, or $5 \times (4 \times 3 \times 2 \times 1)$. Similarly, the number of permutations of 6 colors is 6 times the number of ways for 5 colors, or $6 \times (5 \times 4 \times 3 \times 2 \times 1)$.

Some students may reason that, analagous to a tree diagram for 4 colors beginning with 4 branches, a tree diagram for 6 colors begins with 6 branches, each of which has 5 branches, each of these 4 branches, and so forth, so that the number of ways of arranging 6 colors in a row, i.e., the number of *permutations* of 6 colors is $6 \times 5 \times 4 \times 3 \times 2 \times 1$. Or, the students may reason that for 6 colors, any 1 of the 6 colors can be placed in the 1st position of the row, then any one of the 5 remaining colors can be placed in the 2nd position, etc.

Some students may conjecture about the number of ways that *n* colors can be arranged in a row. If so, rather than affirming or correcting students' conjectures, you might suggest they test and refine them during upcoming actions. Notice that, in Action 4, students are asked to generalize about the number of permutations of *n* different items arranged in row.

Focus Teacher Activity (cont.)

ACTIONS

3 Give each student a copy of Focus Student Activity 16.2 and ask the groups to solve Problems 1 and 2. Invite volunteers to show their group's methods and reasoning and to list conjectures and generalizations on the class poster.

Counting and Probability Diagrams Lesson **16**

Focus Student Activity 16.2

NAME _____ DATE _____

Show how to use a diagram or model to solve each of the following problems. Note: It is okay to generalize from a partial diagram or model without showing all the parts. If you do so, be sure to show your reasoning.

1 In how many ways can 6 people line up to purchase tickets for a concert?

2 In how many ways can 8 students be seated in a row for a school photo?

4 If the students haven't already done so, ask them to use their experiences and observations from Actions 1-3 as the basis for predicting the number of ways *any* number of different items can be arranged in a row, i.e., for predicting the number of permutations of n different items placed in a row. Discuss the students' ideas.

COMMENTS

3 Students may sketch partial diagrams, or reason based on their observations from Actions 1 and 2.

1) Some students may relate this to the problem of arranging 6 different colored cubes in a row. Hence, the number of permutations of 6 people in a line is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.

2) The number of permuations of 8 people seated in a row is $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$.

Note that students complete Focus Student Activity 16.2 during Action 6.

4 Some students' generalizations may be verbal and others may be symbolic; both are appropriate, as long as they are based on understanding.

To arrange a set of n items in a row, there are n possibilities for the 1st position. The number of possibilities left for the 2nd position is $(n - 1)$ because one of the items has been used to fill the 1st position. Based on the Fundamental Counting Principal (see Connector Action 3), the number of ways of filling the 1st 2 positions is $n(n - 1)$.

The number of possibilities for the 3rd position is $(n - 2)$, since this is the number of items remaining. The number of ways of filling the first 3 positions is, therefore, $n(n - 1)(n - 2)$. Continuing in this manner, the number of possibilities for each position is 1 less than the number of possibilities for the previous position, until there is only 1 item left and 1 position for it. Hence, in general, the number of ways that n different items can be arranged in a row is:

$$n \times (n - 1) \times (n - 2) \times \ldots 3 \times 2 \times 1$$

This product is denoted by $n!$ (read "*n factorial*"). For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$. Some calculators have a factorial function. For example, on the TI-83, the factorial function is under the "PRB" menu in the MATH menu.

Focus Teacher Activity (cont.)

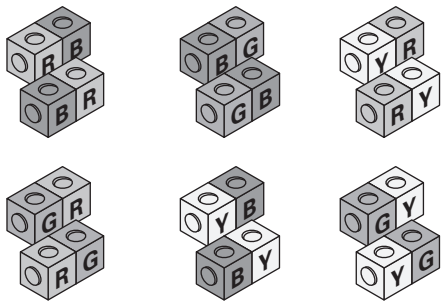
ACTIONS

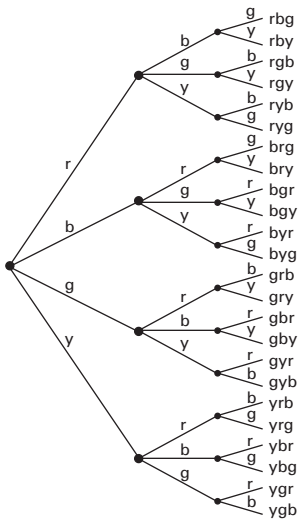
5 Ask the groups to determine the number of possible arrangements of 4 different colored cubes when placed in a row that contains only 2 positions, and the number of possible arrangements of 4 different colored cubes in a row with only 3 positions. Discuss.

COMMENTS

5 If there are 4 different colored cubes and a row has 2 positions, there are $4 \times 3 = 12$ possible arrangements, since any of the 4 colors can be used in the 1st position, and any of the 3 remaining colors in the 2nd position. This is illustrated below for the colors red (R), blue (B), green (G), and yellow (Y).

The 12 arrangements are shown here:





Similarly, if there are 4 colors and 3 positions in a row, $4 \times 3 \times 2 = 24$ different arrangements are possible.

Some students may use a tree diagram, such as the one shown at the left, which shows the number of different arrangements of 4 colors in rows with 3 positions per row. Others may use a rectangle diagram (see below) which shows the 1st rectangle with 4 parts, a 2nd rectangle obtained by dividing each of the 1st 4 parts into 3 parts, and a 3rd rectangle obtained by dividing each of the 12 parts of the 2nd rectangle into 2 parts.

1st Position

r	b	g	y

→

1st and 2nd Positions

rb	br	gr	yr
rg	bg	gb	yb
ry	by	gy	yg

→

1st, 2nd, and 3rd Positions

rbg	brg	grb	yrb
rby	bry	gry	yrg
rgb	bgr	gbr	ybr
rgy	bgy	gby	ybg
ryb	byr	gyr	ygr
ryg	byg	gyb	ygb

Students may conjecture about the general case for situations in which there are more items than positions in the row. If so, you might encourage them to test their conjectures during Action 6.

Focus Teacher Activity (cont.)

ACTIONS

6 Ask the groups to solve Problems 3-10 (or selected problems) from Focus Student Activity 16.2. Discuss their results and reasoning. Remind them to list conjectures and generalizations on the class poster.

Counting and Probability Diagrams Lesson **16**

Focus Student Activity 16.2

NAME _____ DATE _____

Show how to use a diagram or model to solve each of the following problems. Note: It is okay to generalize from a partial diagram or model without showing all the parts. If you do so, be sure to show your reasoning.

1 In how many ways can 6 people line up to purchase tickets for a concert?

2 In how many ways can 8 students be seated in a row for a school photo?

3 A sportswriter makes a preseason prediction about the order of the top 5 teams from among 20 teams in the Women's National Basketball Association. How many different possibilities are there?

4 An agricultural scientist wants to test different combinations of 4 types of soybeans with 7 types of fertilizer and 2 types of insecticides. How many experimental plots are needed?

5 A club of 10 people plans to elect 3 people for the offices of president, vice president, and secretary. In how many ways can these 3 offices be filled?

6 Seven cross-country runners are competing for 1st, 2nd, 3rd, and 4th places. How many different possible outcomes are there for these 4 places?

7 How many 3-letter code words can be formed from the alphabet if the code word can not contain a repeated letter?

Lesson **16** Counting and Probability Diagrams

Focus Student Activity 16.2 (cont.)

8 A contractor wishes to paint 6 houses in a row each with 1 of the colors red, blue, green, and yellow, with the requirement that if 2 houses are side by side they are to have different colors. In how many ways can the houses be painted?

9 Eight members of a fire department are being considered for 3 special awards. In how many ways can 3 of 8 people be selected if no one wins more than 1 award?

10 A school has 683 students. Are there *possibly*/*definitely*/*definitely not* (circle one) any students who have the same pair of initials for their first and last names?

7 If it hasn't already come up, pose the following problem for the groups to investigate:

> **In general, in how many different ways can *n* different items be arranged in a row if there are more items than positions in the row?**

Discuss the groups' conclusions and reasoning.

COMMENTS

6 Notice that Action 7 addresses the general case for arranging objects when there are more objects than places in the row.

The diagrams or models students use to determine the following products will vary.

3) $20 \times 19 \times 18 \times 17 \times 16 = 1{,}860{,}480$

4) $4 \times 7 \times 2 = 56$

5) $10 \times 9 \times 8 = 720$

6) $7 \times 6 \times 5 \times 4 = 840$

7) $26 \times 25 \times 24 = 15{,}600$

8) $4 \times 3 \times 3 \times 3 \times 3 \times 3 = 972$

9) $8 \times 7 \times 6 = 336$

10) There will definitely be students who have the same first and last initials because there are only $26 \times 26 = 676$ different possible pairs of letters.

7 The intent here is for students to reason inductively from their experiences and observations during Actions 5 and 6. Again, verbal or symbolic statements about general relationships are appropriate. Some students may be very challenged by the counting problems in this lesson and need to rely on diagrams and lists for each problem, while other students may be ready to write formulas about products that represent diagrams they can imagine. It is not expected that students

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

7 (continued.)
memorize formulas or use formal notation. It is important to use student comfort with counting concepts as the basis for determining the extent to which symbols and formulas are appropriate.

In general, if there are n colors and k positions in a row, the number of different ways of arranging the colors is given by the following product:

$$n \times (n-1) \times (n-2) \times \ldots \times (n-k+1)$$

Notice that, for n items there are $n!$ different ways of arranging the items in a row with n positions. When the number of positions is limited to k, for $k < n$, the number of arrangements, $n!$, is reduced by a factor of $(n-k)!$ For example, for 6 colors and 4 positions per row (i.e., $n = 6$ and $k = 4$), the number of ways of arranging the colors in rows is:

$$6 \times 5 \times 4 \times 3 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{6!}{(6-4)!} = \frac{6!}{2!}$$

For, $n = 9$ and $k = 3$, the number of ways is:

$$9 \times 8 \times 7 = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{9!}{(9-3)!} = \frac{9!}{6!}$$

If students invent verbal or symbolic generalizations, they may be interested in discussing the following standard formula for the number of permutations of a row of k different items selected from n different items. Note: ${}_nP_k$ reads "a permutation of n items taken k at a time."

$${}_nP_k = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

8 Pose the following problem for the students to investigate:

In how many *distinguishable* ways can colored cubes be arranged in a row when there are several identical cubes? For example, suppose there are:

a) 3 identical red cubes and 1 blue,

b) 4 identical red and 1 blue,

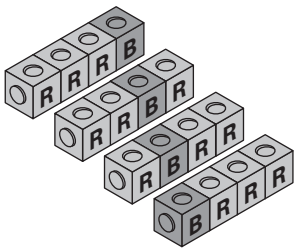
c) 3 identical red and 2 identical blue,

d) 4 identical red and 3 identical blue,

e) (optional) n identical red cubes and k identical blue cubes.

8 You may need to clarify the meaning of *distinguishable arrangements*. For example, while there are 3! = 6 ways of arranging 3 cubes in a row, if the 3 cubes are identical, the 6 arrangements are *not distinguishable*.

a) There are 4 distinguishable ways of arranging 3 identical red cubes and 1 blue cube in a row:



Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

b) For 4 identical red cubes and 1 blue cube there are 5 distinguishable arrangements:

RRRRB RRRBR RRBRR RBRRR BRRRR

c) If there are 3 identical red cubes and 2 identical blue cubes, the situation is more complicated. One way to proceed is to first label the cubes so they are distinguishable, and then arrange them in rows and determine how many of these arrangements appear the same if the labels are removed.

Suppose, for example, that the red cubes are labeled 1, 2, and 3 and the blue cubes are labeled 1 and 2. Then since all 5 cubes are now different, the number of distinguishable ways they can be arranged in a row is $5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$. However, if the labels are removed, these cubes are no longer all distinguishable. For example, both of the arrangements R1 R3 B2 R2 B1 and R2 R1 B1 R3 B2 become RRBRB. As a matter of fact, there are $3 \times 2 \times 1 \times 2 \times 1 = 3! \times 2! = 12$ arrangements which become RRBRB, since the first R can be labeled in 3 ways, and then the second R in 2 ways, the third R in 1 way, the first B in 2 ways, and the second B in 1 way. Thus, once the labels are removed, the 120 distinguishable arrangements form groups of 12 nondistinguishable arrangements. Hence, with the labels removed, there are only $120 \div 12$, or 10, distinguishable arrangements.

d) If there are 4 identical red cubes and 3 identical blue cubes, labeling them so all 7 cubes are distinguishable results in $7! = 5040$ different ways of arranging them in a row. However, if the labels are removed, these arrangements form groups of $4! \times 3! = 144$ nondistinguishable arrangements. Thus, with labels removed, there are only $5040 \div 144$, or 35, distinguishable arrangements.

e) Again, keep emphasis on informal observations and generalizations. Depending on your students' readiness, you might show them the following formulas which are typically used by mathematicians, and ask the groups to develop arguments to show: why the formulas work; what relationships led the mathematicians to use addition, subtraction, multilplication or division as they did; etc.

In general, if r identical red cubes and k identical blue cubes are arranged in a row, the number of distinguishable arrangements is $\frac{(r+k)!}{r!k!}$.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

9 Read aloud the following situation:

Seven students are on the school Science Olympics team. Four of these students will represent the school at the state competition.

Ask the groups to explore a) below. Discuss their solutions and reasoning. Repeat for b).

a) How many different teams of 4 students are possible?

b) Suppose that, as the 4 students are selected, they are assigned 1st, 2nd, 3rd, and 4th positions on the team, to indicate the order in which they will answer questions at the state competition. Under these conditions, in how many different ways can the team of 4 students be formed?

Discuss the students' approaches and solutions.

COMMENTS

8 (continued.)
Stated differently, if there are n total items made up of 2 groups of k identical and $(n - k)$ identical items, there are $\frac{n!}{k!(n-k)!}$ different arrangements of the items.

9 a) Some students may solve this problem by listing combinations. The beginning of such a list is shown below. The 7 names are hypothetical student team members. The underlines indicate the 4 students being selected from each set of 7 students.

Jo	Lu	Sue	Tim	Ke	Ray	Fey
Jo	Lu	Sue	Tim	Ke	Ray	Fey
Jo	Lu	Sue	Tim	Ke	Ray	Fey
Jo	Lu	Sue	Tim	Ke	Ray	Fey

Such a list produces 35 teams of 4 students.

Other students may view Problem a) as a variation of Problem d) from Action 8 in which 4 red and 3 blue cubes are placed in a row: suppose that each student is identified with 1 of 7 positions in a row and that placing a red cube in a position means the student in that position is selected to represent the school; placing a blue cube means the student is not selected. For example, if the students are positioned as above, then the arrangement RBBRRBR means Jo, Tim, Ke, and Fey are selected. The number of different ways a team of 4 can be chosen to represent the school is then the number of distinguishable ways 4 identical red cubes and 3 identical blue cubes can be placed in a row, which, in Comment 8d), is determined to be 35.

b) This problem can be solved by counting the number of ways the students can be selected and ordered, as contrasted to the above solution in which the representatives are chosen but not ordered. There are 7 ways of selecting a student to be the 1st to answer a question; then there are 6 ways of selecting a student to be 2nd, then 5 ways to be 3rd, and 4 ways to be 4th. Hence, there are $7 \times 6 \times 5 \times 4 = 840$ ways of selecting and ordering 4 students.

Another strategy for solving Problem b) is to note that, since there are 35 different combinations of 4 students, and for any group of 4 students, there are 4! ways the students can be ordered 1st, 2nd, 3rd, and 4th, then there are $35 \times 4! = 35 \times 24 = 840$ ways of selecting and ordering 4 students from a group of 7 students.

Focus Teacher Activity (cont.)

ACTIONS

10 Give a copy of Focus Student Activity 16.3 to each student and ask the groups to complete Problem 1. Discuss their results and reasoning. Then repeat for 4 or more of the remaining problems.

Counting and Probability Diagrams Lesson **16**

Focus Student Activity 16.3

NAME _____ DATE _____

For each of the following problems, communicate your methods and reasoning so that it is clear how and why your methods and answers work.

1 How many different code words with 11 letters can be formed by using the letters in "achievement"?

2 A coin is tossed 5 times and the sequence of heads and tails is recorded for each toss.

a) How many different sequences are possible?

b) In how many ways are 3 heads and 2 tails possible?

3 The 5 starting players of a basketball team are to be introduced before the game. In how many orders can they be introduced?

4 One method of signaling on boats is to arrange 3 flags of different colors vertically on a flagpole. If a boat has flags of 6 different colors, how many different 3-flag signals are possible?

5 For the Lincoln High School jazz band concert, the jazz band will play 5 traditional jazz compositions and 3 original compositions. If the concert begins with any 1 of the traditional compositions, in how many ways can the 8 compositions be arranged on the program?

6 From a group of 12 smokers, a researcher wants to randomly select 8 people for a study. How many different combinations of 8 smokers are possible?

(Continued on back.)

COMMENTS

Problems a) and b) illustrate 2 different types of counting problems: *combinations* and *permutations*. Problem a) is an example of a combination—the students are *not ordered* and Problem b) is an example of a permutation—the students are *ordered*. The 2 problems illustrate the difference in counting the ways in which a selection can be made if, on the one hand, ordering the selection is immaterial and, on the other hand, ordering the selection must be taken into account.

10 All or part of these problems could be assigned as homework. However, since some of these problems may be particularly challenging to some students, you might have the students each select only 5-7 problems to solve, and have them classify the problems by difficulty. To aid in problem solving, you might encourage students to: create similar problems involving smaller numbers; ask the class for clues (not answers); and/or share their "stuck points" with each other.

Following are solutions to the problems on Focus Student Activity 16.3. Note that it is not expected that students use formulas to solve these problems; however, some students may do so, and some may invent formulas as they work on these problems.

1) Students may view this situation by imagining 11 colored cubes—3 of one color (representing the e's in the word *achievement*) and 8 each of different colors (representing the remaining letters). There are 11! ways of arranging 11 different colored cubes. However, the 3 identically colored cubes are nondistinguishable, and there are 3! ways of arranging those 3 cubes; thus, the 11! arrangements of 11 cubes form groups of 3! nondistinguishable arrangements. Hence, there are $^{11!}/_{3!}$ $= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 6{,}652{,}800$ distinguishable arrangements of the letters in *achievement*.

2a) Students may model this using (or sketching) rows of 5 coins to represent the possible sequences of outcomes from 5 tosses of a coin. Or, they may represent the possible sequences of tosses using red cubes to represent heads and blue cubes to represent tails. Since there are 2 possibilities for each of the 5 tosses, there are $2 \times 2 \times 2 \times 2 \times 2 = 32$ different sequences of red and/or blue cubes.

2b) If all of the cubes were different colored, there would be 5! different arrangements of 5 cubes. However, since the 3 red cubes (3 heads) are nondistinguishable, and

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **16** Counting and Probability Diagrams

Focus Student Activity 16.3 (cont.)

7 A club of 7 students decides to send 2 of their members to the principal to request a field trip to an art museum.

a) In how many ways can the 2 students be chosen?

b) If the president of the club is to be one of the students, in how many ways can the group of 2 be chosen?

8 A state lottery requires that you pick 6 different numbers from 1 to 99. If you pick all 6 winning numbers, you win $1,000,000.

a) How many ways are there to choose 6 numbers if the order of the numbers is not important? For example, 1,2,3,4,5,6 is the same combination as 2,1,3,4,5,6.

b) How many ways are there to choose 6 numbers if order is important?

9 Two children are allowed to select from 7 baseball cards. The younger child is to select first and can select 2 cards. Then the older child is to select 3 cards from the remaining 5. The older child complains that this is not fair and says that the younger child has more choices. What do you think?

COMMENTS

10 (continued.)
the 2 blue cubes are nondistinguishable, the 5! arrangements form 3!2! nondistinguishable groups. Hence, there are $^{5!}/_{3!2!}$ = 10 distinguishable sequences of 3 heads and 2 tails.

3) This solution could be modeled using a different colored cube to represent each player and determining the number of different arrangements of the 5 different colored cubes in a row. Since there are 5 choices for the 1st position, 4 choices for the 2nd, 3 for the 3rd, 2 for the 4th, and 1 for the 5th, there are $5 \times 4 \times 3 \times 2 \times 1$ = 120 arrangements. This product is a result of the Fundamental Counting Principal (see Connector activity).

4) Any one of the 6 colors may be placed in the top position; then one of the 5 remaining colors in the 2nd position; and finally any one of the 4 remaining colors in the lower position. Hence, there are $6 \times 5 \times 4 = 120$ possible arrangements.

5) Since the 1st piece is traditional, there are 4 traditional and 3 original pieces to be placed in the remaining 7 spots of the program. These can be represented by 4 red cubes and 3 blue cubes. Since there are 7! ways of arranging 7 distinguishable cubes (i.e., if the cubes were all of different colors) and there are 4! nondistinguishable arrangements of the 4 red cubes and 3! nondistinguishable arrangements of the 3 blue cubes, there are $^{7!}/_{4!3!}$ = 35 distinguishable arrangements of traditional and original pieces. Note: this assumes there is no particular order required for the traditional or the original pieces.

6) In terms of colored cubes, consider the number of distinguishable arrangements of 12 cubes that can be formed using 8 red cubes for the 8 chosen smokers, and 4 blue cubes for the 4 smokers not chosen. Since the 8 red cubes are nondistinguishable and the 4 blue cubes are nondistinguishable, the number of distinguishable arrangements is $^{12!}/_{8!4!}$ = 495. Note: this assumes the order in which the smokers are selected does not matter.

7a) Since there are 7 ways the first student can be chosen and 6 ways the 2nd student can be chosen, there are $7 \times 6 = 42$ ways of choosing 2 students. However, since the order of the students is not important (i.e., choosing Student A and then Student B forms the same pair of students as choosing Student B and then Student A), there are $^{(7 \times 6)}/_{2}$ different combinations of 2 students. Notice this is equivalent to $^{7!}/_{2!5!}$, or the number of ways

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

of arranging 7 items in a row with only 2 positions in the row, where the order of items does not matter.

7b) If the president is to be one of the 2 students, then there are only 6 ways to choose the other member. So, there are only 6 different ways of choosing 2 students.

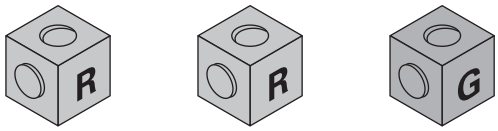
8a) If order is not important there are $^{99!}/_{6!93!}$ = 1,120,529,256 different ways to choose 6 numbers.

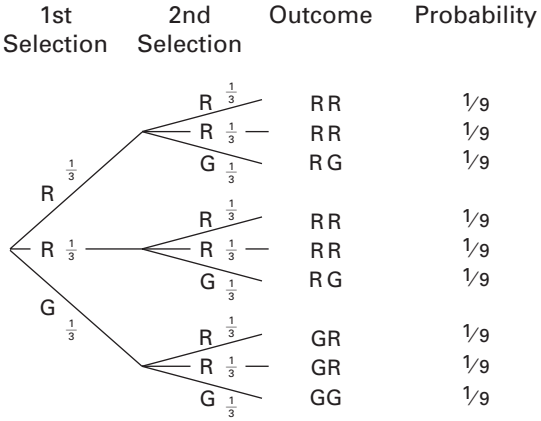
8b) If order is important there are $99 \times 98 \times 97 \times 96 \times 95 \times 94$ different ways to choose 6 numbers.

9) The younger child has $^{7!}/_{2!5!}$ = 21 different pairs as possible choices. From the remaining 5 cards, the older child has $^{5!}/_{3!2!}$ = 10 different sets of cards as choices. So, the younger child has more choices but the older child will have 3 cards compared to the 2 for the younger child.

11 Pose the following problem for groups to solve. Discuss the groups' reasoning and conclusions. Introduce the use of *probability trees* and *probability rectangles*.

> **Two red cubes and 1 green cube are placed in a sack. One cube is randomly selected, its color is recorded, and it is returned to the sack. Then 1 cube is selected again and its color is recorded. What is the probability of getting 2 red cubes?**





11 Some students may suggest computing the experimental probability by randomly selecting from sacks containing 1 green and 2 red cubes. You might have sacks available in the event the class decides to conduct an experiment as the basis for computing the probability.

One erroneous method of reasoning theoretically that may come up is to list RR, GR, RG, and GG as the 4 possible outcomes for the 2 choices and assume that the probability of each is equally likely. This leads to the incorrect conclusion that the probability of getting 2 red cubes is $^1/_4$. However, since the cube is returned to the sack after the first draw, there are 3 possibilities for the 1st draw and 3 possibilities for the 2nd draw. Hence, there are $3 \times 3 = 9$ equally likely outcomes. Since there are 4 of these outcomes in which both cubes are red (see diagram at the left), the probability of getting 2 red cubes is $^4/_9$. Students may determine this by labeling cubes and forming all possible pairs.

A *probability tree* can be sketched, as shown at the left, to list the theoretical outcomes. The probability of each outcome is written on each branch of the tree. The 9 outcomes show there are 4 in which both cubes are red. So, the probability of choosing 2 red cubes is $^4/_9$. Notice that the probability at the end of each path of the tree is the product of the probabilities in the branches which comprise that path. The sum of the probabilities for all 9 paths is 1.

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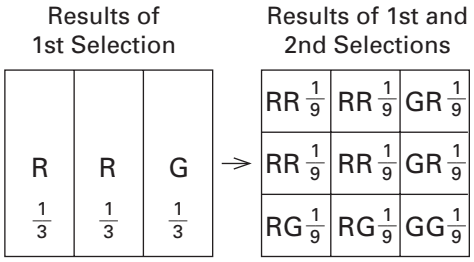
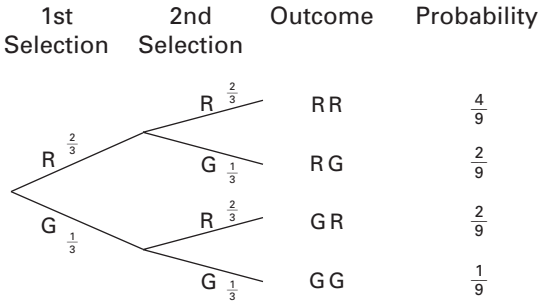
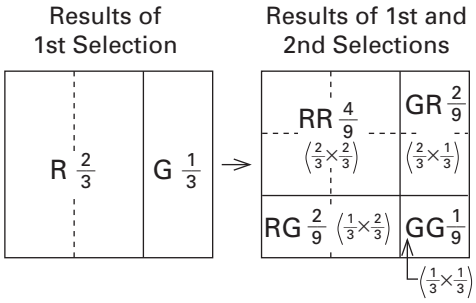
Focus Teacher Activity (cont.)

ACTIONS







12 Ask the students to determine the probability of selecting 2 red cubes from a sack that contains 2 red and 1 green, if the 1st cube selected is *not* replaced for the 2nd selection. Discuss the students conclusions and reasoning.

COMMENTS

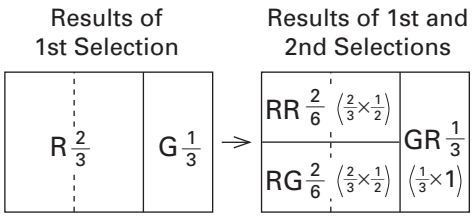
11 (continued.)
Similarly, a *probability rectangle*, which is a rectangle diagram whose parts are labeled with the probabilities associated with a given situation, shows that there are 4 parts with RR (2 red cubes) and each part occurs with probability 1/9, as shown at the left. So, the probability of 2 red cubes is 4/9. Notice that this probability rectangle contains 9 parts of equal area and provides an area model for probability.

Some students may reason that there is a 2/3 chance of obtaining a red cube on each selection, as illustrated by the probability tree shown at the left, so the probability of selecting 2 red cubes is $2/3 \times 2/3 = 4/9$.

A probability rectangle based on this line of reasoning has parts of different sizes, as shown at the left. In the first rectangle, the part representing R has an area that is 2/3 the area of the rectangle and the part representing G is 1/3 of the area. These areas correspond to the 2/3 probability of a red cube on the first selection and a 1/3 probability of a green cube on the first selection. The second rectangle has 4 parts and the area of the part representing 2 red cubes is $2/3 \times 2/3 = 4/9$, which is the probability of 2 red cubes in the 2 selections.

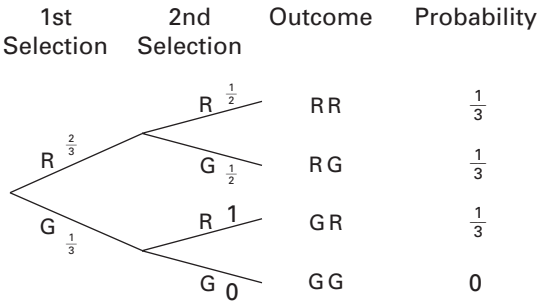
12 In this case, a cube is selected, the color recorded, and then, without replacing the first cube, another cube is selected and its color recorded.

As in Action 11, there is a 2/3 chance of obtaining a red cube on the 1st selection. However, if a red cube is chosen on the 1st selection, and not replaced, then on the 2nd selection there is only a 1/2 chance of obtaining a red cube. So, the probability of selecting 2 red cubes is $2/3 \times 1/2 = 1/3$. This is illustrated by the following probability rectangle and probability tree.



Focus Teacher Activity (cont.)

ACTIONS



13 Give each student a copy of Focus Student Activity 16.4 and ask the groups to complete Problems 1 and 2. Discuss. Then repeat for Problems 3-7 (or selected problems).

Counting and Probability Diagrams Lesson **16**

Focus Student Activity 16.4

NAME _____ DATE _____

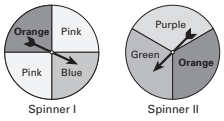
Show your methods and reasoning for each problem.

1 A sack contains 2 blue cubes, 3 yellow cubes, and 5 red cubes. If 1 cube is randomly se1lected and then placed back in the sack and a 2nd cube is selected, what is the probability that both of the cubes will be yellow?

2 Solve Problem 1, but this time assume that the 1st cube selected is not placed back in the sack.

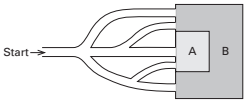
3 There are 2 sacks with cubes and 1 cube will be randomly selected out of each. Sack 1 contains 3 red cubes and 2 blue, and Sack 2 contains 2 yellow, 1 green, and 2 red. What is the probability of selecting 2 red cubes? What is the probability of not selecting 2 red cubes? What is the probability of not selecting any red cubes?

4 The 2 spinners shown at the right are each to be spun once. Determine the probabilities of obtaining the following colors: orange on both spins; pink on one spin and green on the other; blue on one spin and either purple or green on the other.



5 A family has 4 children. Assuming that the chances of having a boy or a girl on each birth are equally likely, determine the probability that the children were born in *either* the order boy, girl, boy, girl *or* the order girl, boy, girl, boy.

6 Assuming that each branch point in the maze at the right is equally likely to be chosen, determine the probability of entering room A.



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Lesson **16** Counting and Probability Diagrams

Focus Student Activity 16.4 (cont.)

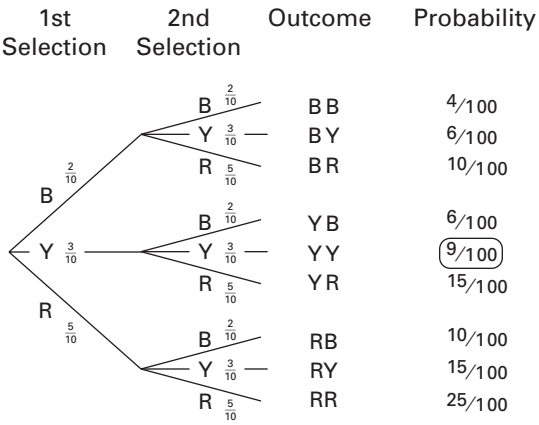
7 A game is to be played in which 1 player holds a sack that contains 5 blue cubes and 5 yellow cubes, and a 2nd player selects 2 cubes from the sack. The player who selects wins the game if 2 cubes of the same color are obtained, and otherwise the player holding the sack wins. Which player has the better chance of winning and what is this player's probability of winning?

COMMENTS

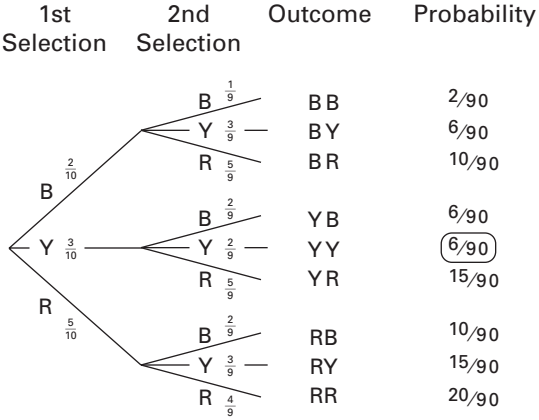
The lower branch of the probability tree shown at the left may need some discussion. If a green cube is chosen on the 1st selection, then the probability of obtaining a red cube on the 2nd selection is 1, because this is certain to happen, and the probability of obtaining a green on the 2nd selection is 0, because this cannot happen. Notice that the GG outcome is not shown in the probability rectangle since it corresponds to an area of zero.

13 Answers and possibilities for diagrams are shown below.

1) $3/10 \times 3/10 = 9/100$. In the following probability tree, B, Y, and R represent blue, yellow, and red, respectively.



2) $3/10 \times 2/9 = 6/90 = 1/15$



3) The probability of selecting 2 red cubes is $6/25$. So, the probability of not selecting 2 red cubes is $1 - 6/25 = 19/25$ which can also be determined by adding the appropriate probabilities from the tree below: $6/25 + 3/25 + 4/25 + 2/25 + 4/25$.

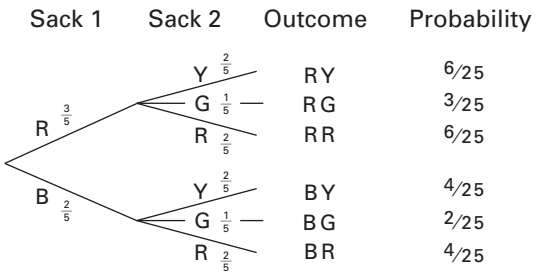
(Continued next page.)

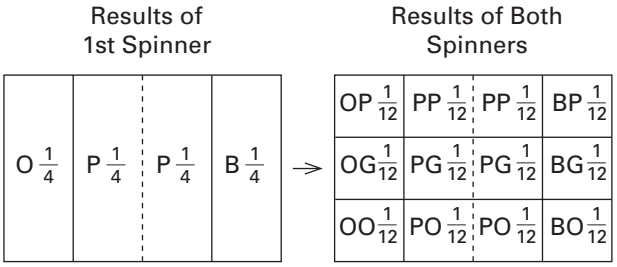
Focus Teacher Activity (cont.)

ACTIONS

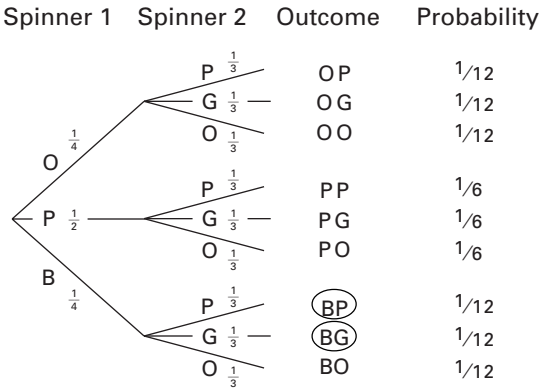
COMMENTS

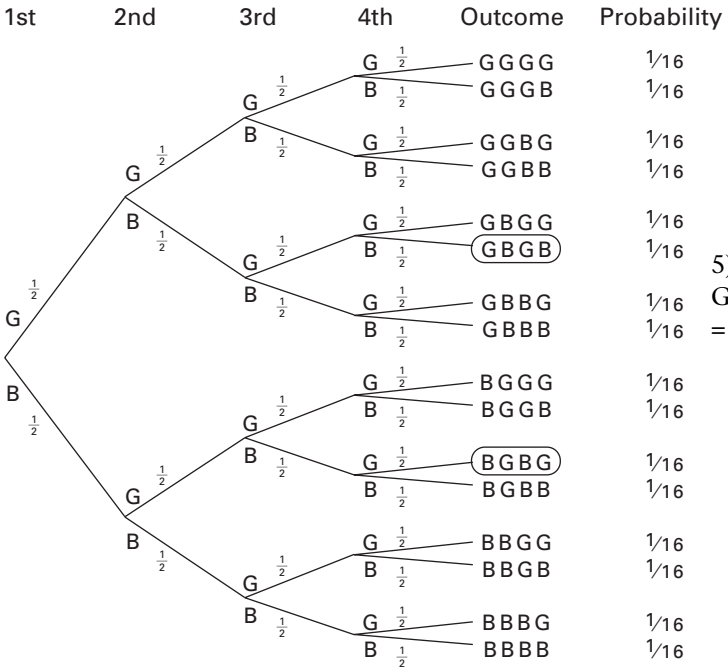
13 (continued.)
The probability of not selecting any red cubes is $\frac{4}{25} + \frac{2}{25} = \frac{6}{25}$, as also can be seen in the following probability tree.





4) The letters in the probability rectangle at the left and the probability tree shown below represent the spinner colors whose names begin with those letters. The probabilities are: orange on both spins, $\frac{1}{12}$; pink on one spinner and green on the other, $\frac{1}{6}$; blue on one and purple or green on the other, $\frac{1}{12} + \frac{1}{12} = \frac{1}{6}$. Notice that the outcomes for the condition "blue on one and purple or green on the other" have been circled on the probability tree.





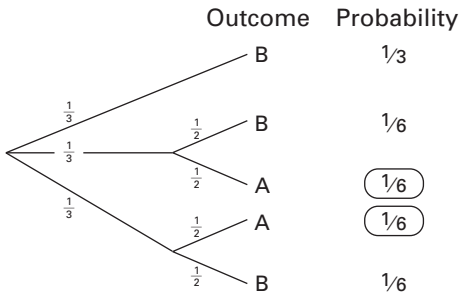
5) The probability of BGBG is $\frac{1}{16}$ and the probability of GBGB is $\frac{1}{16}$. So, the probability of either order is $\frac{1}{16} + \frac{1}{16} = \frac{2}{16} = \frac{1}{8}$, as shown by the probability tree at the left.

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

6) The probability tree below shows that the probability of entering room A is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.



Some students may use a probability rectangle to determine the probability of entering room A. In this case, since each of the 3 paths is equally likely to be chosen at the beginning, the rectangle is first divided into 3 equal parts, as shown in Figure I below. Then, in Figure II the upper path remains $\frac{1}{3}$ of the rectangle since it goes directly to room B, while for the middle path there are 2 equally likely choices: go left to room B or straight to room A. Likewise, there are 2 equally likely choices for the lower path. Hence, the 3 sections formed in Figure I are subdivided as shown in Figure II. The fraction of the rectangle's area that is occupied by region A is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ and this is the probability of entering room A.

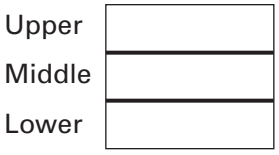


Figure I

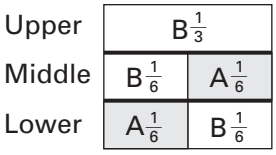
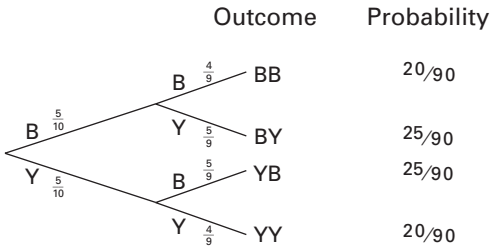


Figure II

7) The player who selects cubes has a $\frac{40}{90}$ or $\frac{4}{9}$ chance of obtaining 2 cubes of the same color, so the player who holds the sack has $\frac{5}{9}$ chance of winning, as shown in the following probability tree.



Alternatively, the methods of Action 6 can be used to determine the outcomes. The number of ways of selecting 2 blue cubes from a group of 5 blue cubes is $\frac{5!}{2!3!} = 10$.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS	COMMENTS
	13 (continued.) Similarly, the number of ways of selecting 2 yellow cubes from a group of 5 yellow cubes is 10. Hence, the number of ways of selecting 2 cubes of the same color is 20. Since the total number of ways of selecting 2 objects from a set of 10 objects is $^{10!}/_{2!8!} = 45$, the probability of selecting 2 cubes of the same color is $^{20}/_{45} = {}^{4}/_{9}$.

Follow-up Student Activity 16.5

NAME _____ **DATE** _____

1 For each of the following problems, communicate your methods and reasoning so that it is clear how and why your methods and answers work.

a) On a sandwich menu, there are 5 choices of bread (French, sourdough, onion, rye, or wheat) and 4 choices of filling (turkey, tuna, egg salad, or beef). How many different sandwiches are possible from these choices?

b) An art gallery has 5 special paintings to display, but space to hang only 4 of them in a row. How many different arrangements of paintings are possible in this row?

c) An area code is a 3-digit number where the 1st digit cannot be 0 or 1. How many different area codes are possible?

d) Five houses in a row are each to be painted with 1 of the colors red, green, blue, or yellow. If no 2 adjacent houses can have the same color, in how many ways can the houses be painted?

e) A little league team is to be formed from 9 children of whom only 1 can be the catcher, only 1 can play 1st base, and only 1 can be the pitcher. The other players can play any of the remaining 6 positions. How many different lineups are possible?

f) A girl dresses each day in a blouse, a skirt, and shoes. She always wears white socks. She wants to wear a different combination on every day of the year. If she has the same number of blouses, skirts, and pairs of shoes, how many of each would she need to have a different combination every day?

g) How many 5-digit ZIP codes are possible in which the ZIP code is an even number?

h) A Girl Scout troop has 12 members. In how many different ways can the scoutmaster appoint 4 members to clean up the camp?

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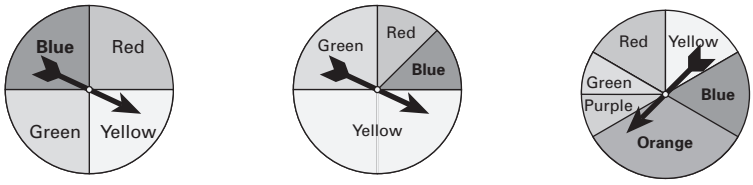
Follow-up Student Activity (cont.)

2 Use these spinners to determine the probabilities of the outcomes listed below.



Spinner A Spinner B Spinner C

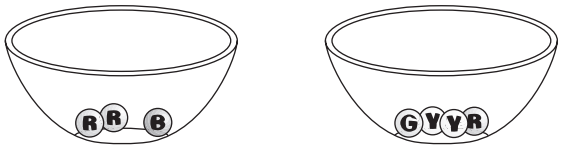
a) blue from Spinner A and red from Spinner B,

b) blue from Spinner B and green from Spinner C,

c) yellow from Spinner A, green from Spinner B, and red from Spinner C,

d) either green or red from Spinner B and either red or yellow from Spinner C.

3 Use these 2 bowls of marbles to determine the probabilities of selecting the following:



Bowl I Bowl II

a) a red from I and a red from II,

b) a blue from I and a yellow from II,

c) 1 marble from each bowl and no red marble,

d) 1 marble from each bowl, no red or green marbles.

4 A gumball machine contains 2 red and 4 white gumballs and no others. If each gumball has an equal chance of being released and 2 are purchased, what is the probability of getting 2 red gumballs?

5 A coin will be tossed 4 times. What is the probability of obtaining 2 or more heads in a row?

6 Write a letter to an adult explaining the "big ideas" from this lesson about counting concepts and strategies and their role in computing probabilities. Be detailed and include examples that help the adult see why ideas work. Next, have an adult read your letter and write comments about what is clear and unclear. Then edit your letter to clarify as needed. Turn in the original letter with the adult's comments and turn in your edited version.

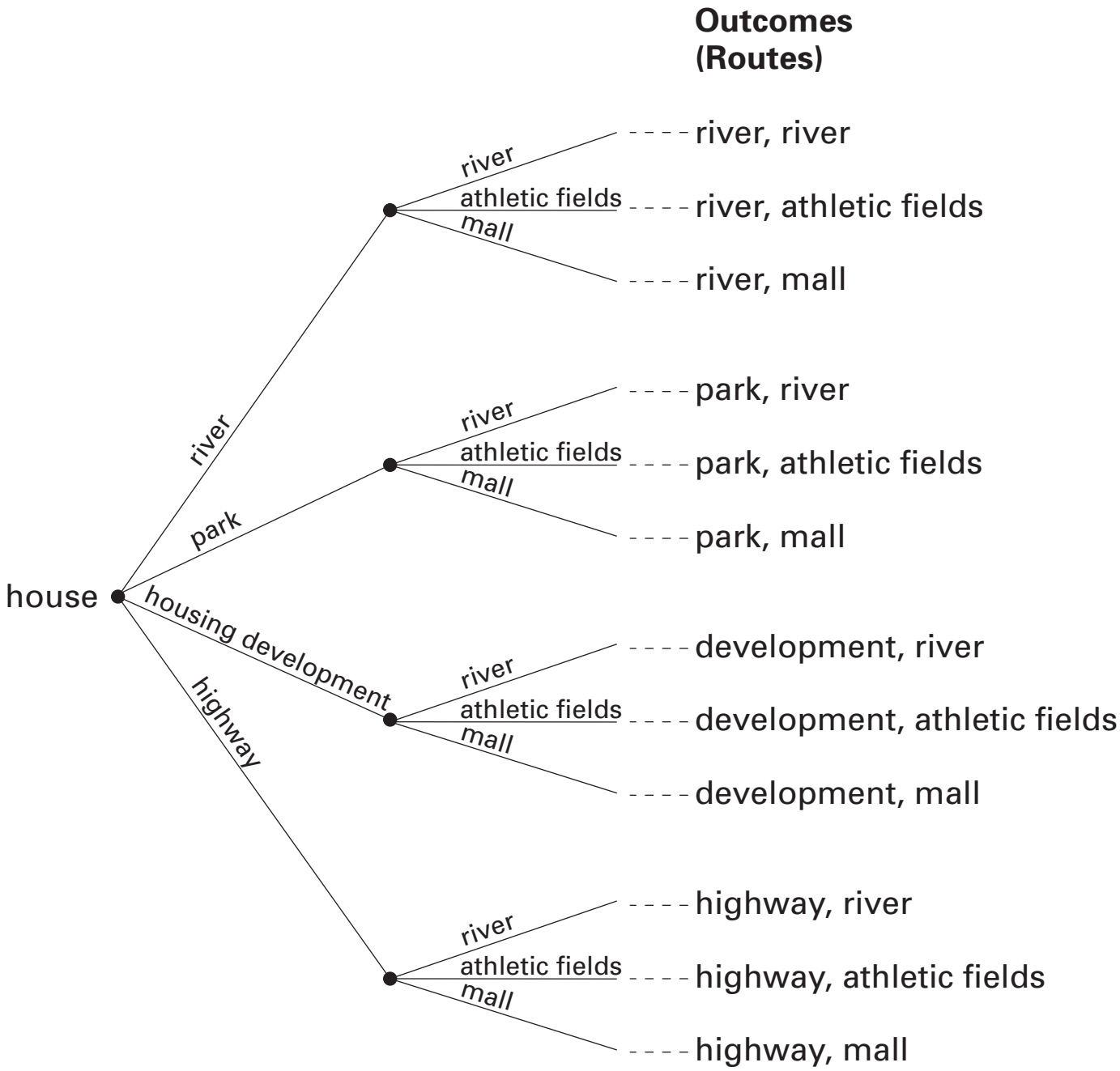
Connector Master A

Marcia's Routes to School

Each morning before school Marcia walks to the post office to mail letters for her parents, and then she continues on to school. Marcia has 4 different routes she can walk from her home to the post office: along the river; through the park; by the new housing development; or on the highway. When she leaves the post office, there are 3 different routes that she can walk to school; along the river; across the athletic fields; or by the mall. How many different routes can Marcia take on her trips to school?

Connector Master B

Tree Diagram



Connector Master C

Routes from home to post office

R	P	D	H

→

Routes from home via post office to school

RR	PR	DR	HR
RF	PF	DF	HF
RM	PM	DM	HM

Connector Student Activity 16.1

NAME _____ **DATE** _____

Solve each problem and show a diagram or write a brief explanation of your reasoning.

1 A house has 3 doors to the outside and 9 windows. In how many ways can a burglar enter the house through a window and leave through a door?

2 A car manufacturer provides 6 different exterior colors, 5 interior colors, and 3 different trims. How many different exterior color/ interior color/trim schemes are available?

3 For breakfast Henry always chooses one of the following drinks: orange juice, tomato juice, or apple juice. For cereal he chooses either corn flakes or bran flakes. For pastry he has either a doughnut, a glazed muffin, or an apple tart. What are all the different possibilities for Henry's breakfasts if he always has cereal, a drink, and a pastry?

4 A teenager posed the following problem: How many different outfits can she wear if she has 2 skirts with different patterns, 3 different colored blouses and 2 different types of shoes?

5 Each student at Athey Creek Middle School has a 4-digit locker number. If the 1st digit of each 7th grader's locker number is a 7 and each of the other 3 digits is one of the digits from 0 through 9, how many different 4-digit numbers are available for the 7th graders? Note: digits can be repeated any number of times in a locker number.

6 A school cafeteria offers a selection of 2 types of meat, 3 types of fish, 4 different vegetables, and 3 desserts. In how many ways can a student select 1 vegetable, 1 dessert, and either 1 type of meat or 1 type of fish?

(Continued on back.)

Connector Student Activity 16.1 (cont.)

7 A building developer may choose from 4 different roofing subcontractors, 5 different electrical subcontractors, 3 different plumbing subcontractors, 2 different carpenters, and 6 different painters. In how many ways can he select 1 of each?

8 The Tigers and the Wildcats are to play a 3-game series in soccer. The first team to win 2 games wins the series. Find all the possible outcomes for the 3 games.

9 The annual Fourth of July raft race in Centerville has 7 entries. If a different prize is offered for each of the following categories; fastest raft, slowest raft, and most original raft, in how many ways is it possible to award the prizes?

10 A store carries 5 styles of backpacks in 4 different sizes. The customer also has a choice of 2 different kinds of material for 3 of the styles, and only 1 material for the other styles. How many different types of backpacks are there to choose from? If Larry is only interested in the 2 largest backpacks, how many different backpacks does he have to choose from?

Focus Student Activity 16.2

NAME _____ **DATE** _____

Show how to use a diagram or model to solve each of the following problems. Note: It is okay to generalize from a partial diagram or model without showing all the parts. If you do so, be sure to show your reasoning.

1 In how many ways can 6 people line up to purchase tickets for a concert?

2 In how many ways can 8 students be seated in a row for a school photo?

3 A sportswriter makes a preseason prediction about the order of the top 5 teams from among 20 teams in the Women's National Basketball Association. How many different possibilities are there?

4 An agricultural scientist wants to test different combinations of 4 types of soybeans with 7 types of fertilizer and 2 types of insecticides. How many experimental plots are needed?

5 A club of 10 people plans to elect 3 people for the offices of president, vice president, and secretary. In how many ways can these 3 offices be filled?

6 Seven cross-country runners are competing for 1st, 2nd, 3rd, and 4th places. How many different possible outcomes are there for these 4 places?

7 How many 3-letter code words can be formed from the alphabet if the code word can not contain a repeated letter?

(Continued on back.)

Focus Student Activity 16.2 (cont.)

8 A contractor wishes to paint 6 houses in a row each with 1 of the colors red, blue, green, and yellow, with the requirement that if 2 houses are side by side they are to have different colors. In how many ways can the houses be painted?

9 Eight members of a fire department are being considered for 3 special awards. In how many ways can 3 of 8 people be selected if no one wins more than 1 award?

10 A school has 683 students. Are there *possibly*/*definitely*/*definitely not* (circle one) any students who have the same pair of initials for their first and last names?

Focus Student Activity 16.3

NAME _____ **DATE** _____

For each of the following problems, communicate your methods and reasoning so that it is clear how and why your methods and answers work.

1 How many different code words with 11 letters can be formed by using the letters in "achievement"?

2 A coin is tossed 5 times and the sequence of heads and tails is recorded for each toss.

a) How many different sequences are possible?

b) In how many ways are 3 heads and 2 tails possible?

3 The 5 starting players of a basketball team are to be introduced before the game. In how many orders can they be introduced?

4 One method of signaling on boats is to arrange 3 flags of different colors vertically on a flagpole. If a boat has flags of 6 different colors, how many different 3-flag signals are possible?

5 For the Lincoln High School jazz band concert, the jazz band will play 5 traditional jazz compositions and 3 original compositions. If the concert begins with any 1 of the traditional compositions, in how many ways can the 8 compositions be arranged on the program?

6 From a group of 12 smokers, a researcher wants to randomly select 8 people for a study. How many different combinations of 8 smokers are possible?

(Continued on back.)

Focus Student Activity 16.3 (cont.)

7 A club of 7 students decides to send 2 of their members to the principal to request a field trip to an art museum.

a) In how many ways can the 2 students be chosen?

b) If the president of the club is to be one of the students, in how many ways can the group of 2 be chosen?

8 A state lottery requires that you pick 6 different numbers from 1 to 99. If you pick all 6 winning numbers, you win $1,000,000.

a) How many ways are there to choose 6 numbers if the order of the numbers is not important? For example, 1,2,3,4,5,6 is the same combination as 2,1,3,4,5,6.

b) How many ways are there to choose 6 numbers if order is important?

9 Two children are allowed to select from 7 baseball cards. The younger child is to select first and can select 2 cards. Then the older child is to select 3 cards from the remaining 5. The older child complains that this is not fair and says that the younger child has more choices. What do you think?

Focus Student Activity 16.4

NAME _____ **DATE** _____

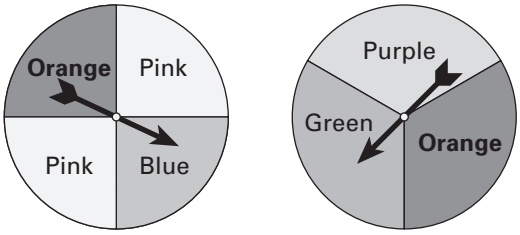
Show your methods and reasoning for each problem.

1 A sack contains 2 blue cubes, 3 yellow cubes, and 5 red cubes. If 1 cube is randomly se1lected and then placed back in the sack and a 2nd cube is selected, what is the probability that both of the cubes will be yellow?

2 Solve Problem 1, but this time assume that the 1st cube selected is not placed back in the sack.

3 There are 2 sacks with cubes and 1 cube will be randomly selected out of each. Sack 1 contains 3 red cubes and 2 blue, and Sack 2 contains 2 yellow, 1 green, and 2 red. What is the probability of selecting 2 red cubes? What is the probability of not selecting 2 red cubes? What is the probability of not selecting any red cubes?

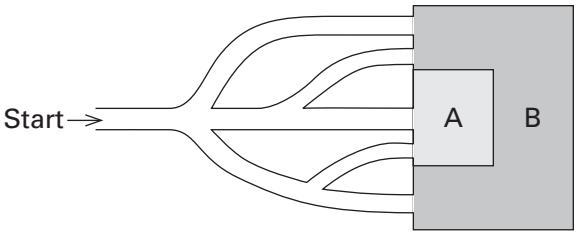
4 The 2 spinners shown at the right are each to be spun once. Determine the probabilities of obtaining the following colors: orange on both spins; pink on one spin and green on the other; blue on one spin and either purple or green on the other.



Spinner I Spinner II

5 A family has 4 children. Assuming that the chances of having a boy or a girl on each birth are equally likely, determine the probability that the children were born in *either* the order boy, girl, boy, girl *or* the order girl, boy, girl, boy.

6 Assuming that each branch point in the maze at the right is equally likely to be chosen, determine the probability of entering room A.



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Focus Student Activity 16.4 (cont.)

7 A game is to be played in which 1 player holds a sack that contains 5 blue cubes and 5 yellow cubes, and a 2nd player selects 2 cubes from the sack. The player who selects wins the game if 2 cubes of the same color are obtained, and otherwise the player holding the sack wins. Which player has the better chance of winning and what is this player's probability of winning?

Follow-up Student Activity 16.5

NAME _____ **DATE** _____

1 For each of the following problems, communicate your methods and reasoning so that it is clear how and why your methods and answers work.

a) On a sandwich menu, there are 5 choices of bread (French, sourdough, onion, rye, or wheat) and 4 choices of filling (turkey, tuna, egg salad, or beef). How many different sandwiches are possible from these choices?

b) An art gallery has 5 special paintings to display, but space to hang only 4 of them in a row. How many different arrangements of paintings are possible in this row?

c) An area code is a 3-digit number where the 1st digit cannot be 0 or 1. How many different area codes are possible?

d) Five houses in a row are each to be painted with 1 of the colors red, green, blue, or yellow. If no 2 adjacent houses can have the same color, in how many ways can the houses be painted?

e) A little league team is to be formed from 9 children of whom only 1 can be the catcher, only 1 can play 1st base, and only 1 can be the pitcher. The other players can play any of the remaining 6 positions. How many different lineups are possible?

f) A girl dresses each day in a blouse, a skirt, and shoes. She always wears white socks. She wants to wear a different combination on every day of the year. If she has the same number of blouses, skirts, and pairs of shoes, how many of each would she need to have a different combination every day?

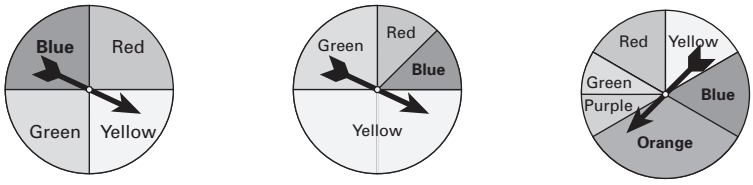
g) How many 5-digit ZIP codes are possible in which the ZIP code is an even number?

h) A Girl Scout troop has 12 members. In how many different ways can the scoutmaster appoint 4 members to clean up the camp?

(Continued on back.)

Follow-up Student Activity (cont.)

2 Use these spinners to determine the probabilities of the outcomes listed below.



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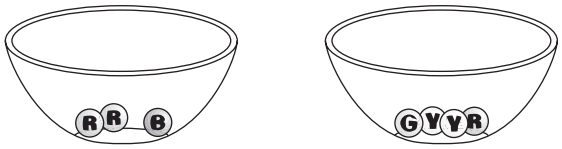

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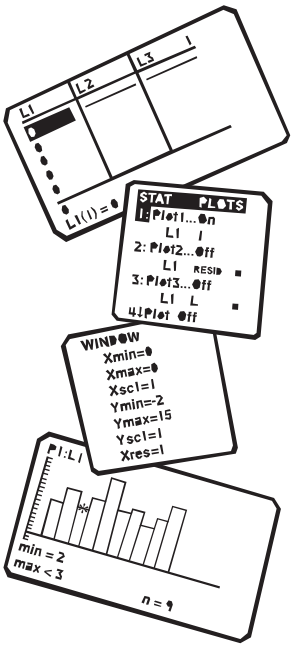
Simulations & Probability

Lesson 17

THE BIG IDEA

Because of their mathematical content or complexity, theoretical solutions to many probability problems can be beyond the reach of middle school students. Further, experimental solutions for many problems are impractical to solve by carrying out the actual conditions of the problems. Frequently simulations make such problems accessible, produce meaningful experimental solutions, and/or lend insights regarding theoretical possibilities.



CONNECTOR

OVERVIEW

Students discuss the idea of randomness and investigate ways to create sets of random numbers. Students use a table of random numbers, calculators, and/or computers to generate random numbers satisfying certain conditions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Masters A and B, 1 copy of each per student and 1 transparency of each.
- ✔ Connector Student Activity 17.1, 1 copy per group and 1 transparency.
- ✔ Graphing calculators, 1 per student.
- ✔ Overhead graphing calculator (optional).
- ✔ Miscellaneous materials (compasses, protractors, bobby pins, paper clips, dice, paper bags or other containers, cubes, tile, game markers, calculators, grid paper, straightedges, marking pens, etc.), accessible to groups as needed.

FOCUS

OVERVIEW

Students identify key components of simulations, and they design and carry out simulations to solve probability problems whose theoretical solutions are complex and/or out of reach for this grade level. The graphing calculator serves as an efficient tool for generating and representing data. Central tendency, range, variation, and confidence intervals provide bases for predictions.

MATERIALS FOR TEACHER ACTIVITY

- ✔ Connector Master A, 1 copy per group and 1 transparency.
- ✔ Focus Masters A and C, 1 transparency of each.
- ✔ Focus Master B, 1 copy per student and 1 transparency.
- ✔ Focus Student Activities 17.2 and 17.3, 1 copy of each per group and 1 transparency.
- ✔ Assorted materials (e.g., protractors, compasses, bobby pins, straightedges, dice, game markers, cubes, tile, grid paper, etc.) available to groups.
- ✔ Graphing calculators, 1 per student.
- ✔ Overhead graphing calculator (optional).
- ✔ ¼" grid paper, 1 transparency.
- ✔ Blank transparencies and overhead pens, 2 per group.

FOLLOW-UP

OVERVIEW

Students generate sets of random numbers that satisfy certain conditions. They design and carry out simulations to solve problems, and they write a letter outlining the important characteristics of a simulation.

MATERIALS FOR STUDENT ACTIVITY

- ✔ Student Activity 17.4, 1 copy per student.

LESSON IDEAS

QUOTE

It is also important for students to understand the difference between, and the advantages associated with, theoretical and simulation techniques. Even more important, students should value both approaches. … What should not be taught is that only the theoretical approach yields the "right" solution.

NCTM Standards

TEACHER NOTES:

SELECTED ANSWERS

1a) Enter randInt(100,999,2) 5 times.

b) Enter randInt(1,217,8) 5 times.

c) Enter randInt(0,360,12) 3 times.

d) Enter randInt(1,100,20) 4 times. Let 1-87 represent successful free throws and 88-100 unsuccessful free throws. Multiply the number of successful free throws in a set by 5 to compute the percentage for that set.

2. Following is one possible strategy for each problem; students' approaches may vary.

a) For 1 trial, enter randInt(1,100,10). Let 1-72 represent correct predictions and 73-100 incorrect predictions. A trial is successful if 8 or more predictions are correct. The probability of making 8 or more correct predictions is the ratio of the number of successful trials to the total number of trials.

b) For 1 trial, enter randInt(1,1000,4). Let 1-293 represent "hits," and 294-1000 represent "no hits." A trial is successful if it has 2 or more hits. Determine what percentage of the total number of trials are successful.

c) Let 0 represent a boy and 1 a girl. For 1 trial, enter randInt(0,1) repeatedly until there is at least 1 boy and 1 girl. Record as data for the trial the number of entries required. Repeat many times. Form a line plot of the data and determine the maximum of the first 90% of the data from the line plot. Find the average (mean, median, or mode as deemed appropriate) of the data from the 90% interval.

d) Enter randInt(1,6,5) for 1 trial. A trial is a success if it contains 2 or more 3's. Repeat for many trials and determine the percentage of successful trials.

e) Enter randInt(1,6,2) to simulate 1 toss of 2 dice. Repeat until the sum for a toss is 10 or greater. As data for 1 trial, record the number of tosses required. Repeat for many trials and make a line plot of the data; determine the maximum of the lower 90% of the data in the line plot.

f) Let 1 and 2 represent belt buckles and 3 represent rings. For 1 trial, enter randInt(1,3,3). Repeat for many trials and determine the percentage of trials with at least one 3 and at least one 1 or 2.

g) Let 1-28 represent 1-child families, and 29-100 represent not 1-child families. For 1 trial, enter randInt(1,100) repeatedly until 2 numbers from 1-28 are obtained. Record as data the number of entries in a trial. Repeat for many trials, and make a line plot of the data. Determine the maximum of the lower 80%, 85%, and 90% of the data.

h) Let the numbers 1, 2, and 3 stand for 6th graders, 4 and 5 for 7th graders, and 6 and 7 for 8th graders. For 1 trial, enter randInt(1,7,3). A trial is successful if the 3 numbers represent 1 student from each grade.

Connector Teacher Activity

OVERVIEW & PURPOSE

Students discuss the idea of randomness and investigate ways to create sets of random numbers. Students use a table of random numbers, calculators, and/or computers to generate random numbers satisfying certain conditions.

MATERIALS

- ✔ Connector Masters A and B, 1 copy of each per student and 1 transparency of each.
- ✔ Connector Student Activity 17.1, 1 copy per group and 1 transparency.
- ✔ Graphing calculators, 1 per student.
- ✔ Overhead graphing calculator (optional).
- ✔ Miscellaneous materials (compasses, protractors, bobby pins, paper clips, dice, paper bags or other containers, cubes, tile, game markers, calculators, grid paper, straightedges, marking pens, etc.), accessible to groups as needed.

ACTIONS

1 Arrange the students in groups. Write the word *random* on the board or overhead and ask each group of students to write a statement to explain how they think about the meaning of random and ways this word is used. Discuss.

2 Write the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 on the overhead or chalkboard. Invite several volunteers to describe possible strategies for randomly selecting one of the digits. Discuss the students' ideas regarding reasons these strategies are likely to produce random selections, and/or any ways such strategies could be biased. Clarify as needed.

Then give a copy of Connector Student Activity 17.1 (see next page) to each group of students and ask them to complete Problems 1 and 2, alerting the groups to a variety of materials they may wish to use for generating random digits. Ask each group to show their line plots to the class. Discuss the students' observations.

COMMENTS

1 In Lessons 16, 24, and 25 of *Math Alive! Course II*, students discussed issues related to randomness in sampling (i.e., issues related to generating *unbiased* data samples). Here are a few ways students may suggest the word *random* is used:

- something that happens by chance;
- haphazard movement;
- something that is done without a plan or system;
- making an arbitrary or unbiased selection;
- a sampling in which all items have an equal probability of being selected.

2 You might have a variety of materials available, such as: compasses and protractors (for constructing spinners); bobby pins or paper clips (for spinner pointers); dice; paper bags or other containers; cubes; tiles; game markers; calculators; grid paper, straightedges, marking pens; etc. Notice that students have opportunity to evaluate their results in Problem 3 of Connector Student Activity 17.1.

1) The students may suggest a variety of strategies for generating random digits, such as the 3 examples given below. Methods that are biased cause one digit to be favored over another, whereas, when a set of random digits is generated without bias, each digit selected is equally likely to be any one of the 10 possible digits, and the selection of each digit is independent of all other selections.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

Simulations and Probability Lesson **17**

Connector Student Activity 17.1

NAME _____ DATE _____

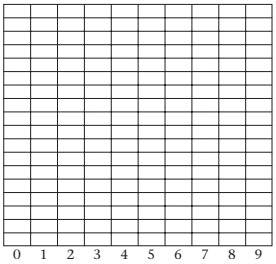
1 Discuss possible methods for creating a list of 100 random digits.

2 Choose a method discussed for Problem 1 to generate a list of 100 random digits.

a) Describe your method of generating 100 random digits:

b) Record your 100 digits, in the order generated, in this table:

c) Make a line plot of your 100 digits here:



3 Do you feel that your set of 100 digits is random? List some conditions that you feel a set of digits should satisfy in order to be considered a set of random digits?

3 Ask the groups to complete Problem 3 on Connector Student Activity 17.1. Discuss.

COMMENTS

2 (continued.)
Container selections: Label 10 slips of paper, 10 tile, or 10 game markers so that each object has a different one of the digits 0-9 written on it. Place the objects in a container. Shake the contents, select one object, record the digit, and return the object to the container.

Spinner or 10-sided die: Spin or roll and record the digits obtained. In place of a 10-sided die, 2 cubes can be numbered, one with the digits 1-6 and the other with 2 blank sides and the digits 7, 8, 9, and 0 on the remaining sides.

Top-of-the-head: one student quickly repeats digits, trying to choose randomly, and another student records the digits.

Some students may discover the random number functions on their graphing calculators. Notice that such functions are introduced in Action 4.

Students' observations may resemble those described in Comment 3.

3 The conditions which students list for the numbers in a set to be random may vary. For example, some students may suggest that each digit should occur about the same number of times, so a line plot should not have many bumps or gaps. Others may suggest that the mean of a set of random digits should be close to 4.5 since the mean of 0-9 is 4.5.

Some students may point out that the bumps and gaps in the shape and spread of some groups' line plots imply there may be bias in the selection methods. Others may suggest that, for a given method, combining several groups' data might produce a more "leveled-off" line-plot. Still others may point out that an even, or level, distribution of the digits selected does not necessarily imply no bias in selection techniques.

While there are statistical tests for evaluating the randomness of a set of numbers (e.g., students may be interested in looking in the library or on the Internet for information about the *chi-square test* or other tests for randomness), the intent here is to illustrate the difficulty of generating sets of numbers that are truly random. There are many ways that bias can occur in meth-

Connector Teacher Activity (cont.)

ACTIONS

4 Give each student a copy of Connector Master A. Then:

a) Discuss the use of a *random number table* to generate sets of random numbers. Have the groups each use this method to randomly generate 10 digits from the set of digits 0-9, and to randomly generate five 10-place decimals that are greater than zero and less than 1. Discuss.

b) Give each student a graphing calculator (or have the students get out their own calculators). Repeat a), but using the random number functions on a graphing calculator to generate the random numbers.

c) (Optional) Repeat a), using computers.

Simulations and Probability Lesson **17**

Connector Master A

Table of Random Numbers

16877	62796	31056	59269	96725	03510	19186	41672	73387	79284	45697	83702
04419	47911	71787	71696	16243	94507	34229	64227	74719	92763	26120	26676
75939	36554	94366	76143	35655	30214	14796	43156	80044	29719	87905	90690
95567	44275	71147	14808	18677	88968	07326	79537	54169	08392	94501	75642
76431	76938	15066	69132	24674	37084	86339	01514	01943	47018	43849	29406
75407	87829	99305	08420	02584	07888	34234	18388	90287	10718	43386	56457
55762	77741	78263	50907	73016	04595	52029	67789	65254	15750	61520	77606
41403	72567	47625	39432	74864	80303	63257	16798	48436	34737	76541	37816
59269	94176	30216	29459	70063	47051	23187	78262	02884	39857	79393	85013
38793	30586	26783	98341	01379	89414	59707	53130	18896	11211	59959	56900
44722	22398	08182	59445	18743	51183	43850	83134	34802	56327	85775	50176
76792	15413	21246	62483	85973	31468	84551	79229	23694	41874	53784	51870
54146	99549	03892	90288	91541	15296	41359	79732	47707	32861	08284	09301
13258	98825	01224	81597	92304	02093	92229	15847	93575	09709	08284	32880
10017	95166	51510	07391	88433	15653	61773	24623	64493	41353	75811	20046
59657	17253	82592	41514	35609	42019	74328	30695	38949	63331	36022	43688
41389	56877	18323	92464	94235	63498	31954	47714	28424	04041	61545	82321
41858	73549	80343	92251	65510	58671	18490	24278	86894	75956	07622	04836
19566	51618	19581	09700	24562	06142	29945	52454	25711	56299	57296	50691
83203	14552	64977	50767	69853	17417	99352	38352	70414	53469	41128	14907
25112	68883	84887	13802	66934	51070	93787	23816	33707	28622	45618	61966
58415	53175	87365	90811	21936	09650	72851	38107	72851	04268	69737	65574
34693	13723	45306	95760	75208	30848	35810	97002	56068	50981	84566	10941
24872	90442	84322	68282	39577	22272	02654	32727	39361	60659	38083	65416
41984	99543	77292	04579	72718	78427	53647	71794	86988	41346	80978	73041
75478	86379	14318	96269	44900	63619	30630	54758	23360	17874	05854	74444
30434	89355	57050	34114	21280	99919	05157	30236	54951	71782	25803	38854
68066	17895	99454	18143	53727	50943	95711	87285	16946	73275	85412	86282
67782	75800	53994	25767	87501	91511	61128	45081	65661	41952	58406	42426
83840	02103	43204	44410	09188	64132	21788	14714	24348	34371	44907	29912
39074	47327	91183	05675	20069	06647	11494	33750	57719	90226	28720	00578
11320	17425	37182	41460	58948	88640	43978	41636	59767	85239	42843	06220
20061	21591	32017	47522	15491	34360	15385	07860	29687	88036	17448	05554
54138	01873	61646	85299	58479	47410	56923	41593	51695	92199	61850	53722
03669	13358	09583	33932	59348	53359	52437	50889	99250	93626	51771	13278

COMMENTS

ods that involve actions such as spinning spinners, rolling dice, and drawing numbers from a container. Further, even when calculators or computers (see Action 4) are used to generate random numbers, there are opportunities for bias in programming the calculators or computers to generate the numbers. For the purposes of this lesson, sets of random numbers generated by the graphing calculator or by most common computer programs are less likely to involve bias and are less time-consuming to produce than by-hand methods such as spinners and dice. In general, the nature of the situation for which a set of random numbers is to be used should dictate the level of sophistication needed for generating the random numbers.

4 a) The Table of Random Numbers shown on Connector Master A gives 540 digits randomly generated by a computer and arranged for convenience in groups of 5 digits. One way to form a sequence of 10 random digits is to start anywhere on the table and then select a sequence of 10 digits in a row or column.

b) Some students may have already discovered the random number functions on their graphing calculators. If so, you might ask them to demonstrate their findings to the class. Or, before class discussion you might have the groups spend some time examining their calculator manuals and/or calculators to determine what random number functions are available.

Note: Students may discover a variety of ways of using a calculator to solve problems in this lesson; only a few possibilities for using the TI-83 are offered in these comments. Consult your calculator manual for details regarding use of specific calculator functions and for troubleshooting.

One way to generate a set of 10 random digits from the digits 0-9 on the TI-83 is to enter the command: randInt(0,9,10), where the function "randInt" is the 5th function under the "PRB" menu of the MATH menu. An alternative command for generating random digits is: randInt(0,9); each time ENTER is pressed, a random digit is selected and displayed.

One way to generate 10-place decimals is to locate the function "rand," which is the 1st command under the "PRB" menu of the MATH menu; each time ENTER is pressed a random 10-place decimal is selected and displayed.

(Continued next page.)

Connector Teacher Activity (cont.)

ACTIONS

COMMENTS

4 (continued.)
Note: It is also possible to activate a TI-83 function by scrolling to locate the function by name in the CATALOG (obtained using the 2nd function and the numeric key 0) and then pressing ENTER.

c) If you have a computer available for demonstration and/or accessible to students, there is probably a random number generator function built into the computer. Using a computer, it is possible to quickly generate very large sets of random numbers of any size.

5 Ask the students to use the Table of Random Numbers (Connector Master A) to randomly generate 20 whole numbers from 1 through 15. Discuss their methods and results. Then repeat for the graphing calculator and (optional) for the computer.

5 Following is one way of using the Table of Random Numbers to generate whole numbers less than or equal to 15:

Beginning, for example, with the 4th row of digits and the 14th and 15th columns, by reading down, the first 15 pairs of random digits are:

47 66 05 63 25 16 83 82 46 92
24 10 92 23 43

Notice that the only numbers from the above list which are less than or equal to 15 are 05 and 10. To produce a set of 20 random whole numbers that are greater than 0 and less than or equal to 15, continue forming the above list.

One way to use the TI-83, to randomly generate 20 integers that are greater than 0 and less than or equal to 15 is to enter the command randInt(1,15,20).

Another method students may suggest is to generate 10-place decimals (see Comment 4) and consider pairs of digits. For example, the digits in the randomly-generated 10-place decimals .6714424448 and .2216170607 can be paired as follows:

67 14 42 44 48 22 16 17 06 07

From the above 10 pairs of digits, 14, 06, and 07 are the only whole numbers greater than zero and less than or equal to 15. The digits from additional random 10-place decimals can be paired to generate more such whole numbers.

Connector Teacher Activity (cont.)

ACTIONS

6 Ask the students to:

a) use the random number functions on their calculators to obtain 100 random digits (from the digits 0-9) and to store the digits in a list;

b) investigate ways to use their calculator to graph a histogram and a box plot of the list formed in a);

c) determine how to use the calculator to compute the mean, median, and sum of the list formed in a).

Discuss the students' results.

7 Pose the following to the groups:

There are 45 sailboats in a race. The boats are numbered from 1 to 45 and the director of the boat race wishes to randomly select 12 different sailboat numbers for prizes. Write a set of calculator commands to generate the winning numbers.

Discuss the groups' methods and results.

COMMENTS

6 a) Entering the command randInt(0,9,100) STO→ L1 on the TI-83 generates a set of 100 random digits and stores them in List 1.

Notes: To store (STO→) in List 1 (L1), notice the name L1 is above the numeric key 1 and obtained using the 2nd function. To view L1 in a table, choose "edit" under the EDIT menu of the STAT menu. To clear List 1, use the command ClrList L1, where "ClrList" is the 4th function of the EDIT menu of the STAT menu.

b) To produce a histogram of the List 1 data, turn on the histogram for Plot 1 in the STAT PLOT menu and then press GRAPH. One can TRACE the histogram to determine the frequency of each digit (i.e., the height of each bar). To produce a box plot of List 1, turn on the box plot for Plot 1 and then press GRAPH. Using the TRACE function on a box-plot displays the minimum, maximum, upper quartile, lower quartile, and median. See your calculator manual for detailed procedures and troubleshooting.

c) To compute the mean of List 1, enter the command mean(L1), where the "mean" function is in the "MATH" menu of the STAT LIST menu. The "median" and "sum" functions are in the same menu as the "mean" function.

7 One purpose of this problem is to introduce the idea of *selection without replacement*. Since there must be 12 different winners, once a boat's number is chosen, it cannot be chosen again. Thus, if in the random selection of 12 numbers, a number is selected more than once, only the 1st occurrence is used and additional numbers must be selected until a total of 12 different numbers are selected. This is called *selection without replacement*.

For example, the following 12 numbers were obtained on the TI-83 using the command randInt(1,45,12): 13, 20, 10, 38, 41, 32, 24, 9, 3, ~~24~~, 5, 22. Since the second 24 must be disregarded, the command randInt(1,45) must be used repeatedly until 12 different numbers are generated.

Connector Teacher Activity (cont.)

ACTIONS

8 Give each student a copy of Connector Master B and ask them to complete Problem a). Discuss their methods and results. Repeat for 3 or more of b)-g).

Simulations and Probability Lesson **17**

Connector Master B

For each of the following, use a calculator to generate the random numbers that are described. Then record the calculator commands that you enter and the data generated by the commands.

a) Last month Lakeway Theater sold tickets numbered from 372500 to 374000. The theater manager plans to give free passes to the first 5 numbers that are randomly selected from these numbers.

b) To practice multiplication facts, each day Rachelle randomly generates a set of 25 pairs of whole numbers, where each number in a pair is randomly selected from the whole numbers 2 through 12.

c) Some combination locks are designed for use with any 3 numbers from 0 to 45. When these locks are manufactured, the numbers used in each combination are selected at random.

d) In a drawing for 2 CD players, tickets are numbered from 41 through 80. The 1st 2 ticket numbers that are randomly selected and divisible by 3 are the winning numbers.

e) Suzanne's teacher asked her to randomly generate a set of even numbers less than 100 and to select the first 3 that are divisible by 5.

f) A researcher randomly selects 15 numbers from the following set: {1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, … 38, 38.5, 39, 39.5, 40}. Then she computes the mean and the sum of the 15 numbers, and she makes a box plot of 15 numbers.

g) Joey randomly selected 3 odd numbers greater than 100 and less than 140.

COMMENTS

8 You might encourage groups that solve these problems quickly to determine more than one possible set of calculator commands for generating each set of random numbers. Or, you could ask them to determine a strategy for using the Table of Random Numbers to generate the numbers for each situation. Following are examples of TI-83 commands for each problem:

a) The command randInt(372500,374000,5) randomly generates 5 numbers from 372,500 through 374,000.

b) The command randInt(2,12,2) generates a pair of numbers where each number is greater than or equal to 2 and less than or equal to 12. Each time ENTER is pressed a new pair of random integers is selected and displayed.

As an alternative method, some students may suggest generating a list of 50 random integers from 2 through 12 and then grouping adjacent pairs of numbers from the list.

c) The command randInt(0,45,3) generates a set of 3 whole numbers randomly selected from 0 through 45.

d) For example, using the command randInt(41,80) and pressing ENTER 6 times generated the following random whole numbers from 41 through 80:

46 49 (72) 52 61 (45)

Notice that, in this example, it was necessary to generate 6 numbers before obtaining 2 numbers which are divisible by 3.

Another approach is to note that, for the whole numbers from 41 through 80, the smallest number divisible by 3 is $3 \times 14 = 42$ and the largest such number divisible by 3 is $3 \times 26 = 78$. Hence, entering the command 3*randInt(14,26,2) generates a pair of random integers between 41 and 80 such that each integer is divisible by 3.

e) The command 2*randInt(1,50) randomly generates even whole numbers less than or equal to 100. In the following example, 15 such numbers were required to obtain 3 numbers which are divisible by 5.

14 92 82 86 68 (70) 86 44 88
66 68 12 (10) 36 (90)

As another approach, since all even numbers are divisible by 2, and since all numbers divisible by both 2 and 5 must be divisible by 10, one can satisfy the conditions

Connector Teacher Activity (cont.)

ACTIONS	COMMENTS
	of this situation by generating a set of 3 random numbers that are divisible by 10. This can be done by entering the command 10*randInt(1,10,3).
	f) To generate and store a list of 15 such random numbers enter the command .5*randInt(2,80,15) STO→ L1. To compute the mean of the 15 numbers, enter mean (L1), using the "mean" function from the "MATH" menu of the LIST menu. To compute the sum of the numbers, enter sum(L1) where the "sum" function is also in the "MATH" menu of the LIST menu. To create a box plot of the 15 numbers, turn on a box plot for LIST 1 from the STAT PLOT menu and then press GRAPH or TRACE (be sure other graphs are turned off).
	g) The odd numbers which are greater than 100 and less than 140 can be randomly generated using the command 2*randInt(51,70)–1.
	As another approach, some students may generate random integers from 100 through 140 and then select the 1st 3 odd numbers.

TEACHER NOTES:

Focus Teacher Activity

OVERVIEW & PURPOSE

Students identify key components of simulations, and they design and carry out simulations to solve probability problems whose theoretical solutions are complex and/or out of reach for this grade level. The graphing calculator serves as an efficient tool for generating and representing data. Central tendency, range, variation, and confidence intervals provide bases for predictions.

MATERIALS

- ✔ Connector Master A, 1 copy per group, 1 transparency.
- ✔ Focus Masters A and C, 1 transparency of each.
- ✔ Focus Master B, 1 copy per student, 1 transparency.
- ✔ Focus Student Activities 17.2 and 17.3, 1 copy per group and 1 transparency.
- ✔ Assorted materials (e.g., protractors, compasses, bobby pins, straightedges, dice, game markers, cubes, tile, grid paper, etc.) available to groups.
- ✔ Graphing calculators, 1 per student.
- ✔ Overhead graphing calculator (optional).
- ✔ ¼" grid paper, 1 transparency.
- ✔ Blank transparencies and overhead pens, 2 per group.

ACTIONS

1 Arrange the students in groups and place a transparency of Focus Master A on the overhead, revealing only the Blood Type Problem. Discuss the students' ideas regarding the meaning of the expression "on the average, 2 out of 5 people." Then ask each group to propose 2 different approaches for using a *simulation* to solve the Blood Type Problem.

Invite volunteers to describe their group's proposals. Ask the students to provide feedback regarding the strengths of the strategies proposed, and ways these strategies may be biased and/or need refinement. Use this as a context for discussing the terms *trial* and *successful trial*.

Simulations and Probability Lesson **17**

Focus Master A

Blood Type Problem

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

COMMENTS

1 Students' interpretations of the expression "on the average, 2 out of 5 people" may vary. Following are 3 examples:

If 5 people are randomly selected, then it may be that any number from 0-5 have blood type O. But if many groups of 5 people are selected, then the average of the numbers of people in each group with blood type O will be close to 2.

Regardless the number of people selected, 40% of the people have blood type O.

If 1 person is randomly selected, there is a 40% chance that person has blood type O.

You may wish to engage students in discussion about the meaning of the word *simulation* and reasons for the use of simulations. In Lesson 8 of this course, students were introduced to simulations—procedures for answering questions about a problem by conducting an experiment whose conditions resemble the conditions of the problem—when the actual conditions of the problem are not practical to carry out. For example, it is inconvenient to actually select groups of 3 people and interview them regarding their blood type, or to examine randomly selected medical records; hence, a simulation can be used to model such selections.

It may be helpful here to remind students that you are only asking them to propose possible simulations, rather

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

1 (continued.)
than to determine solutions, as there will be an opportunity later to carry out strategies and solve the problem.

Some students may suggest determining the theoretical probability; if so, you might suggest that they focus on ideas for simulations now and let them know there will be an opportunity later to investigate the theoretical probability. Note: One reason for simulating this problem is that determining the theoretical probability may be challenging to many students.

Students may propose a variety of simulations for collecting experimental data. For example, one group of students suggested the following:

Place 2 markers of one color, representing blood type O, and 3 markers of another color, representing NOT blood type O, in a sack; select 1 marker, record its color, and return it to the sack. Repeat this until 3 markers have been selected. If exactly 2 markers representing type O blood are selected, this selection of 3 markers is recorded as a successful set. Carry out these selections of sets of 3 markers many times. Divide the number of successful sets by the total number of sets to determine the probability that exactly 2 out of 3 people will have type O blood.

In the above example, a set of 3 selections is a *trial*; a *successful trial* is the selection of 2 people with type 0 blood and 1 with a different blood type.

As another example, the following simulation is based on random numbers (if no one suggests this, you might ask the students to develop a strategy that involves use of random numbers, or you could suggest such a strategy as another possibility):

Use a Table of Random Numbers, letting the digits 1 and 2 represent people with type O blood and the digits 3, 4, and 5 represent people without type O blood. Beginning at any number in the table and reading across a row or down a column, record the first 3 digits that are greater than 0 and less than 6 and denote this set of 3 digits as a successful trial if exactly 2 of the digits are either a 1 or a 2. Carry this out many times and determine the ratio of successful trials to the total number of trials.

A variation of the above simulation is to use the command randInt(1,5,3) on the TI-83 (or the corresponding command on another brand of calculator) to randomly generate sets of 3 digits from 1 through 5. Then examine the sets of numbers as described above.

Focus Teacher Activity (cont.)

ACTIONS

2 Give each group a copy of Focus Student Activity 17.2 and ask them to complete Parts a)-c), alerting them to the location of a variety of materials, e.g., compasses, protractors, straightedges, bobby pins, Table of Random Numbers, paper clips, dice, paper bags or other containers, cubes, tiles, game markers, calculators, marking pens, grid paper, etc., for use in their simulations. Discuss their strategies and results.

Simulations and Probability Lesson **17**

Focus Student Activity 17.2

NAME _____ DATE _____

Blood Type Problem:

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

Design and carry out a simulation to answer the above Blood Type Problem. Collect data for *a minimum of 25 trials*, where a trial is the blood type information for a set of 3 people. On separate paper:

a) Describe, in detail, your simulation procedures.

b) Show all of the data that you collect.

c) Show how you organize, graph, and analyze your data to answer the Blood Type Problem. Be sure to support all conclusions with sound mathematical reasoning and a variety of mathematical evidence. If you form graphs or compute statistics on the calculator, sketch and label the results and include them in your written arguments.

d) Challenge. Determine and explain the theoretical probability for the Blood Type Problem. Show/explain your methods and reasoning so that it is clear why your answer is correct.

COMMENTS

2 Students may simulate this situation by using any of the methods outlined in Comment 1, or they may develop other simulations. One line plot with data generated using the TI-83 is shown below; the experimental probability for each outcome is shown in the top row. For this experiment, the probability of exactly 2 people having blood type O is .32.

Probability:	$\frac{7}{25}$ = .28	$\frac{9}{25}$ = .36	$\frac{8}{25}$ = .32	$\frac{1}{25}$ = .04
		X		
		X	X	
	X	X	X	
	X	X	X	
	X	X	X	
	X	X	X	
	X	X	X	
	X	X	X	
	X	X	X	X
	0	1	2	3

Number of people out of 3 with blood type O

Some students may suggest combining the groups' data or finding the average of the groups' probabilities. As more data is collected, the experimental probability for exactly 2 out of 3 people with blood type O should approach the theoretical probability of 28.8% (see Comment 3).

You might ask the students to use the data from their graphs to predict other probabilities, such as the probability that none of the 3 people selected will have blood type O or the probability that at least one of the 3 people will have blood type O. Note: For the given Blood Type Problem, the theoretical probabilities that 0 people, 1 person, 2 people, and 3 people will have blood type O are 21.6%, 43.2%, 28.8%, and 6.4%, respectively.

Focus Teacher Activity (cont.)

ACTIONS

3 (Optional) Ask the groups to complete Part d) on Focus Student Activity 17.2. Discuss their methods and results.

4 Place a transparency of Focus Master A on the overhead, revealing the Cereal Box Problem. Discuss the students' ideas regarding:

a) What is their "quick guess" of the number of cereal boxes needed?

b) What are some statements/predictions students can make with a high level of confidence about the situation described in the Cereal Box Problem.

c) What aspects of the problem need clarification?

COMMENTS

3 Whether to assign Part d) on Focus Student Activity 17.2 may depend on your students' needs and interests. Regardless, determining the theoretical probability should not detract from the importance of simulations; scientists often rely on simulations as a basis for decision making.

Following is one way of determining the theoretical probability that, for the given conditions, exactly 2 out of 3 people selected at random will have blood type O:

When 3 people are randomly selected, there are 3 possible ways in which 2 of the people selected can have type O blood (let O represent type O and N represent not type O): ONO, OON, or NOO. For each person selected, there is a $2/5$ probability their blood type is O and a $3/5$ probability it is not type O. Hence, the theoretical probabilities for the possibilities involving 2 people with type O blood are:

ONO $2/5 \times 3/5 \times 2/5 = 12/125$
OON $2/5 \times 2/5 \times 3/5 = 12/125$
NOO $3/5 \times 2/5 \times 2/5 = 12/125$

Thus, the theoretical probability that exactly 2 of the 3 people selected have type O blood is $12/125 + 12/125 = 12/125 = 36/125 = 28.8\%$.

4 The intent here is to prompt the students to engage in thought about the problem and to sensitize them to the fact that recognizing and clarifying assumptions are important aspects of designing a simulation.

a)-b) Students' guesses may range from 6 boxes to several hundred; following are examples of statements students commonly make. Remember to avoid giving clues or judging responses!

We are certain we must purchase at least 6 boxes.

We don't think it is very likely that we would get all 6 trading cards by purchasing only 6 boxes.

We think it will take 36 boxes because there are 6 possibilities for each box and we need at least 6 boxes.

c) Following are examples of points that students may raise for clarification/agreement:

Does the cereal company use the same number of each trading card?

Focus Teacher Activity (cont.)

ACTIONS

Simulations and Probability Lesson **17**

Focus Master A

Blood Type Problem

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

Cereal Box Problem

As a special promotion, the Crunchy-Crispy Cereal Company includes a baseball trading card in each box of cereal. There are 6 different trading cards, and each cereal box contains exactly 1 of the 6 cards. How many boxes of cereal would you expect to buy in order to collect all 6 trading cards?

5 Discuss the students' ideas regarding ways to simulate the Cereal Box Problem.

6 Distribute a copy of Focus Student Activity 17.3 (see next page) to each group and ask them to complete Parts a)-d), again alerting students to the location of materials that are available for use in their simulations. When the groups have completed Part d), invite volunteers to share their group's results and confidence levels. Discuss as needed.

COMMENTS

Is there one trading card that is harder to get than the other ones?

Are the boxes randomly mixed when distributed to stores?

For the purposes of this activity, it is important to assume that each trading card has the same chance of being in any one box and that any store will have a random mix of all the trading cards. These are the types of assumptions that need to be made when designing a simulation.

5 There are a variety of possibilities for this simulation, such as: selecting from 6 different colored markers in a sack; using a spinner with 6 equal parts; rolling an ordinary die; and using a random number function on a calculator or a Table of Random Numbers to generate random digits from 1-6.

6 Frequently, when students are involved in establishing guidelines for evaluating their work, they report feeling a higher sense of commitment to producing quality work. Hence, prior to beginning Focus Student Activity 17.3, it may be helpful to have the class create a set of evaluation standards for their written work on the activity. For example, groups may suggest that their simulation procedures should reflect the conditions of the problem. Or, they may require use of specific statistical plots (e.g., histogram, line plot, box plot) or specific statistical measures (e.g., mean, median, mode, range, outliers, average deviation) and a discussion of the relevance of these measures to the data. In other words, evaluation standards can be used to enhance the depth and quality of the groups' investigations and analyses.

You might also have the students establish expectations for the presentation of their results (e.g., create a poster to be evaluated by other groups, a written report that meets certain formatting guidelines, an oral report with visual aids presented by each group to 2 other groups, etc.). See *Starting Points* for other assessment ideas.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Lesson **17** Simulations and Probability

Focus Student Activity 17.3

NAME _____ DATE _____

Cereal Box Problem

As a special promotion, the Crunchy-Crispy Cereal Company includes a baseball trading card in each box of cereal. There are 6 different trading cards, and each cereal box contains exactly 1 of the 6 cards. How many boxes of cereal would you expect to buy in order to collect all 6 trading cards?

Design and carry out 2 simulations to answer the above problem so that one simulation involves the use of a random number function on a graphing calculator or a Table of Random Numbers, and the other simulation does not involve these methods. For each simulation, collect data for a minimum of 20 trials, where one trial consists of selecting until all 6 trading cards are obtained (i.e., the number of selections required for a trial will vary from trial to trial).

a) For each simulation, describe in detail your simulation procedures.

b) For each simulation, show all of the data that you collect.

c) For each simulation, show how you organize, graph, and analyze your data and use the results to answer the Cereal Box Problem. On a scale of 1-100, how confident are you in each solution? Explain.

d) Show how you combine the data from your 2 simulations and analyze the results. Based on this information, now what is your solution to the Cereal Box Problem? On a scale of 1-100, how confident are you now in your solution? Explain.

e) Show how you analyzed the combined data from all groups in your class. Discuss any adjustments this leads you to make in your solution from d). On a scale of 1-100, now how confident are you in your prediction?

7 Discuss the students' ideas regarding ways to combine the class data. Then have the class determine and carry out a process for combining the data. Next, have the groups complete Part e) on Focus Student Activity 17.3. Discuss their results and confidence levels.

8 Pose the following question to the groups:

How many boxes of cereal do you think would have to be purchased in order to feel 90% certain of obtaining all 6 trading cards?

Allow time for the groups to determine an answer to the above question and to build arguments to support their answers. Discuss. Use this as a context for introducing *confidence intervals* and *90% box and whisker plots* (and other % box plots).

COMMENTS

6 (continued.)
Students' simulations need to involve enough trials (e.g., 50 or more trials for this set where 1 trial generates all 6 trading cards) to assure a fairly representative data set, and so that students experience the wide variation possible from 1 trial to another.

The groups' data analysis methods may vary. For example, when analyzing their data for Part c) some groups may rely on the mode, especially if their data shows a given number of boxes occurs with high frequency. Others may feel the median, or mean is a better indicator. Still others may report a range of numbers of boxes to purchase. What is important is that groups have a mathematically sound rationale for basing a conclusion on a particular statistical measure.

7 Students may suggest carrying this out as a whole class activity, or they may suggest that each group compile all the class data and analyze the results. In general, students should feel more confidence in predictions that are based on larger numbers of trials.

Some students may suggest the idea of a "likely range" or "likely spread" of data by choosing a range of values that includes data for most of the trials. To do this, they may suggest that, if a graph shows that most of the experimental data clusters in a certain interval, then the number of cereal boxes to purchase is likely to fall within that interval and the data far away from the cluster are outliers for the experiment.

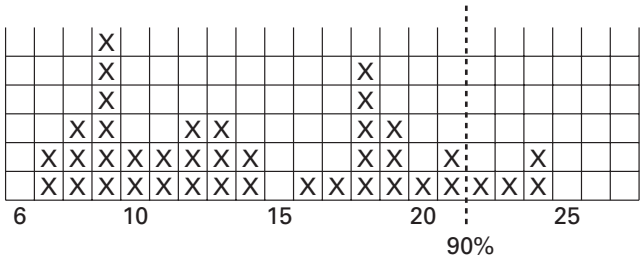
8 The groups' responses may vary, although many may relate their confidence levels to the amount of data collected or the distribution of the data in a line plot or box and whiskers plot. For example, one group used the line plot below to support the following conclusion:

Using the data in this line plot, we feel 90% certain of getting all 6 trading cards with 21 boxes of cereal, because 90% of our data is less than or equal to 21.

Focus Teacher Activity (cont.)

ACTIONS

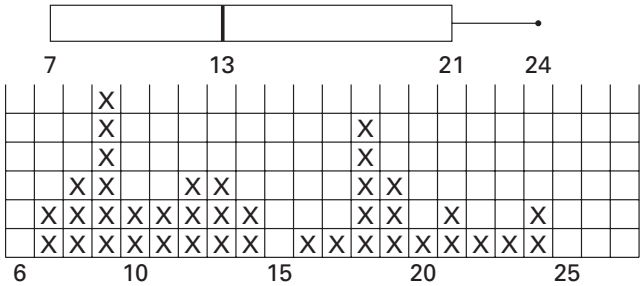
COMMENTS



Notice that, based on the above reasoning and data, other lines can be drawn to indicate other confidence levels. For example, again based the the above data, one could say with 82.5% certainty that purchasing 19 or fewer boxes would assure all 6 trading cards.

Some students may suggest that the above method of locating a *90% confidence interval* is valid only if sufficient data is collected. Ideas about what is sufficient may vary among individuals and can vary depending on the nature of a problem.

Another visual method of representing a 90% confidence interval is to use a *90% box-and-whisker plot*, as shown above the line plot below.



Notice that the box is positioned above the line plot so that the left end of the box corresponds with the minimum data value and the right end corresponds with the maximum data value of the lower 90% of the data. There is no whisker on the left end of this 90% box plot since the box to contains the *lower* 90% of the data, (or as close to 90% as possible, noting that, since a data value equal to the value of an end of a box is considered "in" the box, and since, in some cases there are several data entries equal to the value of an end of a box, a box may contain more than 90% of the data). The right whisker of this 90% box plot extends across the upper 10% of the data. Note: in experiments based on other problems it may be more reasonable for a 90% box plot to include the *middle* 90% of the data and have 2 whiskers; however, that is not reasonable for the Cereal Box Problem.

Focus Teacher Activity (cont.)

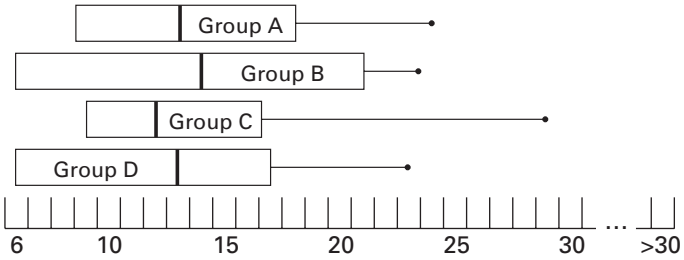
ACTIONS

9 Ask the groups to each form a 90% box plot of their data from Part d) of Focus Student Activity 17.3. Then sketch the base line of a line plot along the edge of a transparency of ¼" grid paper. Invite several groups to sketch box plots for their data for Part d) on the transparency. Discuss the students' observations.

10 Ask the groups to list their ideas about the key stages of the simulation process, in general. Discuss and reach consensus as a class.

COMMENTS

9 Graphing 90% box plots from several groups of students provides a visual summary of the data. Four such plots are shown below.



Students may make observations about the variability in the length of the boxes and/or the whiskers, the relative locations of medians, and the extent to which the box plots overlap. With each observation students make about the box plots, it is helpful to encourage them to relate the observation to information it reveals about the Cereal Box Problem.

10 The following ideas were sugested by one class:

Understand the problem.

Determine what assumptions are imbedded in the problem.

Know what key questions must be answered to solve the problem.

Determine what simulation methods best model the conditions of the problem so the key questions can be answered.

Determine what makes a trial, what it takes for a trial to be successful, and how many trials to complete.

Develop a system for recording data.

Select appropriate statistical measures and plots to analyze the data. Keep confidence levels in mind to determine if more data is needed.

Report conclusions.

Focus Teacher Activity (cont.)

ACTIONS

11 Place a transparency of Focus Master A on the overhead, revealing the Principal Problem. Give each group 2 blank transparencies and an overhead pen. Have the groups design and carry out a simulation to solve the problem, recording the following on the 2 transparencies:

a) all data collected;

b) all relevant statistical graphs formed;

c) all statistical measures computed;

d) their answer to the Principal Problem; and

e) their level of confidence in their answer.

Invite volunteers to show their data and results to the class. Discuss the students' ideas about similarities and differences in their results, and other observations students make about the data.

Simulations and Probability Lesson **17**

Focus Master A

Blood Type Problem

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

Cereal Box Problem

As a special promotion, the Crunchy-Crispy Cereal Company includes a baseball trading card in each box of cereal. There are 6 different trading cards, and each cereal box contains exactly 1 of the 6 cards. How many boxes of cereal would you expect to buy in order to collect all 6 trading cards?

Principal Problem

Mount Hood Middle School has 3 classes of 7th graders: 25 students in Class I, 30 in Class II, and 22 in Class III. If 5 of the 77 students are randomly selected to meet with the principal, what is the probability that at least 1 student from each class will be in the meeting?

COMMENTS

11 Rather than presenting their work on transparencies, you might have the groups use 2 sheets of blank paper. Then groups can exchange completed work and provide feedback to each other. Limiting groups to 2 sheets encourages them to be selective about what/how they report.

One objective of this problem is to illustrate the convenience of using random numbers in simulations of situations involving larger numbers. For example, the whole numbers from 1 to 77 can be assigned to the students in the following way:

1, 2, 3, … 25 represent Class I
26, 27, … 55 represent Class II
56, 57, … 77 represent Class III

If pairs of digits, 01, 02, … 77 are used from a Table of Random Numbers, the numbers greater than 77 must be disregarded. To randomly generate whole numbers from 1 through 77 on the TI-83, use the command: randInt(1,77). A trial ends when 5 different numbers have been selected. A trial is successful if 1 number from each of the 3 classes (see above) is selected.

A second objective of this problem is to illustrate sampling *without replacement*. If a student is selected as 1 of the 5 team members, then this same student can not be selected again for the same team of 5. That is, once a student is selected, that student is not replaced back into the set of 77 students for the next selection. Thus, in generating sets of 5 numbers, if the same number occurs more than once, the repetitions must be crossed out and additional numbers selected until a set of 5 different numbers is formed.

The 20 sets of 5 numbers shown on the next page were generated on the TI-83. Each number from 1-25 has been circled, each number from 26-55 has been boxed, and each number from 56-77 has been underlined. The 16 trials marked with a ✔ were successful, i.e., at least 1 student from each class was represented in the set of 5. All remaining trials were not successful, i.e., all 3 classes were not represented. For these 20 trials the experimental probability that at least 1 student from each of the 3 classes will meet with the principal is $16/20 = .8$.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

11 (continued.)

✔	23	57	43	47	14		6	27	69	19	71	✔
	44	49	27	41	19		15	34	8	61	73	✔
✔	59	56	69	52	13		60	24	44	2	10	✔
	30	43	5	6	~~43~~	23	7	16	3	70	45	✔
✔	73	23	62	38	21		69	51	12	14	67	✔
	67	25	~~25~~	70	24	16	29	21	57	63	57	✔
	45	41	72	56	43		65	8	14	44	50	✔
✔	18	68	60	31	46		71	8	4	57	44	✔
✔	52	16	33	15	74		11	23	10	55	75	✔
✔	56	72	5	50	23		73	53	43	51	5	✔

A variety of observations involving experimental probabilities can be made about the information in this chart. For example, the probability that at least 1 student from each class will not meet with the principal is .2. The probability that at least 2 students from Class I will meet with the principal is $14/20 = .7$. The probability that a student from Class III will not be included is $2/20 = .1$. The probability that 3 or more students from Class II will be included is $3/20 = .15$.

12 Give each student a copy of Focus Master B. Discuss your expectations. When completed, discuss the students' various approaches and conclusions.

12 In most cases the problems on Focus Master B will be too difficult for the students to determine answers theoretically. These problems have been designed to be solved by simulations.

You might base the number of problems you assign and your expectations regarding student reporting of solutions on your time needs and the interest level and needs of your students. You may wish to select some problems for homework and/or 1 or 2 problems for all students to complete as an assessment activity. Or, you might have the students select 1 problem each of low, medium, and high difficulty levels. See Comment 6 for ideas for establishing standards for evaluating students' work.

Results from students' simulations will vary. However, due to the *Law of Large Numbers*, as the amount of data students collect increases, the likelihood of experimental probabilities resembling the theoretical probabilities increases. Following is one possible simulation for each problem; these simulations use the TI-83 to generate data. Note that the number of trials in these examples is

Focus Teacher Activity (cont.)

ACTIONS

Lesson **17** Simulations and Probability

Focus Master B

Solve the following problems by designing and carrying out simulations.

a) Each box of Pops-a-Lot Popcorn contains 1 of 7 different colored pens. How many boxes of popcorn would you purchase in order to be 90% certain that you would obtain a complete set of all 7 colors?

b) Based on his past archery records, the probability that Eric will hit the bulls-eye of a target is .94. If Eric takes 8 shots at the target, what is the probability he will hit the bulls-eye exactly 7 times?

c) Assume that the probability of a randomly selected person having a birthday in a given month is $1/12$.

i) How many people, on the average, would you need to select to be 90% certain that 2 of them will have a birthday in the same month?

ii) If 8 people are randomly selected, what is the probability that at least 3 will have a birthday in the same month?

d) Two students are playing a coin-tossing game. Each player tosses a coin until obtaining 3 heads or 3 tails in a row. The player who requires the fewest number of tosses wins the game. How many tosses of a coin are required on the average to obtain 3 heads or 3 tails in a row?

e) Hoopersville Hospital uses 2 tests to classify blood. Every blood sample is subjected to both tests. Test A correctly identifies blood type with probability .7 and Test B correctly identifies blood type with probability .5. Determine the probability that at least 1 of the tests correctly determines the blood type.

(Continued on back.)

Simulations and Probability Lesson **17**

Focus Master B (cont.)

f) The names of 5 people (all with different names) are placed on 5 separate slips of paper and these slips are placed in a sack. If each person randomly chooses a slip from the sack, on the average, how many people will select their own name?

g) On a quiz show, contestants guess which 1 of 3 envelopes contains a $5000 bill. What is the probability that exactly 4 people out of 8 contestants will select the envelope with $5000?

h) At a certain university it is required that 85% of the students be from within the state. If 6 students are randomly selected from this university's student body, what is the probability that exactly 1 of them will be from outside the state?

i) Assume that the probability of a randomly chosen person having a birthday on a given day of the year is $1/365$. How many people, on the average, would you need select in order to be 90% certain of obtaining exactly 2 people with a birthday on the same day?

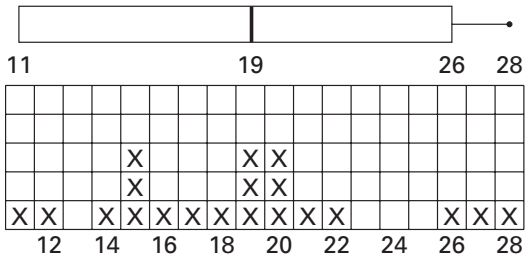
COMMENTS

small due to space limitations. In most cases, students should collect larger samples than shown here.

a) Notice that this problem is a variation of the Cereal Box Problem. Let each of the digits 0-6 represent a different colored pen. One trial consists of repeatedly entering the command: randInt(0,6) until obtaining all 7 of the digits from 0 through 6. For example, the following trial required 15 random digits to obtain all 7 digits:

5 2 5 6 4 3 6 6 2 3
5 0 5 6 1

A line plot and 90% box plot of the results of 20 trials are shown below. The mean of this data is 18.7 boxes and the median is 19 boxes. From this experiment it can be concluded with 90% certainty that the 7 pens will be obtained by purchasing 26 or fewer boxes of popcorn. (Note: the theoretical mean number of boxes to purchase is 18.15.)



b) Use the command randInt(1,100,8) to randomly generate sets of 8 whole numbers from 1 to 100. Let the numbers 1 through 94 represent hitting the target and 95 through 100 represent a miss. In 1 out of the following 3 trials the target was hit exactly 7 times. So for this limited set of trials, the experimental probability is $1/3$.

Trial 1:	72	11	44	12	78	25	13	88	8 hits
Trial 2:	38	77	8	41	70	38	29	91	8 hits
Trial 3:	39	35	55	18	9	36	97	60	7 hits

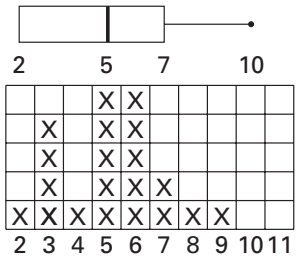
Note: The theoretical probability of hitting the target exactly 7 times out of 8 is .31 to 2 decimal places.

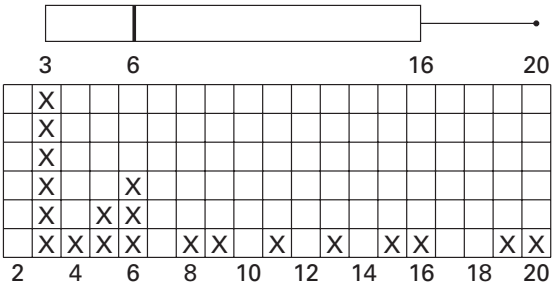
c)i) Let the whole numbers from 1 to 12 represent the months of the year. Using the command randInt(1,12), each of the following 5 trials ended when a number occurred twice, representing 2 people with the same birth months.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS





COMMENTS

12 (continued.)

Trial 1:	2	9	5	11	12	11	
Trial 2:	3	4	3				
Trial 3:	8	5	6	4	11	10	10
Trial 4:	3	1	3				
Trial 5:	6	11	7	8	10	6	

The line plot at the left shows the data for 20 trials. For this data, the mean number of people that must be chosen is 5.2 and the median is 5. The 90% box plot indicates with 90% certainty that 2 people will have a birthday in the same month if 7 or fewer are randomly chosen.

c)ii) The following trials were generated using the command randInt(1,12) 8 times. A trial is a success if the same number occurs 3 or more times. In the 5 trials below, 1 trial was a success; hence, for these 5 trials, the experimental probability of getting a success is $\frac{1}{5}$.

Trial 1:	12	1	8	9	2	8	2	10
Trial 2:	8	3	9	1	10	5	9	10
Trial 3:	12	12	10	10	11	8	1	6
Trial 4:	3	3	4	1	3	2	10	3
Trial 5:	5	1	12	2	3	8	4	7

d) Simulate the tossing of coins by randomly generating the digits 0 and 1, with 0 representing heads and 1 representing tails. A trial ends when either 3 heads or 3 tails are obtained. Five trials are shown below; the numbers of tosses required for these 5 trials are 19, 4, 5, 11, and 3, respectively.

Trial 1:	0	1	0	0	1	0	1	1	0	1	0
	1	0	1	1	0	1	1	1			
Trial 2:	0	1	1	1							
Trial 3:	1	0	1	1	1						
Trial 4:	0	1	0	0	1	0	1	1	0	0	0
Trial 5:	0	0	0								

The line plot at the left contains the data for 20 trials. The mean number of tosses required is 8.05 and the median is 6. The 90% box plot indicates that, for this data, one can be 90% certain of obtaining 3 heads or 3 tails in a row with 16 or fewer tosses.

e) For Test A, let 1, 2, 3, 4, 5, 6, 7 represent a correct identification of blood type and 8, 9, 10 represent an incorrect identification. For Test B let 1, 2, 3, 4, 5 represent a correct identification and 6, 7, 8, 9, 10 an incorrect identification. Use the command randInt(1,10,2) to generate pairs of numbers where the first number repre-

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

sents Test A and the second Test B. A successful trial is a pair of numbers for which at least 1 number indicates a correct identification of blood type. For the following 10 trials, where an underlined number indicates a correct identification of blood type, there were 8 successes; hence for this data, the experimental probability of at least one of the tests correctly determining the blood type is .8.

(2,6), (10,8), (3,4), (6,8), (2,9), (4,10), (1,10), (3,6), (1,4), (9,8)

Note: Although the above sample is small, the experimental probability is close to the theoretical probability of .85.

f) Assign each person a different number from 1 through 5. Enter randInt(1,5,2) 5 times to select 5 pairs of random integers from 1 through 5. Selecting a pair of matching numbers represents a person selecting her own name from the sack.

Trial 1: (5,3) (4,5) (2,3) (4,1) (3,5)	0 matches
Trial 2: (5,4) (5,5) (2,3) (3,3) (3,2)	2 matches
Trial 3: (4,3) (1,3), (1,1) (4,1) (4,5)	1 match
Trial 4: (1,4) (4,5) (5,2) (4,3) (2,3)	0 matches
Trial 5: (2,5) (5,2) (2,3) (4,3) (1,1)	1 match
Trial 6: ((3,3) (3,1) (3,3) (3,3) (4,2)	3 matches
Trial 7: (2,3) (4,5), (3,3) (2,1) (2,4)	1 match

Based on the above 7 trials, on the average, (0 + 2 + 1 + 0 + 1 + 3 + 1)/7 ≈ 1.1 person should select her own name. Since a partial person doesn't make sense, one might round to say that, on the average, 1 person out of 5 should select her own name. This is based on the mean as average. Ordering the results (0, 0, 1, 1, 1, 2, 3) and finding the median affirms 1 out of 5 as the average. And, the mode of this data also suggests an average of 1 out of 5.

Many repeated trials of this experiment should produce a simulated average which is close to the theoretical average of 1 person selecting their own name.

Trail 1:	0	0	2	1	1	2	2	2 ✔
Trial 2:	0	2	0	1	0	2	0	2
Trial 3:	1	2	1	1	0	2	2	1
Trial 4:	0	0	2	1	0	0	0	1
Trial 5:	2	0	0	1	0	0	2	0
Trial 6:	2	1	0	2	1	1	0	0
Trial 7:	2	1	0	0	2	2	0	2 ✔
Trial 8:	0	1	2	1	1	1	0	1
Trial 9:	0	2	2	1	0	1	1	0
Trial 10:	2	1	1	0	2	1	0	0

g) The digits 0, 1, and 2 can be used to represent the 3 envelopes with 2 representing the envelope with $5,000. Use the command randInt(0,3,8) to generate sets of 8 numbers. A trial is a set of 8 numbers and is successful if 4 of the digits are 2. Shown at the left are 10 trials with 2 successes (see ✔ marks). Thus, based on this simulation,

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

Trial 1:	46	40	83	50	94	15	✔
Trial 2:	65	97	83	95	24	45	
Trial 3:	35	27	69	53	32	61	
Trial 4:	60	24	13	38	100	69	✔
Trial 5:	57	75	57	99	82	60	✔
Trial 6:	33	47	32	41	60	5	
Trial 7:	94	97	81	76	90	40	
Trial 8:	49	92	80	68	13	29	✔

COMMENTS

12 (continued.)
the experimental probability of exactly 4 people winning $5000 is $\frac{2}{10}$.

Note: The theoretical probability that exactly 4 people out of 8 people will select the envelope with $5000 is .17 to 2 decimal places.

h) Randomly generate sets of 6 whole number percents from 1 to 100 using the command: randInt(1,100,6). Let the numbers from 1 through 85 represent the in-state students and 86 through 100 represent the out-of-state students. A trial is 1 set of 6 numbers, and a trial is successful if exactly one of the numbers is from the set 86 through 100. Four of the 8 trials shown at the left are successes (see ✔ marks); hence, the experimental probability that exactly 1 student will be from out-of-state is $\frac{4}{8}$ = .5.

The theoretical probability that only 1 of 6 randomly selected students will be from out-of-state is .40 to the nearest 2 decimal places.

i) Use the command randInt(1,365) to randomly generate the whole numbers from 1 through 365. A trial is complete when the same number repeats. For example, the following 4 trials required, respectively, 48, 37, 17, and 15 numbers before obtaining the same number twice.

Trial 1:	267	98	42	220	323	27	280	166	54
	350	44	69	234	275	67	205	191	272
	152	228	18	34	358	167	35	256	237
	95	71	97	305	295	273	193	101	118
	41	360	30	355	132	57	347	296	243
	326	157	18						

Trial 2:	298	276	28	221	220	174
	278	173	232	322	271	262
	226	80	360	125	184	242
	39	306	233	350	22	20
	9	350	64	189	328	32
	158	339	253	247	46	197
	220					

Trial 3:	56	20	128	273	122	108	204	340	127
	292	311	132	326	315	149	161	127	

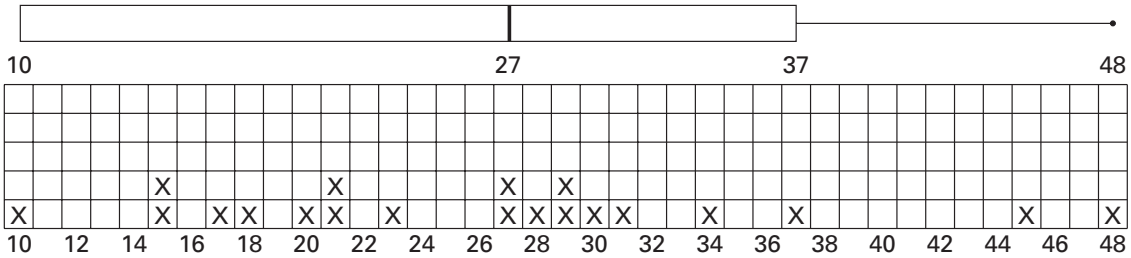
Trial 4:	281	291	304	255	102	245	198	271	56
	277	235	168	74	8	235			

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

The following line plot and 90% box plot show the data for 20 trials. From this data, the mean number of people required in order to obtain 2 people with a birthday on the same day is 26.25 and the median is 27. The 90% box plot for this simulation shows, with 90% confidence, that 2 people will have the same birthday if 37 or fewer are randomly chosen.



Surprisingly, it can be proven that for 23 people the probability that 2 or more will have a birthday on the same day is just over 50%.

13 Place a transparency of Focus Master C on the overhead. Arrange the students in pairs and have the pairs play the Marker Game once. Clarify the game rules as needed. Then ask the students for their predictions as to which player has the better chance of winning. Discuss.

Next ask each pair of students to design and carry out a simulation with at least 20 trials to determine the experimental probability of the game being won by the drawer. Discuss their conclusions and reasoning.

Lesson **17** Simulations and Probability
Focus Master C

Marker Game

A sack is filled with 5 red game markers and 5 green game markers. One player called the *holder*, holds the sack and the other player called the *drawer*, selects 2 markers at a time. The drawer earns a point if both markers have the same color, and the holder earns a point if the 2 markers have different colors. The drawer continues selecting 2 markers at a time, with the drawer and holder earning points as described above, until all the markers have been selected. The player with the most points wins the game.

13 Predicting which player has the advantage of winning in this game is not intuitive, and the theoretical probability is difficult to determine.

Some students may erroneously reason that the drawer has a $\frac{2}{3}$ chance of winning because either the 2 markers will both be red, both be green, or the colors will be different.

Students may wish to play this game using colored markers to become familiar with the game; however, you might remind them that playing the actual game is not a simulation of the conditions.

One way to simulate this game is to use the command randInt(0,9,2) to select pairs of digits from 0 through 9. Let the digits 0-4 represent red markers and 5-9 represent green markers. The game involves selection "without replacement," so if a digit occurs more than once in the data, pairs containing the repeated digit are disregarded. The following 2 trials show data from 2 simulated games. In each trial, the drawer wins 2 points and the holder wins 3 points. Notice that 7 pairs of numbers were selected in Trial 2, since 2 pairs included a digit that had already been selected.

(Continued next page.)

Focus Teacher Activity (cont.)

ACTIONS

COMMENTS

13 (continued.)

Trial 1		Trial 2	
(1,5)	red, green	(4,0)	red, red
(9,0)	green, red	(1,8)	red, green
(4,2)	red, red	~~(8,0)~~	invalid
(3,8)	red, green	(7,9)	green, green
(6,7)	green, green	(5,3)	green, red
		~~(6,4)~~	invalid
		(2,6)	red, green

Repeating this experiment for many trials should show that the holder has the greater probability of winning.

Some students may want to investigate this game for different numbers of markers. For example, for 2 red markers and 2 green markers, the theoretical probability of the holder winning is $\frac{2}{3}$. For 3 red and 3 green, the theoretical probability of the holder winning is $\frac{3}{5}$. Simulations with many trials should produce experimental probabilities which are close to these theoretical probabilities.

Follow-up Student Activity 17.4

NAME _____ **DATE** _____

1 Record your methods and results for each of the following.

a) An electronic lock has digits 0-9, and the code for the lock is 2 3-digit numbers. Randomly generate codes for 5 locks.

b) At a raffle 217 tickets are sold, numbered 1-217. Eight winning numbers are randomly selected. Randomly generate 5 sets of 8 winning numbers.

c) A scientist is studying the directions in which wild animals move and needs to randomly select 12 angles that vary from 0° to 360°. Randomly generate 3 sets of 12 angles.

d) A professional basketball player has a free-throw average of 87% (i.e., 87% of his free throws are successful). Simulate 4 sets of 20 free throws and record his free-throw percentage for each set.

2 Design simulations to solve the following problems. For each problem, explain your simulation procedures, show at least 20 trials, and give statistical evidence to support your answer to the problem.

a) At the school carnival, anyone who correctly predicts 8 or more tosses out of 10 tosses of a coin wins a prize. Luise practiced at home and determined she can predict a coin toss 72% of the time. What is the probability she will win a prize at the school carnival?

b) A baseball player's batting average is the probability of getting a hit each time the player goes to bat. For example, a player with a batting average of .245 has a probability of 24.5% of getting a hit. If a player with a .293 batting average bats 4 times in a game, what is the probability of the player getting 2 or more hits?

c) A newly married couple would like to have a child of each gender. Assuming that the probability of having a girl is 50% and the probability of having a boy is 50%, what is the average number of children the couple must have in order to be 90% certain of having at least 1 girl and 1 boy?

(Continued on back.)

Follow-up Student Activity (cont.)

d) Determine the experimental probability of obtaining a 3 at least twice if a standard die is tossed 5 times.

e) How many times must 2 dice be tossed to be 90% certain of obtaining a sum of 10 or greater?

f) Each box of a certain Kandy Korn contains either a super-hero ring or a super-hero belt buckle. If $\frac{1}{3}$ of the boxes contain a ring and $\frac{2}{3}$ of the boxes contain a belt buckle, what is the probability that a person who buys 3 boxes of Kandy Korn will receive both a ring and a belt buckle?

g) At Kidville Day Care 28% of the children are from 1-child families. How many children must be randomly selected to be 80% certain of obtaining 2 students from 1-child families? to be 85% certain? to be 90% certain?

h) Three 6th graders, two 7th graders, and two 8th graders have been chosen by the student body to receive awards at a school assembly. The principal will randomly select from these awardees to determine the order in which they receive their awards. What is the probability that the first 3 students selected will contain 1 student from each of the 3 grades?

3 Write a letter to Heather, a student from another school, who was absent during all of this lesson. In your letter, explain the following to Heather:

a) the meaning and purpose of a simulation;

b) key points in the design of a simulation;

c) tips for carrying out a simulation;

d) suggestions for analyzing simulation data to solve a problem.

Connector Master A

Table of Random Numbers

16877	62796	31056	59269	96725	03510	19186	41672	73387	79284	45697	83702
04419	47911	71787	71696	16243	94507	34229	64227	74719	92763	26120	26676
75939	36554	94366	76143	35655	30214	14796	43156	80044	29719	87905	90690
95567	44275	71147	14808	18677	88968	07326	79537	54169	08392	94501	75642
76431	76938	15066	69132	24674	37084	86339	01514	01943	47018	43849	29406
75407	87829	99305	08420	02584	07888	34234	18388	90287	10718	43386	56457
55762	77741	78263	50907	73016	04595	52029	67789	65254	15750	61520	77606
41403	72567	47625	39432	74864	80303	63257	16798	48436	34737	76541	37816
59269	94176	30216	29459	70063	47051	23187	78262	02884	39857	79393	85013
38793	30586	26783	98341	01379	89414	59707	53130	18896	11211	59959	56900
44722	22398	08182	59445	18743	51183	43850	83134	34802	56327	85775	50176
76792	15413	21246	62483	85973	31468	84551	79229	23694	41874	53784	51870
54146	99549	03892	90288	91541	15296	41359	79732	47707	32861	08284	09301
13258	98825	01224	81597	92304	02093	92229	15847	93575	09709	08284	32880
10017	95166	51510	07391	88433	15653	61773	24623	64493	41353	75811	20046
59657	17253	82592	41514	35609	42019	74328	30695	38949	63331	36022	43688
41389	56877	18323	92464	94235	63498	31954	47714	28424	04041	61545	82321
41858	73549	80343	92251	65510	58671	18490	24278	86894	75956	07622	04836
19566	51618	19581	09700	24562	06142	29945	52454	25711	56299	57296	50691
83203	14552	64977	50767	69853	17417	99352	38352	70414	53469	41128	14907
25112	68883	84887	13802	66934	51070	93787	23816	33707	28622	45618	61966
58415	53175	87365	90811	21936	09650	72851	38107	72851	04268	69737	65574
34693	13723	45306	95760	75208	30848	35810	97002	56068	50981	84566	10941
24872	90442	84322	68282	39577	22272	02654	32727	39361	60659	38083	65416
41984	99543	77292	04579	72718	78427	53647	71794	86988	41346	80978	73041
75478	86379	14318	96269	44900	63619	30630	54758	23360	17874	05854	74444
30434	89355	57050	34114	21280	99919	05157	30236	54951	71782	25803	38854
68066	17895	99454	18143	53727	50943	95711	87285	16946	73275	85412	86282
67782	75800	53994	25767	87501	91511	61128	45081	65661	41952	58406	42426
83840	02103	43204	44410	09188	64132	21788	14714	24348	34371	44907	29912
39074	47327	91183	05675	20069	06647	11494	33750	57719	90226	28720	00578
11320	17425	37182	41460	58948	88640	43978	41636	59767	85239	42843	06220
20061	21591	32017	47522	15491	34360	15385	07860	29687	88036	17448	05554
54138	01873	61646	85299	58479	47410	56923	41593	51695	92199	61850	53722
03669	13358	09583	33932	59348	53359	52437	50889	99250	93626	51771	13278
69839	06921	78794	89196	23332	98505	13874	96457	89402	13575	18514	48466
15450	65441	81939	81633	69536	07936	88569	32608	20762	25883	84028	66554
44299	68481	28078	12352	19198	54445	48652	27205	91811	61883	59061	69477
04691	72461	17275	77242	64938	97750	50648	89443	56513	85940	34066	17280
45855	40677	54003	67362	03456	95489	06329	58694	87021	47689	18746	22160
58802	98687	39554	56036	38187	51985	90895	09215	14525	68772	53814	57386
86620	73916	17143	42927	10858	57750	78656	42031	73934	79573	75971	49113
09682	00755	70561	45131	16440	14107	17075	78385	64347	51414	56677	97189
99583	07952	12497	58162	33850	93029	49328	87580	50232	71255	82129	39595
01346	62215	94291	63462	06520	27938	83438	41685	59481	39404	49188	44704

Connector Master B

For each of the following, use a calculator to generate the random numbers that are described. Then record the calculator commands that you enter and the data generated by the commands.

a) Last month Lakeway Theater sold tickets numbered from 372500 to 374000. The theater manager plans to give free passes to the first 5 numbers that are randomly selected from these numbers.

b) To practice multiplication facts, each day Rachelle randomly generates a set of 25 pairs of whole numbers, where each number in a pair is randomly selected from the whole numbers 2 through 12.

c) Some combination locks are designed for use with any 3 numbers from 0 to 45. When these locks are manufactured, the numbers used in each combination are selected at random.

d) In a drawing for 2 CD players, tickets are numbered from 41 through 80. The 1st 2 ticket numbers that are randomly selected and divisible by 3 are the winning numbers.

e) Suzanne's teacher asked her to randomly generate a set of even numbers less than 100 and to select the first 3 that are divisible by 5.

f) A researcher randomly selects 15 numbers from the following set: {1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, … 38, 38.5, 39, 39.5, 40}. Then she computes the mean and the sum of the 15 numbers, and she makes a box plot of 15 numbers.

g) Joey randomly selected 3 odd numbers greater than 100 and less than 140.

Connector Student Activity 17.1

NAME _____ **DATE** _____

1 Discuss possible methods for creating a list of 100 random digits.

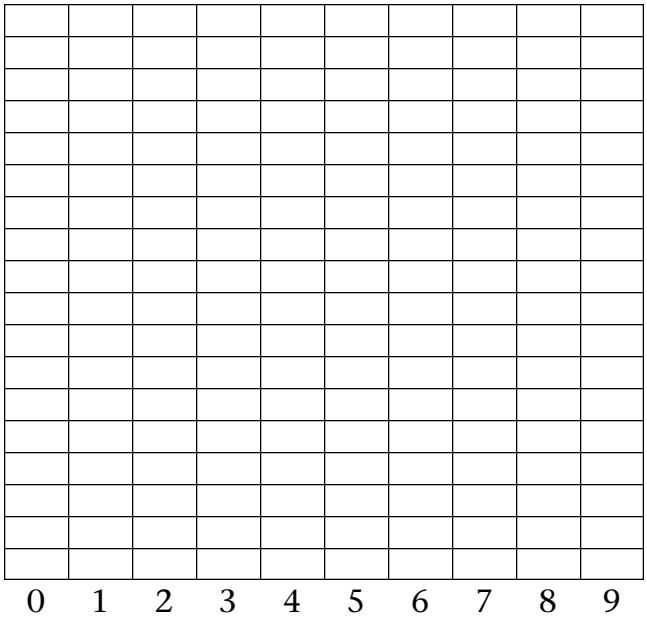
2 Choose a method discussed for Problem 1 to generate a list of 100 random digits.

a) Describe your method of generating 100 random digits:

b) Record your 100 digits, in the order generated, in this table:

—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

c) Make a line plot of your 100 digits here:



3 Do you feel that your set of 100 digits is random? List some conditions that you feel a set of digits should satisfy in order to be considered a set of random digits?

Focus Master A

Blood Type Problem

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

Cereal Box Problem

As a special promotion, the Crunchy-Crispy Cereal Company includes a baseball trading card in each box of cereal. There are 6 different trading cards, and each cereal box contains exactly 1 of the 6 cards. How many boxes of cereal would you expect to buy in order to collect all 6 trading cards?

Principal Problem

Mount Hood Middle School has 3 classes of 7th graders: 25 students in Class I, 30 in Class II, and 22 in Class III. If 5 of the 77 students are randomly selected to meet with the principal, what is the probability that at least 1 student from each class will be in the meeting?

Focus Master B

Solve the following problems by designing and carrying out simulations.

a) Each box of Pops-a-Lot Popcorn contains 1 of 7 different colored pens. How many boxes of popcorn would you purchase in order to be 90% certain that you would obtain a complete set of all 7 colors?

b) Based on his past archery records, the probability that Eric will hit the bulls-eye of a target is .94. If Eric takes 8 shots at the target, what is the probability he will hit the bulls-eye exactly 7 times?

c) Assume that the probability of a randomly selected person having a birthday in a given month is $1/12$.

i) How many people, on the average, would you need to select to be 90% certain that 2 of them will have a birthday in the same month?

ii) If 8 people are randomly selected, what is the probability that at least 3 will have a birthday in the same month?

d) Two students are playing a coin-tossing game. Each player tosses a coin until obtaining 3 heads or 3 tails in a row. The player who requires the fewest number of tosses wins the game. How many tosses of a coin are required on the average to obtain 3 heads or 3 tails in a row?

e) Hoopersville Hospital uses 2 tests to classify blood. Every blood sample is subjected to both tests. Test A correctly identifies blood type with probability .7 and Test B correctly identifies blood type with probability .5. Determine the probability that at least 1 of the tests correctly determines the blood type.

(Continued on back.)

Focus Master B (cont.)

f) The names of 5 people (all with different names) are placed on 5 separate slips of paper and these slips are placed in a sack. If each person randomly chooses a slip from the sack, on the average, how many people will select their own name?

g) On a quiz show, contestants guess which 1 of 3 envelopes contains a $5000 bill. What is the probability that exactly 4 people out of 8 contestants will select the envelope with $5000?

h) At a certain university it is required that 85% of the students be from within the state. If 6 students are randomly selected from this university's student body, what is the probability that exactly 1 of them will be from outside the state?

i) Assume that the probability of a randomly chosen person having a birthday on a given day of the year is $1/365$. How many people, on the average, would you need select in order to be 90% certain of obtaining exactly 2 people with a birthday on the same day?

Focus Master C

Marker Game

A sack is filled with 5 red game markers and 5 green game markers. One player called the *holder*, holds the sack and the other player called the *drawer*, selects 2 markers at a time. The drawer earns a point if both markers have the same color, and the holder earns a point if the 2 markers have different colors. The drawer continues selecting 2 markers at a time, with the drawer and holder earning points as described above, until all the markers have been selected. The player with the most points wins the game.

Focus Student Activity 17.2

NAME _____ **DATE** _____

Blood Type Problem:

If, on the average, 2 out of every 5 people have blood type O, and 3 people are randomly selected, what is the probability that exactly 2 of these 3 people will have blood type O?

Design and carry out a simulation to answer the above Blood Type Problem. Collect data for *a minimum of 25 trials*, where a trial is the blood type information for a set of 3 people. On separate paper:

a) Describe, in detail, your simulation procedures.

b) Show all of the data that you collect.

c) Show how you organize, graph, and analyze your data to answer the Blood Type Problem. Be sure to support all conclusions with sound mathematical reasoning and a variety of mathematical evidence. If you form graphs or compute statistics on the calculator, sketch and label the results and include them in your written arguments.

d) Challenge. Determine and explain the theoretical probability for the Blood Type Problem. Show/explain your methods and reasoning so that it is clear why your answer is correct.

Focus Student Activity 17.3

NAME _____ **DATE** _____

Cereal Box Problem

As a special promotion, the Crunchy-Crispy Cereal Company includes a baseball trading card in each box of cereal. There are 6 different trading cards, and each cereal box contains exactly 1 of the 6 cards. How many boxes of cereal would you expect to buy in order to collect all 6 trading cards?

Design and carry out 2 simulations to answer the above problem so that one simulation involves the use of a random number function on a graphing calculator or a Table of Random Numbers, and the other simulation does not involve these methods. For each simulation, collect data for a minimum of 20 trials, where one trial consists of selecting until all 6 trading cards are obtained (i.e., the number of selections required for a trial will vary from trial to trial).

a) For each simulation, describe in detail your simulation procedures.

b) For each simulation, show all of the data that you collect.

c) For each simulation, show how you organize, graph, and analyze your data and use the results to answer the Cereal Box Problem. On a scale of 1-100, how confident are you in each solution? Explain.

d) Show how you combine the data from your 2 simulations and analyze the results. Based on this information, now what is your solution to the Cereal Box Problem? On a scale of 1-100, how confident are you now in your solution? Explain.

e) Show how you analyzed the combined data from all groups in your class. Discuss any adjustments this leads you to make in your solution from d). On a scale of 1-100, now how confident are you in your prediction?

Follow-up Student Activity 17.4

NAME _____ **DATE** _____

1 Record your methods and results for each of the following.

a) An electronic lock has digits 0-9, and the code for the lock is 2 3-digit numbers. Randomly generate codes for 5 locks.

b) At a raffle 217 tickets are sold, numbered 1-217. Eight winning numbers are randomly selected. Randomly generate 5 sets of 8 winning numbers.

c) A scientist is studying the directions in which wild animals move and needs to randomly select 12 angles that vary from 0° to 360°. Randomly generate 3 sets of 12 angles.

d) A professional basketball player has a free-throw average of 87% (i.e., 87% of his free throws are successful). Simulate 4 sets of 20 free throws and record his free-throw percentage for each set.

2 Design simulations to solve the following problems. For each problem, explain your simulation procedures, show at least 20 trials, and give statistical evidence to support your answer to the problem.

a) At the school carnival, anyone who correctly predicts 8 or more tosses out of 10 tosses of a coin wins a prize. Luise practiced at home and determined she can predict a coin toss 72% of the time. What is the probability she will win a prize at the school carnival?

b) A baseball player's batting average is the probability of getting a hit each time the player goes to bat. For example, a player with a batting average of .245 has a probability of 24.5% of getting a hit. If a player with a .293 batting average bats 4 times in a game, what is the probability of the player getting 2 or more hits?

c) A newly married couple would like to have a child of each gender. Assuming that the probability of having a girl is 50% and the probability of having a boy is 50%, what is the average number of children the couple must have in order to be 90% certain of having at least 1 girl and 1 boy?

(Continued on back.)

Follow-up Student Activity (cont.)

d) Determine the experimental probability of obtaining a 3 at least twice if a standard die is tossed 5 times.

e) How many times must 2 dice be tossed to be 90% certain of obtaining a sum of 10 or greater?

f) Each box of a certain Kandy Korn contains either a super-hero ring or a super-hero belt buckle. If $\frac{1}{3}$ of the boxes contain a ring and $\frac{2}{3}$ of the boxes contain a belt buckle, what is the probability that a person who buys 3 boxes of Kandy Korn will receive both a ring and a belt buckle?

g) At Kidville Day Care 28% of the children are from 1-child families. How many children must be randomly selected to be 80% certain of obtaining 2 students from 1-child families? to be 85% certain? to be 90% certain?

h) Three 6th graders, two 7th graders, and two 8th graders have been chosen by the student body to receive awards at a school assembly. The principal will randomly select from these awardees to determine the order in which they receive their awards. What is the probability that the first 3 students selected will contain 1 student from each of the 3 grades?

3 Write a letter to Heather, a student from another school, who was absent during all of this lesson. In your letter, explain the following to Heather:

a) the meaning and purpose of a simulation;

b) key points in the design of a simulation;

c) tips for carrying out a simulation;

d) suggestions for analyzing simulation data to solve a problem.

Materials

NECESSARY MANIPULATIVES AND MATERIALS

INSTRUCTIONAL MATERIALS

Math Alive! Visual Mathematics Course III (Lessons 1-30)
Starting Points for Implementing Visual Mathematics
Math Alive! Visual Mathematics Course III Blackline Masters

Blackline Masters include Student Activities Grids.

MAKE FROM BLACKLINE MASTERS OR PURCHASE FROM MLC

Algebra Pieces
1 set per 4 students

Algebra Pieces, Overhead
1 set per teacher

Bicolored Counting Pieces, Overhead
1 set per teacher

MATERIALS AVAILABLE FROM MLC

Cubes, Wood (3/4" or 2-cm)
30 per student

Dice, 6-sided
1 die per student

Game Markers
1 bag per 15 students

Modeling Clay
1 box per 16 students

Coffee Stirrers
15 per student

Rulers (metric/imperial)
1 per student

Lunch Bags
1 per student

Overhead Pens
1 set per teacher

Protractors

Visit catalog.mathlearningcenter.org to order these and other materials.

MATERIALS NOT AVAILABLE FROM MLC

Blank Transparencies
Poster Paper
Tape, Glue, Glue Sticks
Colored Markers
Scissors
Graphing Calculators
Compasses
String or Yarn

Hamburger Patty Paper
6" square sheets of translucent waxed tissue like that used between frozen hamburger patties. About 4,000 sheets (four 1,000-sheet boxes) needed per classroom.

OPTIONAL MATERIALS

ADDITIONAL VISUAL MATHEMATICS BOOKS

Visual Mathematics, Course III Student Activities Book
Visual Mathematics Student Journal

VIDEOS FROM MLC

Math and the Mind's Eye Video
A Change of Course: Implementing Visual Mathematics in the Classroom Video

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